Centro de Investigación en Matemáticas A.C.

# Evolution families in the space of $\rho$-nonexpansive mappings and harmonic Loewner chains in $\mathbb{R}^2$

T E S I S

Que presentada como requisito parcial para obtener el grado de

**Doctor en Ciencias**

con Orientación en

**Matemáticas Aplicadas**

P R E S E N T A:

**Luis Enrique Benítez Babilonia**

Director de Tesis:

Dr. Lázaro Raúl Felipe Parada

GUANAJUATO, GTO., MÉXICO AGOSTO 2012

CIMAT

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*To*

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# Abstract

The purpose of this work is to present a version of the Loewner theory, without the requirement of analyticity of the functions belonging to the chain and the evolution families or transition functions. This approach is made in two parts.

In the first part, we introduce two new concepts of subordination in the class of harmonic univalent functions. Then, we introduce two new types of harmonic Loewner chains according to the already given concepts of subordination. A characterization for one type of these harmonic Loewner chains is presented. The compactness of this family of harmonic Loewner chains is proved. For the other type of harmonic Loewner chains an ordinary differential equation is established in a particular case. Actually, a Loewner theory for complex-valued harmonic functions, whose real and imaginary parts not necessarily satisfy the Cauchy-Riemann equations, is constructed.

The second part is taken into consideration, because enforcing that the composition of two harmonic functions is harmonic, is a very restrictive condition. Therefore, we have obtained interesting, but not very "fruitful" results. In this part, we just study the evolution families or transition functions in the class of nonexpansive functions with respect to a certain metric. We shown that these evolution families can be obtained by solving an ordinary differential equation for a certain vector field. The concept of an infinitesimal generator for these evolution families is given. Some characteristics of such infinitesimal generators are established. The nonlinear resolvent for a kind of functions is treated. An ordinary differential equation, which is satisfied by these evolution families, is obtained with some additional assumptions.

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# Introduction

The relationship between embedding simply-connected regions in the complex plane, the Hele-Shaw free boundary problem, Laplacian growth and their applications, is well-known [1]. In fact, The Hele-Shaw free boundary problem consists of describing the evolution of a region occupied by a fluid that is surrounded by another fluid, both fluids in a Hele-Shaw cell [39]. For incompressible fluids in addition with Darcy's law imply that the pressure of the fluid is a harmonic function. It is worth recalling that a Hele-Shaw cell consists of two flat plates that are parallel to each other and separated by a small distance. At least one of the plates is transparent and has a hole. On the other hand, Laplacian growth is a process in which the growth of the domain is governed by a harmonic function, with appropriate boundary conditions. Nowadays, the Hele-Shaw cell is widely used as a powerful tool in several fields of natural sciences and engineering, for example, material science, crystal growth and, of course, fluid mechanics [29, 30, 39]. For example, one of the basic manufacturing processes used in the plastics industry is injection moulding.

Another kind of domain evolution is related to the well-known Loewner theory. This theory was developed in 1923 as a tool to embed a slit domain of the complex plane into a family of domains also in the complex plane with a certain order. We are going to recall some basic facts, and recent contributions about this growing theory below. For example, an interesting application is the Stochastic Loewner Evolution (SLE) introduced by O. Schramm, replacing the driving term in the radial and chordal Loewner equation with a Brownian motion. The Stochastic Loewner Evolution was used to prove the Mandelbrot conjecture about the fractal dimension of a planar Brownian motion [22, 23], more applications can be found in [1].

All these applications motivated this work, as well as the theoretical aspects. For example, the connection with integrable systems and other branches of mathematics such as differential equations, potential theory, variational calculus, et al [4, 18, 24, 26, 35, 45, 46].

Charles Loewner began his research in the theory of conformal mappings, then he focussed on the composition semigroups of conformal mappings. At this point, he introduced the most celebrated result of his investigation, the now well-known Loewner parametric methods and the well-known Loewner differential equations. Later, P. P. Kufarev [21] and C. Pommerenke [34] fully developed the original theory. We believe that this theory was developed in order to solve the Bieberbach conjecture, which states or provides estimates for the Taylor coefficients $a_n$ of functions $f : D = \{|z| < 1\} \longrightarrow \mathbb{C}$ in the class:

$$S = \left\{ f(z) = z + \sum_{n\geq 2} a_n z^n : f \text{ is a one-to-one analytic map in } D \right\}.$$

More explicitly, this conjecture states that $|a_n| \leq n$ for each $n \geq 2$ if $f \in S$. Loewner was able to prove that $|a_3| \leq 3$. This conjecture was proved by L. de Branges [12] in 1985 based on some ideas of the Loewner theory.

The idea of the Loewner method, was to enclose simply-connected domains $\Omega_t$ in the complex plane with a certain order to estimate the Taylor coefficients $a_n$ of each function $f$ in $S$ linearly associated to the Riemann mapping of each simply-connected domain. The Riemann mappings of a complex domain need not belong to $S$, but they can be written as $w_0 + r_0 e^{i\alpha} f$, with $f \in S$. In [34] we can find a one-to-one relationship between Loewner chains, which we denominate *classical Loewner chains*, and families of domains $\Omega_t$ satisfying:

1. $0 \in \Omega_s \subset \Omega_t$, if $0 \leq s < t < +\infty$, and

2. $\Omega_{t_n} \longrightarrow \Omega_{t_0}$, if $t_n \longrightarrow t_0 < +\infty$; $\quad \Omega_{t_n} \longrightarrow \mathbb{C}$, if $t_n \longrightarrow +\infty$, $\quad$ as $n \longrightarrow +\infty$. In the sense of the Carathéodory kernel convergence (see [34]).

The classical Loewner chains are functions $f(z,t) = e^t z + a_2(t) z^2 + \cdots$ $(z \in D, 0 \leq t < +\infty)$, analytic and one-to-one in $D$ for each $t \geq 0$, such that $f(z,s) \prec_{\mathscr{H}} f(z,t)$, if $0 \leq s \leq t < +\infty$, i.e., $f(z,s)$ is subordinate to $f(z,t)$, if $0 \leq s \leq t < +\infty$ (see Definition 1.3.1). The simplest example of classical Loewner chain is $f(z,t) = e^t z$ with $z \in D$, and $0 \leq t < +\infty$.

These classical Loewner chains can be described by a differential equation, the well-known Loewner-Kufarev equation. Also, there are close connections between the classical Loewner chains and the functions which make the subordination, sometimes called transition functions or the evolution family (see Theorem 1.2.2). Moreover, the class $\mathscr{P}$ consisting of analytic functions of positive real part, normalized by $P(0) = 1$, is also related with the classical Loewner chains (see Theorem 1.2.1). On the other hand, the transition functions, or the evolution families, satisfy a semigroup property, which Loewner used as the starting point of his theory.

Now, we are going to recall some facts about semigroups of analytic functions, which are related to the classical Loewner theory and recent works about this topic.

A one-parameter semigroup in the class of functions defined on $U$, $\mathscr{F}$ closed under the composition, is a mapping $t \in \mathbb{R}^* \longrightarrow \phi_t \in \mathscr{F}$, with $\mathbb{R}^* = \mathbb{R}^+ \cup \{0\}$, that satisfies the following three conditions: (see [5])

S1. $\phi_0$ is the identity in $\mathscr{F}$,

S2. $\phi_s \circ \phi_t = \phi_{s+t}$, if $s,t \geq 0$,

S3. For each $x_0 \in U$, the function $\phi_t(x_0)$ is continuous on $\mathbb{R}^*$.

In [5], Berkson and Porta proved that every semigroup of analytic functions in an open set $U \subset \mathbb{C}$ has an infinitesimal generator. This means that if $\{\phi_t\}$ is a semigroup of analytic functions in $U$, then there exists an analytic function $G$, which is called the infinitesimal generator of $\{\phi_t\}$, such that

$$\frac{\partial \phi(z,t)}{\partial t} = G(\phi(z,t)) \quad \text{for} \quad t \in \mathbb{R}^+, \quad z \in U.$$

Moreover, if $U = D$, is the unit disc then, there exist $\beta \in \overline{D}$ and $P \in \mathscr{P}$, such that

$$G(z) = (z-\beta)(\bar{\beta}z - 1)P(z).$$

In [6], the authors F. Bracci, M. Contreras and S. Díaz-Madrigal introduced the concept of $L^d$-*evolution families* with the help of an integral condition instead of the locally uniform condition used by Goryaĭnov [15] in his work with evolution families. These $L^d$-evolution families are more general than semigroups and the families used by Loewner, Kufarev and Pommerenke in their studies. The authors related such families to the Herglotz vector field of order $d \geq 1$, which have the form of the more general Berkson-Porta infinitesimal generator in $D$ [6]. On the other hand, M. Contreras, S. Díaz-Madrigal and P. Gumenyuk [11] proposed a general setting for the Loewner theory. They called these new objects *$L^d$-Loewner chains*, with $d \geq 1$, which work also on complete hyperbolic manifolds, and encloses the classical theory as a special case, when $d = +\infty$. A one-to-one correspondence between $L^d$-Loewner chains and $L^d$-evolution families was shown [11]. Furthermore, a relationship with the Herglotz functions of order $d \geq 1$ was also obtained. The Herglotz functions of order $d \geq 1$ play the role of the functions in $\mathscr{P}$. This Loewner theory has been extended to complete hyperbolic complex manifolds [2, 7] and to the higher dimensional unit ball in $\mathbb{C}^n$ [16, 17, 32, 33].

The aims of this work is to present a version of the Loewner theory, without the requirement of analyticity of the functions belonging to the chain and the evolution families or the transition functions. This goal is made in two parts. First, the functions belonging to the chain are considered to be harmonic functions and we consider two cases for the evolution families or transition functions. Second, we just study the evolution families or transition functions in the class of non-expansive functions with respect to a certain metric.

The first part is based on the work by J. Clunie and T. Sheil-Small [8], W. Hengartner and G. Schober [19], et al., [13, 14, 31, 36]. An analogue class to $S$ for the case of harmonic functions was defined in [8] and also was proved other results. In [36], some necessary and sufficient conditions for the composition of harmonic mappings to be harmonic were found. Further, in [31], the analogue to the class $\mathscr{P}$ was introduced and studied. Finally, in [19], a version of the Riemann mapping theorem for complex-valued univalent harmonic functions was proved. In Chapter 1, we develop a theory of harmonic Loewner chains following the ideas of Ch. Pommerenke [34]. First, we introduce two new concepts of subordination of harmonic functions in Section 1.3. Then, in Section 1.4, we study examples of semigroups and evolution families of harmonic functions. We finish the chapter, in Section 1.5, introducing and studying the two types of harmonic Loewner chains.

In the second part, we approach our investigation following the results about semigroups in the class $N_\rho(D)$, given in [37, 38, 41]. In Chapter 2, we introduce the concepts of evolution families of nonexpansive functions, and $\rho-$monotone weak vector fields. Also, under a certain condition, we establish a one to one relation between these two latter concepts. In Section 2.3, the evolution families of nonexpansive functions is defined. The $\rho-$monotone weak vector fields and their properties are given in Section 2.4. The infinitesimal generators of $\rho-$nonexpansive evolution families and the nonlinear resolvent is studied in Section 2.5. Finally, in Section 2.6 we establish a condition that an evolution family has an infinitesimal generator, which is a $\rho-$monotone weak vector fields. We finish this thesis with some conclusions and a list of problems that we consider as future work.

# Chapter 1

## Harmonic Loewner Chains in $\mathbb{R}^2$

### 1.1 Introduction

Let us first recall basic objects of Loewner theory. Let $f$ and $g$ be analytic in $D := \{|z| < 1\}$. We say that $f$ is subordinate to $g$ if there exists a function $\phi(z)$ analytic (not necessarily univalent) in $D$ satisfying $\phi(0) = 0$ and $|\phi(z)| < 1$, such that $f(z) = g(\phi(z))$, if $|z| < 1$. Subordination is denoted by $f(z) \prec g(z)$ [34].

The function $f(z,t)$, $z \in D$, $0 \leq t < +\infty$, is called a classical Loewner chain if

$$f(z,t) = e^t z + a_2(t)z^2 + a_3 z^3 + \ldots, |z| < 1, \tag{1.1}$$

is analytic and univalent in $D$ for each $t \in [0, +\infty)$, and if $f(z,s) \prec f(z,t)$ whether $0 \leq s \leq t < +\infty$ (subordinated) [34].

The subordination condition means that there exists a function $\varphi(z,s,t)$, sometimes called a transitive function or an evolution family, such that $f(z,s) = f(\varphi(z,s,t),t)$, $0 \leq s \leq t < +\infty$. The transitive function $\varphi(z,s,t) = e^{s-t}z + \alpha_2(s,t)z^2 + \ldots$ is univalent in $D$ and satisfies $|\varphi(z,s,t)| < |z|$. Moreover, it satisfies a semigroup property: $\varphi(z,s,\tau) = \varphi(\varphi(z,s,t),t,\tau)$. See [34] for more details.

In this chapter, we present a first attempt to a harmonic setting for the Loewner theory. This construction is made in two parts, according to two new concepts of subordination of harmonic mappings. In the first part, the transitive function is considered to be an analytic function. In the second part, the functions are harmonic functions. We start in Section 1.2 with definitions, notation and results that will be used throughout this work. Particularly, we recall two theorems, 1.2.1 and 1.2.2, which characterize the classical Loewner chains by means of a differential equation and the transitive function or evolution family associated to them, respectively. In Section 1.3, two new concepts about subordination are introduced. A characterization for each case is also given here. The definition of conjugation is given. Two results which generalize Theorems 1.2.1 and 1.2.2 are proved. The section finishes with the study of composition of harmonic functions using Dirichlet and Neumann problems. Section 1.4 is devoted to examples of semigroups and evolution families of harmonic functions. In Section 1.5, the concept of classical Loewner chains is extended to the case of harmonic functions. A partial differential equation, which is satisfied by this family of functions, is established. The compactness of this new class of functions is proved.

## 1.2 Preliminary Results

We start by introducing some essential facts as well as the notation that will be used in the remaining sections.

The complex-valued harmonic functions in a domain $U$ (open and connected), i.e., functions $f : U \longrightarrow \mathbb{C}$ such that $f = u + iv$ and $\Delta f = \Delta u + i\Delta v = 0$ in $U$, are our main object of study. We are differing from the analytic case, because we are omitting the Cauchy-Riemann equations. So, we denote the set of complex-valued harmonic functions in $U$ by $\mathscr{A}(U,\mathbb{C})$, whereas $\mathscr{H}(U,\mathbb{C})$ will denote the set of complex-valued analytic functions in $U$. If $U$ is a simply-connected domain we can write (see [8, 13])

$$\mathscr{A}(U,\mathbb{C}) = \mathscr{H}(U,\mathbb{C}) \oplus \bar{\mathscr{H}}^0(U,\mathbb{C}), \tag{1.2}$$

where we denote $\bar{\mathscr{H}}^0(U,\mathbb{C}) := \{\overline{\omega} : U \longrightarrow \mathbb{C} : \omega \in \mathscr{H}(U,\mathbb{C}),\ \omega(0) = 0\}$ and $\bar{\omega}$ denotes the function $z \longrightarrow \overline{\omega(z)}$. The decomposition (1.2) means that if $f \in \mathscr{A}(U,\mathbb{C})$ then $f = h + \bar{g}$ where $h, g$ are analytic. From a result of Lewy (see [13]) it can be shown that $f \in \mathscr{A}(U,\mathbb{C})$ is locally one-to-one and sense-preserving if and only if the Jacobian of $f$ is positive, i.e.,

$$J_f(z) = |h'(z)|^2 - |g'(z)|^2 > 0.$$

We call such mappings *locally univalent*, and we say that $f$ is *univalent* in $U$ if $f$ is one-to-one and sense-preserving in $U$. If $f = h + \bar{g}$ we call $h$ and $\bar{g}$ the *analytic part* and *co-analytic part* of $f$, respectively.

We finish this section by recalling two well-known theorems. The first theorem relates the classical Loewner chains with the Loewner-Kufarev partial differential equation. The second theorem establishes the Loewner-Kufarev ordinary differential equation. Moreover, the evolution family associated to the classical Loewner chain satisfies this ordinary differential equation. Furthermore, that the classical Loewner chains can be obtained form this family by solving an initial value problem (IVP) is established.

**Theorem 1.2.1** ([34]). *The function $f(z,t)$ is a classical Loewner chain if and only if the following two conditions are satisfied:*

1. *There exist $r_0 > 0$ and $K_0 > 0$ such that the function $f(z,t) = e^t z + a_2(t)z^2 + \cdots$ is analytic in $|z| < r_0$ for each $t \geq 0$, absolutely continuous in $t \geq 0$ for each $|z| < r_0$ and satisfies*

$$|f(z,t)| \leq K_0 e^t \quad |z| < r_0,\ t \geq 0.$$

2. *There exists a function $P(z,t)$ analytic in $D$ and measurable in $t \geq 0$ that satisfies $Re(P) > 0$, and such that*

$$\frac{\partial f(z,t)}{\partial t} = z \frac{\partial f(z,t)}{\partial z} P(z,t); \qquad |z| < r_0, \quad \text{for almost every } t \geq 0. \tag{1.3}$$

Equation (1.3) is called the linear Loewner-Kufarev partial differential equation.

**Theorem 1.2.2** ([34]). *Suppose that $P(\cdot, t)$ belongs to $\mathscr{P}$ for each $t \geq 0$ and is measurable in $t \geq 0$. Then for $z \in D$ and $s \geq 0$, the initial value problem*

$$\begin{cases} \dfrac{\partial W}{\partial t} = -WP(W,t), & \text{for almost every } t \geq s \\ W(s) = z. \end{cases} \tag{1.4}$$

*has a unique absolutely continuous solution $W(t) = \varphi(z;s,t)$. Moreover, the functions $\varphi(z;s,t)$ are* *univalent in $z \in D$ and*

$$f(z,s) := \lim_{t\to\infty} e^t \varphi(z;s,t) \quad z \in D, s \geq 0, \tag{1.5}$$

*exists locally uniformly in $D$ and is a classical Loewner chain satisfying (1.3).*

*Conversely, if $f(z,t)$ is a classical Loewner chain and $\varphi(z;s,t)$ is determined by $f(z,s) = f(\varphi(z;s,t),t)$, then $W(t) = \varphi(z;s,t)$ is a solution of (1.4) and (1.5) is satisfied.*

Equation in (1.4) is also called Loewner-Kufarev equation. We recommend to see [34] for more details about these two theorems, where the reader can also find their proofs.

## 1.3 New concepts of subordination

In this section, we introduce two new types of subordination. Hence, the section is divided in two subsections. In the first subsection, we consider how the subordination of two harmonic functions is made by means of an analytic function. In the second subsection, we use the Dirichlet problem and Neumann problem in order to find classes of harmonic functions which are closed under composition. Then, we define a type of subordination using such classes.

### 1.3.1 $\mathscr{H}-$Subordination

This subsection begins with a definition of subordination between harmonic functions which need not be analytic.

**Definition 1.3.1.** *Let $f,g : D \longrightarrow \mathbb{C}$ be complex-valued harmonic functions. We say that $f$ is* $\mathscr{H}-$subordinate *or* analytically subordinate *to $g$ if:*

*There exists an analytic function $\phi : D \longrightarrow D$ satisfying $\phi(0) = 0$, such that*

$$f(z) = g(\phi(z)), \quad z \in D.$$

*This fact will be denoted by $f(z) \prec_{\mathscr{H}} g(z)$.*

**Example 1.** *A example of $\mathscr{H}-$subordination is the classical subordination between analytic functions.*

**Example 2.** *If $S$ is an injective analytic function, then $f(z) \prec_{\mathscr{H}} g(z)$ implies*

$$(f \circ S)(z) \prec_{\mathscr{H}} (g \circ S)(z), \quad (f \circ S^{-1})(z) \prec_{\mathscr{H}} (g \circ S^{-1})(z).$$

*In particular, this holds for the mappings*

$$S_\beta(z) = \frac{z-\beta}{1-\bar{\beta}z}, \qquad S_\beta^{-1}(w) = \frac{w+\beta}{1+\bar{\beta}w}, \quad \beta \in D, \tag{1.6}$$

*which are analytic automorphisms of $D$.*

**Example 3.** *Let us suppose $a,b,c,p,q,r \in \mathbb{C}$. Let us consider*

$$f(z) = az + c + b\bar{z}, \text{ and } w = g(z) = pz + r + q\bar{z}, \text{ with } |p|^2 - |q|^2 \neq 0.$$

*We can find the inverse function of $g$ by solving a system of algebraic equations, and we get*

$$g^{-1}(w) = \frac{\bar{p}w + (q\bar{r} - \bar{p}r) - q\bar{w}}{|p|^2 - |q|^2}.$$

*Thus, for the composition $g^{-1} \circ f$, we obtain*

$$\phi(z) = g^{-1}(f(z)) = \frac{(a\bar{p} - \bar{b}q)z + (b\bar{p} - \bar{a}q)\bar{z} + \bar{p}(c-r) - q\overline{(c-r)}}{|p|^2 - |q|^2}.$$

*Hence, the last mapping is analytic if and only if $b\bar{p} - \bar{a}q = 0$.*

Now, we proceed with a lemma which gives a condition that generalizes Example 3.

**Lemma 1.3.1.** *Let $f$, $g$ be harmonic in $D$ and such that $g^{-1} \circ f$ is well defined and differentiable on $D$. Then, $\phi = g^{-1} \circ f$ is analytic if and only if*

$$f_2'(z) = \frac{g_2'(z)}{g_1'(z)} f_1'(z), \quad z \in D, \tag{1.7}$$

*where $f = f_1 + \overline{f_2}$ and $g = g_1 + \overline{g_2}$ (according to the decomposition (1.2)).*

*Proof.* Let us suppose that $f$ and $g$ satisfy all the conditions in order for $\phi := g^{-1} \circ f$ to be a well defined and differentiable function. Then, if $w = f(z)$, we have

$$\partial_{\bar{z}}\phi = \partial_w g^{-1}\partial_{\bar{z}} f + \partial_{\bar{w}} g^{-1}\partial_{\bar{z}}\bar{f} = \partial_w g^{-1}\overline{f_2'} + \partial_{\bar{w}} g^{-1}\overline{f_1'}. \tag{1.8}$$

On the other hand, by solving a system of algebraic equations and using the Inverse Function Theorem, see for example [13] page 146, we obtain

$$\partial_w g^{-1}(w) = \frac{\overline{g_1'}}{|g_1'|^2 - |g_2'|^2} = \frac{\overline{g_1'}}{J_g(z)},$$
$$\partial_{\bar{w}} g^{-1}(w) = -\frac{\overline{g_2'}}{|g_1'|^2 - |g_2'|^2} = -\frac{\overline{g_2'}}{J_g(z)}.$$

Then, by replacing the last two expressions in (1.8), we have

$$\partial_{\bar{z}}\phi = \frac{\overline{g_1' f_2'} - \overline{g_2' f_1'}}{|g_1'|^2 - |g_2'|^2}. \tag{1.9}$$

Hence, Equation (1.9) implies Lemma 1.3.1. □

In [19], the Riemann mapping theorem in the harmonic case is studied. This theorem stablishes the existence, under additional conditions, of a harmonic mapping for a given domain. This mapping can be considered as the harmonic Riemann mapping for such a domain, and satisfies a similar equation as (1.7). Hence, the existence of functions satisfying (1.7) is assured.

On the other hand, it is not difficult to see that if $\Omega_0 \subset \mathbb{C}$ is a simply-connected domain with $g(0) \in \Omega_0 \subset g(D)$ for a given harmonic univalent function $g$ then, there is at least one harmonic function $f(z)$ with $f(D) = \Omega_0$, such that $f(z) \prec_{\mathscr{H}} g(z)$ (See [8] page 9). For a more general case we have the following proposition.

**Proposition 1.3.2.** *Let $g$ be a univalent harmonic in $D$. Then $f(z) \prec_{\mathscr{H}} g(z)$ if and only if*

1. $f(D) \subset g(D)$, $f(0) = g(0)$,
2. *Equation (1.7) holds.*

*Proof.* If we assume that $f(z) \prec_{\mathscr{H}} g(z)$ then, $\phi(D) \subset D$ and $\phi(0) = 0$, for some analytic function

$\phi$. Thus, it follows that $f(D) = g(\phi(D)) \subset g(D)$, $f(0) = g(0)$ and $\phi = g^{-1} \circ f$. Since $\partial_{\bar{z}}\phi = 0$, Equation (1.7) follows from Equation (1.9).

Conversely, since $g$ is a univalent function then, its inverse function exists $g^{-1}$. Also, since $f(D) \subset g(D)$ then, we can define the function $\phi(z) := g^{-1}(f(z))$ for $z \in D$. Evidently $|\phi(z)| < 1$ and $\phi(0) = 0$. From Equations (1.9) and (1.7), it follows that $\partial_{\bar{z}}\phi = 0$. Therefore, $\phi$ is analytic and $g(\phi(z)) = f(z)$. $\square$

Proposition 1.3.2 gives us the sufficient and necessary conditions under which the $\mathscr{H}-$subordination occurs.

**Definition 1.3.2.** *Let $f$ be a function defined on $D$. For each $\beta \in D$, the function*

$$h^{(\beta)}(z) := f(S_\beta(z)), \tag{1.10}$$

*defined for all $z \in D$ is called the* conjugation *of a given function $f(z)$, with respect to $\beta \in D$. Where $S_\beta(z)$ is given by Equation (1.6).*

Note that from the previous Definition 1.3.2, Example 2, we have that if $f_1(z) \prec_{\mathscr{H}} f_2(z)$ then, their conjugations with respect to every $\beta \in D$ are also $\mathscr{H}-$subordinated, that is, $h_1^{(\beta)}(z) \prec_{\mathscr{H}} h_2^{(\beta)}(z)$.

Now, we establish a partial differential equation for the conjugation of classical Loewner chains. This equation has the form of a linear Loewner-Kufarev equation where the right hand side has the form of the more general Berkson-Porta infinitesimal generator. Also, we establish an ordinary differential equation for the transitive function, with the last-mentioned infinitesimal generator.

**Theorem 1.3.3.** *Let $\beta \in D$ be given. The univalent function $h^{(\beta)}(z,t)$ is the conjugation of a classical Loewner chain with respect to $\beta$ if and only if it satisfies the two following conditions:*

*a.) There exist $z_0 \in D$, $R_0 > 0$ and $K > 0$ such that $h^{(\beta)}(z,t)$ is analytic in $D(z_0,R_0)$ for each $t \geq 0$, absolutely continuous in $t \geq 0$ for each $z \in D(z_0,R_0)$ and*

$$|h^{(\beta)}(z,t)| \leq Ke^t, \quad \text{for } t \geq 0,\ z \in D(z_0,R_0),$$

*where* $D(z_0,R_0)=\{z\ :\ |z-z_0|<R_0\}$.

*b.) There exists an analytic function* $P(z,t)$ *in* $D$ *and measurable in* $t\geq 0$ *satisfying*

$$Re(P(z,t))>0 \quad z\in D,\ t\geq 0,$$

*such that, for almost every* $t\geq 0$

$$\frac{\partial h^{(\beta)}(z,t)}{\partial t}=(z-\beta)(1-\bar{\beta}z)P_\beta(z,t)\frac{\partial h^{(\beta)}(z,t)}{\partial z},\quad z\in D(z_0,R_0), \tag{1.11}$$

*with* $P_\beta(z,t)=\frac{P(S_\beta(z),t)}{1-|\beta|^2}$.

**Theorem 1.3.4.** *Suppose that* $P(\,\cdot\,,t)$ *belongs to* $\mathscr{P}$ *for each* $t\geq 0$*, and is measurable in* $t$ *and* $\beta\in D$*. Then, for* $z\in D$ *and* $s\geq 0$*, the following initial value problem*

$$\begin{cases} \dfrac{\partial V}{\partial t}=(V-\beta)(\overline{\beta}V-1)P_\beta(V,t), & \text{for almost every } t\geq s,\\ V(s)=z, \end{cases} \tag{1.12}$$

*with* $(1-|\beta|^2)P_\beta(z,t)=P(S_\beta(z),t)$ *has a unique absolutely continuous solution* $V_z(t)=\psi(z;s,t)$ *in* $t\geq s$ *for each* $z\in D$*. Moreover, the functions* $\psi(z;s,t)$ *are univalent in* $z\in D$*,*

$$h^{(\beta)}(z,s)=\lim_{t\to\infty}e^tS_\beta(\psi(z;s,t))\quad z\in D, s\geq 0, \tag{1.13}$$

*exists locally uniformly in* $D$ *and* $h^{(\beta)}(z,s)$ *is the conjugation of a classical Loewner chain with respect to* $\beta\in D$ *and also satisfying (1.11).*

*Conversely, if* $h(z,t)$ *is the conjugation of a classical Loewner chain given with respect to* $\beta\in D$ *and* $\psi(z;s,t)$ *is determined by* $h(z,s)=h(\psi(z;s,t),t)$ *then* $V_z(t)=\psi(z;s,t)$ *is the solution of (1.12) and satisfies (1.13).*

Before giving the proof of these two theorems, we state a lemma, which relates the solution of the IVP (1.4) with the more general case of the Berkson-Porta infinitesimal generator. The lemma is based on the fact:

- If $\{\phi_t\}_{t\geq 0}$ is a semigroup on $U \subset \mathbb{C}$ and $T : U \longrightarrow \mathbb{C}$ is one-to-one, then $\{T \circ \phi_t \circ T^{-1}\}_{t\geq 0}$ is also a semigroup over $T(U) \subset \mathbb{C}$.

**Lemma 1.3.5.** *Let $P(\cdot,t) : D \longrightarrow \mathbb{C}$ satisfy $Re(P(z,t)) > 0$ in $D$ and $P(0,t) = 1$ for each $t \geq 0$. Let us denote $P_\beta(z,t) = \dfrac{P(S_\beta(z),t)}{1-|\beta|^2}$, and let $s \geq 0$ be fixed.*

*(a) $U_z(t)$ is a solution of (1.4) if and only if $V_z(t) = S_\beta^{-1}(U_{S_\beta(z)}(t))$ is a solution of (1.12).*

*(b) The function $U_z(t)$ is absolutely continuous in $t \geq s$ for each $z \in D$ if and only if $V_z(t) = S_\beta^{-1}(U_{S_\beta(z)}(t))$ is absolutely continuous in $t \geq s$ for each $z \in D$ as well.*

*Proof.* (a) Let us suppose that $V_z(t)$ is a solution of (1.12), $w = S_\beta(z)$ and consider $U_w(t) = S_\beta(V_z(t))$. We are going to show that $U_z(t)$ is a solution of (1.4). In fact,

$$\begin{aligned}
\frac{\partial U_z(t)}{\partial t} &= S'_\beta(V_z(t))\frac{\partial V_z(t)}{\partial t} \\
&= -\frac{(1-|\beta|^2)\,(V_z(t)-\beta)\left(1-\bar{\beta}V_z(t)\right)}{(1-\bar{\beta}V_z(t))^2}P_\beta(V_z(t),t) \\
&= -\left(\frac{V_z(t)-\beta}{1-\bar{\beta}V_z(t)}\right)\left(1-|\beta|^2\right)P_\beta(V_z(t),t) \\
&= -S_\beta(V_z(t))\left(1-|\beta|^2\right)P_\beta(V_z(t),t) \\
&= -U_z(t)P(U_z(t),t).
\end{aligned}$$

In addition, if $V_z(s) = z$, then $U_w(s) = w$. Therefore, $U_z(t) = S_\beta(V_z(t))$ is a solution of (1.4). In a similar way we can show the converse.

(b) It is not difficult to see that $S_\beta^{-1}(z)$ satisfies

$$|(S_\beta^{-1})'(z)| \leq \frac{1-|\beta|^2}{(1-|\beta|)^2} = \frac{1+|\beta|}{1-|\beta|} =: K_\beta, \quad \text{for } z \in D.$$

A similar inequality holds if we use $S'_\beta(z)$ instead of $(S_\beta^{-1})'(z)$. Furthermore, from the convexity of $D$ we have

$$|S_\beta^{-1}(z_1) - S_\beta^{-1}(z_2)| \leq |(S_\beta^{-1})'(z_2+\lambda_0(z_1-z_2))||z_1-z_2| \leq K_\beta|z_1-z_2| \quad \text{for } z_1, z_2 \in D.$$

Also, the function $S_\beta$ satisfies the same inequality. Now, it is easy to check that $U_z(t)$ is absolutely continuous in $t \geq s$ for each $z \in D$ if and only if $V_z(t) = S_\beta^{-1}(U_{S_\beta(z)}(t))$ is also absolutely continuous in $t \geq s$ for each $z \in D$.

$\square$

*Proof of the theorem 1.3.3.*

It is a straightforward consequence of Theorem 1.2.1 and Lemma 1.3.5. $\square$

*Proof of the theorem 1.3.4.*

It is a straightforward consequence of Theorem 1.2.2 and Lemma 1.3.5. $\square$

We would like to indicate that to the best of our knowledge the present work and [6, 11] constitute the first works, in which one can find an explicit relation, between the solutions of the more general Berkson-Porta equation and some types of Loewner chains and evolution families. We differ from [6, 11] because the authors of these works added an integral condition which is not used here. Instead of that condition, we do it by means of conjugations.

### 1.3.2 $\mathscr{A}-$subordination

It is clear that the composition of two analytic functions is always analytic. In the harmonic case, it is easy to verify that if $f$ and $g$ are harmonic mappings, with domain $g \supseteq$ range $f$, then $g \circ f$ is not "in general" harmonic. Trivial exception occur when $f$ or its conjugate are analytic and $g$ is an arbitrary harmonic mapping. We have a similiar result when $f$ is an arbitrary harmonic mapping and $g$ is affine, i.e., $g(z) = az + c + b\bar{z}$, where $a, b, c$ are complex constants. See [36] for more non-trivial examples. However, it is not easy to find families of harmonic functions which are closed under composition. In this section we use the Dirichlet and Neumann problem to find families of harmonic functions closed under composition.

#### Study of the composition of harmonic functions using the Dirichlet problem

Let us recall that the complex-valued Dirichlet problem in the unit disc consists of finding a continuous complex-valued function in $\bar{D}$, which is harmonic in $D$ and its boundary value is given.

This means, for $u_0(e^{is}) \in C[0,2\pi] \subset L^2[0,2\pi]$, we want to find a function $u : D \longrightarrow \mathbb{C}$ and $u \in C(\bar{D}) \cap C^2(D,\mathbb{C})$, such that

$$\begin{cases} \Delta u = 0, & \text{in } D; \\ u = u_0, & \text{on } \partial D. \end{cases} \tag{1.14}$$

This problem has a unique solution of the form:

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} \mathbb{P}(z,s)u_0(e^{is})ds = \sum_{k=0}^{\infty} a_k z^k + \sum_{k=1}^{\infty} a_{-k}\bar{z}^k,$$

with

$$u_0(e^{is}) = \sum_{k=-\infty}^{\infty} a_k e^{iks} \quad \text{and} \quad a_k = \frac{1}{2\pi}\int_0^{2\pi} u_0(e^{is})e^{-iks}ds,$$

where $a_k$ are the Fourier coefficients of $u_0$, and

$$\mathbb{P}(z,s) = \frac{1-|z|^2}{|e^{is}-z|^2} = \mathbb{P}_{\mathscr{H}}(z,s) + \mathbb{P}_{\bar{\mathscr{H}}^0}(z,s),$$

is the so-called **Poisson kernel**.

On the other hand, P. Duren and G. Schober [13, 14] established that the closure of the family of all sense-preserving harmonic mappings on $D$ onto itself, is precisely the family of all functions $\omega$ of the form

$$\omega(z) = \frac{1}{2\pi}\int_0^{2\pi} \mathbb{P}(z,s)e^{i\theta_\omega(s)}ds, \tag{1.15}$$

where $\theta_\omega(s)$ is a circle mapping, defined as a left continuous nondecreasing function on $[0,2\pi)$ with $\theta_\omega(2\pi-) - \theta_\omega(0) \leq 2\pi$, see [13, 14] et al. Furthermore, the functions $\omega$ of the form (1.15) are a particular case when $\Omega = \omega(D)$ is a convex set, and $\partial\Omega$ is a Jordan curve. In this more general case, we have

$$\omega(z) = \frac{1}{2\pi}\int_0^{2\pi} \mathbb{P}(z,s)\varphi_\omega(e^{is})ds, \tag{1.16}$$

where $\varphi_\omega$ is a homeomorphism from $\partial D$ onto $\partial\Omega$, and $\omega$ is univalent in $D$. Moreover, $\omega$ is the solution of the Dirichlet problem with initial condition $u_0(e^{is}) = \varphi_\omega(e^{is})$, see for example, [13]. Let us denote by $\mathscr{K}$ the family of functions $\omega$ of the form (1.16).

Also, it is worthy recalling the harmonic Hardy space on $D$, denoted by $\mathfrak{h}^2(D)$, containing all the harmonic functions $u$ on $D$, such that

$$\sup_{r\in[0,1)} M_2(u,r) < +\infty, \quad \text{with} \quad M_2(u,r) = \frac{1}{2\pi}\int_0^{2\pi} |u(re^{i\theta})|^2 d\theta.$$

This space $\mathfrak{h}^2(D)$, can also be identified as the class of power series

$$\sum_{k=0}^{\infty} a_k z^k + \sum_{k=1}^{\infty} a_{-k}\bar{z}^k, \text{ with } \sum_{k=-\infty}^{\infty} |a_k|^2 < +\infty,$$

that is, $\{a_k\} \in l^2(\mathbb{Z})$. Furthermore, it is shown (see [3]) that the application $f \longrightarrow P[f]$ is a linear isometry from $L^2[0,2\pi]$ onto $\mathfrak{h}^2(D)$, where

$$P[f](z) = \frac{1}{2\pi}\int_0^{2\pi} \mathbb{P}(z,s) f(e^{is}) ds, \quad \text{for } f(e^{is}) \in L^2[0,2\pi].$$

This integral $P[f](z)$ is called the Poisson Integral of $f(e^{is}) \in L^2[0,2\pi]$.

Let $\mathscr{F}_D$ be the family of functions which are solutions of a Dirichlet problem in $D$, this means

$$\mathscr{F}_D := \{u \text{ is a solution of (1.14) for some } u_0(e^{is}) \in C[0,2\pi]\},$$

and let $\tilde{\mathscr{F}}_D$ be the set of functions belonging to $\mathscr{F}_D$ such that $u(D) \subsetneq D$.

Now, let us consider $\Lambda_D$ the closure in $L^2[0,2\pi]$ of the subspace generated by

$$A_D = \left\{ \phi(s) = \partial_{\mu\mu}\mathbb{P}(\mu(z),s) = \frac{e^{is}}{(e^{is}-\mu(z))^3} : \mu(z) \in \tilde{\mathscr{F}}_D, |z| < 1 \right\},$$

and $\Lambda_D^{\perp}$ its orthogonal complement. It is not difficult to show that $A_D \subset L^2[0,2\pi]$ (see [13]). Finally, let us consider the following set

$$\mathscr{G}_D = \{\omega \in \tilde{\mathscr{F}}_D : \alpha(s), \beta(s) \in \Lambda_D^{\perp}, \text{ where } \omega_0(e^{is}) = \alpha(s) + i\beta(s)\}.$$

We show some examples of functions in $\mathscr{G}_D$.

**Example 4.** *Let $a,b \in \mathbb{R}$ be fixed, and satisfy $\max\{|a+b|,|a-b|\} < 1$. If we define the function*

*$\omega(z) = az + b\bar{z}$, for $z \in \overline{D}$ then, $\omega(z) \in \mathscr{G}_D$. In fact, we know that*

$$\mathbb{P}(z,s) = \sum_{n=0}^{\infty} e^{-ins} z^n + \sum_{n=1}^{\infty} e^{ins} \bar{z}^n.$$

*Then,*

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} \mathbb{P}(z,s)(ae^{is} + be^{-is})ds &= \frac{a}{2\pi}\int_0^{2\pi} \mathbb{P}(z,s)e^{is}ds + \frac{b}{2\pi}\int_0^{2\pi} \mathbb{P}(z,s)e^{-is}ds \\
&= \frac{a}{2\pi}\int_0^{2\pi} \left(\sum_{n=0}^{\infty} e^{-ins} z^n + \sum_{n=1}^{\infty} e^{ins} \bar{z}^n\right) e^{is}ds \\
&\quad + \frac{b}{2\pi}\int_0^{2\pi} \left(\sum_{n=0}^{\infty} e^{-ins} z^n + \sum_{n=1}^{\infty} e^{ins} \bar{z}^n\right) e^{-is}ds \\
&= \sum_{n=0}^{\infty} \frac{az^n}{2\pi}\int_0^{2\pi} e^{-i(n-1)s}ds + \sum_{n=1}^{\infty} \frac{a\bar{z}^n}{2\pi}\int_0^{2\pi} e^{i(n+1)s}ds \\
&\quad + \sum_{n=0}^{\infty} \frac{bz^n}{2\pi}\int_0^{2\pi} e^{-i(n+1)s}ds + \sum_{n=1}^{\infty} \frac{b\bar{z}^n}{2\pi}\int_0^{2\pi} e^{i(n-1)s}ds \\
&= az + b\bar{z} = \omega(z).
\end{aligned}$$

*Therefore, $\omega_0(e^{is}) = ae^{is} + be^{-is}$. Now, let us suppose that $\mu = \mu(z) \in \tilde{\mathscr{F}}_D$ and $|z| < 1$. From simple calculations we obtain*

$$\partial_{zz}\mathbb{P}(z,s) = \sum_{n=2}^{\infty} n(n-1)e^{-ins} z^{n-2} = \overline{\partial_{\bar{z}\bar{z}}\mathbb{P}(z,s)}.$$

*Then,*

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} \partial_{\mu\mu}\mathbb{P}(\mu(z),s)\,\omega_0(e^{is})ds &= \frac{1}{2\pi}\int_0^{2\pi} \sum_{n\geq 2} n(n-1)e^{-ins}\mu^{n-2}(ae^{is} + be^{-is})ds \\
&= \sum_{n\geq 2} n(n-1)\left(\frac{1}{2\pi}\int_0^{2\pi} e^{-ins}(ae^{is} + be^{-is})ds\right)\mu^{n-2} \\
&= \sum_{n\geq 2} n(n-1)\left(\frac{a}{2\pi}\int_0^{2\pi} e^{-i(n-1)s}ds + \frac{b}{2\pi}\int_0^{2\pi} e^{-i(n+1)s}ds\right)\mu^{n-2} \\
&= 0.
\end{aligned}$$

*In a similar way we have*

$$\frac{1}{2\pi}\int_0^{2\pi} \partial_{\bar{\mu}\bar{\mu}}\mathbb{P}(\mu(z),s)\,\omega_0(e^{is})ds = 0.$$

*Therefore,*

$$\frac{1}{2\pi}\int_0^{2\pi} \partial_{\mu\mu}\mathbb{P}(\mu(z),s)\,\omega_0(e^{is})ds = 0 = \frac{1}{2\pi}\int_0^{2\pi} \partial_{\bar{\mu}\bar{\mu}}\mathbb{P}(\mu(z),s)\,\omega_0(e^{is})ds,$$

*which implies that* $Re(\omega_0(e^{is})), Im(\omega_0(e^{is})) \in \Lambda_D^{\perp}$. *Thus,* $\omega(z) \in \mathscr{G}_D$.

**Example 5.** *In a similar way as in Example 4, we can show that* $\omega(z) = az + c + b\bar{z} \in \mathscr{G}_D$, *if* $a, b, c \in \mathbb{R}$ *satisfy* $\max\{|a+b+c|, |a-b+c|\} < 1$.

We present now a property that has the family $\mathscr{G}_D$, which establishes that the composition of functions in $\mathscr{G}_D$ is harmonic.

**Theorem 1.3.6.** *The family* $\mathscr{G}_D$ *satisfies the following conditions,*

1. *If* $u, v \in \mathscr{G}_D$, *then* $u \circ v \in \mathscr{A}(D,D)$,
2. $\mathscr{G}_D = \{u(z) = az + c + b\bar{z} : a, b, c \in \mathbb{C},\ u(D) \subsetneq D\}$,
3. $\mathscr{G}_D$ *is closed under composition.*

*Proof.*

1. Let us suppose that

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} \mathbb{P}(z,s)u_0(e^{is})ds, \quad \text{and} \quad \mu = v(z).$$

Since $\mu \in \tilde{\mathscr{F}}_D$, from the definition of $\mathscr{G}_D$ it follows

$$\int_0^{2\pi} \partial_{\mu\mu}\mathbb{P}(\mu(z),s)u_0(e^{is})ds = 0 = \int_0^{2\pi} \partial_{\bar{\mu}\bar{\mu}}\mathbb{P}(\mu(z),s)u_0(e^{is})ds.$$

Here, we have used that $\partial_{zz}\mathbb{P} = \frac{1}{2}(\partial_{xx}^2\mathbb{P} - i\partial_{xy}^2\mathbb{P})$ and $\partial_{\bar{z}\bar{z}}\mathbb{P} = \overline{\partial_{zz}\mathbb{P}}$.

Also, we know that $\Delta = \frac{1}{4}\partial_z\partial_{\bar{z}} = \frac{1}{4}\partial_{z\bar{z}}$ then,

$$\begin{aligned}
\partial_{z\bar{z}}(u\circ v)(z) &= \frac{1}{2\pi}\int_0^{2\pi}\left\{\partial_{\bar{z}}\mu\partial_z\mu\partial_{\mu\mu}\mathbb{P}(\mu,s)+\partial_z\bar{\mu}\partial_{\bar{z}}\bar{\mu}\partial_{\bar{\mu}\bar{\mu}}\mathbb{P}(\mu,s)\right\}u_0(e^{is})ds,\\
&= \left(\frac{\partial_z\mu\partial_{\bar{z}}\mu}{2\pi}\right)\int_0^{2\pi}\partial_{\mu\mu}\mathbb{P}(\mu,s)u_0(e^{is})ds+\left(\frac{\partial_z\bar{\mu}\partial_{\bar{z}}\bar{\mu}}{2\pi}\right)\int_0^{2\pi}\partial_{\bar{\mu}\bar{\mu}}\mathbb{P}(\mu,s)u_0(e^{is})ds,\\
&= 0.
\end{aligned}$$

This implies that $(u\circ v)(z)$ is harmonic.

2. Now, we want to show the second statement. From the definition of $\mathscr{G}_D$, we have that $u\in\mathscr{G}_D$ implies

$$\frac{1}{2\pi}\int_0^{2\pi}\partial_{\mu\mu}\mathbb{P}(\mu,s)u_0(e^{is})ds = 0 = \frac{1}{2\pi}\int_0^{2\pi}\partial_{\bar{\mu}\bar{\mu}}\mathbb{P}(\mu,s)u_0(e^{is})ds,$$

for every $\mu(z)\in\tilde{\mathscr{F}}_D$. But, as we have seen before

$$\partial_{zz}\mathbb{P}(z,s) = \sum_{n=2}^{\infty}n(n-1)e^{-ins}z^{n-2} = \overline{\partial_{\bar{z}\bar{z}}\mathbb{P}(z,s)}.$$

Thus, for every $\mu\in\tilde{\mathscr{F}}_D$, we have

$$\begin{aligned}
0 &= \frac{1}{2\pi}\int_0^{2\pi}\partial_{\mu\mu}\mathbb{P}(\mu,s)u_0(e^{is})ds = \frac{1}{2\pi}\int_0^{2\pi}\sum_{n\geq 2}n(n-1)e^{-ins}\mu^{n-2}u_0(e^{is})ds\\
&= \sum_{n\geq 2}n(n-1)\left(\frac{1}{2\pi}\int_0^{2\pi}e^{-ins}u_0(e^{is})ds\right)\mu^{n-2}.
\end{aligned}$$

Therefore,

$$\sum_{n\geq 2}n(n-1)\left(\frac{1}{2\pi}\int_0^{2\pi}e^{-ins}u_0(e^{is})ds\right)\mu^{n-2} = 0. \tag{1.17}$$

In a similar way, it can be shown

$$\sum_{n\geq 2}n(n-1)\left(\frac{1}{2\pi}\int_0^{2\pi}e^{ins}u_0(e^{is})ds\right)\overline{\mu}^{n-2} = 0. \tag{1.18}$$

Furthermore, if $u$ is the solution of a Dirichlet problem then,

$$u(z) = \sum_{k\geq 0} a_k z^k + \sum_{k\geq 1} a_{-k}\bar{z}^n, \quad \text{with} \quad u_0(e^{is}) = \sum_{k=-\infty}^{\infty} a_k e^{iks}.$$

Thus,

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} e^{-ins} u_0(e^{is})ds &= \frac{1}{2\pi}\int_0^{2\pi} e^{-ins}\left(\sum_{k=-\infty}^{\infty} a_k e^{iks}\right) ds \\ &= \sum_{k=-\infty}^{\infty} a_k \left(\frac{1}{2\pi}\int_0^{2\pi} e^{i(k-n)s} ds\right) = a_n. \end{aligned} \tag{1.19}$$

From the particular case $\mu(z) = z \in \tilde{\mathscr{F}}_D$, and Equations (1.17), (1.18), and (1.19), we obtain that

$$\begin{aligned}\partial_{zz}u(z) &= \sum_{n\geq 2} n(n-1)a_n z^{n-2} = 0,\\ \partial_{\bar{z}\bar{z}}u(z) &= \sum_{n\geq 2} n(n-1)b_n \bar{z}^{n-2} = 0.\end{aligned}$$

Therefore,

$$u(z) = a_1 z + a_0 + b_1\bar{z}.$$

Thus, we conclude that

$$\mathscr{G}_D = \{u(z) = az + c + b\bar{z} : a,b,c \in \mathbb{C}, u(D) \subsetneq D\}.$$

3. Note that if $u, v \in \mathscr{G}_D$ then $u \circ v \in \mathscr{G}_D$. Thus, $\mathscr{G}_D$ is a family of harmonic functions which is closed under compositions. $\square$

**Remark 1.1.** From Example 5, we have that $|a+c| + |b| < 1$ is a sufficient condition in order to a function $u = az + c + b\bar{z}$, with real coefficients, belongs to $\mathscr{G}_D$. In the case, when the coefficients $a, b, c$ are complex numbers, a sufficient condition that the function $u = az + c + b\bar{z}$ belongs to $\mathscr{G}_D$, is $|a| + |b| + |c| < 1$.

**Study of the composition of harmonic functions using the Neumann problem**

Here, we are going to consider the Laplace equation with a Neumann boundary condition instead of a Dirichlet condition. We reason in a similar way as before, in order to find a family which is closed under composition. Note that these two boundary conditions are different and in principle (see Theorem 1.3.8) lead to different harmonic functions.

Let $\mathscr{L}$ be the family of solutions of the Neumann problem for the Laplace operator for some function $h(e^{is}) \in C[0,2\pi]$, that is, $u \in \mathscr{L}$ if there exists a function $h(e^{is}) \in C[0,2\pi]$, such that

$$\begin{cases} \Delta u = 0, & \text{in } D; \\ \\ \dfrac{\partial u}{\partial \mathbf{n}} = h, & \text{on } \partial D, \end{cases}$$

where $\dfrac{\partial u}{\partial \mathbf{n}}$ denotes the normal derivative of $u$. In this case, we write $h_u(e^{is}) := h(e^{is})$.

Since $h_u(e^{is}) \in C[0,2\pi]$, we can write this function in the following form

$$h_u(e^{is}) = \sum_{n=-\infty}^{\infty} a_n e^{ins}, \quad \text{with } a_0 = 0,$$

where $a_n$ are the Fourier coefficient of $h_u$. Therefore, the functions in $\mathscr{L}$ have the form

$$u(z) = \sum_{n=1}^{\infty} \frac{a_n}{n} z^n + \sum_{n=1}^{\infty} \frac{a_{-n}}{n} \bar{z}^n.$$

On the other hand,

$$\begin{aligned} u(z) &= \sum_{n=1}^{\infty} \left( \frac{1}{2\pi} \int_0^{2\pi} h_u(e^{is}) e^{-ins} ds \right) \frac{z^n}{n} + \sum_{n=1}^{\infty} \left( \frac{1}{2\pi} \int_0^{2\pi} h_u(e^{is}) e^{ins} ds \right) \frac{\bar{z}^n}{n} \\ &= \frac{1}{2\pi} \int_0^{2\pi} \left( \sum_{n=1}^{\infty} \frac{z^n}{n} e^{-ins} + \sum_{n=1}^{\infty} \frac{\bar{z}^n}{n} e^{ins} \right) h(e^{is}) ds \\ &= \frac{1}{2\pi} \int_0^{2\pi} \mathbb{Q}(z,s) h_u(e^{is}) ds, \end{aligned}$$

where

$$\mathbb{Q}(z,s) = \sum_{n=1}^{\infty} \frac{(ze^{-is})^n}{n} + \sum_{n=1}^{\infty} \frac{(\bar{z}e^{is})^n}{n}.$$

Note that

$$\partial_z \mathbb{Q}(z,s) = e^{-is}\mathbb{P}_{\mathscr{H}}(z,s), \quad \text{and} \quad \partial_{\bar{z}} \mathbb{Q}(z,s) = e^{is}\overline{\mathbb{P}_{\mathscr{H}}(z,s)},$$

where $\mathbb{P}(z,s) = \mathbb{P}_{\mathscr{H}}(z,s) + \mathbb{P}_{\bar{\mathscr{H}}^0}(z,s)$ is the Poisson kernel.

Also, note that if $u(z)$ is a solution of some Neumann problem, then $u(z)+c$ is a solution of the same problem as well.

Actually, we are interested in subfamilies of $\mathscr{L}$, which are closed under composition. To get a family with this property, let us define $\widetilde{\mathscr{L}}$ to be the family of function $u \in \mathscr{L}$ such that $u(D) \subsetneq D$. Now, let us consider $\Lambda_N$ to be the closure in $L^2[0,2\pi]$ of the subspace generated by

$$A_N = \left\{ \varphi(s) = \partial_{\mu\mu}\mathbb{Q}(\mu(z),s) = \frac{e^{is}}{(e^{is}-\mu(z))^3} : \mu(z) \in \tilde{\mathscr{L}}, |z| < 1 \right\},$$

and $\Lambda_N^{\perp}$ its orthogonal complement. It is not difficult to see that also $A_N \subset L^2[0,2\pi]$.

Finally, let $\mathscr{G}_N$ be the following set

$$\mathscr{G}_N = \{u \in \tilde{\mathscr{L}} : a(s), b(s) \in \Lambda_N^{\perp}, \text{ where } h_u(e^{is}) = a(s) + ib(s)\}.$$

As a consequence of this definition we have,

**Proposition 1.3.7.** *If $\phi, \psi \in \mathscr{G}_N$, then $\phi \circ \psi \in \mathscr{A}(D,D)$.*

*Proof.* It can be shown in a similar way as Proposition 1.3.6. □

**Theorem 1.3.8.** *The set $\mathscr{G}_N$ is closed under the composition. Moreover, $\mathscr{G}_N = \mathscr{G}_D$.*

*Proof.* From the definition of $\mathscr{G}_N$, we have that for functions $u \in \mathscr{G}_N$ it follows

$$\frac{1}{2\pi}\int_0^{2\pi} \partial_{\mu\mu}\mathbb{Q}(\mu,s)h_u(e^{is})ds = 0 = \frac{1}{2\pi}\int_0^{2\pi} \partial_{\bar{\mu}\bar{\mu}}\mathbb{Q}(\mu,s)h_u(e^{is})ds$$

for every $\mu \in \tilde{\mathscr{L}}$ and $|z| < 1$. But, these equations are equivalent to:

$$\frac{1}{2\pi}\int_0^{2\pi} \partial_\mu \mathbb{P}(\mu,s)e^{-is}h_u(e^{is})ds = 0 = \frac{1}{2\pi}\int_0^{2\pi} \overline{\partial_\mu \mathbb{P}(\mu,s)}e^{is}h_u(e^{is})ds.$$

Then,

$$\begin{aligned}
0 &= \frac{1}{2\pi}\int_0^{2\pi} \partial_\mu \mathbb{P}(\mu,s)e^{-is}h_u(e^{is})ds = \frac{1}{2\pi}\int_0^{2\pi}\sum_{n=0}^{\infty} ne^{-ins}\mu^{n-1}e^{-is}h_u(e^{is})ds \\
&= \sum_{n=0}^{\infty} n\left(\frac{1}{2\pi}\int_0^{2\pi} e^{-i(n+1)s}h_u(e^{is})ds\right)\mu^{n-1} \\
&= \sum_{n=0}^{\infty} n\left(\frac{1}{2\pi}\int_0^{2\pi} e^{-i(n+1)s}\left(\sum_{k=-\infty}^{\infty} a_k e^{iks}\right)ds\right)\mu^{n-1} \\
&= \sum_{n=0}^{\infty} n\left(\sum_{k=-\infty}^{\infty} a_k\left(\frac{1}{2\pi}\int_0^{2\pi} e^{-i(n+1-k)s}ds\right)\right)\mu^{n-1} \\
&= \sum_{n=0}^{\infty} na_{n+1}\mu^{n-1}.
\end{aligned}$$

In particular, for $\mu(z) = z$, we have

$$\partial_{zz}u(z) = \sum_{n=0}^{\infty} na_{n+1}z^{n-1} = 0.$$

In a similar way it can be shown that

$$\sum_{n=0}^{\infty} na_{-(n+1)}\bar{\mu}^{n-1} = 0.$$

Thus,

$$\partial_{\bar{z}\bar{z}}u(z) = \sum_{n=0}^{\infty} na_{-(n+1)}\bar{z}^{n-1} = 0.$$

Therefore, $u(z) = a_1 z + a_{-1}\bar{z}$. But, $u(z)+c$ is a solution of the same Neumann problem as well. Thus,

$$\mathscr{G}_N = \{u(z) = a_1 z + c + a_{-1}\bar{z} : a_1, c, a_{-1} \in \mathbb{C},\ u(D) \subsetneq D\},$$

which is closed under composition and $\mathscr{G}_N = \mathscr{G}_D$. $\square$

### Subordination

By considering the family $\mathscr{G} = \{u(z) = a_1 z + c + a_{-1}\bar{z} : a_1, c, a_{-1} \in \mathbb{C}\}$, we are able to introduce a type of subordination for harmonic functions. In the previous section, the fucntion which makes the subordination, was considered to be analytic. In this case, that function is harmonic.

**Definition 1.3.3.** *Let $f, g : D \longrightarrow \mathbb{C}$ be complex-valued harmonic functions. We say that $f$ is $\mathscr{A}-$subordinate or harmonically subordinate to $g$ if:*

*There exists a harmonic mapping $\psi$ on $D$ satisfying $|\psi| < 1$ and $\psi(0) = 0$, such that*

$$f(x,y) = g(\psi(x,y)), \quad (x,y) \in D.$$

*This will be denoted by $f \prec_{\mathscr{A}} g$.*

**Remark 1.2.** *Note that for every harmonic function $f$ with $f(0) = 0$ and $f(D)$ bounded, there is at least one $g \in \mathscr{G}$ such that $f \prec_{\mathscr{A}} g$.*

Furthermore, we have the following lemma.

**Lemma 1.3.9.** *Let $g \in \mathscr{G}$ be given. Then, $f \prec_{\mathscr{A}} g$ if and only if $f(0) = g(0)$ and $f(D) \subset g(D)$.*

*Proof.* Assuming that $f(z) \prec_{\mathscr{A}} g(z)$ then $\phi(D) \subset D$ and $\phi(0) = 0$ from Definition 1.3.3. Thus, it follows that $f(D) = g(\phi(D)) \subset g(D)$ and $f(0) = g(0)$.

Conversely, we introduce the function

$$\phi(z) := u(z) + \overline{v(z)}, \quad \text{with} \quad u(z) = \frac{af_1(z) - bf_2(z)}{|a|^2 - |b|^2},\ v(z) = \frac{af_2(z) - bf_1(z)}{|a|^2 - |b|^2},$$

where $f(z) = f(0) + f_1(z) + \overline{f_2(z)}$, and $g(z) = az + g(0) + b\bar{z}$. Note that $\phi$ is harmonic in $D$, $\phi(0) = 0$ and $\phi(D) \subset D$. Further, $f(z) = g(\phi(z))$. Therefore, $f \prec_{\mathscr{A}} g$. □

Let us see an example of this subordination.

**Example 6.** *Let us consider the functions:*

$$f(z) = \frac{e^{\alpha_1} + e^{\beta_1}}{2} z + \frac{e^{\alpha_1} - e^{\beta_1}}{2} \bar{z} = e^{\alpha_1} x + i e^{\beta_1} y,$$

$$g(z) = \frac{e^{\alpha_2} + e^{\beta_2}}{2} z + \frac{e^{\alpha_2} - e^{\beta_2}}{2} \bar{z} = e^{\alpha_2} x + i e^{\beta_2} y,$$

*with* $\alpha_1 < \alpha_2$ *and* $\beta_1 < \beta_2$, *then* $f \prec_{\mathscr{A}} g$. *More precisely,* $\psi(x,y) = e^{\alpha_1 - \alpha_2} x + i e^{\beta_1 - \beta_2} y$.

We will continue dealing with this concept in posterior sections, where it will be used to define a certain class of harmonic chains.

## 1.4 Semigroups and evolution families in $\mathscr{G}_D$

In the 3 we established the definition of a one-parameter semigroup. Here, we study a semigroup of harmonic functions, more precisely in $\mathscr{G}_D$. Also, we present the more general concept, that is, the concept of evolution families and examples.

**Definition 1.4.1.** *A family* $\{\varphi_{s,t}\}_{0 \le s \le t < +\infty}$ *of harmonic functions on* $D$, *is an evolution family of harmonic functions (in short, an* $\mathscr{A}-$*evolution family), if*

*EF1.* $\varphi_{s,s} = Id_D$,

*EF2.* $\varphi_{s,t} = \varphi_{\tau,t} \circ \varphi_{s,\tau}$, *for all* $0 \le s \le \tau \le t < +\infty$,

*EF3. For each* $z \in D$, *the function* $\varphi^*(s,t) = \varphi_{s,t}(z)$ *is continuous in* $\mathbb{R}^+ \times \mathbb{R}^+$ *if* $s \le t$.

**Example 7.** *It is clear that if* $\{\phi_t\}_{t \ge 0}$ *is a semigroup of harmonic functions then, the set* $\{\varphi_{s,t}(z) = \phi_{t-s}(z)\}_{0 \le s \le t < +\infty}$ *is an evolution family of harmonic functions. In fact, EF1 and EF2 follow from S1 and S2, respectively. Note that S3 implies that for each* $z \in D$ *and* $l_0 > 0$, $\phi_l(z) \longrightarrow \phi_{l_0}(z)$ *as* $l \longrightarrow l_0$. *So, if* $(s_0, t_0) \in \mathbb{R}^+ \times \mathbb{R}^+$ *with* $s_0 \le t_0$ *and setting* $l = t - s > 0$ *and* $l_0 = t_0 - s_0$, *we obtain EF3. If* $l_0 = 0$, *we can only consider the lateral limit* $l \longrightarrow l_0^+$.

We have introduced the family $\mathscr{G}_D$ in order to guarantee that the composition of two harmonic mappings is harmonic. What follows is to present examples of semigroups and evolution families of harmonic functions.

Consider $\{\phi_t(z)\}_{t \ge 0} \subset \mathscr{G}_D$ such that $\phi_t(D) \subset D$ and satisfying the semigroup conditions S1, S2 and S3. Then,

$$\phi_t(z) = a(t)z + c(t) + b(t)\bar{z},$$

where $a(t), b(t), c(t)$ are continuous functions at zero and $a(0) = 1$, $b(0) = 0$ and $c(0) = 0$. We are interested in studying the next question in a explicit way.

What are the conditions that $a(t)$, $b(t)$ and $c(t)$ must satisfy in order to obtain the equality $\phi_{t+s}(z) = \phi_t(\phi_s(z))$?

To answer this question first of all, we have

$$\begin{aligned}\phi_t(\phi_s(z)) &= a(t)\phi_s(z) + c(t) + b(t)\overline{\phi_s(z)} \\ &= a(t)[a(s)z + c(s) + b(s)\bar{z}] + c(t) + b(t)[\overline{a(s)z + c(s) + b(s)\bar{z}}] \\ &= [a(t)a(s) + b(t)\overline{b(s)}]z + [a(t)c(s) + c(t) + b(t)\overline{c(s)}] + [a(t)b(s) + \overline{a(s)}b(t)]\bar{z}.\end{aligned}$$

On the other hand, we have

$$\phi_{t+s}(z) = a(t+s)z + c(t+s) + b(t+s)\bar{z}.$$

Then,

$$\begin{aligned}a(t+s) &= a(t)a(s) + b(t)\overline{b(s)}, \\ c(t+s) &= a(t)c(s) + c(t) + b(t)\overline{c(s)}, \\ b(t+s) &= a(t)b(s) + \overline{a(s)}b(t).\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{a(t+s) - a(t)}{s} &= a(t)\left[\frac{a(s) - 1}{s}\right] + b(t)\frac{\overline{b(s)}}{s}, \\ \frac{c(t+s) - c(t)}{s} &= a(t)\left[\frac{c(s)}{s}\right] + b(t)\left[\frac{\overline{c(s)}}{s}\right], \\ \frac{b(t+s) - b(t)}{s} &= a(t)\frac{b(s)}{s} + \overline{\left[\frac{a(s) - 1}{s}\right]}b(t).\end{aligned}$$

Let us assume that $\dot{a}(t)$, $\dot{c}(t)$, and $\dot{b}(t)$ exist, for every $t \geq 0$. Since $a(0) = 1$, $c(0) = 0$, and

$b(0) = 0$, we get the following system of equations

$$\dot{a}(t) = a(t)\dot{a}(0) + b(t)\overline{\dot{b}(0)},$$
$$\dot{b}(t) = a(t)\dot{b}(0) + b(t)\overline{\dot{a}(0)},$$
$$\dot{c}(t) = a(t)\dot{c}(0) + b(t)\overline{\dot{c}(0)}.$$

This means,

$$\frac{d}{dt}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a_0 & \overline{b_0} & 0 \\ b_0 & \overline{a_0} & 0 \\ c_0 & \overline{c_0} & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix}, \quad \text{with} \quad a_0 = \dot{a}(0),\ b_0 = \dot{b}(0),\ c_0 = \dot{c}(0).$$

At this point, we are going to present two cases:

1. The first case, when the all coefficients are real-valued functions.

   Let us assume $a(t), b(t), c(t) \in \mathbb{R}$, we obtain

$$\frac{d}{dt}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a_0 & b_0 & 0 \\ b_0 & a_0 & 0 \\ c_0 & c_0 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix}, \quad \text{with} \quad a_0 = \dot{a}(0),\ b_0 = \dot{b}(0),\ c_0 = \dot{c}(0).$$

   Then,

$$a(t) = \frac{e^{(a_0+b_0)t} + e^{(a_0-b_0)t}}{2} = e^{a_0 t}\cosh(b_0 t),$$
$$b(t) = \frac{e^{(a_0+b_0)t} - e^{(a_0-b_0)t}}{2} = e^{a_0 t}\sinh(b_0 t),$$
$$c(t) = \frac{c_0}{a_0+b_0}\left(e^{(a_0+b_0)t} - 1\right).$$

   Moreover, a straightforward calculation shows that $\phi_t(z)$ satisfies the following equation

$$\partial_t \phi_t(z) = G(\phi_t(z)), \quad \text{with} \quad G(z) = a_0 z + c_0 + b_0\bar{z}.$$

2. For the second case, we consider that the coefficients are complex and $\phi_t(0) = 0$.

   Now, if we suppose that $a(t), b(t) \in \mathbb{C}$ and $c(t) \equiv 0$, we obtain

$$\frac{d}{dt}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a_0 & \overline{b_0} \\ b_0 & \overline{a_0} \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} \quad \text{with} \quad a_0 = \dot{a}(0),\ b_0 = \dot{b}(0).$$

   But,

$$\begin{bmatrix} a_0 & \overline{b_0} \\ b_0 & \overline{a_0} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -\frac{b_0(\Gamma+i\beta)}{|b_0|^2} & \frac{b_0(\Gamma-i\beta)}{|b_0|^2} \end{bmatrix}\begin{bmatrix} \alpha-\Gamma & 0 \\ 0 & \alpha+\Gamma \end{bmatrix}\begin{bmatrix} \frac{\Gamma-i\beta}{2\Gamma} & -\frac{\overline{b_0}}{2\Gamma} \\ \frac{\Gamma+i\beta}{2\Gamma} & \frac{\overline{b_0}}{2\Gamma} \end{bmatrix},$$

   where $a_0 = \alpha + i\beta$ and $\Gamma = \sqrt{|b_0|^2 - \beta^2}$. Thus,

$$\begin{aligned} a(t) &= \frac{(\Gamma - i\beta)e^{(\alpha-\Gamma)t} + (\Gamma+i\beta)e^{(\alpha+\Gamma)t}}{2\Gamma} = \frac{e^{\alpha t}}{\Gamma}\left\{\Gamma\cosh(\Gamma t) + i\beta\sinh(\Gamma t)\right\}, \\ b(t) &= \frac{b_0(\Gamma^2+\beta^2)(e^{(\alpha+\Gamma)t} - e^{(\alpha-\Gamma)t})}{2\Gamma|b_0|^2} = \frac{b_0 e^{\alpha t}}{\Gamma}\sinh(\Gamma t). \end{aligned} \tag{1.20}$$

   Also, we have that $\{\phi_t(z)\}$ satisfies the following equation

$$\partial_t\phi_t(z) = G(\phi_t(z)), \quad \text{with} \quad G(z) = a_0 z + b_0\bar{z}.$$

In both cases, we have seen that the semigroup $\{\phi_t(z)\}_{t\geq 0} \subset \mathscr{G}_D$ satisfies an evolution equation of the following form

$$\partial_t\phi_t(z) = G(\phi_t(z)).$$

Furthermore, $\{\phi_t(z)\}$ is a group, since it can be defined for all $t \in \mathbb{R}$.

Now, let us consider $\{\varphi_{s,t}(z)\}_{0\leq s\leq t<\infty} \subset \mathscr{G}_D$ such that $\varphi_{s,t}(D) \subset D$, satisfying EF1, EF2, and EF3, and

$$\varphi_{s,t}(z) = a(s,t)z + b(s,t)\bar{z}.$$

Then $a(s,t)$ and $b(s,t)$ are continuous with $a(s,s) = 1$ and $b(s,s) = 0$. Thus, working in a similar way as before we can obtain that: from EF2, if $s \leq \tau \leq t$, we have

$$\begin{aligned}
(\varphi_{\tau,t} \circ \varphi_{s,\tau})(z) &= a(\tau,t)\varphi_{s,\tau}(z) + b(\tau,t)\overline{\varphi_{s,\tau}(z)} \\
&= a(\tau,t)[a(s,\tau)z + b(s,\tau)\bar{z}] + b(\tau,t)\overline{[a(s,\tau)z + b(s,\tau)\bar{z}]} \\
&= a(\tau,t)a(s,\tau)z + a(\tau,t)b(s,\tau)\bar{z} + b(\tau,t)\overline{a(s,\tau)}\bar{z} + b(\tau,t)\overline{b(s,\tau)}z \\
&= [a(\tau,t)a(s,\tau) + b(\tau,t)\overline{b(s,\tau)}]z + [a(\tau,t)b(s,\tau) + b(\tau,t)\overline{a(s,\tau)}]\bar{z}
\end{aligned}$$

We want this equal to

$$\varphi_{s,t}(z) = a(s,t)z + b(s,t)\bar{z}.$$

Thus, we need that

$$a(s,t) = a(\tau,t)a(s,\tau) + b(\tau,t)\overline{b(s,\tau)}, \tag{1.21}$$

$$b(s,t) = a(\tau,t)b(s,\tau) + \overline{a(s,\tau)}b(\tau,t). \tag{1.22}$$

If we add a suitable term in each previous equation and divide by $t-\tau$, we obtain

$$\begin{aligned}
\frac{a(s,t)-a(s,\tau)}{t-\tau} &= a(s,\tau)\left\{\frac{a(\tau,t)-a(\tau,\tau)}{t-\tau}\right\} + \overline{b(s,\tau)}\left\{\frac{b(\tau,t)-b(\tau,\tau)}{t-\tau}\right\}, \\
\frac{b(s,t)-b(s,\tau)}{t-\tau} &= b(s,\tau)\left\{\frac{a(\tau,t)-a(\tau,\tau)}{t-\tau}\right\} + \overline{a(s,\tau)}\left\{\frac{b(\tau,t)-b(\tau,\tau)}{t-\tau}\right\}.
\end{aligned}$$

Setting $\Delta\tau = t-\tau$, we obtain

$$\begin{aligned}
\frac{a(s,\tau+\Delta\tau)-a(s,\tau)}{\Delta\tau} &= a(s,\tau)\left\{\frac{a(\tau,\tau+\Delta\tau)-a(\tau,\tau)}{\Delta\tau}\right\} + \overline{b(s,\tau)}\left\{\frac{b(\tau,\tau+\Delta\tau)-b(\tau,\tau)}{\Delta\tau}\right\}, \\
\frac{b(s,\tau+\Delta\tau)-b(s,\tau)}{\Delta\tau} &= b(s,\tau)\left\{\frac{a(\tau,\tau+\Delta\tau)-a(\tau,\tau)}{\Delta\tau}\right\} + \overline{a(s,\tau)}\left\{\frac{b(\tau,\tau+\Delta\tau)-b(\tau,\tau)}{\Delta\tau}\right\}.
\end{aligned}$$

Let us assume that $\partial_2 a(s,t)$ and $\partial_2 b(s,t)$ exist, for every $t \geq s$, where $\partial_2$ denotes the derivative with respect to the second variable. If we let $\Delta\tau \longrightarrow 0$, we have

$$\partial_2 a(s,\tau) = a(s,\tau)\partial_2 a(\tau,\tau) + \overline{b(s,\tau)}\partial_2 b(\tau,\tau), \tag{1.23}$$

$$\partial_2 b(s,\tau) = b(s,\tau)\partial_2 a(\tau,\tau) + \overline{a(s,\tau)}\partial_2 b(\tau,\tau), \tag{1.24}$$

with, $\partial_2 a(\tau,\tau) = \partial_2 a(\zeta,\tau)|_{\zeta=\tau}$, and similarly for $\partial_2 b(t,t)$. Equations (1.23), and (1.24) imply that $\varphi_{s,t}(z)$ satisfies the following equation:

$$\partial_t \varphi_{s,t}(z) = G(\varphi_{s,t}(z),t), \quad \text{where} \quad G(z,t) = \partial_2 \varphi_{\sigma,t}(z)|_{\sigma=t} = \partial_2 \varphi_{t,\sigma}(z)|_{\sigma=t}.$$

Let us consider $x_s(\tau) = a(s,\tau)$, and $y_s(\tau) = b(s,\tau)$ for an arbitrary $s \in \mathbb{R}$ fixed. Then, we can rewrite Equations (1.23), and (1.24) as follows

$$\frac{d}{d\tau}x_s(\tau) = \alpha(\tau)x_s(\tau) + \beta(\tau)\overline{y_s(\tau)},$$
$$\frac{d}{d\tau}y_s(\tau) = \alpha(\tau)y_s(\tau) + \beta(\tau)\overline{x_s(\tau)},$$

with $\alpha(\tau) = \partial_2 a(\tau,\tau)$, and $\beta(\tau) = \partial_2 b(\tau,\tau)$. Equivalently, these equations can be written as a non-autonomous linear system of differential equations

$$\frac{d}{d\tau}\begin{bmatrix} x_s(\tau) \\ y_s(\tau) \\ \overline{x_s(\tau)} \\ \overline{y_s(\tau)} \end{bmatrix} = \begin{bmatrix} \alpha(\tau) & 0 & 0 & \beta(\tau) \\ 0 & \alpha(\tau) & \beta(\tau) & 0 \\ 0 & \overline{\beta(\tau)} & \overline{\alpha(\tau)} & 0 \\ \overline{\beta(\tau)} & 0 & 0 & \overline{\alpha(\tau)} \end{bmatrix} \begin{bmatrix} x_s(\tau) \\ y_s(\tau) \\ \overline{x_s(\tau)} \\ \overline{y_s(\tau)} \end{bmatrix} = B(\tau)\vec{x}(\tau), \tag{1.25}$$

with $\vec{x}(\tau) = (x_s(\tau), y_s(\tau), \overline{x_s(\tau)}, \overline{y_s(\tau)})^T$. Therefore, for each $s \geq 0$ we have the IVP

$$\begin{cases} \frac{d}{d\tau}\vec{x}(\tau) = B(\tau)\vec{x}(\tau), \quad \tau \geq s, \\ \\ \vec{x}(s) = e_1 + e_3, \end{cases} \tag{1.26}$$

where $e_j = (\delta_{ij})$.

In a similar way, but now for the first variable, we add other suitable terms in Equations (1.21), and (1.22) and dividing by $s-\tau$, also assuming that $\partial_1 a(s,t)$ and $\partial_1 b(s,t)$ exist, for every $0 \leq s \leq t$, where $\partial_1$ denotes the derivative with respect to the first variable, we obtain

$$\frac{a(s,t)-a(\tau,t)}{s-\tau} = a(\tau,t)\left\{\frac{a(s,\tau)-a(\tau,\tau)}{s-\tau}\right\} + b(\tau,t)\overline{\left\{\frac{b(s,\tau)-b(\tau,\tau)}{s-\tau}\right\}},$$
$$\frac{b(s,t)-b(\tau,t)}{s-\tau} = a(\tau,t)\left\{\frac{b(s,\tau)-b(\tau,\tau)}{s-\tau}\right\} + b(\tau,t)\overline{\left\{\frac{a(s,\tau)-a(\tau,\tau)}{s-\tau}\right\}}.$$

Since $a(\tau,\tau)=1$, and $b(\tau,\tau)=0$. If we set $\Delta\tau = s-\tau$, we obtain

$$\frac{a(\tau+\Delta\tau,t)-a(\tau,t)}{\Delta\tau} = a(\tau,t)\left\{\frac{a(\tau+\Delta\tau,\tau)-a(\tau,\tau)}{\Delta\tau}\right\} + b(\tau,t)\overline{\left\{\frac{b(\tau+\Delta\tau,\tau)-b(\tau,\tau)}{\Delta\tau}\right\}},$$
$$\frac{b(\tau+\Delta\tau,t)-b(\tau,t)}{\Delta\tau} = a(\tau,t)\left\{\frac{b(\tau+\Delta\tau,\tau)-b(\tau,\tau)}{\Delta\tau}\right\} + b(\tau,t)\overline{\left\{\frac{a(\tau+\Delta\tau,\tau)-a(\tau,\tau)}{\Delta\tau}\right\}}.$$

Let us denote $\partial_1 a(\tau,\tau) = \partial_1 a(\tau,\zeta)|_{\zeta=\tau}$, and $\partial_1 b(t,t) = \partial_1 b(\tau,\zeta)|_{\zeta=\tau}$, as before. If we let $\Delta\tau \longrightarrow 0$, we have

$$\partial_1 a(\tau,t) = a(\tau,t)\partial_1 a(\tau,\tau) + b(\tau,t)\overline{\partial_1 b(\tau,\tau)}, \tag{1.27}$$
$$\partial_1 b(\tau,t) = a(\tau,t)\partial_1 b(\tau,\tau) + b(\tau,t)\overline{\partial_1 a(\tau,\tau)}. \tag{1.28}$$

Since $\partial_1 a(\tau,\tau)+\partial_2 a(\tau,\tau)=0$, and $\partial_1 b(\tau,\tau)+\partial_2 b(\tau,\tau)=0$, it follows

$$\partial_1 a(\tau,t) = -\alpha(\tau)a(\tau,t) - \overline{\beta(\tau)}b(\tau,t), \tag{1.29}$$
$$\partial_1 b(\tau,t) = -\beta(\tau)a(\tau,t) - \overline{\alpha(\tau)}b(\tau,t). \tag{1.30}$$

So, from Equations (1.29), and (1.30) it follows that $\varphi_{s,t}$ also satisfies another evolution equation

$$\partial_s \varphi_{s,t}(z) = -\{G(z,s)\partial_z \varphi_{s,t}(z) + \overline{G(z,s)}\partial_{\bar{z}} \varphi_{s,t}(z)\}. \tag{1.31}$$

Now, if $u_t(\tau) = a(\tau,t)$, and $v_t(\tau) = b(\tau,t)$ with a fixed $t \in \mathbb{R}$, then

$$\frac{d}{d\tau}u_t(\tau) = -\alpha(\tau)u_t(\tau) - \overline{\beta(\tau)}v_t(\tau),$$
$$\frac{d}{d\tau}v_t(\tau) = -\beta(\tau)u_t(\tau) - \overline{\alpha(\tau)}v_t(\tau).$$

Equivalently,

$$\frac{d}{d\tau}\begin{bmatrix} u_t(\tau) \\ v_t(\tau) \end{bmatrix} = -\begin{bmatrix} \alpha(\tau) & \overline{\beta(\tau)} \\ \beta(\tau) & \overline{\alpha(\tau)} \end{bmatrix}\begin{bmatrix} u_t(\tau) \\ v_t(\tau) \end{bmatrix}. \tag{1.32}$$

In this way, we obtain another non-autonomous linear system of differential equations. But, this $2\times 2$ system can also be written as the following $4\times 4$ system

$$\frac{d}{d\tau}\begin{bmatrix} u_t(\tau) \\ v_t(\tau) \\ \overline{u_t(\tau)} \\ \overline{v_t(\tau)} \end{bmatrix} = -\begin{bmatrix} \alpha(\tau) & \overline{\beta(\tau)} & 0 & 0 \\ \beta(\tau) & \overline{\alpha(\tau)} & 0 & 0 \\ 0 & 0 & \overline{\alpha(\tau)} & \beta(\tau) \\ 0 & 0 & \overline{\beta(\tau)} & \alpha(\tau) \end{bmatrix}\begin{bmatrix} u_t(\tau) \\ v_t(\tau) \\ \overline{u_t(\tau)} \\ \overline{v_t(\tau)} \end{bmatrix} = -A(\tau)\vec{u(\tau)}, \tag{1.33}$$

with $\vec{u}(\tau) = (u_t(\tau), v_t(\tau), \overline{u_t(\tau)}, \overline{v_t(\tau)})^T$. Thus, once again for each $t \geq 0$ we have the following IVP

$$\begin{cases} \frac{d}{d\tau}\vec{u}(\tau) = -A(\tau)\vec{u}(\tau), \quad \tau \leq t, \\ \\ \vec{u}(t) = e_1 + e_3. \end{cases} \tag{1.34}$$

In the autonomous case of the IVP (1.34), and (1.26), we can show that $a(s,t)$ and $b(s,t)$ have the following form

$$a(s,t) = \frac{e^{\lambda_1(s-t)}}{\Gamma}\left\{\Gamma\cosh(\Gamma(s-t)) + i\lambda_2\sinh(\Gamma(s-t))\right\}, \tag{1.35}$$
$$b(s,t) = \frac{\beta e^{\lambda_1(s-t)}}{\Gamma}\sinh(\Gamma(s-t)),$$

where $\alpha = \lambda_1 + i\lambda_2, \beta \in \mathbb{C}$ and $\Gamma = \sqrt{|\beta|^2 - \lambda_2^2}$.

In the more general case, note that

$$A(\tau) = JB(\tau)J, \quad \text{with} \quad J = J^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}. \tag{1.36}$$

Also, there are inverse matrices $\Pi_B(s,\tau)$, $\Pi_A(\tau,t)$ (see [44]), called *the principal matrices solution* associated to the IVP (1.26), and (1.34), respectively. The principal matrix solution is defined in such a way that

$$\begin{aligned}
\partial_\tau \Pi_A(\tau,t) &= -A(\tau)\Pi_A(\tau,t), \\
\partial_\tau \Pi_B(s,\tau) &= B(\tau)\Pi_B(s,\tau), \\
\Pi_B(s,s) &= \mathbb{I}_{Id} = \Pi_A(t,t), \quad \mathbb{I}_{Id} \text{ is the matrix identity.}
\end{aligned}$$

For more details see G. Teschl [44]. Then, Equation (1.36) implies that

$$\Pi_B(s,t) = J\Pi_A^{-1}(s,-t)J = J\Pi_A(-t,s)J.$$

Moreover, the solutions $\phi_A(\tau,t)$, and $\phi_B(s,\tau)$ for the IVP's (1.34), and (1.26), respectively, are given by

$$\begin{aligned}
\phi_A(\tau,t) &= \Pi_A(\tau,t)(e_1+e_3) \\
\phi_B(s,\tau) &= \Pi_B(s,\tau)(e_1+e_3) = J\Pi_A(-\tau,s)J(e_1+e_3) \\
&= J\Pi_A(-\tau,s)(e_1+e_3) = J\phi_A(-\tau,s).
\end{aligned}$$

The latter equations guarantee the solutions of both systems by simply solving the IVP (1.32).

**Example 8.** *If $\alpha(t) = \beta(t) \in C(\mathbb{R},\mathbb{R})$, then the IVP (1.32) is written as*

$$\frac{d}{d\tau}\begin{bmatrix} u_t(\tau) \\ v_t(\tau) \end{bmatrix} = -\alpha(\tau)\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} u_t(\tau) \\ v_t(\tau) \end{bmatrix}, \quad \begin{bmatrix} u_t(t) \\ v_t(t) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

*The solution of this system is given by*

$$\begin{bmatrix} u(s,t) \\ v(s,t) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} p(s,t)+1 \\ p(s,y)-1 \end{bmatrix}, \quad p(s,t) = e^{-2\int_s^t \alpha(\sigma)d\sigma}.$$

*Finally,*

$$\varphi_{s,t}(z) = u(s,t)z + v(s,t)\bar{z} = xp(s,t) + iy, \quad z = x+iy,$$

*is an evolution family in $\mathscr{G}$.*

**Example 9.** *If $\alpha(t), \beta(t) \in C(\mathbb{R},\mathbb{R})$ then, the IVP (1.32) is rewritten as*

$$\frac{d}{d\tau}\begin{bmatrix} u_t(\tau) \\ v_t(\tau) \end{bmatrix} = -\begin{bmatrix} \alpha(\tau) & \beta(\tau) \\ \beta(\tau) & \alpha(\tau) \end{bmatrix}\begin{bmatrix} u_t(\tau) \\ v_t(\tau) \end{bmatrix}, \quad \begin{bmatrix} u_t(t) \\ v_t(t) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

*The solution of this system is given by*

$$\begin{bmatrix} u(s,t) \\ v(s,t) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} e^{-\int_s^t[\alpha(\sigma)+\beta(\sigma)]d\sigma} + e^{-\int_s^t[\alpha(\sigma)-\beta(\sigma)]d\sigma} \\ e^{-\int_s^t[\alpha(\sigma)+\beta(\sigma)]d\sigma} - e^{-\int_s^t[\alpha(\sigma)-\beta(\sigma)]d\sigma} \end{bmatrix}.$$

*Finally,*

$$\varphi_{s,t}(z) = u(s,t)z + v(s,t)\bar{z} = xp(s,t) + iyq(s,t), \quad z = x+iy,$$

*where $p(s,t) = e^{-\int_s^t[\alpha(\sigma)+\beta(\sigma)]d\sigma}$ and $q(s,t) = e^{-\int_s^t[\alpha(\sigma)-\beta(\sigma)]d\sigma}$. This $\varphi_{s,t}(z)$ is also an evolution family in $\mathscr{G}$.*

## 1.5 Harmonic Loewner chains

In this section we introduce a notion of Loewner chains for harmonic functions taking into account the decomposition (1.2) given in [8]. Note that, the classical case is obtained if the co-analytic part is null. This idea is developed in two cases. The first case is made considering the evolution family of analytic functions. The second case, the evolution family is in the space of harmonic functions.

### 1.5.1 $\mathscr{H}-$Loewner chains and $\mathscr{A}-$Loewner chains

We have two types of subordination then, according to each type, we can introduce a notion of chains. Firstly, we attempt to develop an analogous of the classical Loewner theory in the case when only the elements of the chain are harmonic mappings. Then, we assume that the associated evolution families are also considered harmonic mappings.

#### $\mathscr{H}-$Loewner chains

**Definition 1.5.1.** *The sense-preserving harmonic function $f_t(z) = f(z,t)$, with $z \in D$, and $t \geq 0$, such that*

$$f(z,t) = \sum_{n\geq 1} a_n(t)z^n + \alpha\left(\sum_{n\geq 1} \overline{a_{-n}(t)}\bar{z}^n\right) \text{ with } |\alpha| < 1, \tag{1.37}$$

*with $a_1(t) = e^t$ for $t \geq 0$, is called a $\mathscr{H}-$*Loewner chain, *if the following two conditions hold,*

1. *For each $t \geq 0$, its analytic part is one-to-one,*
2. *$f_s(z) \prec_{\mathscr{H}} f_t(z)$, if $0 \leq s \leq t < +\infty$.*

The following results are consequences of the definition.

**Proposition 1.5.1.** *Let $f(z,t) = h(z,t) + \overline{g(z,t)}$ be an $\mathscr{H}-$Loewner chain. Then,*

1. *For $s \leq t$, $h(z,s) \prec_{\mathscr{H}} h(z,t)$ and $g(z,s) \prec_{\mathscr{H}} g(z,t)$,*
2. *The function $h(z,t)$ is a classical Loewner chain,*
3. *There exists an evolution family $\phi(z,s,t)$ of one-to-one analytic functions from $D$ into $D$,*

4. *For* $0 \le s \le t \le \tau < +\infty$,

$$|h(z,t) - h(z,s)| \le \frac{8|z|}{(1-|z|)^4}(e^t - e^s), \quad |z| < 1, \tag{A}$$

$$|g(z,t) - g(z,s)| < \frac{8|z|}{(1-|z|)^4}(e^t - e^s), \quad |z| < 1, \tag{B}$$

$$|\phi(z,s,\tau) - \phi(z,t,\tau)| \le \frac{2|z|}{(1-|z|)^2}(1 - e^{s-t}), \quad |z| < 1. \tag{C}$$

*Proof.* Assertion 1. follows from Definition 1.5.1 together with the decomposition (1.2). The proof of 2. is obtained from the subordination in 1. and Definition 1.5.1. The existence of an evolution family $\phi(z,s,t)$ and the inequalities (A) and (C) are consequences of the fact that $h(z,t)$ is a classical Loewner chain (see [34] page 157). Moreover, it was proved [34] that for $t \ge 0$, and $|z| < 1$ we have

$$\frac{e^t|z|}{(1+|z|)^2} \le |h(z,t)| \le \frac{e^t|z|}{(1-|z|)^2}, \tag{A*}$$

$$|h'(z,t)| \le \frac{e^t(1+|z|)}{(1-|z|)^3} \le \frac{2e^t}{(1-|z|)^3}, \tag{B*}$$

$$|z - \phi(z,s,t)| \le 2|z|\frac{1+|z|}{1-|z|}(1 - e^{s-t}) \le \frac{4(1-e^{s-t})}{1-|z|}. \tag{C*}$$

Finally, the inequality (B) will be proved following the steps used in [34] to prove (A), and using simultaneously that $f$ is sense-preserving, and the inequalities (B*) and (C*).

First of all, we have $|g'(z,t)| < |h'(z,t)|$ since $f$ is sense-preserving. From (B*) and (C*) we obtain

$$\begin{aligned} |g(z,t) - g(z,s)| &= \left| \int_{\phi(z,s,t)}^{z} g'(w,t)\,dw \right| < \frac{2e^t}{(1-|z|)^3}|z - \phi(z,s,t)| \\ &\le \frac{8|z|}{(1-|z|)^4}(e^t - e^s). \end{aligned}$$

Thus, we conclude the proof of (B). □

**Example 10.** *The classical Loewner chains are clearly examples of* $\mathscr{H}$*−Loewner chains.*

Convenient representation of the $\mathscr{H}$−Loewner chains is established in the next theorem.

**Theorem 1.5.2.** *Let* $f(z,t) = h(z,t) + \overline{g(z,t)}$ *be an* $\mathscr{H}-$*Loewner chain, then there is an analytic function* $F : \bigcup_{t\geq 0} h(D,t) \longrightarrow \mathbb{C}$ *such that*

$$f(z,t) = h(z,t) + \overline{F(h(z,t))} \quad z \in D, t \geq 0$$

*Proof.* Let $\{\phi(z,s,t)\}$ be the evolution family which makes the subordination. From Remark 1.5.1, the function $h(z,s)$ is a classical Loewner chain and one-to-one. Thus, $\phi(z,s,t) = h^{-1}(h(z,s),t)$. If we study the co-analytic part of $f(z,s)$, we obtain

$$g(z,s) = g(h^{-1}(h(z,s),t),t), \quad \text{if } s \leq t. \tag{1.38}$$

Let us define $F(\cdot,t) = g(h^{-1}(\cdot,t),t) : \mathscr{N} \longrightarrow \mathbb{C}$, with $\mathscr{N} := \bigcup_{t\geq 0} h(D,t)$. Then, (1.38) implies $F(z,s) = F(z,t)$. Thus, $F$ is independent of $t$ and $g(z,t) = F(h(z,t))$. □

**Remark 1.3.** From the Koebe 1/4 theorem we have that $\cup_{t\geq 0} h(D,t) = \mathbb{C}$, see, for examples, [34] pages 22-23. Therefore, $F$ is an entire function.

We now present some analogous results to the classical case, including the compactness of the family of $\mathscr{H}-$Loewner chains, and a partial differential equation satisfied by the $\mathscr{H}-$Loewner chains.

This first proposition states that the set of $\mathscr{H}-$Loewner chains is compact.

**Proposition 1.5.3.** *Let* $\{f_n(z,t)\}$ *be a sequence of* $\mathscr{H}-$*Loewner chains. Then, there exists a subsequence of* $\{f_n(z,t)\}$ *that converges to an* $\mathscr{H}-$*Loewner chain locally uniformly in* $D$ *for each fixed* $t \geq 0$.

*Proof.* Suppose that $f_n(z,t) = h_n(z,t) + \overline{F_n(h_n(z,t))}$ is a sequence of $\mathscr{H}-$Loewner chains. Then, $\{h_n(z,t)\}$ is a sequence of classical Loewner chains and contains a subsequence $\{h_{n_k}(z,t)\}$ converging to a classical Loewner chain $h(z,t)$ locally uniformly in $D$ for each fixed $t \geq 0$ (see [34] page 158). Then, in particular $h(z,t)$ is one-to-one.

Now, we are going to prove that $\{F_n\}_{n\in\mathbb{N}}$ is a normal family. In fact, the function $f_n(z,s)$ is sense-preserving, for every $n \in \mathbb{N}$. Thus,

$$|h_n'(z,t)|^2 - |F_n'(h_n(z,t))|^2 |h_n'(z,t)|^2 > 0.$$

Since $h_n$ is one-to-one, for every $n \in \mathbb{N}$, we have $|F_n'(w)| < 1$ for every $n$ and $w \in \bigcup_{n\in\mathbb{N}} \mathscr{N}_n$ fixed. Thus, from Montel's Theorem $\{F_n'\}_{n\in\mathbb{N}}$ is normal. But, if $w \in \bigcup_{n\in\mathbb{N}} \mathscr{N}_n$ and $r > 0$, such that $D(w,r) \subset \bigcup_{n\in\mathbb{N}} \mathscr{N}_n$ then,

$$F_n(w) = \frac{1}{2\pi i} \int_{\partial D(w,r)} \frac{F_n'(\zeta)}{(w-\zeta)^2} d\zeta, \quad \text{for each } n \in \mathbb{N},$$

which implies that $|F_n(w)| < 1/r^2$ for every $n \in \mathbb{N}$. Therefore, once again Montel's Theorem implies that the family $\{F_n\}_{n\in\mathbb{N}}$ is normal. Moreover, $\{F_n\}_{n\in\mathbb{N}}$ is compact because $F_n(0) = 0$.

Thus, the subsequence $\{f_{n_k}(z,t) = h_{n_k}(z,t) + \overline{F_{n_k}(h_{n_k}(z,t))}\}_{k\in\mathbb{N}}$ contains a subsequence converging to $h(z,t) + \overline{F(h(z,t))}$, which is an $\mathscr{H}-$Loewner chain. □

The representation of the $\mathscr{H}-$Loewner chains as the sum

$$f(z,t) = h(z,t) + \overline{F(h(z,t))},$$

where $h(z,t)$ is a classical Loewner chain, helps us to show an analogue differential equation to the Loewner-Kufarev equation in the following form. In fact, if we derive in both sides with respect to $t$, we obtain

$$\begin{aligned}
\frac{\partial f(z,t)}{\partial t} &= \frac{\partial h(z,t)}{\partial t} + \frac{\partial \overline{F(h(z,t))}}{\partial t} \\
&= \frac{\partial h(z,t)}{\partial t} + \overline{F'(h(z,t))}\,\overline{\frac{\partial h(z,t)}{\partial t}} \\
&= G(h(z,t),t)\frac{\partial h(z,t)}{\partial z} + \overline{G(h(z,t),t)F'(h(z,t))}\,\overline{\frac{\partial h(z,t)}{\partial z}} \\
&= G(h(z,t),t)\frac{\partial h(z,t)}{\partial z} + \overline{G(h(z,t),t)}\,\overline{F'(h(z,t))\frac{\partial h(z,t)}{\partial z}} \\
&= G(h(z,t),t)\frac{\partial f(z,t)}{\partial z} + \overline{G(h(z,t),t)}\frac{\partial f(z,t)}{\partial \bar{z}}.
\end{aligned} \tag{1.39}$$

Now, we establish a partial differential equation type Loewner-Kufarev equation which is satisfied by these $\mathscr{H}-$Loewner chains.

**Theorem 1.5.4.** *Suppose that the harmonic function $f(z,t)$ defined by (1.37) is sense-preserving for each $t \geq 0$. Then $f(z,t)$ is an $\mathscr{H}-$Loewner chain if and only if the following two conditions are satisfied:*

*CL. I.) There exist $K_0 \geq 0$ and $r_0 > 0$ such that $h(z,t)$ is analytic in $|z| < r_0$ for each $t \geq 0$, absolutely continuous in $t \geq 0$ for each $|z| < r_0$ and satisfies*

$$|h(z,t)| \leq K_0 e^t, \qquad |z| < 1;\ t \geq 0. \tag{1.40}$$

*CL. II.) There exists an analytic function $G(z,t) = zP(z,t)$ with $P(z,t) \in \mathscr{P}$ and measurable on $t \geq 0$ for each $z \in D$, such that*

$$\frac{\partial f(z,t)}{\partial t} = G(z,t)\frac{\partial f(z,t)}{\partial z} + \overline{G(z,t)}\frac{\partial f(z,t)}{\partial \bar{z}}, \tag{1.41}$$

*for $|z| < r_0$ and almost every $t \geq 0$.*

*Proof.* Let us suppose $f(z,t) = h(z,t) + g(z,t)$ is a harmonic function for each $t \geq 0$ which satisfies CL. I.) and CL. II.). We will reason as in the proof of Theorem 1.2.1 Pommerenke [34]. Let $\phi(z,s,t)$ be the solution of the IVP (1.4) according to Theorem 1.2.2 where $s \leq t$. From (1.4) and (1.41) we have

$$\begin{aligned}
\frac{\partial}{\partial t}\{f(\phi(z,s,t),t)\} &= \frac{\partial f(\phi,t)}{\partial z}\frac{\partial \phi(z,s,t)}{\partial t} + \frac{\partial f(\phi,t)}{\partial \bar{z}}\frac{\partial \overline{\phi(z,s,t)}}{\partial t} + \frac{\partial f(\phi,t)}{\partial t} \\
&= \frac{\partial f(\phi,t)}{\partial z}\frac{\partial \phi(z,s,t)}{\partial t} + \frac{\partial f(\phi,t)}{\partial \bar{z}}\overline{\frac{\partial \phi(z,s,t)}{\partial t}} + \frac{\partial f(\phi,t)}{\partial t} \\
&= -\frac{\partial f(\phi,t)}{\partial z}G(z,t) - \frac{\partial f(\phi,t)}{\partial \bar{z}}\overline{G(z,t)} + \frac{\partial f(\phi,t)}{\partial t} \\
&= 0.
\end{aligned}$$

Here, we have used (1.40) and the sense-preserving property of $f$ to guarantee the existence of $h'(z,t)$ and $g'(z,t)$ uniformly in every bounded interval of $t \geq 0$. This implies that the right hand side does not depend on $t$. In particular,

$$f(\phi(z,s,t),t) = f(z,s), \qquad s \le t.$$

Thus, $h(z,s) = h(\phi(z,s,t),t)$. Moreover, from CL.II) if $|z| < 1$,

$$\frac{\partial h(z,t)}{\partial t} = G(z,t)h'(z,t) = zP(z,t)h'(z,t) \text{ for almost every } t \ge 0,$$

for an analytic function $P(z,t) \in \mathscr{P}$.

By the condition CL. I.) and Theorem 1.2.1, it follows that $h(z,t)$ is a classical Loewner chain and $h(z,t)$ is one-to-one.

The reciprocal follows from Definitions 1.5.1 and Equation (1.39). □

Although one might suspect that the evolution family, which makes the subordination of an $\mathscr{H}-$Loewner chain, and at the same time the subordination of its analytic part (that is a classical Loewner chain), could satisfy two different partial differential equations, this is not the case. The evolution family essentially satisfies only one equation, as is established in the following proposition.

**Proposition 1.5.5.** *The evolution family $\phi(z,s,t)$ associated to the subordination of an $\mathscr{H}-$Loewner chain $f(z,t)$ and its analytic part $h(z,t)$ which is a classical Loewner chain, satisfies essentially only one equation. That is, there exists essentially a unique $P \in \mathscr{P}$ measurable in $t \ge 0$ for each $z \in D$, such that*

$$\frac{\partial \phi(z,s,t)}{\partial t} = -\phi(z,s,t)P(\phi(z,s,t),t), \quad z \in D, \ \textit{for almost every } \ t \ge 0.$$

In this context, the term *essentially* means that $P \in \mathscr{P}$ is unique up to a null set (set of measure zero).

*Proof.* Since $f(z,t)$ is an $\mathscr{H}-$Loewner chain then, by Theorem 1.5.4 there exists an analytic function $G(z,t) = zP_1(z,t)$ for some $P_1 \in \mathscr{P}$ measurable in $t \geq 0$ for each $z \in D$, such that

$$\partial_s f(z,s) = zP_1(z,s)h'(z,s) + \overline{zP_1(z,s)g'(z,s)}, \quad |z| < 1;\ s \in \mathbb{R}^+ \setminus E_1, \tag{1.42}$$

where $E_1$ is a null set. Also, we have that $h(z,t)$ is a classical Loewner chain. So, by Theorem 1.2.1, there exists another function $P_2 \in \mathscr{P}$ measurable in $t \geq 0$ for each $z \in D$, such that

$$\partial_s h(z,s) = zP_2(z,s)h'(z,s), \quad |z| < 1;\ s \in \mathbb{R}^+ \setminus E_2, \tag{1.43}$$

where $E_2$ is a null set.

On the other hand, there exists another null set $E_3$, such that $\partial_s g(z,s)$ exists for every $s \in \mathbb{R}^+ \setminus E_3$, and the following equality holds

$$\partial_s f(z,s) = \partial_s h(z,s) + \overline{\partial_s g(z,s)}. \tag{1.44}$$

Now, if $s \in \mathbb{R}^+ \setminus \{E_1 \cup E_2 \cup E_3\}$, from Equations (1.42), and (1.44), we obtain that

$$\partial_s h(z,s) = zP_1(z,s)h'(z,s), \tag{1.45}$$

we have used the decomposition (1.2). Finally, from Equations (1.43), and (1.45), we have $P_1(z,s) = P_2(z,s)$ for almost every $s \in \mathbb{R}^+$. □

There exists a one-to-one relation between the conjugation of $\mathscr{H}-$Loewner chains and Equation (1.11) as follows:

**Corollary 1.5.6.** *The univalent harmonic function $W^{(\beta)}(z,t) = u^{(\beta)}(z,t) + \overline{v^{(\beta)}(z,t)}$ is the conjugation of an $\mathscr{H}-$Loewner chain with respect to $\beta \in D$ given if and only if the following two conditions are satisfied:*

1. *There exist $z_0 \in D$, $R_0 > 0$ and $K > 0$ such that $u^{(\beta)}(z,t)$ is analytic in $D(z_0,R_0)$ for each $t \geq 0$, absolutely continuous in $t \geq 0$ for each $z$ in $D(z_0,R_0)$ and*

$$|u^{(\beta)}(z,t)| \leq Ke^t, \quad \text{for } t \geq 0,\ z \in D(z_0,R_0).$$

2. *There exists an analytic function* $G(z,t) = (z-\beta)(1-\bar{\beta}z)P_\beta(z,t)$ *with* $P(z,t)$ *and* $P_\beta(z,t)$ *as in Theorem 1.3.3, such that*

$$\frac{\partial W(z,t)}{\partial t} = G(z,t)\frac{\partial W(z,t)}{\partial z} + \overline{G(z,t)}\frac{\partial W(z,t)}{\partial \bar{z}}, \tag{1.46}$$

*for* $|z| < r_0$ *and almost every* $t \geq 0$.

*Proof.* The result follows from Theorems 1.3.3 and 1.5.4. □

### $\mathscr{A}-$Loewner chains

We proceed, in this part, to introduce the Loewner chains when all the elements are considered to be harmonic functions. In particular, the evolution families.

**Definition 1.5.2.** *The sense-preserving harmonic function* $f_t(z) = f(z,t)$, *with* $z \in D$, *and* $t \geq 0$, *such that*

$$f(z,t) = \sum_{n\geq 1} a_n(t)z^n + \alpha\left(\sum_{n\geq 1}\overline{a_{-n}(t)}\bar{z}^n\right), \text{ with } |\alpha| < 1, \tag{1.47}$$

*where* $a_1(t) \neq 0$ *for* $t \geq 0$, *and* $\lim_{t\longrightarrow\infty} a_1(t) = +\infty$, *is called an* $\mathscr{A}-$Loewner chain *if the following two conditions hold,*

1. *For each* $t \geq 0$, $f(z,t)$ *is univalent,*

2. $f_s(z) \prec_{\mathscr{A}} f_t(z)$, *if* $0 \leq s \leq t < +\infty$.

**Remark 1.4.** *There exists an evolution family* $\phi(z,s,t)$ *of one-to-one harmonic functions from* $D$ *into* $D$ *associated to each* $\mathscr{A}-$*Loewner chain. Moreover,* $\phi(z,s,t) = f^{-1}(f(z,s),t)$.

**Example 11.** *The following family of functions is an example of an* $\mathscr{A}-$*Loewner chain:*

$$f(z,t) = \frac{e^{\alpha t}+e^{\beta t}}{2}z + \frac{e^{\alpha t}-e^{\beta t}}{2}\bar{z}, \quad \text{with } \alpha,\beta > 0.$$

*Moreover, the evolution family in* $\mathscr{G}_D$ *associated to the* $\mathscr{A}-$*subordination of these functions is:*

$$\phi(z,s,t) = \frac{e^{\alpha(s-t)}+e^{\beta(s-t)}}{2}z + \frac{e^{\alpha(s-t)}-e^{\beta(s-t)}}{2}\bar{z}.$$

Now, we establish a partial differential equation, which is satisfied by the $\mathscr{A}-$Loewner chains, assuming additional conditions.

**Theorem 1.5.7.** *Let $f(z,t)$ be an $\mathscr{A}-$Loewner chain that satisfies the following two conditions*

*A1. Its associated evolution family is in $\mathscr{G}_D$,*

*A2. There exist $K_0 \geq 0$ and $r_0 > 0$ such that $h(z,t)$ is analytic in $|z| < r_0$ for each $t \geq 0$, absolutely continuous in $t \geq 0$ for each $|z| < r_0$, and satisfies*

$$|h(z,t)| \leq K_0 e^t, \qquad |z| < r_0;\ t \geq 0. \tag{1.48}$$

*Then, there exists a harmonic function $G(z,t)$ measurable in $t \geq 0$ for each $z \in D$, such that*

$$\frac{\partial f(z,t)}{\partial t} = G(z,t)\frac{\partial f(z,t)}{\partial z} + \overline{G(z,t)}\frac{\partial f(z,t)}{\partial \bar{z}},$$

*for $|z| < r_0$ and almost every $t \geq 0$.*

*Proof.* Note that condition A2 implies that $h(z,t)$ and $g(z,t)$ are absolutely continuous in $t$ for each $|z| < 1$. Then, $f(z,s)$ is absolutely continuous in $t$ for each $|z| < 1$. Thus, $\partial_t f(z,t)$ exists for almost every $t \geq 0$.

Defining $t_n = s + 1/n$ with $n \in \mathbb{N}$, and fixed $s > 0$. Now, using the definition of subordination we can write

$$\begin{aligned}\frac{f(z,t_{n_k}) - f(z,s)}{t_{n_k} - s} &= \frac{h(z,t_{n_k}) - h(z,s)}{t_{n_k} - s} + \overline{\left(\frac{g(z,t_{n_k}) - g(z,s)}{t_{n_k} - s}\right)} \\ &= \frac{h(z,t_{n_k}) - h(\phi,t_{n_k})}{z - \phi}\,\frac{z-\phi}{t_{n_k} - s} + \overline{\frac{z-\phi}{t_{n_k} - s}}\,\overline{\left(\frac{g(z,t_{n_k}) - g(\phi,t_{n_k})}{z-\phi}\right)}.\end{aligned} \tag{1.49}$$

We have that $h(z,t_{n_k}) \longrightarrow h(z,s)$ and $g(z,t_{n_k}) \longrightarrow g(z,s)$ locally uniformly in $|z| < r_0$ as $n_k \longrightarrow \infty$, where $\{h(z,t_{n_k})\}$ and $\{g(z,t_{n_k})\}$ are the subsequences of $\{h(z,t_n)\}$ and $\{g(z,t_n)\}$, respectively, which convergent locally uniformly due to the Montel theorem applied to the families $\{h(z,t_n)\}$ and $\{g(z,t_n)\}$, taking into account that they are normal families. This latter statement follows from A2, and the Cauchy integral formula. Thus, we obtain $h'(z,t_{n_k}) \longrightarrow h'(z,s)$, and $g'(z,t_{n_k}) \longrightarrow g'(z,s)$ as $n_k \longrightarrow \infty$. Also, $\phi(z,s,t_{n_k}) \longrightarrow z$ as $n_k \longrightarrow \infty$. Therefore,

$$\frac{h(z,t_{n_k}) - h(\phi,t_{n_k})}{z - \phi(z,s,t_{n_k})} = \int_0^1 h'(z+\gamma[\phi(z,s,t_{n_k}) - z])d\gamma \longrightarrow h'(z,s), \text{ as } n_k \longrightarrow \infty,$$

$$\frac{g(z,t_{n_k}) - g(\phi,t_{n_k})}{z - \phi(z,s,t_{n_k})} = \int_0^1 g'(z+\gamma[\phi(z,s,t_{n_k}) - z])d\gamma \longrightarrow g'(z,s), \text{ as } n_k \longrightarrow \infty.$$

Let $s \geq 0$ be fixed such that $\frac{\partial f(z,s)}{\partial s}$ exists. If we let $n_k \longrightarrow \infty$ in Equations (1.49), and (1.31), we obtain

$$\begin{aligned}\frac{\partial f(z,s)}{\partial s} &= \lim_{n_k \longrightarrow \infty}\left\{\frac{z-\phi}{t_{n_k}-s}\frac{h(z,t_{n_k})-h(\phi,t_{n_k})}{z-\phi(z,s,t_{n_k})}\right\} + \lim_{n_k \longrightarrow \infty}\left\{\overline{\frac{z-\phi}{t_{n_k}-s}}\overline{\left(\frac{g(z,t_{n_k})-g(\phi,t_{n_k})}{z-\phi(z,s,t_{n_k})}\right)}\right\}\\ &= h'(z,s)G(z,s) + \overline{g'(z,s)G(z,s)}\\ &= \left\{G(z,s)\frac{\partial f(z,s)}{\partial z} + \overline{G(z,s)}\frac{\partial f(z,s)}{\partial \bar{z}}\right\}.\end{aligned}$$

□

This theorem extends and establishes a partial differential equation of Loewner-Kufarev type for the case of harmonic functions. Although, we have supposed that the associated evolution family is in the class $\mathscr{G}_D$, we hope to extend this result to a more general case.

# Chapter 2

## Evolution families in the space of $\rho-$nonexpansive mappings

### 2.1 Introduction

This chapter is inspired by the results given in the previous chapter. We are actually interested in extending the Loewner theory to non-analytic functions. This chapter is written following the setting given by F. Bracci, M. Contreras, and S. Díaz-Madrigal [6, 7], where the evolution mappings are introduced and studied before introducing the Loewner chains. Although here, we only study the evolution families of $\rho-$nonexpansive functions, we hope to extand aur study in future works.

In [41] Shoikhet studied the semigroups of analytic functions, and $\rho-$nonexpansive functions. He characterized the infinitesimal generator of both types of semigroups. In this study, he used the nonlinear resolvent of continuous functions on $D$. Furthermore, one characteristic of the infinitesimal generator of semigroups is the following autonomous IVP

$$\begin{cases} \frac{d}{dt}u(t) = F(u(t)), \quad t \in [0, +\infty), \\ u(0) = z, \end{cases}$$

where the solution is a semigroup and the vector field is its infinitesimal generator.

On the other hand, the main result in [6, 7] is to relate the concept of evolution families of analytic functions defined on $D$ or on a complex hyperbolic manifold, respectively, with a certain type of holomorphic functions, that the authors called Herglotz vector fields. This relation is done by means of the following non-autonomous IVP

$$\begin{cases} \dot{\omega} = G(\omega,t), & \text{for almost every } t \in [s,+\infty), \\ \omega(s) = z_0, \end{cases} \tag{2.1}$$

where the solution is an evolution family and $G(z,t)$ is a Herglotz vector field.

In this chapter we define the space $N_\rho(D)$ of $\rho-$nonexpansive functions on $D$, and the concept of a $\rho-$nonexpansive evolution family in $D$ is introduced as well, where $\rho$ is the Poincaré hyperbolic metric on $D$. These sets of functions are more general than the sets of harmonic functions and analytic functions. Also, we define a type of weak vector fields, which we call $\rho-$monotone weak vector field ($\rho-$WVF). The purpose of this chapter is to study the relation between these two given concepts by means of a non-autonomous IVP as before. Also, we introduce the infinitesimal generators of $\rho-$nonexpansive evolution families and the nonlinear resolvent for a certain class of functions. We establish some of their characteristics.

We begin by recalling some properties of the Poincaré hyperbolic metric and the $\rho-$nonexpansive mappings in Section 2.2. In Section 2.3 the concept of a $\rho-$nonexpansive evolution family in $D$ is introduced, and some properties of these families are shown. Section 2.4 is devoted to study the $\rho-$monotone weak vector fields. The introduction and the analysis of the infinitesimal generators of $\rho-$nonexpansive evolution families, and the nonlinear resolvent is made in Section 2.5. Finally, in Section 2.6, we establish a condition for the evolution family to have an infinitesimal generator.

## 2.2 Preliminaries

Let $\rho$ denote the Poincaré hyperbolic metric. This section is devoted to recall the space of $\rho-$nonexpansive functions, some properties of the Poincaré hyperbolic metric, and results that will be used in the following sections.

**Definition 2.2.1.** *A mapping $F : D \longrightarrow D$, is said to be $\rho-$nonexpansive if for each pair $z,w \in D$,*

$$\rho(F(z),F(w)) \leq \rho(z,w).$$

*Here, $\rho(z,w)$ denotes the Poincaré hyperbolic metric and it is defined as*

$$\rho(z,w) = \tanh^{-1}\left(\left|\frac{z-w}{1-z\bar{w}}\right|\right).$$

*Let us denote by $N_\rho(D)$ the class of all $\rho-$nonexpansive mappings on $D$.*

An equivalent formula that defines $\rho(z,w)$, and is useful, is the following

$$\rho(z,w) = \tanh^{-1}(|S_{-z}(w)|) = \rho(0,|S_{-z}(w)|),$$

where as before $S_{-z}(w) = \frac{w-z}{1-\bar{z}w}$ is a Möbius transformation.

Some properties of the Poincaré hyperbolic metric are listed in the next propositions.

**Proposition 2.2.1 (see [41]).** *The Poincaré metric $\rho$ on the unit disc $D$ satisfies the following properties:*

1. *If $z,w \in D$ are different points, then for each $k \in (0,1)$*

$$\rho(kz,kw) < k\rho(z,w).$$

2. *If $z,w,u \in D$ are different points, then for each $k \in (0,1)$*

$$\rho((1-k)z+ku,(1-k)w+ku) < \alpha\rho(z,w),$$

*where $\alpha = (1-k)+k|u| < 1$.*

3. *If $z,w,u,v \in D$ are different points, then for each $k \in [0,1]$*

$$\rho((1-k)z+ku,(1-k)w+kv) \leq \max\{\rho(z,w),\rho(u,v)\}.$$

*Proof.* See [41], page 47. □

The previous four properties can be considered as geometrical properties. The next list contains more topological properties. Their proofs can be found in [41] as well.

**Proposition 2.2.2 (see [41]).** *The Poincaré metric $\rho$ on the unit disc $D$ satisfies the following topological properties*

1. *If $a \in D$ and $r > 0$ then,*

$$|S_{-a}(w)| < r, \quad \textit{if and only if} \quad |w - S_{-r^2a}(a)| < S_{-r|a|^2}(r).$$

2. *If we define $B_\rho(a,R) = \{z \in D : \rho(z,a) < R\}$, and $B(a,r) = \{z \in D : |z-a| < r\}$ then, $B_\rho(a,R) = B(\lambda a, \eta \tanh R)$, where*

$$\lambda = \frac{1-(\tanh R)^2}{1-(\tanh R)^2|a|^2}, \quad \eta = \frac{1-|a|^2}{1-(\tanh R)^2|a|^2}.$$

*Moreover, $B_\rho(a,R) = S_a(B(0,r))$.*

3. *For every compact set $K \subset D$, there are constants $\mathbf{m}_K, \mathbf{M}_K > 0$, such that*

$$\mathbf{m}_K\,|z-w| \leq \rho(z,w) \leq \mathbf{M}_K\,|z-w|, \quad \textit{for all } z,w \in K. \tag{2.2}$$

*That is, the metrics $|\cdot|$ and $\rho$ are equivalent on compact sets.*

4. *If $z_n \in D$, for all $n \in \mathbb{N}$ then, $\lim_{n \longrightarrow +\infty} \rho(0,z_n) = +\infty$ if and only if $|z_n| \longrightarrow 1^-$ as $n \longrightarrow +\infty$, i.e., if $z_n \longrightarrow \partial D$.*

5. *If $z, z_n \in D$, for all $n \in \mathbb{N}$ then,*

$$\lim_{n \longrightarrow +\infty} \rho(z_n, z) = 0, \quad \textit{if and only if} \quad \lim_{n \longrightarrow +\infty} |z_n - z| = 0.$$

**Remark 2.1.** *1. Let us denote by $C^0(U,V)$ the space of all continuous mappings on $U$ into $V$, and $C^0(U) = C^0(U,U)$. Then, the class $N_\rho(D)$ is a closed convex subset of the space $C^0(D)$. Further, $N_\rho(D)$ is a semigroup with respect to the composition, see [41] for more details.*

*2. The Schwartz-Pick Lemma, implies that $\mathscr{H}(D)$, the set of all holomorphic self-mappings of $D$, is a subset of $N_\rho(D)$. Also, it is not difficult to see that $\bar{f} \in N_\rho(D)$ if $f \in N_\rho(D)$. Then,* *the convexity of $N_\rho(D)$ implies that* Re$[f] = \frac{f+\bar{f}}{2} \in N_\rho(D)$ *if* $f \in \mathscr{H}(D)$, *and* $\frac{f+\bar{g}}{2} \in N_\rho(D)$ *if* $f, g \in \mathscr{H}(D)$. *Thus,* $\mathscr{A}(D)$ *the set of all complex-valued harmonic self-mappings of* $D$ *is contained in* $N_\rho(D)$.

## 2.3 Evolution families in $N_\rho(D)$

**Definition 2.3.1.** *The family* $\{\varphi_{s,t}\}_{0\le s\le t<+\infty}$ *in* $N_\rho(D)$ *is called an evolution family if*

*EF1.* $\varphi_{s,s} = I_{id}$,

*EF2.* $\varphi_{s,t} = \varphi_{\tau,t} \circ \varphi_{s,\tau}$, *for all* $0 \le s \le \tau \le t < +\infty$,

*EF3. For each compact set* $K \subset D$ *and* $T > 0$, *there exists a constant* $C_{K,T} = C(K,T)$ *depending on* $K$ *and* $T$ *such that*

$$\rho\left(\varphi_{s,t}(z), \varphi_{s,\tau}(z)\right) \le C_{K,T}\,(t-\tau),$$

*for every* $z \in K$, *and* $0 \le s \le \tau \le t \le T$.

Direct consequences of the definition are given in the next remark.

**Remark 2.2.** *Let* $\{\varphi_{s,t}\}_{0\le s\le t<+\infty}$ *be an evolution family. Then,*

1. *From Remark 2.1 we have that, for every* $s \le t$, *the mapping* $\varphi_{s,t}(\cdot)$ *is continuous in* $D$. *That is, each element of an evolution family is continuous in* $D$.

2. *Condition EF3 implies that, for all* $z \in D$ *and for all* $s \ge 0$, *the mapping*

$$\begin{aligned} \varphi_{s,\cdot}(z) : [s,+\infty) &\longrightarrow \mathbb{C} \\ t \quad &\longrightarrow \varphi_{s,t}(z), \end{aligned}$$

*is continuous in* $[s,+\infty)$. *Furthermore, it is absolutely continuous on* $[s,T]$, *for all* $T > s$. *In fact, if* $z \in D$ *and* $s \ge 0$ *are given, we define* $y(t) = \varphi_{s,t}(z)$ *with* $t \ge s$. *To prove the first part, let us suppose* $t > s$, *and* $\varepsilon > 0$. *Now, let us consider a compact set* $K \subset D$ *containing* $z$, *and*

*$T > 2t - s$. Then, there exists a constant $C_{K,T} > 0$, such that*

$$\rho(y(t), y(\tau)) = \rho(\varphi_{s,t}(z), \varphi_{s,\tau}(z)) \leq C_{K,T}(t - \tau), \quad \text{if } 0 \leq s \leq \tau \leq t \leq T.$$

*Let $\delta = \min\{C_{K,T}^{-1}\varepsilon, t - s\} > 0$ be fixed. If $|\tau - t| < \delta$ then, in both cases either $\tau \leq t$ or $t \leq \tau$, we obtain*

$$\rho(y(t), y(\tau)) < \varepsilon.$$

*Thus, $y(t)$ is continuous in $t > s$. If $t = s$, all calculations work as well, but in this case, we only have $t \leq \tau$ obtaining a similar result.*

*The second assumption follows from the fact that the metric $\rho(z,w)$ and $|z-w|$ are equivalent on compact sets. Since $K = \{\varphi_{s,t}(z) : t \in [s,T]\}$ is a compact set then, $0 \leq s \leq \tau \leq t \leq T$ implies*

$$|\varphi_{s,t}(z) - \varphi_{s,\tau}(z)| \leq \mathbf{m}_K^{-1} \rho(\varphi_{s,t}(z), \varphi_{s,\tau}(z)) \leq \mathbf{m}_K^{-1} C_{K,T}(t - \tau).$$

*Thus, $\varphi_{s,\cdot}(z)$ is absolutely continuous on $[s,T]$.*

**Lemma 2.3.1.** *Let $\{\varphi_{s,t}\}_{0 \leq s \leq t < +\infty}$ be an evolution family, $s_0 \geq 0$, and $t_0 > 0$ be fixed.*

1. *If $t_n \to t_0^-$ as $n \to +\infty$ then, $\varphi_{t_n,t_0} \to I_{id}$ as $n \to +\infty$ uniformly on compact sets.*
2. *If $s_n \to s_0^+$ as $n \to +\infty$ then, $\varphi_{s_0,s_n} \to I_{id}$ as $n \to +\infty$ uniformly on compact sets.*

*Proof.* Let $\{\varphi_{s,t}\}_{0 \leq s \leq t < +\infty}$ be an evolution family, $s_0 \geq 0$, and $t_0 > 0$ be fixed. Then,

1. Let us suppose $t_n \longrightarrow t_0^-$ as $n \longrightarrow +\infty$. Let $K$ be a compact subset of $D$ and $T > t_0$. Then, for $z \in K$, and $n \geq 1$, we have

   $$\rho(\varphi_{t_n,t_0}(z), z) = \rho(\varphi_{t_n,t_0}(z), \varphi_{t_n,t_n}(z)) \leq C_{K,T}(t_0 - t_n).$$

   Setting $n \to +\infty$, we obtain $\rho(\varphi_{t_n,t_0}(z), z) \to 0$. Equivalently, $|\varphi_{t_n t_0}(z) - z| \to 0$.

2. Let us suppose that $s_n \longrightarrow s_0^+$ as $n \longrightarrow +\infty$. Since $\{s_n\}$ converges, then it is bounded. Let $K$

be a compact subset of $D$ and $T > \max\{s_n, s_0 : n \in \mathbb{N}\}$. Then, for $z \in K$, and $n \geq 1$, we have

$$\rho(\varphi_{s_0,s_n}(z), z) = \rho(\varphi_{s_0,s_n}(z), \varphi_{s_0,s_0}(z)) \leq C_{K,T}(s_n - s_0, ).$$

If we set $n \longrightarrow +\infty$, we obtain $\rho(\varphi_{s_0,s_n}(z), z) \longrightarrow 0$. Equivalently, we have $|\varphi_{s_0,s_n}(z) - z| \longrightarrow 0$.

□

The following proposition states the uniform joint continuity on compact sets of evolution families.

**Proposition 2.3.2.** *Let $\{\varphi_{s,t}\}_{0\leq s\leq t<+\infty}$ be an evolution family. For each compact set $K \subset D$ and two convergent sequences $\{s_n\}$ and $\{t_n\}$ in $[0, +\infty)$, with $0 \leq s_n \leq t_n$, such that $s_n \longrightarrow s$, and $t_n \longrightarrow t$, as $n \longrightarrow +\infty$. Then, $|\varphi_{s_n,t_n} - \varphi_{s,t}| \longrightarrow 0$ uniformly on $K$, as $n \longrightarrow +\infty$.*

*Proof.* Let $\{s_n\}$ and $\{t_n\}$ be two sequences in $[0, +\infty)$, with $0 \leq s_n \leq t_n$, such that $s_n \longrightarrow s$, and $t_n \longrightarrow t$, as $n \longrightarrow +\infty$. Since $\{s_n\}$ and $\{t_n\}$ are convergent, they are bounded. Let us take $T > 2\max\{\{s_n\}, \{t_n\}, s, t\} > 0$ fixed, and $K \subset D$ be a compact set. At this point, we have that $\{s_n\}$ and $\{t_n\}$ satisfy one of the following three options:

**Case I.** $s_n \leq t_n \leq s$ for all $n \in \mathbb{N}$,

**Case II.** $s \leq s_n \leq t_n$ for all $n \in \mathbb{N}$,

**Case III.** $s_n \leq s \leq t_n$ for all $n \in \mathbb{N}$.

In Case I, we have $s = t$. Then, EF1 and EF3 imply,

$$\rho(\varphi_{s_n,t_n}(z), \varphi_{s,t}(z)) = \rho(\varphi_{s_n,t_n}(z), \varphi_{s_n,s_n}(z)) \leq C_{K,T}(t_n - s_n) \longrightarrow 0 \text{ as } n \longrightarrow +\infty,$$

uniformly on $K$. Therefore, $|\varphi_{s_n,t_n}(z) - \varphi_{s,t}(z)| \longrightarrow 0$ uniformly on $K$.

In Case II, using EF1, EF2, EF3 and the $\rho$-nonexpansivity of $\varphi_{s,t}$, we have

$$
\begin{aligned}
\rho(\varphi_{s_n,t_n}(z),\varphi_{s,t}(z)) &= \rho(\varphi_{s_n,t_n}(z),\varphi_{s,t_n}(z)) + \rho(\varphi_{s,t_n}(z),\varphi_{s,t}(z)) \\
&= \rho(\varphi_{s_n,t_n}\circ\varphi_{s,s}(z),\varphi_{s_n,t_n}\circ\varphi_{s,s_n}(z)) + \rho(\varphi_{s,t_n}(z),\varphi_{s,t}(z)) \\
&\leq \rho(\varphi_{s,s}(z),\varphi_{s,s_n}(z)) + \rho(\varphi_{s,t_n}(z),\varphi_{s,t}(z)) \\
&\leq C_{K,T}\{(s_n-s)+|t_n-t|\} \longrightarrow 0 \text{ as } n\longrightarrow+\infty,
\end{aligned}
$$

uniformly on $K$. Therefore, $|\varphi_{s_n,t_n}(z)-\varphi_{s,t}(z)| \longrightarrow 0$ uniformly on $K$.

In Case III, using $\varphi_{s_n,s_n}=I_{id}$, EF2, EF3 and the $\rho$-nonexpansivity of $\varphi_{s,t}$, we obtain

$$
\begin{aligned}
\rho(\varphi_{s_n,t_n}(z),\varphi_{s,t}(z)) &= \rho(\varphi_{s_n,t_n}(z),\varphi_{s,t_n}(z)) + \rho(\varphi_{s,t_n}(z),\varphi_{s,t}(z)) \\
&= \rho(\varphi_{s,t_n}\circ\varphi_{s_n,s}(z),\varphi_{s,t_n}\circ\varphi_{s_n,s_n}(z)) + \rho(\varphi_{s,t_n}(z),\varphi_{s,t}(z)) \\
&\leq \rho(\varphi_{s_n,s}(z),\varphi_{s_n,s_n}(z)) + \rho(\varphi_{s,t_n}(z),\varphi_{s,t}(z)) \\
&\leq C_{K,T}\{(s-s_n)+|t_n-t|\} \longrightarrow 0 \text{ as } n\longrightarrow+\infty,
\end{aligned}
$$

uniformly on $K$. Therefore, $|\varphi_{s_n,t_n}(z)-\varphi_{s,t}(z)| \longrightarrow 0$ uniformly on $K$.

Thus, we obtain that the mapping $(s,t)\longrightarrow\varphi_{s,t}$ is jointly continuous. $\square$

**Remark 2.3.** *If $\{\varphi_{s,t}(z)\}_{0\leq s\leq t<+\infty}$ is a evolution family in $N_\rho(D)$. Then, for each compact $K\subset D$ and $T>0$, there is a constant $C_{K,T}>0$ such that*

$$\rho(\varphi_{s,t}(z),\varphi_{r,u}(z)) \leq C_{K,T}(|r-s|+|t-u|),$$

*for all $z\in K$ and $r,s,t,u\in[0,T]$, with $s\leq t$ and $r\leq u$. For instance, if $r,s,t,u\in[0,T]$, with $s\leq t$ and $r\leq u$ then, we have three options:*

1. $0\leq s\leq r\leq u\leq t\leq T$,
2. $0\leq s\leq t\leq r\leq u\leq T$,
3. $0\leq s\leq r\leq t\leq u\leq T$.

*The other cases are equivalent to these cases. Let us suppose $0 \leq s \leq r \leq t \leq u \leq T$. The other two cases can be proved in a similar way as this case.*

$$\begin{aligned}
\rho(\varphi_{s,t}(z), \varphi_{r,u}(z)) &\leq \rho(\varphi_{s,t}(z), \varphi_{r,t}(z)) + \rho(\varphi_{r,t}(z), \varphi_{r,u}(z)) \\
&\leq \rho(\varphi_{r,t}(\varphi_{s,r}(z)), \varphi_{r,t}(\varphi_{s,s}(z))) + C_{K,T}(u-t) \\
&\leq \rho(\varphi_{s,r}(z), \varphi_{s,s}(z)) + C_{K,T}(u-t) \\
&\leq C_{K,T}(r-s) + C_{K,T}(u-t) = C_{K,T}(|r-s| + |t-u|).
\end{aligned}$$

**Remark 2.4.** *For each $z \in D$, and $t > 0$, the function $s \in [0,t] \longrightarrow \varphi_{s,t}(z)$ is absolutely continuous on $[0,t]$.*

## 2.4 $\rho-$Monotone weak vector fields

In this section, we introduce the vector field, corresponding to the right hand side of the Loewner-Kufarev ordinary differential equation, associated to the evolution families in $N_\rho(D)$.

First, we are going to recall the well-known Carathéodory Theorem (see [9]), which guarantees the existence of an interval and a solution in an extended sense of an initial value problem, as follows:

Suppose $f$ is a complex-valued (not necessarily continuous) function defined in some set $\mathbf{R}$ of the $(z,t)-$space, containing the given point $(z_0, s) \in D \times [0, +\infty)$. We can formulate the next problem.

**Problem E.** *To find an absolutely continuous function $\omega$ defined on a interval $I$, such that*

$$\begin{cases} (\omega(t), t) \in \mathbf{R}, \ \textit{whenever} \ t \in I; \\ \dot{\omega} = f(\omega, t), \ \textit{for almost every} \ t \in I; \\ \omega(s) = z_0. \end{cases} \tag{2.3}$$

If such an interval $I$ and such a function $\omega$ exist then, $\omega$ is said to be a solution of (E) in the extended sense on $I$.

To solve this problem, we can reformulate the Carathéodory Theorem as follows, where $\mathbb{R}^* = \mathbb{R}^+ \cup \{0\}$.

**Theorem 2.4.1** (Carathéodory Theorem). *Let $f$ be a function defined on $\mathbf{R} = K \times [a,b] \subset D \times \mathbb{R}^*$, and $\mathbf{R}$ containing $(z_0, s)$ in its interior. Assume that $f$ is measurable in the variable $t$ on $[a,b]$, for each fixed $z \in D$, continuous in $z$ for each fixed $t \in [a,b]$.*

*If there exists a non-negative Lebesgue-integrable function $m(t)$ on $[a,b]$, such that*

$$|f(z,t)| \leq m(t), \quad (z,t) \in \mathbf{R}, \tag{2.4}$$

*then, there exist $I_{\mathbf{R}}(z_0, s) > s$, and a function $\phi$ such that the initial value problem (2.3) holds. Furthermore, if $f(\cdot, t)$ is Lipschitz on $K$ then, this solution is unique.*

See (A.2) for the proof and [9] for more details.

Next, we introduce the definition of the corresponding vector fields. Finally, we apply the Carathéodory Theorem to show that these vector fields have always associated a unique evolution family in $N_\rho(D)$.

**Definition 2.4.1.** *A $\rho$-monotone weak vector field ($\rho-WVF$, for short) on the unit disc $D$ is a function $G : D \times [0, +\infty) \longrightarrow \mathbb{C}$ which satisfies the following properties:*

*VF1. For all $z \in D$, the function $G(z, \cdot)$ is measurable on $[0, +\infty)$,*

*VF2. The function $G(0, \cdot) : [0, +\infty) \to \mathbb{C}$ is bounded on $[0, T]$ for all $T > 0$,*

*VF3. For all $t \geq 0$, the function $G(\cdot, t)$ is $\rho$-monotone, that is, if for each pair $z, w \in D$ the following condition holds:*

$$\rho(z - rG(z,t), w - rG(w,t)) \geq \rho(z, w),$$

*for all $r \geq 0$ such that $z - rG(z,t), w - rG(w,t) \in D$ and each fixed $t \geq 0$.*

*VF4. For any compact set $K \subset D$ and $T > 0$, there exists a constant $F_{K,T}$ such that*

$$|G(z,t) - G(w,t)| \leq F_{K,T}|z - w|, \quad \text{for all } z, w \in K, \ t \in [0, T]. \tag{2.5}$$

*Let us denote by $H_\rho(D)$ the set of all $\rho$-monotone weak vector field on the unit disc $D$.*

Actually, in the last definition we have thought of imposing, instead of VF4, the following condition:

VF4* *For each compact set $K$ and $T > 0$, there exists a bounded, non-negative (therefore Lebesgue-integrable) function $M_{K,T} : [0,T] \longrightarrow \mathbb{R}^+$, such that*

$$|G(z,t)| \leq M_{K,T}(t), \quad \textit{for all } \ z \in K, \ t \in [0,T]. \tag{2.6}$$

This condition, in order to apply Carathéodory's Theorem to an initial value problem as (2.3) where the right hand side is a $\rho-$WVF. We could note that (2.6) ensures the existence of "a" solution of such an IVP. But, we do not have uniqueness of the solution.

In the analytic case, it is possible to show that condition VF4 follows from VF4*. This implication is due to the Cauchy integral formula. In our case, there is no result such as the Cauchy formula. But, condition VF4* follows from VF4. The following lemma states such a implication.

**Lemma 2.4.2.** *Let $G(z,t)$ be a $\rho-$WVF. For any compact set $K \subset D$ and $T > 0$, there exists $M_{K,T}$ : $[0,T] \longrightarrow \mathbb{R}^+$ bounded, non-negative, and measurable function (and therefore Lebesgue-integrable) such that inequality (2.6) holds.*

*Proof.* If $K \subset D$ is a compact set and $T > 0$, let us consider $K_1 = K \cup \{0\}$. Then, VF4 implies the existence of a constant $F_{K_1,T}$ such that (2.5) holds. For $z \in K \subset K_1$ and $t \in [0,T]$, Equation (2.5) implies

$$\begin{aligned} |G(z,t)| &\leq |G(z,t) - G(0,t)| + |G(0,t)| \leq F_{K,T}|z| + |G(0,t)| \\ &\leq F_{K,T} + |G(0,t)|. \end{aligned}$$

Thus, Equation (2.6) holds with $M_{K,T}(t) = F_{K,T} + |G(0,t)|$, which is bounded and measurable on $[0,T]$, therefore Lebesgue-integrable on $[0,T]$, because $|G(0,t)|$ is bounded and measurable on $[0,T]$. $\square$

**Remark 2.5.** *Carathéodory's Theorem and Lemma 2.4.2 imply that if $G \in H_\rho(D)$ then, for every $z \in D$ and $s \geq 0$ there exist a unique number $I(z,s) > s$, and a unique function $\omega : [s, I(z,s)) \longrightarrow \mathbb{C}$, such that $(\omega(t), t) \in D \times [s, I(z,s))$, satisfying $\omega(t) \longrightarrow \partial D$, as $t \longrightarrow I(z,s)$ (see, for example, Theorem 1.3, page 47 [9]) such that*

$$\begin{cases} \dot{\omega} = G(\omega, t), & \text{for almost every } t \in [s, I(z,s)) \\ \omega(s) = z. \end{cases}$$

*The number $I(z,s)$ is called the* escaping time *and $\omega$ is called the positive trajectory of $G$. We define*

$$I(z,s) := \sup I_{K_r \times [0,N]}(z,s),$$

*the sup is taken over all compact sets $K_r := \{z \in D : |z| \leq r\}$ containing $z$ in its interior with $r \in (0,1)$, and $N \in \mathbb{N}$. Note that $I(z,s)$ could be infinity.*

**Remark 2.6.** *If $G \in H_\rho(D)$ and $z, w \in D$ are different points, note that condition VF3 implies*

$$(d\rho)_{(z,w)}(G(z,t), G(w,t)) \leq 0, \quad \text{for each } t \geq 0, \tag{2.7}$$

*where $(d\rho)$ denotes the total differential of $\rho$. Indeed, let $z, w \in D$ be given, with $z \neq w$. Let us choose $r > 0$ fixed, such that $z - rG(z,t), w - rG(w,t) \in D$. By convexity,*

$$h(\lambda) = \rho(z - \lambda G(z,t), w - \lambda G(w,t)), \quad \lambda \in [0,r]$$

*is well defined and differentiable at $0$. Then,*

$$h'(0) = (d\rho)_{(z,w)}(-G(z,t), -G(w,t)) = -(d\rho)_{(z,w)}(G(z,t), G(w,t)).$$

*But VF3 implies that $h(\lambda) \geq h(0)$, for all $\lambda \in [0,r]$, so that $h'(0) \geq 0$. Therefore, Equation (2.7) holds.*

The next theorem establishes a first relation between the elements of $H_\rho(D)$ and evolution families. More precisely, we show that every $G \in H_\rho(D)$ is a semi-complete vector field, i.e.,

$I(z,s)=+\infty$, and its positive trajectory is an evolution family.

**Theorem 2.4.3.** *Let $G(z,t)\in H_\rho(D)$ be a $\rho-$WVF. Then, there exists a unique evolution family $\varphi_{s,t}(z)$ in $N_\rho(D)$, which is the solution of the initial value problem*

$$\begin{cases} \dot{\omega}=G(\omega,t), & \text{for almost every } t\in[s,+\infty) \\ \omega(s)=z_0, \end{cases} \tag{2.8}$$

*for all given $s\geq 0$, and $z_0\in D$.*

*Proof.* From Remark 2.5, we guarantee the existence of $I(z,s)$ and the positive trajectory $\omega(t)=\varphi_s(z,t)$ of $G$, with $t\in[s,I(z,s))$, for each given $z\in D$ and $s\geq 0$. Our aim is to verify $I(z,s)=+\infty$ for all $z\in D$ and $s\geq 0$.

**1.** First, we are going to prove that $I(z,s)=I(w,s)$ for all $z,w\in D$, and $s\geq 0$. In fact, let us consider $z,w\in D$, $s\geq 0$, and define

$$g(t)=\rho(\varphi_s(z,t),\varphi_s(w,t)), \quad t\in[s,\min\{I(z,s),I(w,s)\}).$$

Then, $g$ is absolutely continuous on $[s,\min\{I(z,s),I(w,s)\})$. In fact, if $\{(x_i,y_i)\}_{i=1}^n$ is a finite collection of non-overlapping intervals of $[s,\min\{I(z,s),I(w,s)\})$, then there exists a compact set $K_0\subset D$, such that

$$\{\varphi_s(z,x_i),\varphi_s(z,y_i),\varphi_s(w,x_i),\varphi_s(w,y_i):i=1,2,\dots,n\}\subset K_0.$$

Moreover,

$$\begin{aligned}
|g(x_i)-g(y_i)| &= |\rho(\varphi_s(z,x_i),\varphi_s(w,x_i))-\rho(\varphi_s(z,y_i),\varphi_s(w,y_i))| \\
&\leq |\rho(\varphi_s(z,x_i),\varphi_s(w,x_i))-\rho(\varphi_s(z,y_i),\varphi_s(w,x_i))| \\
&\qquad\qquad + |\rho(\varphi_s(z,y_i),\varphi_s(w,y_i))-\rho(\varphi_s(z,y_i),\varphi_s(w,x_i))| \\
&\leq \rho(\varphi_s(z,x_i),\varphi_s(z,y_i))+\rho(\varphi_s(w,x_i),\varphi_s(w,y_i)) \\
&\leq \mathbf{M}_{K_0}\{|\varphi_s(z,x_i)-\varphi_s(z,y_i)|+|\varphi_s(w,x_i)-\varphi_s(w,y_i)|\}.
\end{aligned}$$

Since $\varphi_s(z,\cdot)$ and $\varphi_s(w,\cdot)$ are absolutely continuous on $[s,\min\{I(z,s),I(w,s)\})$, then for $\varepsilon>0$

there exists $\delta > 0$ such that if $\sum_{i=1}^{n} |x_i - y_i| < \delta$, then $\sum_{i=1}^{n} |\varphi_s(z,x_i) - \varphi_s(z,y_i)| < \varepsilon/2\mathbf{M}_{K_0}$ and $\sum_{i=1}^{n} |\varphi_s(w,x_i) - \varphi_s(w,y_i)| < \varepsilon/2\mathbf{M}_{K_0}$. Therefore,

$$\sum_{i=1}^{n} |g(x_i) - g(y_i)| < \varepsilon.$$

Now, for almost every $t \in [s, \min\{I(z,s), I(w,s)\})$, we have that $g'(t)$ exists, and

$$\begin{aligned} g'(t) &= (d\rho)_{(\varphi_s(z,t),\varphi_s(w,t))}(\dot{\varphi}_s(z,t), \dot{\varphi}_s(w,t)) \\ &= (d\rho)_{(\varphi_s(z,t),\varphi_s(w,t))}(G(\varphi_s(z,t),t), G(\varphi_s(w,t),t)) \leq 0. \end{aligned}$$

Hence $g$ is a decreasing function in $[s, \min\{I(z,s), I(w,s)\})$. Therefore,

$$\rho(\varphi_s(z,t), \varphi_s(w,t)) \leq \rho(z,w). \tag{2.9}$$

If we assume that $I(z,s) < I(w,s)$ then, $\varphi_s(z,t) \longrightarrow \partial D$ from Remark 2.5, while $\varphi_s(w,t) \longrightarrow \varphi_s(w, I(z,s))$, as $t \longrightarrow I(z,s)^-$. Therefore, $\rho(\varphi_s(z,t), \varphi_s(w,t)) \longrightarrow +\infty$ as $t \longrightarrow I(z,s)^-$, due to Proposition 2.2.2 4., which contradicts (2.9). If we suppose $I(w,s) < I(z,s)$ and interchanging the rolls, $z$ with $w$, and $w$ with $z$, in a similar way we also obtain a contradiction. Therefore, $I(z,s) = I(w,s)$ or equivalently, $I$ does not depend on $z$, i.e., $I(s) = I(z,s)$.

**2.** Let us define $I = I(0)$. The second step is to show that $I(s) = I$, if $s < I$. For instance, let us take $s < I$, and $w = \varphi_0(z,s) \in D$. Then, $\varphi_s(w,t)$ and $\varphi_0(z,t)$ are the solutions of an IVP with the same initial condition $w$ then, $\varphi_s(w,t) = \varphi_0(z,t)$ from the uniqueness of the solution. From Equation (2.9) we obtain

$$\rho(\varphi_s(z,t), \varphi_0(z,t)) = \rho(\varphi_s(z,t), \varphi_s(w,t)) \leq \rho(z,w).$$

Arguing as in Step 1. we show that $I = I(s)$.

**3.** In this step we show that $I = +\infty$. Let us suppose that $I < +\infty$, and let us choose $r < 1$. Applying Lemma 2.4.2 and VF4 with $T = I + 2$, and $K = \{|z| \leq r\}$, we have a constant $F_{K,T}$, and a function

$M_{K,T} : [0,T] \longrightarrow \mathbb{R}^+$, such that

$$|G(z,t) - G(w,t)| \leq F_{K,T}|z-w|, \quad \text{and} \quad |G(z,t)| \leq M_{K,T}(t), \tag{2.10}$$

for all $z,w \in K$, and a.e. $t \in [0,T]$.

Let us define $M : [0,I+2] \longrightarrow \mathbb{R}^+$, and $m : [0,I+2] \longrightarrow \mathbb{R}^+$, as follow

$$M(t) = \int_0^t F_{K,T}d\tau = F_{K,T}t, \quad m(t) = \int_0^t M_{K,T}(\tau)d\tau.$$

Note that, $M$ and $m$ are non-decreasing on $[0,T]$. From the integrability of $M_{K,T}$ on $[0,T]$, we have that $m(t)$ is absolutely continuous on $[0,T]$, and $M(t)$ is absolutely continuous on $[0,T]$ as well. Then, there exits a $\delta_1 > 0$ such that

$$\int_s^{s+\delta} F_{K,T}d\tau = M(s+\delta) - M(s) \leq r, \tag{2.11}$$

$$\int_s^{s+\delta} M_{K,T}(\tau)d\tau = m(s+\delta) - m(s) \leq r, \tag{2.12}$$

for all $\delta < \delta_1$, and $s + \delta_1 \leq T$. In particular, for $\delta_0 = \frac{1}{2}\min\{\delta_1, 1\}$, and $s \in [0,I+1]$, we always have that $s + \delta_0 \leq T$.

On the other hand, for a fixed $s \in [0,I+1]$, let us define by induction

$$\begin{cases} x_0^s(t) = 0, \\ x_n^s(t) = \int_s^t G(x_{n-1}^s(\tau),\tau)d\tau, \quad t \in [s,s+\delta_0]. \end{cases}$$

Now, by induction we prove that $|x_n^s(t)| \leq r$ for all $n \geq 0$ and $t \in [s,s+\delta_0]$. In fact, $|x_0^s(t)| = |0| \leq r$. Assuming that $|x_{n-1}^s(t)| \leq r$ for all $t \in [s,s+\delta_0]$ then, from (2.10) and (2.12), we have

$$|x_n^s(t)| \leq \int_s^t |G(x_{n-1}^s(\tau),\tau)|d\tau \leq \int_s^t M_{K,T}(\tau)d\tau \leq r.$$

Moreover, the last equation shows that $\{x_n^s(t)\}_{n\geq 0}$ is well defined for all $t \in [s,s+\delta_0]$.

If we define $\alpha_n = \max\limits_{t\in[s,s+\delta_0]}\{|x_n^s(t) - x_{n-1}^s(t)|\}$ for $n \geq 1$, we obtain $\alpha_n \leq \alpha_1 r^{n-1} \leq r^n$. In fact, note that if $n \geq 1$, Equations (2.10) and (2.11) imply

$$\begin{aligned}|x_n^s(t) - x_{n-1}^s(t)| &\leq \int_s^t |G(x_{n-1}^s(\tau),\tau) - G(x_{n-2}^s(\tau),\tau)| d\tau \\ &\leq \int_s^t F_{K,T}(\tau)|x_{n-1}^s(\tau) - x_{n-2}^s(\tau)| d\tau \\ &\leq \alpha_{n-1} \int_s^t F_{K,T}(\tau) d\tau \leq \alpha_{n-1} r.\end{aligned}$$

Thus, $\alpha_n \leq \alpha_{n-1} r$. By repeating the same argument, we obtain $\alpha_n \leq \alpha_{n-1} r \leq \alpha_{n-2} r^2 \leq \ldots \leq \alpha_1 r^{n-1} \leq r^n$, which implies our assertion.

Since $r < 1$ then, the $\{x_n^s(t)\}_{n\geq 0}$ is a Cauchy sequence in the Banach space $C^0([s, s+\delta_0], \mathbb{C})$. Therefore, $\{x_n^s(t)\}_{n\geq 0}$ converges uniformly on $[s, s+\delta_0]$, let us say to $x^s(t)$. By Equation (2.10) and the Lebesgue Dominated Converge Theorem it follows that

$$x^s(t) = \int_s^t G(x^s(\tau),\tau) d\tau, \; t \in [s, s+\delta_0].$$

By the uniqueness of the solution of ODEs,

$$\varphi_s(0,t) = x^s(t) \text{ for all } \; t \in [s, s+\delta_0],$$

which proves that $I(s) \geq s + \delta_0$ with $\delta_0 > 0$ and $s \leq I+1$. But, in Step 2. it was proved that $I(s) = I$ for all $s < I$. Then, taking $I - \delta_0 < s < I$ we have $I < s + \delta_0 \leq I(s) = I$ obtaining a contradiction. Therefore, $I = +\infty$.

**4.** The family $\varphi_{s,t}(z) = \varphi_s(z,t)$ is in fact an evolution family in $N_\rho(D)$. Indeed, Equation (2.9) implies that $\varphi_{s,t} \in N_\rho(D)$ for all $0 \leq s \leq t < +\infty$. On the other hand, $\varphi_{s,s}(z) = z$ for all $z \in D$ and for all $s \geq 0$ by the initial condition of the IVP. If $0 \leq s \leq \tau \leq t < +\infty$ and $z \in D$, we define $u(t) = \varphi_s(z,t)$ and $v(t) = \varphi_\tau(\varphi_{s,\tau}(z),t)$. Note that, both functions are the solution of the following IVP:

$$\begin{cases} \dot{\omega} = G(\omega,t), & \text{for almost every } t \in [s,+\infty), \\ \omega(\tau) = \varphi_{s,\tau}(z). \end{cases}$$

By the uniqueness of the solution of ODE's we obtain $\varphi_{s,t}(z) = \varphi_{\tau,t}(\varphi_{s,\tau}(z))$.

Let $K \subset D$ be a compact set and $T > 0$. For fixed $s \in [0,T)$, let us prove that there is $R = R_{K,T} \in (0,1)$, such that

$$|\varphi_{s,l}(z)| \leq R, \quad \text{for all } z \in K, \text{ and } l \in [s,T]. \tag{2.13}$$

If this assertion is not true, then there exist two sequences $\{z_n\} \subset K$ and $\{l_n\} \subset [s,T]$, such that $|\varphi_{s,l_n}(z_n)| \longrightarrow 1^-$ as $n \longrightarrow +\infty$. Moreover, there exist two convergent subsequences $z_{n_k} \longrightarrow z_0$ and $l_{n_k} \longrightarrow l_0$ as $k \longrightarrow +\infty$. Since $G(\cdot,t)$ is $\rho-$monotone, from Inequality 2.9, we have

$$\rho(\varphi_{s,l_{n_k}}(z_{n_k}), \varphi_{s,l_{n_k}}(z_0)) \leq \rho(z_{n_k}, z_0) \longrightarrow 0, \text{ as } k \longrightarrow +\infty.$$

Then,

$$1 > |\varphi_{s,l_{n_k}}(z_0)| \geq |\varphi_{s,l_{n_k}}(z_{n_k})| - |\varphi_{s,l_{n_k}}(z_{n_k}) - \varphi_{s,l_{n_k}}(z_0)| \longrightarrow 1^-, \text{ as } k \longrightarrow +\infty.$$

Thus, $|\varphi_{s,l_{n_k}}(z_0)| \longrightarrow 1^-$, as $k \longrightarrow +\infty$. Furthermore, the mapping $t \longrightarrow \varphi_{0,t}(z_0)$ is continuous and

$$\rho(\varphi_{0,l_{n_k}}(z_0), \varphi_{s,l_{n_k}}(z_0)) = \rho(\varphi_{s,l_{n_k}}(\varphi_{0,s}(z_0)), \varphi_{s,l_{n_k}}(z_0)) \leq \rho(\varphi_{0,s}(z_0), z_0) < +\infty.$$

But, this is a contradiction because $\varphi_{0,l_{n_k}}(z_0) \longrightarrow \varphi_{0,l_0}(z_0) \in D$, and $|\varphi_{s,l_{n_k}}(z_0)| \longrightarrow 1^-$, as $k \longrightarrow +\infty$. Therefore, (2.13) holds. Consider $K_1 = \{z \in D : |z| \leq R\}$ and $s \leq \tau \leq t \leq T$. Since $\varphi_s(z,\cdot)$ is absolutely continuous then, it is differentiable almost everywhere. Thus, Equation (2.6) implies

$$\begin{aligned}
|\varphi_{s,t}(z) - \varphi_{s,\tau}(z)| &= \left| \int_\tau^t \frac{\partial \varphi_{s,\sigma}(z)}{\partial \sigma} d\sigma \right| \\
&\leq \int_\tau^t |G(\varphi_{s,\sigma}(z), \sigma)| d\sigma \\
&\leq \int_\tau^t M_{K_1,T}(\sigma) d\sigma \leq C_{K_1,T}(t - \tau).
\end{aligned}$$

Hence, condition EF3 follows from Equation (2.2). Therefore, by the uniqueness of the solution of ODEs this $\{\varphi_{s,t}(z)\}_{0 \leq s \leq t < +\infty}$ is the unique evolution family in $N_\rho(D)$ associated to $G(z,t)$. $\square$

We state in the following proposition one property that evolution families, obtained by means of (2.8) and the $\rho-$WVFs, satisfies.

**Proposition 2.4.4.** *Let $\{\varphi_{s,t}(z)\}_{0\le s\le t<+\infty}$ be an evolution family obtained by means of (2.8) and a $\rho-$WVF. Then, for each compact set $K\subset D$ and $T>0$, there exists a constant $M_{K,T}>0$ such that*

$$|\varphi_{s,t}(z)-\varphi_{s,t}(w)|\le(1+M_{K,T}(t-s))|z-w|,$$

*for all $z,w\in K$, and $s,t\in[0,T]$.*

*Proof.* In fact, if $K\subset D$ is a compact set and $T>0$ and $z,w\in K$, and $s,t\in[0,T]$, we have

$$\varphi_{s,t}(z)=z+\int_s^t G(\varphi_{s,\sigma}(z),\sigma)d\sigma.$$

Then,

$$\begin{aligned}|\varphi_{s,t}(z)-\varphi_{s,t}(w)|&\le|z-w|+\int_s^t|G(\varphi_{s,\sigma}(z),\sigma)-G(\varphi_{s,\sigma}(w),\sigma)|\,d\sigma\\&\le|z-w|+F_{K,T}\int_s^t|\varphi_{s,\sigma}(z)-\varphi_{s,\sigma}(w)|\,d\sigma\\&\le|z-w|+\frac{F_{K,T}\,\mathbf{M}_K}{\mathbf{m}_K}|z-w|\int_s^t d\sigma=\left(1+\frac{F_{K,T}\,\mathbf{M}_K}{\mathbf{m}_K}(t-s)\right)|z-w|.\end{aligned}$$

Therefore, the proof is complete. □

We have shown that every $\rho-$monotone weak vector field has a unique evolution family associated by means of the initial value problem (2.8). In a Section 2.6 we are going to establish a condition to show the reciprocal implication.

## 2.5 Infinitesimal generator of evolution families, and nonlinear resolvent

At this point, we want to approach this work with the help of the nonlinear resolvent, as in the approach by D. Shoikhet in his study of semigroups and their infinitesimal generators. He characterizes the continuous functions which are infinitesimal generators of a semigroup. This approach was also used to study semigroups of holomorphic functions, and $\rho-$nonexpansive functions on certain domains in complex Banach spaces, and their infinitesimal generators [37, 38]. We recall some basic definitions used in the case of semigroups given in [41], and at the same time we introduce the analogue to such definitions for evolution families.

Firstly, we are interested in some types of infinitesimal generators as in the case of one-parameter semigroups. So, we give the next definition.

**Definition 2.5.1.** *Let $\{\varphi_{s,t}\}_{0\le s\le t<\infty}$ be an evolution family in $N_\rho(D)$. If for almost every $t\in[0,+\infty)$ there exists a sequence $\{h_n\}_{n\in\mathbb{N}}\subset(0,1)$ satisfying $h_n\longrightarrow 0^+$ as $n\longrightarrow+\infty$, such that the limit*

$$G(z,t)=\lim_{n\to+\infty}G_{h_n}(z,t),\quad \text{with}\quad G_h(z,t)=\frac{\varphi_{t,t+h}(z)-z}{h}, \tag{2.14}$$

*exists uniformly on compact sets $K\subset D$, we say that the family is generated by $G$ and $G$ is called the infinitesimal generator of the family. By $\mathscr{G}_{EF}N_\rho(D)$ we denote the set of all continuous functions for almost everywhere $t\in[0,+\infty)$ and all $z\in D$, which are generators of evolution families in $N_\rho(D)$.*

F. Bracci, M. Contreras, and S. Díaz-Madrigal [6] show that if $\{\varphi_{s,t}\}_{0\le s\le t<\infty}$ is an evolution family of holomorphic functions then there exists the infinitesimal generator $G(z,t)$ of this family, where $G(\cdot,t)$ is holomorphic on $D$ and $G(z,\cdot)$ is measurable on $[0,+\infty)$.

A direct consequence of this definition is the following proposition, which establishes that the infinitesimal generators of an evolution family can be considered as vector fields of the evolution equations.

**Proposition 2.5.1.** *If $G\in\mathscr{G}_{EF}N_\rho(D)$ then, for every $z\in D$, and $s\ge 0$ the IVP*

$$\begin{cases}\dot{\omega}=G(\omega,t), & \text{for almost every } t\in[s,+\infty)\\ \omega(s)=z,\end{cases} \tag{2.15}$$

*has a solution, which is an evolution family in $N_\rho(D)$.*

*Proof.* Since $G\in\mathscr{G}_{EF}N_\rho(D)$, there exist an evolution family $\{\varphi_{s,t}\}_{0\le s\le t<\infty}$ in $N_\rho(D)$, and a null set $I^*\subset[0,+\infty)$ such that 2.14 holds. Let us prove that this evolution family is the solution of the given IVP. Let $z\in D$ and $s\ge 0$ be given. Let us consider $K\subset D$ a compact set containing $z$, $T\in\mathbb{N}$ such that

$T > s$, and the compact set $K_1 := \{\varphi_{s,t}(z) : z \in K,\ t \in [s,T]\}$. Since, $\varphi_{s,\cdot}(z)$ is absolutely continuous on $[s,T]$ then, it is differentiable almost everywhere on $[s,T]$, i.e., there exists a null set $I_T \subset [s,T]$ such that $\frac{\partial \varphi_{s,t}(z)}{\partial t}$ exists, whenever $t \in [s,T] \setminus I_T$. Let us consider the null set $I := \cup_{T\in\mathbb{N}} I_T$. Therefore, $\varphi_{s,\cdot}(z)$ is differentiable almost every where on $[s,+\infty)$. If $t \in [s,T] \setminus \{I \cup I^*\}$ and $\varphi_{s,t}(z) \in K_1$ then,

$$\begin{aligned}
\frac{\partial \varphi_{s,t}(z)}{\partial t} &= \lim_{h\longrightarrow 0} \frac{\varphi_{s,t+h}(z) - \varphi_{s,t}(z)}{h} \\
&= \lim_{h_n\longrightarrow 0^+} \frac{\varphi_{t,t+h_n}(\varphi_{s,t}(z)) - \varphi_{s,t}(z)}{h_n} \\
&= \lim_{n\longrightarrow +\infty} G_{h_n}(\varphi_{s,t}(z),t) \\
&= G(\varphi_{s,t}(z),t).
\end{aligned}$$

Further, $\varphi_{s,s}(z) = z$. Therefore, $\omega(t) := \varphi_{s,t}(z)$ is a solution of the IVP. $\square$

Now, we recall the nonlinear resolvent of a continuous function $F : D \longrightarrow \mathbb{C}$ given in [41]. Then, we extend this definition for continuous functions $G : D \times [0,+\infty) \longrightarrow \mathbb{C}$.

**Definition 2.5.2.** *Let $F : D \longrightarrow \mathbb{C}$ be a continuous function. We say that $F$ satisfies the range condition (RC) if, for each $r > 0$ the nonlinear resolvent $J_r := (I - rF)^{-1}$ is well defined on $D$ and belongs to $N_\rho(D)$.*

Now, we extend the previous definition as follows.

**Definition 2.5.3.** *Let $G : D \times [0,+\infty) \longrightarrow \mathbb{C}$ be a continuous function. We say that $G$ satisfies the strong range condition (SRC) if $G$ satisfies the range condition for all fixed $t \geq 0$, and for each compact subset $K \subset D$, $T > 0$, and for $r > 0$ small enough, we have*

$$|J_r^t(z) - J_r^s(z)| \leq |t - s|, \tag{2.16}$$

*for all $s,t \in [0,T]$, and $z \in K$.*

In other words, the function $G$ satisfies the strong range condition if, for each $t \geq 0$, $r > 0$, and $z \in D$ the equation

$$w - rG(w,t) = z,$$

has a unique solution $w = J_r^t(z)$ in $D$, such that

$$\rho(J_r^t(z_1), J_r^t(z_2)) \leq \rho(z_1, z_2),$$

if $z_1, z_2 \in D$. Also, there exists $\delta_{K,T} > 0$, such that if $r \in (0, \delta_{K,T})$, the function $J_r^{(\cdot)}(z)$ satisfies the uniform Lipschitz condition (2.16) in the variable $t$ on compact sets.

**Remark 2.7.** *Let $G$ be a continuous function on $D \times [0, +\infty)$ such that for each $t \geq 0$, $G^t(\cdot) \equiv G(\cdot, t)$ satisfies the RC. For each compact $K \subset D$ and $T > 0$, there is another compact set $K \subset K^* \subset D$ and $\delta > 0$, such that*

$$J_r^t(z), J_{\frac{r}{n}}^{t^{(k)}}(z) \in K^*, \quad \text{for all} \quad r \in (0, \delta),\ n \in \mathbb{N},\ k = 1, 2, \ldots, n,$$

*whenever $z \in K$ and $t \in [0, T]$. Here, $J_r^{t^{(k)}}$ denotes the $k-$fold iterate of the mapping $J_r^t$. For instance, let $K \subset D$ be a compact set and $T > 0$. Let us consider $\delta = \min\{\frac{d}{M_1}, R_0\}$, where $0 < d < dist\{K, \partial D\}$,*

$$\begin{aligned} M &:= \sup\{|G(z,t)| : z \in K, t \in [0,T]\}, \\ R_0 &:= \max\{r : z - rG(z,t) \in D,\ |G(z,t)| \leq M, z \in K\}. \end{aligned}$$

*Let us set $w = z - rG(z,t) \in D$ for $z \in K$, $t \in [0,T]$ and $r \in (0, \delta)$, we have $z = J_r^t(w)$ and $|z - w| < rM < d$. Then,*

$$\rho(J_r^t(z), z) = \rho(J_r^t(z), J_r^t(w)) \leq \rho(z, w) \leq \mathbf{M}_{K^*} |z - w| \leq \mathbf{M}_{K^*} Mr,$$

*where $K^* := \overline{\bigcup_{z \in K} B(z, d)} \cup K$ and $M_1 := \max\{M, M\mathbf{M}_{K^*}\}$. Furthermore, if $l \in \mathbb{N}$ then,*

$$\rho(J_r^{t^{(l)}}(z), z) \leq \sum_{i=0}^{l-1} \rho(J_r^{t^{(i+1)}}(z), J_r^{t^{(i)}}(z)) \leq \sum_{i=0}^{l-1} \rho(J_r^t(z), z) = l\rho(J_r^t(z), z) \leq l\mathbf{M}_{K^*} Mr.$$

**Definition 2.5.4.** *Let $F : D \longrightarrow \mathbb{C}$ be a continuous function. The function $F$ is called $\rho$-*monotone *on $D$ if for each pair $z, w \in D$ the following condition holds:*

$$\rho(z - rF(z), w - rF(w)) \geq \rho(z, w), \tag{2.17}$$

*for all $r > 0$ such that $z - rF(z), w - rF(w) \in D$.*

The last definition can be extended as follows.

**Definition 2.5.5.** *Let $G : D \times [0, +\infty) \longrightarrow \mathbb{C}$ be a continuous function. We say that $G$ is strongly $\rho$-monotone on $D \times [0, +\infty)$ if for fixed $t \geq 0$, the function $G^t(\cdot) \equiv G(\cdot, t)$ is $\rho$-monotone on $D$, and for each compact subset $K \subset D$ and $T > 0$, there is a constant $F_{K,T} > 0$ such that*

$$|G(z,t) - G(w,s)| \leq F_{K,T}(|z - w| + |t - s|), \tag{2.18}$$

*for all $z, w \in K$ and $s, t \in [0, T]$.*

The reader can find the following three results and more details about these results in Shoikhet [41].

**Lemma 2.5.2 ([41]).** *For any given four points $u, v, z, w$ in $D$ the following statements are equivalent:*

1. *The function $\phi : [0,1] \to \mathbb{R}^+$, $\phi(t) = \rho((1-t)z + tu, (1-t)w + tv)$ is not decreasing on $[0,1]$;*
2. *$\phi(0) \leq \phi(t)$, $t \in [0,1]$;*
3. *$\phi'(t) \geq 0$, $t \in [0,1]$;*
4. *$Re\left[\frac{(z-u)\bar{z}}{1-|z|^2} + \frac{(w-v)\bar{w}}{1-|w|^2}\right] \leq Re\left[\frac{\bar{z}(w-v) + w\overline{(z-u)}}{1-\bar{z}w}\right]$.*

The next result is called the numerical range lower bound (see, Lemma 3.4.1 page 80, D. Shoikhet [41]).

**Lemma 2.5.3 (The numerical range lower bound [41]).** *Let $\alpha : [0,1] \to \mathbb{R}$ be a continuous function on $[0,1]$ such that $\alpha(0) \leq 0$ and the equation*

$$s + r\alpha(s) = \lambda$$

*has a unique solution $s = s(\lambda) \in [0,1)$ for each $\lambda \in [0,1)$ and $r > 0$. Suppose that $f : D \to \mathbb{C}$ is a continuous function on $D$ which satisfies the following condition:*

$$Re[f(w)\bar{w}] \geq \alpha(|w|)|w|, \quad w \in D.$$

*Then, for each $z \in D$ and $r \geq 0$, the equation $w + rf(w) = z$ has a unique solution $w = w(z)$, such that*

$$|w(z)| \leq s(\lambda),$$

*for $|z| \leq \lambda$.*

**Proposition 2.5.4.** *Let $G : D \times [0,+\infty) \longrightarrow \mathbb{C}$ be a continuous function. Then, for fixed $t \geq 0$, $G^t(\cdot) \equiv G(\cdot,t)$ is $\rho$-monotone if and only if it satisfies the range condition.*

*Proof.* Let $t \geq 0$ be fixed and $G^t$ satisfy the range condition. Then, for all $r > 0$ the nonlinear resolvent $J_r^t := (I - rG^t)^{-1}$ is well defined and $\rho-$nonexpansive self mapping of $D$, i.e.,

$$\rho((I - rG^t)^{-1}(\xi_1), (I - rG^t)^{-1}(\xi_2)) = \rho(J_r^t(\xi_1), J_r^t(\xi_2)) \leq \rho(\xi_1, \xi_2),$$

for all $\xi_1, \xi_2 \in D$. Now, if we take $z, w \in D$ and let $r > 0$ be such that $\xi_1 := z - rG(z,t) = (J_r^t)^{-1}(z)$ and $\xi_2 := w - rG(w,t) = (J_r^t)^{-1}(w)$ belong to $D$. Then, by definition $z = J_r^t(\xi_1)$ and $w = J_r^t(\xi_2)$. Thus,

$$\rho(z,w) = \rho(J_r^t(\xi_1), J_r^t(\xi_2)) \leq \rho(\xi_1, \xi_2) = \rho(z - rG(z,t), w - rG(w,t))$$

Therefore, $G$ is $\rho$-monotone.

Conversely, let us fix $t \geq 0$. If $z, w \in D$ denote $u = z - G(z,t)$ and $v = w - G(w,t)$. For $r > 0$ sufficiently small we have

$$\rho(z,w) \leq \rho(z - rG(z,t), w - rG(w,t)) = \rho((1-r)z + ru, (1-t)w + rv).$$

Let us denote $\phi(r) := \rho((1-r)z + ru, (1-r)w + rv)$. Thus, $\phi(r) \geq \phi(0)$. By Lemma 2.5.2, the last inequality is equivalent to

$$\mathrm{Re}\left[\frac{(z-u)\bar{z}}{1-|z|^2}+\frac{(w-v)\bar{w}}{1-|w|^2}\right]\leq \mathrm{Re}\left[\frac{\bar{z}(w-v)+w\overline{(z-u)}}{1-\bar{z}w}\right].$$

Substituting $z-u=G(z,t)$ and $w-v=G(w,t)$ in the last equation, we have

$$\mathrm{Re}\left[\frac{\bar{z}G(z,t)}{1-|z|^2}+\frac{G(w,t)\bar{w}}{1-|w|^2}\right]\leq \mathrm{Re}\left[\frac{\bar{z}G(w,t)+w\overline{G(z,t)}}{1-\bar{z}w}\right].$$

Now, if we consider $z=0$, we obtain

$$\mathrm{Re}\left[\frac{G(w,t)\bar{w}}{1-|w|^2}\right]\leq \mathrm{Re}\left[w\overline{G(0,t)}\right]=\mathrm{Re}\left[\bar{w}G(0,t)\right],$$

for all $w\in D$. Then,

$$\mathrm{Re}\left[G(w,t)\bar{w}\right]\leq \mathrm{Re}\left[\bar{w}G(0,t)\right](1-|w|^2),\quad w\in D.$$

But, $\mathrm{Re}[z]\leq |z|$. Thus,

$$\mathrm{Re}\left[-G(w,t)\bar{w}\right]\geq -|w||G(0,t)|(1-|w|^2)=|w|\alpha_t(|w|),\quad w\in D,$$

with $\alpha_t(s)=-|G(0,t)|(1-s^2)$. Then, applying the numerical range lower bound with $f\equiv -G^t$, we obtain that for each $z\in D$ and $r>0$ equation

$$w-rG(w,t)=z,$$

has a unique solution $w=J_r^t(z)=(I-rG^t)^{-1}(z)$. Furthermore, the function $J_r^t$ is $\rho-$nonexpansive. Indeed, if $z,w\in D$ consider $u=J_r^t(z)$ and $v=J_r^t(w)$. Thus, $z=u-rG(u,t)$ and $w=v-rG(v,t)$. Since $G$ is $\rho$-monotone we have

$$\rho(z,w)=\rho(u-rG(u,t),v-rG(v,t))\geq \rho(u,v)=\rho(J_r^t(z),J_r^t(w)).$$

Thus, the proof is completed. $\square$

This result can be extended to functions depending on two parameters, and the strong definitions given above, assuming an additional condition.

**Theorem 2.5.5.** *Let $G : D \times [0,+\infty) \to \mathbb{C}$ be a continuous function. Then, $G$ is strongly $\rho$-monotone if and only if the following two conditions hold:*

***SRC.*** *$G$ satisfies the strong range condition.*

***ASRC.*** *For each compact subset $K \subset D$, $T > 0$, and $R > 0$, there exists a function $L_{K,T,R} : (0,R) \longrightarrow \mathbb{R}^+$ satisfying $0 < \lim_{r \longrightarrow 0^+} L_{K,T,R}(r) < +\infty$ such that*

$$L_{K,T,R}(r)\rho(z,w) \leq \rho(J_r^t(z), J_r^t(w)), \tag{2.19}$$

*for all $t \in [0,T]$, and $z,w \in K$.*

*Proof.* From Proposition 2.5.4, we only have to show that Equation (2.18) is equivalent to Equations (2.16) and (2.19).

Let us suppose that Equation (2.18) holds. Let $K \subset D$ be a compact subset of $D$, $T > 0$, and $R > 0$. To show that Equation (2.16) holds, let us consider $s,t \in [0,T]$ and $z \in K$ then,

$$z = J_r^t(z) - rG(J_r^t(z),t), \quad z = J_r^s(z) - rG(J_r^s(z),s).$$

Subtracting these equations, we have

$$\begin{aligned} |J_r^t(z) - J_r^s(z)| &= r|G(J_r^t(z),t) - G(J_r^s(z),s)| \\ &\leq rF_{K,T}\left(|J_r^t(z) - J_r^s(z)| + |t-s|\right) \\ &= rF_{K,T}|J_r^t(z) - J_r^s(z)| + rF_{K,T}|t-s|. \end{aligned}$$

The last inequality implies that,

$$|J_r^t(z) - J_r^s(z)| \leq \frac{rF_{K,T}}{1 - rF_{K,T}}|t-s|, \quad \text{if } r < \frac{1}{F_{K,T}}.$$

But, if we assume $r \leq \frac{1}{2F_{K,T}}$ we obtain

$$|J_r^t(z) - J_r^s(z)| \leq |t - s|.$$

Thus, Equation (2.16) holds. Now, we show that Equation (2.19) holds. If $t \in [0,T]$ and $z,w \in K$ then, for $r \in (0,R)$ we have

$$z = J_r^t(z) - rG(J_r^t(z),t), \quad w = J_r^t(w) - rG(J_r^t(w),t).$$

Subtracting these equations, we have

$$\begin{aligned}
\frac{1}{\mathbf{M}_{K^*}}\rho(z,w) \leq |z-w| &\leq |J_r^t(z) - J_r^t(w)| + r|G(J_r^t(z),t) - G(J_r^t(w),t)| \\
&\leq |J_r^t(z) - J_r^t(w)| + rF_{K,T}|J_r^t(z) - J_r^t(w)| \\
&= (1 + rF_{K,T})|J_r^t(z) - J_r^t(w)| \leq \frac{(1 + rF_{K,T})}{\mathbf{m}_{K^*}}\rho(J_r^t(z), J_r^t(w)).
\end{aligned}$$

Thus,

$$\frac{\mathbf{m}_{K^*}}{\mathbf{M}_{K^*}(1 + rF_{K,T})}\rho(z,w) \leq \rho(J_r^t(z), J_r^t(w)).$$

Therefore, the first part is shown.

Conversely, let $K \subset D$ be a compact subset and $T \geq 0$. We have to show that there is a constant $F_{K,T} > 0$, such that

$$|G(z,t) - G(w,s)| \leq F_{K,T}(|z-w| + |t-s|),$$

for all $z,w \in K$ and $s,t \in [0,T]$. If $z,w \in K$ and $s,t \in [0,T]$, let us consider $z_0 = (J_{r_0}^t)^{-1}(z)$, and $w = J_{r_0}^s(w_0)$ with fixed $r_0 \in (0, \min\{1,\delta\})$, where $\delta > 0$ is given by the SRC and $R = 1$. Then,

$$z_0 = z - r_0 G(z,t), \qquad w_0 = w - r_0 G(w,s).$$

Thus, $r_0|G(z,t)-G(w,s)| \leq |z-w|+|z_0-w_0|$. Then, by SRC and ASRC, we obtain

$$\begin{aligned} r_0|G(z,t)-G(w,s)| &\leq |z-w| + \tfrac{\mathbf{M}_{K^*}}{\mathbf{m}_{K^*}L_{K,T}(r_0)}|J^t_{r_0}(z_0)-J^t_{r_0}(w_0)| \\ &\leq |z-w| + \tfrac{\mathbf{M}_{K^*}}{\mathbf{m}_{K^*}L_{K,T}(r_0)}\left(|J^t_{r_0}(z_0)-J^s_{r_0}(w_0)|+|J^s_{r_0}(w_0)-J^t_{r_0}(w_0)|\right) \\ &\leq |z-w| + \tfrac{\mathbf{M}_{K^*}}{\mathbf{m}_{K^*}L_{K,T}(r_0)}\left(|z-w|+|t-s|\right) \\ &\leq \left(1+\tfrac{\mathbf{M}_{K^*}}{\mathbf{m}_{K^*}L_{K,T}(r_0)}\right)(|z-w|+|t-s|). \end{aligned}$$

Therefore, setting $F_{K,T} \geq \dfrac{1}{r_0}\left(1+\tfrac{\mathbf{M}_{K^*}}{\mathbf{m}_{K^*}L_{K,T}(r_0)}\right)$ we complete the proof. □

In the next proposition we establish one way to construct functions satisfying SRC.

**Proposition 2.5.6.** *Let $\{\phi_t : t \geq 0\} \subset N_\rho(D)$ be a family of $\rho-$nonexpansive functions on $D$ that satisfies the following condition: for each compact subset $K \subset D$ and $T > 0$ there exists a constant $C_{K,T} > 0$, such that*

$$|\phi_t(z)-\phi_s(z)| \leq C_{K,T}|t-s|, \quad \textit{for all } z \in K,\ s,t \in [0,T].$$

*Then, the functions $G(z,t) := \phi_t(z)-z$, and more generally $\gamma G(z,t)$ for $\gamma > 0$, satisfy the strong range condition.*

*Proof.* For each $t \geq 0$, $z \in D$ and $r > 0$ the equation

$$z = w - r(\phi_t(w)-w), \tag{2.20}$$

has a unique solution $w = J^t_r(z)$, and $J^t_r : D \longrightarrow D$ is a $\rho-$nonexansive mapping. In fact, if $t \geq 0$, $z \in D$ and $r > 0$ the last equation can be written as

$$w = (1-\lambda)z + \lambda\phi_t(w) = H^t_r(w), \quad \lambda = \frac{r}{r+1}.$$

Then,

$$\begin{aligned}\rho(H_r^t(w_1),H_r^t(w_2)) &= \rho((1-\lambda)z+\lambda\phi_t(w_1),(1-\lambda)z+\lambda\phi_t(w_2))\\ &\leq ((1-\lambda)|z|+\lambda)\rho(\phi_t(w_1),\phi_t(w_2))\\ &\leq ((1-\lambda)|z|+\lambda)\rho(w_1,w_2)=\alpha\rho(w_1,w_2), \quad \alpha<1.\end{aligned}$$

From the Banach Fixed Point Theorem we have that the function $H_r^t$ has a unique fixed point in $D$, which is the unique solution to Equation (2.20) and we denote by $J_r^t$. Now, let us see that for fixed $r>0$, and $t\geq 0$, the mapping $J_r^t : D \longrightarrow D$ is a $\rho-$nonexansive mapping. Indeed, this fixed point $w=J_r^t(z)$ can be obtained by the iteration:

$$w^{(n+1)}(z,t)=H_r^t(w^{(n)}(z,t))=\ldots=(H_r^t)^{(n+1)}(w^{(0)}(z)),$$

where $w^{(0)}(z)$ is an arbitrary element in $D$. Then, setting $w_h^{(0)}(z)=z$, we obtain that

$$J_r^t(z)=\lim_{n\to\infty}(H_r^t)^{(n)}(z).$$

By induction and Proposition 2.2.1 (3), we have

$$\begin{aligned}&\rho\left((H_r^t)^{(n+1)}(z_1),(H_r^t)^{(n+1)}(z_2)\right)\\ &\qquad= \rho\left((1-\lambda)z_1+\lambda\phi_t\left((H_r^t)^{(n)}(z_1)\right),(1-\lambda)z_2+\lambda\phi_t\left((H_r^t)^{(n)}(z_2)\right)\right)\\ &\qquad\leq \max\left\{\rho(z_1,z_2),\rho\left(\phi_t\left((H_r^t)^{(n)}(z_1)\right),\phi_t\left((H_r^t)^{(n)}(z_2)\right)\right)\right\}\\ &\qquad\leq \max\left\{\rho(z_1,z_2),\rho\left((H_r^t)^{(n)}(z_1),(H_r^t)^{(n)}(z_2)\right)\right\}\\ &\qquad\leq \ldots\leq \max\left\{\rho(z_1,z_2),\rho\left((H_r^t)^{(0)}(z_1),(H_r^t)^{(0)}(z_2)\right)\right\}=\rho(z_1,z_2),\end{aligned}$$

which implies that $J_r^t(\cdot)$ is a $\rho-$nonexansive mapping.

It remains to show the uniform Lipschitz condition in the variable $t$ on compact sets. In fact, let

$K \subset D$ be a compact set $T > 0$, $z \in K$, and $s,t \in [0,T]$. Since,

$$J_r^t(z) = \lambda \phi_t(J_r^t(z)) + (1-\lambda)z, \quad \lambda = \frac{r}{r+1}.$$

Then, $w = J_r^t(z) \in K^*$, and

$$\begin{aligned}
|J_r^t(z) - J_r^s(z)| &= \lambda |\phi_t(J_r^t(z)) - \phi_s(J_r^s(z))| \\
&\leq r \left\{ |\phi_t(J_r^t(z)) - \phi_s(J_r^t(z))| + |\phi_s(J_r^t(z)) - \phi_s(J_r^s(z))| \right\} \\
&\leq r \left\{ |\phi_t(w) - \phi_s(w)| + |\phi_s(J_r^t(z)) - \phi_s(J_r^s(z))| \right\} \\
&\leq r \left\{ C_{K^*,T}|t-s| + \frac{1}{\mathbf{m}_{K^*}} \rho\left(\phi_s(J_r^t(z)), \phi_s(J_r^s(z))\right) \right\} \\
&\leq r \left\{ C_{K^*,T}|t-s| + \frac{\mathbf{M}_{K^*}}{\mathbf{m}_{K^*}} \left|J_r^t(z) - J_r^s(z)\right| \right\}.
\end{aligned}$$

Thus, if $r < \frac{\mathbf{m}_{K^*}}{\mathbf{M}_{K^*}}$, we have

$$|J_r^t(z) - J_r^s(z)| \leq \frac{rC_{K^*,T}\,\mathbf{m}_{K^*}}{\mathbf{m}_{K^*} - r\mathbf{M}_{K^*}}|t-s|.$$

Finally, considering $r \leq \frac{\mathbf{m}_{K^*}}{C_{K^*,T}\,\mathbf{m}_{K^*} + \mathbf{M}_{K^*}}$, we obtain

$$|J_r^t(z) - J_r^s(z)| \leq |t-s|,$$

which implies the strong range condition for the function $G(z,t)$. To prove the assertion for the function $\gamma G(z,t)$, is just to replace $\lambda = \frac{r}{r+1}$ with $\lambda = \frac{\gamma r}{\gamma r+1}$ or equivalently $\gamma r$ instead of $r$. □

**Remark 2.8.** *The functions constructed in the previous proposition also satisfy the condition ASRC given in Proposition 2.5.5. In fact, let $K \subset D$ be a compact subset, $T > 0$, and $R > 0$ be given. If $z,w \in K$ and $t \in [0,T]$, for $r < 1$, we have*

$$J_r^t(z) = \lambda \phi_t(J_r^t(z)) + (1-\lambda)z, \quad J_r^t(w) = \lambda \phi_t(J_r^t(w)) + (1-\lambda)w.$$

*Then,*

$$\begin{aligned}
(1-\lambda)\rho(z,w) &\leq (1-\lambda)\,\mathbf{M}_{K^*}\,|z-w| \\
&\leq \mathbf{M}_{K^*}\,|J_r^t(z)-J_r^t(w)| + \mathbf{M}_{K^*}\,\lambda\,|\phi_t(J_r^t(z))-\phi_t(J_r^t(w))| \\
&\leq \frac{\mathbf{M}_{K^*}}{\mathbf{m}_{K^*}}\rho(J_r^t(z),J_r^t(w)) + \frac{\lambda\,\mathbf{M}_{K^*}}{\mathbf{m}_{K^*}}\rho(\phi_t(J_r^t(z)),\phi_t(J_r^t(w))) \\
&\leq \frac{\mathbf{M}_{K^*}}{\mathbf{m}_{K^*}}\rho(J_r^t(z),J_r^t(w)) + \frac{\lambda\,\mathbf{M}_{K^*}}{\mathbf{m}_{K^*}}\rho(J_r^t(z),J_r^t(w)) \\
&\leq \frac{2\,\mathbf{M}_{K^*}}{\mathbf{m}_{K^*}}\rho(J_r^t(z),J_r^t(w)).
\end{aligned}$$

*Therefore,*

$$\frac{\mathbf{M}_{K^*}}{2\,\mathbf{m}_{K^*}(r+1)}\rho(z,w) \leq \rho(J_r^t(z),J_r^t(w)).$$

*Thus, the desired result is proved.*

We follow with some interesting results which will be used in the following section and it will help us to associate evolution families with $\rho-$monotone vector fields.

**Theorem 2.5.7.** *If $G \in \mathscr{G}_{EF}N_\rho(D)$ then, for each $h \in (0,1)$, the functions $G_h(z,t)$ given by Definition 2.5.1, satisfy the strong range condition, and the function $G(\cdot,t)$ satisfies the range condition for almost every fixed $t \geq 0$.*

*Proof.* Let us suppose that the function $G \in \mathscr{G}_{EF}N_\rho(D)$ then, it is continuous on $D$ and almost every $t \in [0,+\infty)$, and has associated an evolution family $\{\varphi_{s,t}\}_{0\leq s\leq t<\infty}$ in $N_\rho(D)$, for which, for almost every $t \in [0,+\infty)$ there is a sequence $\{h_n(t)\}_{n\in\mathbb{N}} \subset (0,1)$ satisfying $h_n(t) \longrightarrow 0^+$ as $n \longrightarrow +\infty$, such that the limit

$$G(z,t) = \lim_{n\longrightarrow+\infty} G_{h_n(t)}(z,t), \quad \text{with} \quad G_h(z,t) = \frac{\varphi_{t,t+h}(z)-z}{h}, \tag{2.21}$$

exists uniformly on compact sets $K \subset D$. Then, for $t \geq 0$, $h \in (0,1)$ and $z \in D$, let us consider

$$\phi_{t,h}(z) = \varphi_{t,t+h}(z), \quad \text{then} \quad G_h(z,t) = \frac{\phi_{t,h}(z)-z}{h}.$$

Note that $\phi_{t,h} \in N_\rho(D)$ for all $t \geq 0$ and $h \in (0,1)$. Further, from Remark 2.3, for each $h \in (0,1)$

$$|\phi_{t,h}(z) - \phi_{s,h}(z)| = |\varphi_{t,t+h}(z) - \varphi_{s,s+h}(z)| \leq 2C_{K,T}|t-s|, \quad \text{for all } z \in K,\ s,t \in [0,T].$$

By Proposition 2.5.6, for each $h \in (0,1)$ the function $G_h$ satisfies the strong range condition then, $J^t_{r,h} : D \longrightarrow D$ is well defined, belongs to $N_\rho(D)$, and for each compact subset $K \subset D$, $T > 0$, and exist $\delta_h > 0$ such that if $r \in (0, \delta_h)$ then,

$$|J^t_{r,h}(z) - J^s_{r,h}(z)| \leq |t-s|, \tag{2.22}$$

for all $s,t \in [0,T]$, $z \in K$.

Let us see that $G(\cdot,t)$ satisfies the range condition for every $t \in [0,+\infty)$, that is, for every fixed $r > 0$, $t \in [0,+\infty)$, and $z \in D$, the equation

$$w - rG(w,t) = z, \tag{2.23}$$

has a unique solution $w = J^t_r(z)$, where $J^t_r : D \longrightarrow D$ belongs to $N_\rho(D)$.

Since $J^t_{r,h}(z)$ is the fixed point of the function

$$H^t_{r,h}(w) := \lambda \phi_{t,h}(w) + (1-\lambda)z, \qquad \lambda = \frac{r}{r+h}.$$

Then, this fixed point can be obtained by the iteration:

$$w_h^{(n+1)}(z,t) = H^t_{r,h}(w_h^{(n)}(z,t)) = \ldots = (H^t_{r,h})^{(n+1)}(w_h^{(0)}(z)),$$

where $w_h^{(0)}(z)$ is an arbitrary element in $D$. Then, setting $w_h^{(0)}(z) = z$ we obtain that

$$J^t_{r,h}(z) = \lim_{n\to\infty} (H^t_{r,h})^{(n)}(z).$$

Now, we claim that for every $z \in D$, exists $A_z < +\infty$ such that

$$\rho(z, J^t_{r,h}(z)) \leq A_z, \text{ as } h \longrightarrow 0^+.$$

In fact, since

$$J^t_{r,h}(z) = H^t_{r,h}(J^t_{r,h}(z)) = \lambda \phi_{t,h}(J^t_{r,h}(z)) + (1-\lambda)z,$$

we have, from Proposition 2.2.1 3. with $\alpha = \lambda + (1-\lambda)|z|$

$$\begin{aligned}
\rho(z, J^t_{r,h}(z)) &= \rho(\lambda z + (1-\lambda)z, \lambda \phi_{t,h}(J^t_{r,h}(z)) + (1-\lambda)z) \\
&\leq \alpha \rho(z, \phi_{t,h}(J^t_{r,h}(z))) \\
&\leq \alpha[\rho(z, \phi_{t,h}(z)) + \rho(\phi_{t,h}(z), \phi_{t,h}(J^t_{r,h}(z)))] \\
&\leq \alpha[\rho(z, \phi_{t,h}(z)) + \rho(z, J^t_{r,h}(z))].
\end{aligned}$$

Then, this implies

$$\rho(z, J^t_{r,h}(z)) \leq \frac{\alpha}{1-\alpha}\rho(z, \phi_{t,h}(z)) = \frac{r+h|z|}{1-|z|}\frac{\rho(\varphi_{t,t}(z), \varphi_{t,t+h}(z))}{h} \leq C_{K,T}\frac{r+h|z|}{1-|z|},$$

where $K \subset D$ is a compact set containing $z$ and $t \leq T$. Consequently, we have the estimate

$$\limsup_{h \longrightarrow 0^+} \rho(z, J^t_{r,h}(z)) \leq \frac{rC_{K,T}}{1-|z|}, \tag{2.24}$$

which implies our assertion. Further, this implies that $J^t_{r,h}(D) \subset D$, for all $h \in (0,1)$.

We now prove that for each $r > 0$, and almost every $t \in [0, +\infty)$, the sequence $\{f_{h_n}\}_{n \in \mathbb{N}}$, contains a uniform convergent subsequence on every compact set $K \subset D$, where the sequence $\{h_n\}_{n \in \mathbb{N}} \subset (0,1)$ is the sequence given by Definition 2.5.1, which satisfies $h_n \longrightarrow 0$ as $n \longrightarrow +\infty$, and $f_{h_n} := J^t_{r,h_n} : D \longrightarrow D$. Indeed, if $K \subset D$ is a compact set, and $z \in K$, inequality (2.24) implies that the closure of $\{f_{h_n}(z) : n \in \mathbb{N}\}$ is a compact subset of $D$. Moreover, $\{f_{h_n}\}_{n \in \mathbb{N}}$ is equicontinuous on $K$. Then, $\{f_{h_n}\}_{n \in \mathbb{N}}$ contains a uniform convergent subsequence on $K$ (see, for example, [40]). Let

$\{f_{h_{n_j}}\}_{j\in\mathbb{N}}$ be such a subsequence, and let us consider

$$J_r^t(z) = \lim_{j\longrightarrow+\infty} J^t_{r,h_{n_j}}(z).$$

Note that $J_r^t(z)$ is a solution of

$$w - rG(w,t) = z.$$

Furthermore, $J_r^t(\cdot)$ is a $\rho-$nonexpansive function on $D$. Therefore, $G(\cdot,t)$ satisfies the range condition. $\square$

Considering $K \subset D$ a compact set and $T > 0$ then, for each $h \in (0,1)$, we have that for all $s,t \in [0,T]$, $z \in K$,

$$\begin{aligned}
|J^t_{r,h}(z) - J^s_{r,h}(z)| &= \lambda\,|\varphi_{t,t+h}(J^t_{r,h}(z)) - \varphi_{s,s+h}(J^s_{r,h}(z))| \\
&\le \lambda\left\{|\varphi_{t,t+h}(J^t_{r,h}(z)) - \varphi_{s,s+h}(J^t_{r,h}(z))| + |\varphi_{s,s+h}(J^t_{r,h}(z)) - \varphi_{s,s+h}(J^s_{r,h}(z))|\right\} \\
&\le \lambda\left\{2C_{K^*,T}|t-s| + \frac{\mathbf{M}_{K^*}}{\mathbf{m}_{K^*}}|J_r^t(z) - J_r^s(z)|\right\} \\
&\le \frac{r}{h}\left\{2C_{K^*,T}|t-s| + \frac{\mathbf{M}_{K^*}}{\mathbf{m}_{K^*}}|J_r^t(z) - J_r^s(z)|\right\}.
\end{aligned}$$

Hence, if $r \le \alpha_\varepsilon h < \frac{\mathbf{m}_{K^*}}{\mathbf{M}_{K^*}}h$, with $\alpha_\varepsilon = \frac{\mathbf{m}_{K^*}}{2C_{K^*,T}\,\mathbf{m}_{K^*}/\varepsilon + \mathbf{M}_{K^*}} < 1$, and $\varepsilon > 0$, we obtain

$$|J^t_{r,h}(z) - J^s_{r,h}(z)| \le \varepsilon|t-s|.$$

**Theorem 2.5.8.** *If $G$ is strongly $\rho$-monotone on $D\times[0,+\infty)$ then, $G \in \mathcal{G}_{EF}N_\rho(D)$.*

*Proof.* Note that if $G$ is strongly $\rho$-monotone on $D\times[0,+\infty)$ then, $G \in H_\rho(D)$. Theorem 2.4.3, ensures the existence of an evolution family $\varphi_{s,t}(z)$, such that (2.8) holds. It remains to show the uniform convergence of the Definition 2.5.1. I in fact, if $K \subset D$ is a compact set and $T > 0$, let us consider $z \in K$ and $t \in [0,T]$, then we have

$$\varphi_{t,t+h}(z) = z + \int_t^{t+h} G(\varphi_{t,\sigma}(z),\sigma)\,d\sigma.$$

Thus,

$$\begin{aligned}
\frac{1}{h}\left|\varphi_{t,t+h}(z)-z-hG(z,t)\right| &= \frac{1}{h}\left|\int_t^{t+h} G(\varphi_{t,\sigma}(z),\sigma)d\sigma - \int_t^{t+h} G(z,t)d\sigma\right| \\
&\leq \frac{1}{h}\int_t^{t+h} |G(\varphi_{t,\sigma}(z),\sigma)-G(z,t)|\,d\sigma \\
&\leq \frac{F_{K,T}}{h}\int_t^{t+h} (|\varphi_{t,\sigma}(z)-z|+|\sigma-t|)\,d\sigma \\
&\leq \frac{F_{K,T}}{h}\int_t^{t+h} \left(\frac{C_{K,T}}{\mathbf{m}_K}|t-\sigma|+|\sigma-t|\right)d\sigma \\
&\leq \frac{F_{K,T}}{h}\left(\frac{C_{K,T}}{\mathbf{m}_K}+1\right)\int_t^{t+h} |\sigma-t|d\sigma \\
&\leq F_{K,T}\left(\frac{C_{K,T}}{\mathbf{m}_K}+1\right)\int_t^{t+h} d\sigma \\
&= F_{K,T}\left(\frac{C_{K,T}}{\mathbf{m}_K}+1\right)h \longrightarrow 0, \quad h\longrightarrow 0^+.
\end{aligned}$$

This implies the uniform convergence on $K\subset D$, and almost every $t\in[0,T]$. Therefore, $G(z,t)$ is the infinitesimal generator of the evolution family $\varphi_{s,t}(z)$. □

**Corollary 2.5.9.** *Let $G : D\times[0,+\infty)\longrightarrow\mathbb{C}$ be a function defined on $D\times[0,+\infty)$, satisfying that for any compact set $K$ and $T>0$, there exists a constant $F_{K,T}$ such that*

$$|G(z,t)-G(w,s)|\leq F_{K,T}(|z-w|+|t-s|), \quad \text{for all } z,w\in K,\ a.e., s,t\in[0,T]. \tag{2.25}$$

*Then, the following conditions are equivalent:*

1. *The function $G\in\mathscr{G}_{EF}N_\rho(D)$.*

2. *For almost every $t\geq 0$, the function $G(\cdot,t)$ satisfies the range condition.*

3. *For almost every $t\geq 0$, the function $G(\cdot,t)$ is $\rho$-monotone on $D$.*

4. *The Cauchy problem*

$$\begin{cases} \dfrac{\partial u(z,t)}{\partial t} = G(u(z,t),t), & \text{for almost every } t\geq s \\ u(z,s)=z, \end{cases}$$

*has a unique solution $u(z,t) = \varphi_{s,t}(z)$ for all $s \geq 0$ and $z \in D$; where $\varphi_{s,t}(z)$ is an evolution family in $N_\rho(D)$.*

*Proof.*

(2) $\Leftrightarrow$ (3) It follows from Proposition 2.5.4.

(3) $\Rightarrow$ (4) It follows from Theorem 2.4.3.

(4) $\Rightarrow$ (1) It follows from proof of Theorem 2.5.8.

(1) $\Rightarrow$ (2) It follows from proof of Theorem 2.5.7. $\square$

**Proposition 2.5.10.** *Let $G : D \times [0,+\infty) \longrightarrow \mathbb{C}$ be a continuous function which satisfies the strong range condition. Then, the sequence $\{J_{\frac{r}{n}}^{t^{(n)}}\}_{n\in\mathbb{N}}$ is a Cauchy sequence on compact sets of $D$ in the Poincaré metric in $D$ with fixed $t \geq 0$. Therefore, the limit*

$$\lim_{n\longrightarrow+\infty} J_{\frac{r}{n}}^{t^{(n)}}(z),$$

*exists uniformly on compact sets of $D$.*

*Proof.* Let us assume that $G$ is a continuous function which satisfies the strong range condition. We claim that for each $r > 0$ small enough, the sequence $\{J_{\frac{r}{n}}^{t^{(n)}}\}_{n\in\mathbb{N}}$ is a Cauchy sequence on compact subsets of $D$ in the Poincaré metric in $D$ with fixed $t \geq 0$.

Let $K \subset D$ be a compact set, $T > 0$. From Remark 2.7 we have, for $z \in K$, $r \in (0,\delta)$ and $t \in [0,T]$,

$$\rho(J_r^t(z),z) = \rho(J_r^t(z),J_r^t(w)) \leq \rho(z,w) \leq \mathbf{M}_{K^*}|z-w| \leq \mathbf{M}_{K^*}Mr.$$

This shows that the family $\{J_r^t\}_{r\in(0,\delta)}$ converges to the identity uniformly on compact sets of $D$ as $r \longrightarrow 0^+$. Furthermore,

$$\rho(J_r^{t^{(l)}}(z),z) \leq \sum_{i=0}^{l-1}\rho(J_r^{t^{(i+1)}}(z),J_r^{t^{(i)}}(z)) \leq \sum_{i=0}^{l-1}\rho(J_r^t(z),z) = l\rho(J_r^t(z),z) \leq l\,\mathbf{M}_{K^*}Mr.$$

In a similar way, we obtain

$$\rho(J_{r/n}^{t^{(l)}}(z),z) \leq \frac{r\cdot l}{n}\mathbf{M}_{K^*}M, \quad \text{and} \quad \rho(J_r^{t^{(l)}}(z),J_r^{t^{(k)}}(z)) \leq |l-k|\,\mathbf{M}_{K^*}Mr.$$

Let us take $R > 0$ such that $K^* \subset B_\rho(a,R)$ with fixed $a \in D$. Consider $\varepsilon > 0$ and $z \in K$. By Lemma 3.3.3, (D. Shoikhet, page 73 [41]), there are $\mu > 0$, $n_0, m_0 \in \mathbb{N}$ such that

$$\rho(J^{t}_{\frac{r}{n}}(z), J^{t^{(m)}}_{\frac{r}{mn}}(z)) < \frac{2r\varepsilon}{n}, \quad \text{and} \quad \rho(J^{t}_{\frac{r}{m}}(z), J^{t^{(n)}}_{\frac{r}{mn}}(z)) < \frac{2r\varepsilon}{m},$$

when $n \geq n_0$, $m \geq m_0$ and $z \in B_\rho(a,R)$.

On the other hand, we have

$$\begin{aligned}
\rho(J^{t^{(n)}}_{\frac{r}{n}}(z), J^{t^{(mn)}}_{\frac{r}{mn}}(z)) &\leq \sum_{j=0}^{n-1} \rho\left(J^{t^{(n-j)}}_{\frac{r}{n}}(J^{t^{(jm)}}_{\frac{r}{mn}}(z)), J^{t^{(n-j-1)}}_{\frac{r}{n}}(J^{t^{((j+1)m)}}_{\frac{r}{mn}}(z))\right) \\
&= \sum_{j=0}^{n-1} \rho\left(J^{t^{(n-j-1)}}_{\frac{r}{n}}(J^{t}_{\frac{r}{n}}(J^{t^{(jm)}}_{\frac{r}{mn}}(z))), J^{t^{(n-j-1)}}_{\frac{r}{n}}(J^{t^{((j+1)m)}}_{\frac{r}{mn}}(z))\right) \\
&\leq \sum_{j=0}^{n-1} \rho\left(J^{t}_{\frac{r}{n}}(J^{t^{(jm)}}_{\frac{r}{mn}}(z)), J^{t^{((j+1)m)}}_{\frac{r}{mn}}(z)\right) \\
&= \sum_{j=0}^{n-1} \rho\left(J^{t}_{\frac{r}{n}}(J^{t^{(jm)}}_{\frac{r}{mn}}(z)), J^{t^{(m)}}_{\frac{r}{mn}}(J^{t^{(jm)}}_{\frac{r}{mn}}(z))\right) \\
&= \sum_{j=0}^{n-1} \rho\left(J^{t}_{\frac{r}{n}}(w), J^{t^{(m)}}_{\frac{r}{mn}}(w)\right) = \sum_{j=0}^{n-1} \frac{2r\varepsilon}{n} = 2r\varepsilon.
\end{aligned}$$

In a similar way we obtain

$$\rho(J^{t^{(m)}}_{\frac{r}{m}}(z), J^{t^{(mn)}}_{\frac{r}{mn}}(z)) \leq 2r\varepsilon.$$

Hence,

$$\rho(J^{t^{(n)}}_{\frac{r}{n}}(z), J^{t^{(m)}}_{\frac{r}{m}}(z)) \leq \rho(J^{t^{(n)}}_{\frac{r}{n}}(z), J^{t^{(mn)}}_{\frac{r}{mn}}(z)) + \rho(J^{t^{(m)}}_{\frac{r}{m}}(z), J^{t^{(mn)}}_{\frac{r}{mn}}(z)) \leq 4r\varepsilon.$$

Therefore, the sequence $\{J^{t^{(n)}}_{\frac{r}{n}}(z)\}$ is a Cauchy sequence on $K$ and since $(D,\rho)$ is a complete metric space then,

$$F(z,r,t) = \lim_{n \longrightarrow +\infty} J^{t^{(n)}}_{\frac{r}{n}}(z),$$

exists uniformly on compact sets of $D$. $\square$

We can show the continuity of $J_r^t$ in the variable $r > 0$. In fact, if $r+s < \delta$

$$
\begin{aligned}
\rho(J_r^t(z), J_{r+s}^t(z)) &\le \rho\left(J_r^t(z), J_r^t\left(\frac{r}{r+s}z + \frac{s}{r+s}J_{r+s}^t(z)\right)\right) \\
&\le \rho\left(z, \frac{r}{r+s}z + \frac{s}{r+s}J_{r+s}^t(z)\right) \\
&\le \mathbf{M}_{K^*}\left|z - \frac{r}{r+s}z + \frac{s}{r+s}J_{r+s}^t(z)\right| \\
&\le \frac{s\,\mathbf{M}_{K^*}}{r+s}\left|z - J_{r+s}^t(z)\right| \le \frac{s\,\mathbf{M}_{K^*}}{(r+s)\,\mathbf{m}_{K^*}}\rho\left(z, J_{r+s}^t(z)\right) \\
&\le \frac{s\,\mathbf{M}_{K^*}}{(r+s)\,\mathbf{m}_{K^*}}\mathbf{M}_{K^*}M(r+s) = s\tilde{C}_{K^*} \longrightarrow 0,
\end{aligned}
$$

as $s \longrightarrow 0$. Also, we have for $t, t_1 \in [0,T]$, and $r > 0$ small enough

$$
\begin{aligned}
\rho(J_r^t(z), J_{r+s}^{t_1}(z)) &\le \rho(J_r^t(z), J_r^{t_1}(z)) + \rho(J_r^{t_1}(z), J_{r+s}^{t_1}(z)) \\
&\le \rho(J_r^t(z), J_r^{t_1}(z)) + s\tilde{C}_{K^*} \\
&\le \mathbf{M}_{K^*}\left|J_r^t(z) - J_r^{t_1}(z)\right| + s\tilde{C}_{K^*} \\
&\le \mathbf{M}_{K^*}\left|t - t_1\right| + s\tilde{C}_{K^*}.
\end{aligned}
$$

**Definition 2.5.6.** *We say that a family $\{F_r^t : 0 \le r \le t < +\infty)\}$, of self-mappings of $D$ satisfies the approximate evolution property if for each compact subset $K \subset D$ and $T > 0$ the following conditions hold:*

1. *For each $\varepsilon > 0$, there is a positive $\delta_1 = \delta(D, \varepsilon) < T$ such that*

$$\sup_{z \in K} \rho\left(F_r^t(z), (F_{\frac{r}{n}}^t)^{(n)}(z)\right) < \varepsilon\delta_1$$

   *for all positive integers $n$ and all $r \in (0, \delta_1)$;*

2. *For each $0 \le s \le \tau \le t \le T$, there exist $L = L_{K,T}$ and $\delta_2 = \delta(K,T) > 0$, such that*

$$\sup_{z \in K} \rho\left(F_{t-s}^t(z), F_{t-\tau}^t\left(F_{\tau-s}^\tau(z)\right)\right) < L\sqrt{pq}, \quad p = t-\tau,\ q = \tau - s.$$

*if* $|p|,|q| < \delta_2$.

We have the next result.

**Proposition 2.5.11.** *If $G : D \times [0,+\infty) \to \mathbb{C}$ satisfies the strong range condition then, $\{J_r^t\}$ satisfies the approximate evolution property. Moreover,*

$$F(z,r,t) = \lim_{n \longrightarrow +\infty} J_{\frac{r}{n}}^{t^{(n)}}(z),$$

*exists uniformly on compact sets of $D \times [0,+\infty)$ and fixed $r > 0$.*

*Proof.* By Proposition 2.5.10, we have only to prove the condition (2). If $s < \tau < t \le T$, consider $a = t-s$, $b = t-\tau$ and $c = \tau - s$, with $|t-s| < \delta_2$, and $\delta_2 > 0$ given by Definition 2.5.3. Then, $b \le a$ and $a = b+c$. Since

$$w = \frac{b}{a}z + \left(1 - \frac{b}{a}\right) J_a^t(z) = \frac{c}{a}\left(J_a^t(z) - z\right) + z.$$

Then,

$$\begin{aligned}
\rho\left(J_{t-s}^t(z), J_{t-\tau}^t\left(J_{\tau-s}^\tau(z)\right)\right) &= \rho\left(J_a^t(z), J_b^t\left(J_c^\tau(z)\right)\right) \\
&= \rho\left(J_b^t(w), J_b^t\left(J_c^\tau(z)\right)\right) \\
&\le \rho\left(w, J_c^\tau(z)\right) \le \mathbf{M}_{K^*}\left|w - J_c^\tau(z)\right| \\
&= \frac{\mathbf{M}_{K^*}}{a}\left|bz + cJ_a^t(z) - (b+c)J_c^\tau(z)\right| \\
&\le \frac{\mathbf{M}_{K^*}}{a}\left[c\left|J_a^t(z) - J_c^\tau(z)\right| + b\left|z - J_c^\tau(z)\right|\right] \\
&\le \frac{\mathbf{M}_{K^*}}{a}\left[c\left|J_a^t(z) - J_a^\tau(z)\right| + c\left|J_a^\tau(z) - J_c^\tau(z)\right| + b\left|z - J_c^\tau(z)\right|\right] \\
&\le \frac{\mathbf{M}_{K^*}}{a}\left[c|t-\tau| + c\frac{\tilde{C}_{K^*}}{\mathbf{m}_{K^*}}|a-c| + b\frac{\mathbf{M}_{K^*}M}{\mathbf{m}_{K^*}}c\right] \\
&\le \frac{bc}{a}\mathbf{M}_{K^*}\left[1 + \frac{\tilde{C}_{K^*}}{\mathbf{m}_{K^*}} + \frac{\mathbf{M}_{K^*}M}{\mathbf{m}_{K^*}}\right] \le \frac{1}{2}F_{K^*,T}\sqrt{bc}.
\end{aligned}$$

Thus, the desired result is obtained. $\square$

## 2.6 The property LB

In this section, we want to establish a condition under which the infinitesimal generator, for a given evolution family in $N_\rho(D)$, exists in $H_\rho(D)$ or $\mathscr{G}_{EF}N_\rho(D)$.

Let $\varphi_{s,t}(z)$ be an evolution family in $N_\rho(D)$, $K \subset D$, and $T > 0$. Let us recall from page 65

$$G_h(z,t) := \frac{\varphi_{t,t+h}(z) - z}{h}, \quad z \in K_0,\ t \in [0,T],\ h \in (0,1).$$

From Proposition 2.5.6, we have that for each $h \in (0,1)$, $G_h(z,t)$ satisfies the strong range condition. Then, $J^t_{r,h} : D \longrightarrow D$ is well defined and belongs to $N_\rho(D)$. Furthermore,

$$J^t_{r,h}(z) = \lambda \varphi_{t,t+h}(J^t_{r,h}(z)) + (1-\lambda)z, \quad \lambda = \frac{r}{r+h}.$$

Furthermore, if $K \subset D$ is a compact subset, and $T > 0$ then, for $z, w \in K$, $t \in [0,T]$, $r > 0$, and $h \in (0,1)$, we have

$$\begin{aligned}
\frac{(1-\lambda)}{\mathbf{M}_{K^*}}\rho(z,w) &\leq (1-\lambda)|z-w| \\
&\leq |J^t_{r,h}(z) - J^t_{r,h}(w)| + \lambda|\varphi_{t,t+h}(J^t_{r,h}(z)) - \varphi_{t,t+h}(J^t_{r,h}(w))| \\
&\leq \frac{1}{\mathbf{m}_{K^*}}\rho(J^t_{r,h}(z), J^t_{r,h}(w)) + \frac{\lambda}{\mathbf{m}_{K^*}}\rho(\varphi_{t,t+h}(J^t_{r,h}(w)), \varphi_{t,t+h}(J^t_{r,h}(z))) \\
&\leq \frac{1}{\mathbf{m}_{K^*}}\rho(J^t_{r,h}(z), J^t_{r,h}(w)) + \frac{\lambda}{\mathbf{m}_{K^*}}\rho(J^t_{r,h}(w), J^t_{r,h}(z)) \\
&< \frac{2}{\mathbf{m}_{K^*}}\rho(J^t_{r,h}(z), J^t_{r,h}(w)).
\end{aligned}$$

Hence,

$$\frac{\mathbf{m}_{K^*}\, h}{2\mathbf{M}_{K^*}(r+h)}\rho(z,w) < \rho(J^t_{r,h}(z), J^t_{r,h}(w)).$$

Note that the lower bound of the factor $\frac{\rho(J^t_{r,h}(z), J^t_{r,h}(w))}{\rho(z,w)}$ goes to cero as $h \longrightarrow 0^+$, but we do not require that. Actually, we need that the limit results to be positive. So that, we impose a lower bound (**LB**) condition on the family of nonlinear resolvent $\{J^t_{r,h}(z)\}_{h\in(0,1)}$, in order to show that an evolution family is associated to a $\rho-$WVF in $H_\rho(D)$ by means of an ordinary differential equation.

**Definition 2.6.1.** *We say that the evolution family $\{\varphi_{s,t}(z)\}_{0\le s\le t<+\infty}$ satisfies the property **LB**, if*

**LB.** *For each compact subset $K\subset D$ and $T>0$ there exist $r_0>0$ and a constant $L_{K,T,r_0}>0$ such that*

$$L_{K,T,r_0}\rho(z,w)\le\rho(J^t_{r,h}(z),J^t_{r,h}(w)), \tag{2.26}$$

*for all $r\in(0,r_0)$, $z,w\in K$, $s,t\in[0,T]$, and $h\in(0,1)$.*

Examples of evolution families which satisfy the Property **LB** are the evolution families obtained by means of $\rho$-WVFs. In fact, in such a case

$$\varphi_{t,t+h}(z)=z+\int_t^{t+h}G(\varphi_{t,\tau}(z),\tau)d\tau.$$

Then,

$$(1-\lambda)J^t_{r,h}(z)=\lambda\int_t^{t+h}G(\varphi_{t,\tau}(J^t_{r,h}(z)),\tau)d\tau+(1-\lambda)z.$$

Thus,

$$\begin{aligned}
\frac{1}{\mathbf{M}_{K^*}}\rho(z,w)&\le|z-w|\\
&\le|J^t_{r,h}(z)-J^t_{r,h}(w)|\\
&\qquad+\frac{\lambda}{1-\lambda}\int_t^{t+h}|G(\varphi_{t,\tau}(J^t_{r,h}(z)),\tau)-G(\varphi_{t,\tau}(J^t_{r,h}(w)),\tau)|d\tau\\
&\le|J^t_{r,h}(z)-J^t_{r,h}(w)|+\frac{rF_{K,T}}{h}\int_t^{t+h}|\varphi_{t,\tau}(J^t_{r,h}(z))-\varphi_{t,\tau}(J^t_{r,h}(w))|d\tau\\
&\le\frac{1}{\mathbf{m}_{K^*}}\rho(J^t_{r,h}(z),J^t_{r,h}(w))+\frac{rF_{K,T}}{\mathbf{m}_{K^*}h}\int_t^{t+h}\rho(\varphi_{t,\tau}(J^t_{r,h}(z)),\varphi_{t,\tau}(J^t_{r,h}(w)))d\tau\\
&\le\frac{1}{\mathbf{m}_{K^*}}\rho(J^t_{r,h}(z),J^t_{r,h}(w))+\frac{rF_{K,T}}{\mathbf{m}_{K^*}h}\rho(J^t_{r,h}(z),J^t_{r,h}(w))\int_t^{t+h}d\tau\\
&\le\left(\frac{1}{\mathbf{m}_{K^*}}+\frac{rF_{K,T}}{\mathbf{m}_{K^*}}\right)\rho(J^t_{r_0,h}(z),J^t_{r,h}(w)).
\end{aligned}$$

Therefore,

$$\frac{\mathbf{m}_{K^*}}{\mathbf{M}_{K^*}(1+r_0F_{K,T})}\rho(z,w)\le\frac{\mathbf{m}_{K^*}}{\mathbf{M}_{K^*}(1+rF_{K,T})}\rho(z,w)\le\rho(J^t_{r,h}(z),J^t_{r,h}(w)),$$

for all $r \in (0, r_0)$.

We present the following theorem that states that the evolution families which satisfy the Property **LB**, have associated a $\rho$-monotone weak vector field.

**Theorem 2.6.1.** *Let $\{\varphi_{s,t}(z)\}_{0\leq s\leq t<+\infty}$ be an evolution family in $N_\rho(D)$, which satisfies the Property **LB**. Then, there exists a $\rho-WVF$, $G(z,t) \in H_\rho(D)$, such that for $z \in D$ and $s \geq 0$, $\omega(t) = \varphi_{s,t}(z)$ is the solution of the initial value problem*

$$\begin{cases} \dot{\omega} = G(\omega, t), & \text{for almost every } t \in [s, +\infty) \\ \omega(s) = z_0. \end{cases} \tag{2.27}$$

*Proof.* Let $U$ be an open set of $D$, $K_0 \subset\subset U$ a relatively compact open subset of $U$ and $T > 0$. By the uniform joint continuity on compact sets of $\varphi_{s,t}(z)$, there exists $h_0 = h_0(\overline{K}_0, T) \leq 1$, such that

$$\{\varphi_{t,t+h}(z) : \; z \in K_0, \; t \in [0,T], \; h \in (0, h_0]\} \subset U.$$

**1.** Let us consider

$$G_h(z,t) := \frac{\varphi_{t,t+h}(z) - z}{h} \quad z \in K_0, \; t \in [0,T], \; h \in (0, h_0].$$

**2.** The function $G_h(z, \cdot) : [0,T] \to \mathbb{C}$ is continuous for each $h \in (0, h_0]$ and $z \in K_0$. In fact, for fixed $h \in (0, h_0]$ and $z \in K_0$, let $\varepsilon > 0$ be given. Then, by Proposition 2.3.2 there exists $\delta_1 > 0$ such that if $\|(s,t) - (s_0,t_0)\| < \delta_1$ implies $|\varphi_{s,t}(z) - \varphi_{s_0,t_0}(z)| < \varepsilon h$. Considering $\delta = \delta_1/\sqrt{2}$ we have that $|t - s| < \delta$ implies

$$|G_h(z,s) - G_h(z,t)| = \frac{|\varphi_{s,s+h}(z) - \varphi_{t,t+h}(z)|}{h} < \varepsilon.$$

**3.** For every compact set $K \subset K_0$, there exists $A_{K,T} > 0$ such that $|G_h(z,t)| \leq A_{K,T}$ for all $h \in (0, h_0]$; $z \in K_0$ and $t \in [0,T]$. Indeed, if $K$ is a compact set and $T > 0$, condition EF3 implies the existence of $C_{K,T}$ such that $\rho(\varphi_{s,t}(z), \varphi_{s,\tau}(z)) \leq C_{K,T}(t - \tau)$, for every $z \in K$ and $0 \leq s \leq \tau \leq t \leq T$. Now, if $h \in (0, h_0]$; $z \in K$ and $0 \leq t \leq T$ then,

$$|G_h(z,t)| = \frac{|\varphi_{t,t+h}(z) - \varphi_{t,t}(z)|}{h} \leq \mathbf{m}_K^{-1} \frac{\rho(\varphi_{t,t+h}(z), \varphi_{t,t}(z))}{h} \leq \mathbf{m}_K^{-1} C_{K,T} =: A_{K,T}.$$

The equivalence of metrics on the compact set $\{\varphi_{t,t+h}(z) : z \in K, t \in [0,T], t+h \in [0,T+h_0]\}$ guarantees the existence of $\mathbf{m}_K^{-1} > 0$.

**4.** For every compact set $K \subset K_0$ and for every fixed $h \in (0,h_0]$, there exists $P_{K,T} > 0$ such that

$$|G_h(z,t) - G_h(w,t)| \leq \frac{P_{K,T}}{h}|z-w|,$$

for all $z,w \in K$, and $t \in [0,T]$. In fact, if $z,w \in K$ and $t \in [0,T]$,

$$\begin{aligned}
|G_h(z,t) - G_h(w,t)| &= \frac{|\varphi_{t,t+h}(z) - z - \varphi_{t,t+h}(w) + w|}{h} \\
&\leq \frac{1}{h}\{|\varphi_{t,t+h}(z) - \varphi_{t,t+h}(w)| + |z-w|\} \\
&\leq \frac{1}{h}\{\mathbf{m}_K^{-1}\rho(\varphi_{t,t+h}(z), \varphi_{t,t+h}(w)) + |z-w|\} \\
&\leq \frac{1}{h}\{\mathbf{m}_K^{-1}\rho(z,w) + |z-w|\} \\
&\leq \frac{1}{h}\{\mathbf{m}_K^{-1}\mathbf{M}_K + 1\}|z-w| := \frac{P_{K,T}}{h}|z-w|.
\end{aligned}$$

**5.** For every $h \in (0,h_0]$, the map $G_h : D \times [0,+\infty) \longrightarrow \mathbb{C}$ satisfies the strong range condition. It follows immediately from the definition of $G_h(z,t)$ and Proposition 2.5.6. In particular, for each $t \in [0,T]$, we have $\{G_h(\cdot,t)\}_{h\in(0,h_0]} \subset \mathscr{G}N_\rho(D)$, where $\mathscr{G}N_\rho(D)$ is set of all continuous infinitesimal generators of one-parameter semigroups in $N_\rho(D)$ (see [41]).

**6.** For every compact set $K \subset K_0$, there exists $F_{K,T} > 0$ such that

$$|G_h(z,t) - G_h(w,t)| \leq F_{K,T}|z-w|, \tag{2.28}$$

for all $z,w \in K$ and $t \in [0,T]$. Indeed, if $z,w \in K$ and $t \in [0,T]$, let us consider

$$z_1 = z - r_1 G_h(z,t), \quad \text{and} \quad w_1 = w - r_1 G_h(w,t),$$

where $r_1 \in (0,r_0)$ is fixed, and taken such that $z_1, w_1 \in D$. Then,

$$z = J^t_{r_1,h}(z_1) \quad \text{and} \quad w = J^t_{r_1,h}(w_1).$$

Thus, by the property **LB**, we have

$$\begin{aligned} r_1|G_h(z,t)-G_h(w,t)| &\leq |z-w|+|z_1-w_1| \leq |z-w|+\frac{1}{\mathbf{m}_K}\rho(z_1,w_1) \\ &\leq |z-w|+\frac{L_{K,T}}{\mathbf{m}_K}\rho(J^t_{r,h}(z_1),J^t_{r,h}(w_1)) \\ &\leq |z-w|+\frac{L_{K,T}\,\mathbf{M}_K}{\mathbf{m}_K}\left|J^t_{r,h}(z_1)-J^t_{r,h}(w_1)\right| \\ &= \left(1+\frac{L_{K,T}\,\mathbf{M}_K}{\mathbf{m}_K}\right)|z-w|. \end{aligned}$$

Therefore, Equation (2.28) holds with

$$F_{K,T}=\frac{\mathbf{m}_K\,r_1+L_{K,T}\,\mathbf{M}_K}{\mathbf{m}_K\,r_1}.$$

**7.** Let $\{K_j\}_{j\in\mathbb{N}}$ be a compact exhaustion of $K_0$ and $T\in\mathbb{N}$. Let us define

$$\Upsilon_T := \left\{ f\in C^0(K_0,\mathbb{C})\cap\mathscr{G}N_\rho(D) \;\middle/\; \begin{array}{lll} \textbf{a.} & \sup\limits_{z,w\in K_j}\frac{|f(z)-f(w)|}{|z-w|}\leq F_{K_j,T}, & j\in\mathbb{N} \\ \textbf{b.} & \sup\limits_{z\in K_j}|f(z)|\leq A_{K_j,T}, & j\in\mathbb{N} \end{array} \right\}$$

where $A_{K_j,T}$ and $F_{K_j,T}$ are given by the steps **3** and **6**, respectively. The space $\Upsilon_T$ is equipped with the compact open topology $\mathscr{U}_0$. We are going to show that $(\Upsilon_T,\mathscr{U}_0)$ is a metrizable space and we also use the Ascoli-Arzelà Theorem to prove that $(\Upsilon_T,\mathscr{U}_0)$ is a compact space, see [20] page 234.

Recall that $\mathscr{U}_0$ is the topology of uniform convergence on compact sets. Moreover, $\mathscr{U}_0$ is the topology generated by the collection of semi-norms $\{p_{K_j}:K_j\subset K_0\}$, where

$$p_{K_j}(f):=\sup_{z\in K_j}|f(z)|.$$

For every $f\in\Upsilon_T$, the set $f+\Sigma:=\{f+V:V\in\Sigma\}$ forms a neighborhood base at $f$, where $\Sigma:=\{V_{K_j,\varepsilon}:\varepsilon>0\}$ and $V_{K_j,\varepsilon}:=\{g\in\Upsilon_T:p_{K_j}(g)<\varepsilon\}$. Then, $(\Upsilon_T,\mathscr{U}_0)$ is a locally convex topological vector space. In addition, $(\Upsilon_T,\mathscr{U}_0)$ is a Hausdorff and metrizable space.

Another form to introduce the topology $\mathscr{U}_0$ is that generated by the subbase

$$\{W(K,U) : K \text{ compact set of } K_0 \text{ and } U \text{ is open of } \mathbb{C}\},$$

with $W(K,U) := \{f \in C^0(K_0,\mathbb{C}) : f(K) \subset U\}$. Then, $(\Upsilon_T, \mathscr{U}_0)$ is a locally convex topological space. Since $\mathbb{C}$ is regular and completely regular then, $(\Upsilon_T, \mathscr{U}_0)$ is regular and completely regular.

Let $K \subset K_0$ be a compact set. If $z \in K$, there exists an open set $U_z \subset \mathrm{Int}(K_{j_z}) \subset K_0$ containing $z$. Note that

$$\sup\left\{\sup_{z\in U_z} |f(z)| : f \in \Upsilon_T\right\} \leq A_{K_{j_z},N}.$$

Since $K$ is a compact set, there exist $U_{z_1}, U_{z_2}, \ldots, U_{z_k}$ such that $K \subset U_{z_1} \cup U_{z_2} \cup \cdots \cup U_{z_k}$. Then,

$$\sup\{p_K(f) : f \in \Upsilon_T\} \leq \max\{A_{K_j,N} : j = j_{z_1}, j_{z_2}, \ldots, j_{z_k}\}.$$

This is equivalent to say that $(\Upsilon_T, \mathscr{U}_0)$ is bounded.

If $\{f_n\}_{n\in\mathbb{N}} \subset \Upsilon_T$, such that $f_n \longrightarrow f$ as $n \longrightarrow +\infty$ uniformly on compact sets of $K_0$. Since $\mathscr{G}_{EF}N_\rho(D)$ is a closed real cone with the uniform convergence on compact sets of $D$, see [41], then, $f$ is a continuous function, and $f \in \mathscr{G}_{EF}N_\rho(D)$. Let $K_j$ be an element of the exhaustion of $K_0$, and $z \in K_j$. Then,

$$|f(z)| \leq |f_n(z) - f(z)| + |f_n(z)| \leq |f_n(z) - f(z)| + A_{K_j,T}.$$

Setting $n \longrightarrow +\infty$, the last inequality implies that $|f(z)| \leq A_{K_j,N}$ for all $z \in K_j$. Now, if $z, w \in K_j$ then,

$$\begin{aligned} |f(z) - f(w)| &\leq |f_n(z) - f(z)| + |f_n(z) - f_n(w)| + |f_n(w) - f(w)| \\ &\leq |f_n(z) - f(z)| + F_{K_j,T}|z - w| + |f_n(w) - f(w)|. \end{aligned}$$

Setting $n \longrightarrow +\infty$, the last inequality implies that $|f(z) - f(w)| \leq F_{K_j,N}|z - w|$ for all $z, w \in K_j$. Thus, $f \in \Upsilon_T$. Therefore $(\Upsilon_T, \mathscr{U}_0)$ is closed.

Let $K \subset K_0$ be a compact set, $\varepsilon > 0$ and $z \in K$. Then, there exist $j_0 \in \mathbb{N}$ such that $K \subset \mathrm{Int}(K_{j_0})$.

Thus, there is a $\delta_1 > 0$ such that $B(z,\delta_1) \subset \mathrm{Int}(K_{j_0})$. Let us consider $\delta := \min\{\delta_1, \frac{\varepsilon}{F_{K_{j_0},T}}\}$. Now, if $|z-w| < \delta$ then,

$$|f(z) - f(w)| \leq F_{K_{j_0},T}|z-w| \leq \delta F_{K_{j_0},T} \leq \varepsilon, \quad \text{for all } f \in \Upsilon_T.$$

Therefore, $\Upsilon_T$ is equicontinuous on compact sets.

**8.** For each $t \in [0,+\infty)$ there exists a subsequence $m_k(t) \longrightarrow 0^+$ as $k \longrightarrow +\infty$, such that

$$G(z,t) := \lim_{k\to+\infty} G_{m_k(t)}(z,t). \tag{2.29}$$

Furthermore, $G(z,t) \in H_\rho(D)$. Indeed, for each $T \in \mathbb{N}$, we have that $(\Upsilon_T, \mathscr{U}_0)$ is a compact metrizable subspace of $(C^0(K_0), \mathscr{U}_0)$. We are going to apply the result of H. Luschgy [25] (see Appendix A.1), with $X = \Upsilon_T$, endowed with the natural structure of Fréchet space and $\mathscr{M} = \mathscr{B}(X)$ the $\sigma-$algebra of all Borel subsets of $X$. Thus, we have that there exist a measurable selection (see Apendix A.1) $\sigma_2 : \Upsilon_T^{\mathbb{N}} \longrightarrow \Upsilon_T$, defined by

$$\sigma_2(\{x_n\}) = \begin{cases} \lim\limits_{n\longrightarrow+\infty} \sigma_1(x)_n, & \{x_n\} \in \Theta^*; \\ I_{id}, & \{x_n\} \notin \Theta^*, \end{cases}$$

where $\{\sigma_1(x)_n\} = \sigma_1(\{x_n\})$ is a convergent subsequence of $\{x_n\}$.

Now, since for each $t \in [0,T]$, we have $G_{1/n}(\cdot,t) \in \Upsilon_T$ for all $n \in \mathbb{N}$. Thus, if we set

$$\begin{aligned} x_n : [0,T] &\longrightarrow \Upsilon_T; & \mathbf{x} : [0,T] &\longrightarrow \Upsilon_T^{\mathbb{N}} \\ t &\longrightarrow x_n(t) := G_{1/n}(\cdot,t) & t &\longrightarrow \mathbf{x}(t) := \{x_n(t)\}_{n\in\mathbb{N}}, \end{aligned}$$

we have that for all $n \in \mathbb{N}$, the mapping $x_n(\cdot)$ is continuous on $[0,T]$ from Proposition 2.3.2 and therefore, $\mathbf{x}(\cdot)$ is continuous on $[0,T]$ as well. Applying the last result to the sequence $\{x_n(t)\}_{n\in\mathbb{N}} \in \Upsilon_T^{\mathbb{N}}$, we obtain that for each $t \in [0,T]$ there exists a subsequence $\{n_k(t)\} \subset \mathbb{N}$, such that $\sigma_1(\{x_n(t)\}) =$

$\{x_{n_k(t)}(t)\}$. Let us define

$$G^T(z,t) := \sigma_2(\mathbf{x}(t)) = \sigma_2(\{G_{\frac{1}{n}}(z,t)\}) = \lim_{k\longrightarrow+\infty} G_{\frac{1}{n_k(t)}}(z,t). \tag{2.30}$$

From the definition of measurable selection and the definition of $G^T(z,t)$, we have that $G^T(z,\cdot)$ is measurable on $[0,T]$, in particular on $[T-1,T)$, for each $z \in K_0$. Further, $G^T(\cdot,t) \in \Upsilon_T$ for each $t \in [0,T]$. Now, let us define the function $G : D \times [0,+\infty) \longrightarrow \mathbb{C}$, as follows

$$G(z,t) = G^T(z,t), \quad \text{if } t \in [T-1,T).$$

Note that, for each $t \geq 0$; $G(\cdot,t) \in \cup_{T\in\mathbb{N}}\Upsilon_T$, and $G(z,\cdot)$ is measurable on $[0,+\infty)$, for all $z \in K_0$. Thus, we have that $G(z,t)$ satisfies VF1, VF3 and VF4. The condition VF2 is satisfied if we consider $K_0$ containing 0. Therefore, $G(z,t) \in H_\rho(D)$.

**9.** Let $z \in K_0$ be fixed and $s \geq 0$ then,

$$\frac{\partial \varphi_{s,t}(z)}{\partial t} = G(\varphi_{s,t}(z),t), \text{ for almost every } t \in [s,+\infty). \tag{2.31}$$

Let $z \in K_0$ be fixed and $s \geq 0$. Since $\varphi_{s,\cdot}(z)$ is absolutely continuous on $[s,T]$, hence for almost every $t \in [s,+\infty)$ (and $\varphi_{s,\tau}(z) \in K_0$ whenever $\tau \in [s,t]$), we have, choosing $m_k(t) = 1/n_k(t)$

$$\begin{aligned}
\frac{\partial \varphi_{s,t}(z)}{\partial t} &= \lim_{h\longrightarrow 0^+} \frac{\varphi_{s,t+h}(z) - \varphi_{s,t}(z)}{h} \\
&= \lim_{h\longrightarrow 0^+} \frac{\varphi_{t,t+h}(\varphi_{s,t}(z)) - \varphi_{s,t}(z)}{h} \\
&= \lim_{k\longrightarrow+\infty} \frac{\varphi_{t,t+m_k(t)}(\varphi_{s,t}(z)) - \varphi_{s,t}(z)}{m_k(t)} \\
&= \lim_{k\longrightarrow+\infty} G_{m_k(t)}(\varphi_{s,t}(z),t) \\
&= G(\varphi_{s,t}(z),t).
\end{aligned}$$

Therefore, the proof is completed. $\square$

**Example 12.** *Note that the property* ***LB*** *is satisfied when $\{\varphi_{s,t}(z)\}_{0\leq s\leq t<+\infty}$ is an evolution family* *of holomorphic or analytic functions, because in this case $G_h(z,t)$ is an analytic function for each $h \in (0,1)$, and the family $\{G_h(z,t)\}_{h\in(0,1)}$ is uniformly bounded on compact sets $K \times [0,T]$. Similarly, the family $\{z - rG_h(z,t)\}_{h\in(0,1)}$ is uniformly bounded on compact sets, for each $r > 0$. Then, the Cauchy formula, implies*

$$|[z - rG_h(z,t)] - [w - rG_h(w,t)]| \leq (1 + rC_{K,T})|z - w|.$$

*Since $J^t_{r,h} := [I - rG_h(\cdot,t)]^{-1}$, and considering $u = (J^t_{r,h})^{-1}(z)$ and $v = (J^t_{r,h})^{-1}(w)$, we have*

$$|u - v| = |[z - rG_h(z,t)] - [w - rG_h(w,t)]| \leq (1 + rC_{K,T})|z - w| = (1 + rC_{K,T})|J^t_{r,h}(u) - J^t_{r,h}(v)|.$$

*Therefore, the equivalence of the hyperbolic and Euclidean metrics on compact sets implies that for every $r \in (0,r_0)$, we have*

$$\frac{\mathbf{m}_{K^*}}{\mathbf{M}_{K^*}(1 + r_0 C_{K,T})}\rho(u,v) \leq \frac{\mathbf{m}_{K^*}}{\mathbf{M}_{K^*}(1 + rC_{K,T})}\rho(u,v) \leq \rho(J^t_{r,h}(u), J^t_{r,h}(v)).$$

*Hence, the analytic evolution families have the property **LB**.*

An interesting property of evolution families is that every element of an evolution family is injective under certain conditions, as is stated in the next proposition.

**Proposition 2.6.2.** *Let $\{\varphi_{s,t}\}_{0\leq s\leq t<+\infty}$ be an evolution family, which satisfies the Property **LB**. Then, for all $0 \leq s \leq t < +\infty$ the mapping $\varphi_{s,t}(\cdot)$ is injective.*

*Proof.* We are going to proceed by contradiction. So, let us suppose that there exist $0 < s_0 < t_0$, and $z \neq w$ in $D$, such that $\varphi_{s_0,t_0}(z) = \varphi_{s_0,t_0}(w)$. Let us set

$$A = \{\tau \in [s_0,t_0] : \varphi_{s_0,\tau}(z) = \varphi_{s_0,\tau}(w)\}.$$

Since $t_0 \in A$ then, $A \neq \emptyset$. Further, $s_0$ is a lower bound of $A$ and $s_0 \notin A$. Then, $r_A = \inf\{A\}$ exists and $s_0 \leq r_A$.

First, notice that $s_0 < r_A$. In fact, if $s_0 = r_A$ then, there is a sequence $\{s_n\}_{n\in\mathbb{N}} \subset A$ with $s_n \longrightarrow s_0^+$ as $n \longrightarrow +\infty$. From Lemma 2.3.1, we have $|\varphi_{s_0,s_n} - I_{id}| \longrightarrow 0$ as $n \longrightarrow +\infty$ uniformly on compact sets. Let $K \subset D$ be a compact set, such that $z, w \in K$ then, $|\varphi_{s_0,s_n}(z) - z| \longrightarrow 0$ and $|\varphi_{s_0,s_n}(w) - w| \longrightarrow 0$ as $n \longrightarrow +\infty$. Then,

$$|z - w| \leq |\varphi_{s_0,s_n}(z) - z| + |\varphi_{s_0,s_n}(w) - w|.$$

Letting $n \longrightarrow +\infty$ in the previous equation we obtain $z = w$, which is a contradiction to the hypothesis $z \neq w$. Therefore, $s_0 < r_A$.

Moreover, $r_A \in A$. In fact, since $r_A = \inf\{A\}$, there exists a sequence $\{r_n\}_{n\in\mathbb{N}} \subset A$, such that $r_n \longrightarrow r_A$ as $n \longrightarrow +\infty$. Then, by EF3,

$$\begin{aligned}\rho(\varphi_{s_0,r_A}(z), \varphi_{s_0,r_A}(w)) &\leq \rho(\varphi_{s_0,r_A}(z), \varphi_{s_0,r_n}(z)) + \rho(\varphi_{s_0,r_n}(w), \varphi_{s_0,r_A}(w)) \\ &\leq 2C_{K,T}|r_n - r_A| \longrightarrow 0, \quad \text{as} \quad n \longrightarrow +\infty.\end{aligned}$$

This implies that $\varphi_{s_0,r_A}(z) = \varphi_{s_0,r_A}(w)$.

Now, let us consider $z_1(\tau) = \varphi_{s_0,\tau}(z)$ and $w_1(\tau) = \varphi_{s_0,\tau}(w)$ for $\tau \in (s_0, r_A)$. Then, $z_1(\tau) \neq w_1(\tau)$ for all $\tau \in (s_0, r_A)$. Furthermore, condition EF2. implies

$$\varphi_{\tau,r_A}(z_1(\tau)) = \varphi_{s_0,r_A}(z) = \varphi_{s_0,r_A}(w) = \varphi_{\tau,r_A}(w_1(\tau)). \tag{2.32}$$

Thus, for every $\tau \in (s_0, r_A)$, the function $\varphi_{\tau,r_A}$ is not injective on any compact set containing the open curve $z_1(\tau), w_1(\tau)$ with $\tau \in (s_0, r_A)$.

But, if $K \subset D$ is a compact set containing $z$ and $w$ in its interior, from Proposition 2.3.2 we choose $\tau_0 \in (s_0, r_A)$ such that $z_0 := z_1(\tau_0), w_0 := w_1(\tau_0) \in K$ and $\varphi_{\tau_0,\tau}(K)$ is contained in $K_1$, a compact subset of $D$, for all $\tau \in [\tau_0, r_A]$. Then, for the function $\gamma(\tau) := |\varphi_{\tau_0,\tau}(z_0) - \varphi_{\tau_0,\tau}(w_0)|$, with $\tau \in [\tau_0, r_A]$, due to Equation 2.32 and Theorem 2.6.1, we have

$$\begin{aligned}\gamma(\tau) &= |(\varphi_{\tau_0,\tau}(z_0) - \varphi_{\tau_0,r_A}(z_0)) - (\varphi_{\tau_0,\tau}(w_0) - \varphi_{\tau_0,r_A}(w_0))| \\ &= \left| \int_\tau^{r_A} \left( G(\varphi_{\tau_0,\xi}(z_0), \xi) - G(\varphi_{\tau_0,\xi}(w_0), \xi) \right) d\xi \right| \\ &\leq \int_\tau^{r_A} F_{K_1,t_0} \left| \varphi_{\tau_0,\xi}(z_0) - \varphi_{\tau_0,\xi}(w_0) \right| d\xi \\ &= F_{K_1,t_0} \int_\tau^{r_A} \gamma(\xi) d\xi.\end{aligned}$$

Hence, the well-known Gronwall inequality implies that $\gamma(\tau) = 0$, for all $\tau \in [\tau_0, r_A]$. Therefore, $z_0 = w_0$, which is a contradiction because $r_A$ is the minimun of $A$. $\square$

**Corollary 2.6.3.** Let $H(z,t)$ be another $\rho-$WVF, in $H_\rho(D)$, such that

$$\frac{\partial \varphi_{s,t}(z)}{\partial t} = H(\varphi_{s,t}(z),t), \tag{2.33}$$

for almost every $t \in [s,+\infty)$ and $s \leq t$ so that $\varphi_{s,\tau}(z) \in K_0$ whenever $\tau \in [s,t]$. Then, $G(z,t) = H(z,t)$ for all $z \in K_0$ and almost every $t \in [s,+\infty)$, with $K_0$ as before.

*Proof.* Let us suppose $z \in K_0$ and $s > 0$. Then, there exists a null set $N_1(z,s) \subset [s,+\infty)$ such that $\frac{\partial \varphi_{s,t}(z)}{\partial t} = H(\varphi_{s,t}(z),t)$, for all $t \in [s,+\infty) \setminus N_1(z,s)$. From Theorem 2.27, there is a null set $N_2(z,s) \subset [s,+\infty)$ such that $\frac{\partial \varphi_{s,t}(z)}{\partial t} = G(\varphi_{s,t}(z),t)$, for all $t \in [s,+\infty) \setminus N_2(z,s)$. Now, if $t \in [s,+\infty) \setminus (N_1(z,s) \cup N_2(z,s))$ then, for $z = \varphi_{s,t}(w)$, by the injectivity

$$G(z,t) = G(\varphi_{s,t}(w),t) = \frac{\partial \varphi_{s,t}(w)}{\partial t} = H(\varphi_{s,t}(w),t) = H(z,t).$$

$\square$

# Conclusions

A very good first attempt to the harmonic Loewner theory is made. Though, we have emphasized in two cases, in order to maintain the property of *harmonicity*, similar results to the classical case have been obtained as a direct consequences. This same consideration has led us to study the cases, for example, the linear case, et all. Based on [36], we could ask if the unique families $\mathscr{F}$ of harmonic functions which are closed under the composition are:

1. The union of the families of analytic and co-analytic functions, that is,

$$\mathscr{F} = \{f : D \longrightarrow f(D) \subset D : f \in \mathscr{H}(U,\mathbb{C}) \text{ or } \bar{f} \in \mathscr{H}(U,\mathbb{C})\}$$

2. The family $\mathscr{F} = \mathscr{G}_D$.

Although we defined the Loewner chain in a more general setting, these later two cases were only considered in this work.

The condition of maintaining the property of *harmonicity*, resulted to be a restrictive, therefore, we do not had diverse and fruitful results. Hence, this also guided us to focus on a wider class of functions.

Following the ideas given by M. Contreras and S. Díaz-Madrigal, et al, [6, 7], and D. Shoikhet [41], the concept of $\rho-$nonexpansive evolution families was introduced. Under an additional condition an analogue theorem to the main result of F. Bracci, M. Contreras and S. Díaz-Madrigal [6, 7] was proved. This result can be considered as a starting point in the setting for a non analytic Loewner theory.

The additional condition depends on the nonlinear resolvent for functions in a certain class of functions. Further properties of the nonlinear resolvent had been proved. But, a Hille-Yoshida type theorem, characterizing the infinitesimal generator of evolution families, could not be proved.

Further work in this direction is remaining. So, we consider the following points as future work arising from our investigation.

1. To work in a higher dimension, and we have the following example of a type of subordination which serves us as a motivation.

   **Example 13.** Let us consider the following family of surface of revolution

   $$r = f(z,t) = a(z-t)(1+t-z), \quad t \geq 0;\ z \in [t, t+1];\ a : \text{constant}, \tag{2.34}$$

   where $r = \sqrt{x^2+y^2}$. Note that associated to this family of surface of revolution, we can find the following family of functions

   $$\phi(z;s,t) = z+t-s, \quad \text{for } s \leq t,$$

   which help us to write a member of the family of surface of revolution in terms of another, and this family satisfies the conditions of semigroups. More precisely, we have the followings properties:

   (a) If we define $\phi_l(z) := \phi(z;s,t)$ with $l = t-s$. Then, $\{\phi_l\}$ with $l \geq 0$ form a semigroup of functions. In fact, $\phi_0(x) = \phi(z;s,s) = z$ for every $z \in \mathbb{R}$. Further, note that $\phi_l(z) = z+l$, then $\phi_{l+k}(z) = z+l+k = \phi_k(z)+l = \phi_l(\phi_k(z))$.

   (b) If $s \leq t$, then $f(\phi(z;s,t),t) = f(z,s)$. Indeed,

   $$\begin{aligned} f(\phi(z;s,t),t) &= a(\phi(z;s,t)-t)(1+t-\phi(z;s,t)) \\ &= a(z+t-s-t)(1+t-(z+t-s)) \\ &= a(z+t-s-t)(1+t-z-t+s) \\ &= a(z-s)(1+s-z) = f(z,s). \end{aligned}$$

The family of surface of revolution given by (2.34) could be consider as a Loewner chain in $\mathbb{R}^3$ in the following sense:

*We can put in a certain order the family $f(z,t)$ with respect to $t$, and reclaim a member of the family from the another taking into account such order.*

2. Since the analyticity is considered only in the cases of $\mathbb{C}^n$ or $\mathbb{R}^{2n}$, and the condition of being harmonic can be studied in whatever $\mathbb{R}^k$, our work could be a good starting point in order to extend the harmonic Loewner theory to harmonic functions defined on any domain of $\mathbb{R}^k$. On the other hand, we would like to say that we intend to generalize this work to the case in which one replaces the Laplace operator by a more general elliptic partial differential operator of order two.

3. In the analytic case, it is well-known the one-to-one relation between two embedded simply-connected domains $\Omega_1 \subset \Omega_2$ and the subordination of their associated conformal Riemann mappings $f_1(z) \prec_{\mathscr{H}} f_2(z)$. Now, based on the existence of the harmonic Riemann mapping associated to certain domains which was established in [19], we can formulate the same question for two domains given $\Omega_1 \subset \Omega_2$. In the harmonic case we have a similar relation if $\Omega_1 \subset \Omega_2$ are bounded. In fact, the principal result in [19] is:

   **Theorem** (Harmonic Riemann mapping theorem)**.** Let $\Omega \subset \mathbb{C}$ be a bounded simply-connect domain whose boundary is locally connected. Fix $z_0 \in \Omega$ and let $\omega \in \mathscr{H}(D,\mathbb{C})$ satisfy $|\,\omega\,| < 1$. Then, there exists a univalent, harmonic, sense-preserving mapping $f$ with the following properties:

   (a) $f$ maps $D$ into $\Omega$ and $f(0) = z_0$, $\partial_z f(0) > 0$;
   (b) $f$ is a solution of $\overline{\partial_{\bar{z}} f} = \omega\, \partial_z f$;
   (c) the limits $\lim_{r \longrightarrow 1} f(re^{i\theta})$ belong to $\partial\Omega$ for almost every $\theta$.

   Furthermore, if $\|\,\omega\,\| := \sup\{\omega(z) : z \in D\} < 1$ then $f(D) = \Omega$.

   Note that the $f$ is not unique, because it depends on the choice of $\omega$. Thus, if $f_1$, $f_2$ are the harmonic Riemann mappings guaranteed by the previous theorem for $\Omega_1, \Omega_2$ respectively with $\| \omega \| < 1$ given, then from Proposition 1.3.2 we have that $f_1(z) \prec_{\mathscr{H}} f_2(z)$. Moreover, this relation can be extended to a family of domains. But, we have to take care that Carathéodory's kernel theorem is not true in the harmonic case.

4. One problem, which is closely related with 3.), is the following. In [35] a Hamiltonian formulation of the Loewner-Kufarev equation leading to an integrable system is shown (see also [26, 35, 46]). In our case, we intend to establish a similar relationship between Equation (1.41) and the integrable systems.

5. According to [2, 11], and the results given here, to develop a non analytic Loewner theory. The next step is to introduce a type of subordination or subordination chain in a certain space of functions. Then, we have to study this space of function in order to establish a relation between this same space of functions and the evolution families in $N_\rho(D)$.

6. In the case of getting a non analytic Loewner theory with respect to the later point, it will be interesting to apply this theory in order to solve solve the *harmonic analogue of the Bieberbach conjecture* [8]:

$$
\begin{aligned}
||a_n(f)| - |a_{-n}(f)|| &\leq n, \quad n = 2, 3, \dots \\
|a_{-n}(f)| &\leq \frac{1}{6}(n-1)(2n-1), \quad n = 2, 3, \dots .
\end{aligned}
$$

if $f(z) = \sum\limits_{n=0}^{\infty} a_n(f) z^n + \sum\limits_{n=1}^{\infty} a_{-n}(f) \bar{z}^n$ is a harmonic function on $D$.

# Appendix A

## Further preliminary results

### A.1 Measurable Selectors

Let us recall that a multifunction or multivalued function $\Gamma$ from $A$ to nonempty set $B$, is a relation $\Gamma : A \longrightarrow 2^B$, such that $\mathrm{Dom}\,\Gamma = A$, where $2^B$ represents the family of all subsets of $B$. That is, a multifunction is a relation that associates every element in $A$ with a subset of $B$. Denote by $\Gamma^{-}(V) := \{x \in A : \Gamma(x) \cap V \neq \emptyset\}$, if $V \in 2^B$.

Let us suppose that $(A, \mathscr{M})$ is a measurable space, and $B$ is a complete separable metric space. Then we say that a multifunction $\Gamma$ is measurable if $\Gamma^{-}(U) \in \mathscr{M}$, for every open set $U \in 2^B$.

A selection or selector of the multifunction $\Gamma$ is a map a $\sigma : A \longrightarrow B$ such that $\sigma(x) \in \Gamma(x)$, for all $x \in A$. The selector $\sigma$ of $\Gamma$ is said to be measurable is $\sigma^{-1}(U) \in \mathscr{M}$, for every open set $U \in 2^B$.

Now, let us consider the following cases of multifunction: Let us suppose that $(X, \mathscr{M})$ is a compact metrizable space, and $(X^{\mathbb{N}}, \mathscr{M}^{\mathbb{N}})$, with $\mathscr{M}^{\mathbb{N}}$ the product $\sigma$-algebra $\bigotimes_{1}^{\infty} \mathscr{M}$ on $X^{\mathbb{N}}$. Let us define the following multifunctions $\Gamma_1 : X^{\mathbb{N}} \longrightarrow X^{\mathbb{N}}$ and $\Gamma_2 : X^{\mathbb{N}} \longrightarrow X$, where

$$\Gamma_1(\{x_n\}) = \begin{cases} M(x_n), & \{x_n\} \in \Theta^*; \\ X^{\mathbb{N}}, & \{x_n\} \notin \Theta^*, \end{cases} \qquad \Gamma_2(\{x_n\}) = \begin{cases} L(x_n), & \{x_n\} \in \Theta^*; \\ X, & \{x_n\} \notin \Theta^*. \end{cases}$$

Here, we have used the following notation: $X^{\mathbb{N}} := \{\{x_n\} : x_n \in X\}$, i.e., the set of all sequences of element in $X$; $\Theta := \{\{x_n\} \in X^{\mathbb{N}} : L(x_n) \neq \emptyset\}$, where $L(x_n)$ is the set of the limit points of the sequence $\{x_n\}$, more precisely,

$$L(x_n) := \{x \in X : \text{ there is } \{x_{n_k}\} \text{ a subsequence of } \{x_n\}, \text{ such that } x_{n_k} \longrightarrow x\},$$
$$M(x_n) := \{\{x_{n_k}\} \in X^{\mathbb{N}} : \{x_{n_k}\} \text{ is a convergent subsequence of } \{x_n\}\}.$$

In [25] Luschgy proved that there exist measurable selections of these multifunctions $\Gamma_1 : X^{\mathbb{N}} \longrightarrow X^{\mathbb{N}}$, and $\Gamma_2 : X^{\mathbb{N}} \longrightarrow X$, (see Lemma 1. and Proposition 3 [25]). Let $\sigma_1$ and $\sigma_2$ be such measurable selectors of $\Gamma_1$ and $\Gamma_2$, respectively. Then,

$$\begin{aligned} \sigma_1 : X^{\mathbb{N}} &\longrightarrow X^{\mathbb{N}}; \\ \{x_n\} &\longrightarrow \sigma_1(\{x_n\}) = \{\sigma_1(x)_n\} \in \Gamma_1(\{x_n\}) \end{aligned} \qquad \begin{aligned} \sigma_2 : X^{\mathbb{N}} &\longrightarrow X \\ \{x_n\} &\longrightarrow \sigma_2(\{x_n\}) \in \Gamma_2(\{x_n\}). \end{aligned}$$

Moreover, it was shown that if $x_0 \in X$ is fixed, in particular for $x_0 = I_{id}$, we have

$$\sigma_2(\{x_n\}) = \begin{cases} \lim\limits_{n \longrightarrow +\infty} \sigma_1(x)_n, & \{x_n\} \in \Theta^*; \\ x_0, & \{x_n\} \notin \Theta^*. \end{cases}$$

## A.2 Carathéodory Theorem

Let us suppose that $f$ is a complex-valued (not necessarily continuous) function defined in some set $\mathbf{R}$ of the $(z,t)$ space, containing $(z_0,s) \in D \times [0,+\infty)$. We can formulate the next problem

**Problem E.** *To find an absolutely continuous function $\omega$ defined on a interval $I$, such that*

$$\begin{cases} (\omega(t),t) \in \mathbf{R}, & \textit{whether } t \in I \\ \dot{\omega} = f(\omega,t), & \textit{for almost every } t \in I \\ \omega(s) = z_0. \end{cases} \tag{A.1}$$

If such an interval $I$ and such a function $\omega$ exist then, $\omega$ is said to be a solution of (E) in the extended sense on $I$.

If $f \in C^0(\mathbf{R}, \mathbb{C})$, and $\phi$ is a solution of (E) in the above sense, then from differential equation $\dot{\phi} \in C^0(I, \mathbb{C})$, and therefore the more general notion of the equation (E), and of solution $\phi$, reduces to the classical definition of (E). That is way we say *in the extended sense*.

**Theorem A.2.1** (Carathéodory Theorem)**.** *Let $f$ be a function defined on $\mathbf{R} = K \times [a,b] \subset D \times \mathbb{R}^+$, and $\mathbf{R}$ containing $(z_0, s)$ in its interior. Assume that $f$ is measurable in the variable $t$ on $[a,b]$, for each fixed $z \in D$, continuous in $z$ for each fixed $t \in [a,b]$. If there exists a non-negative Lebesgue-integrable function $m(t)$ on $[a,b]$, such that*

$$|f(z,t)| \leq m(t), \quad (z,t) \in \mathbf{R}. \tag{A.2}$$

*Then, there exist $I_{\mathbf{R}}(z_0, s) > s$ and a function $\phi$ such that the initial value problem (A.1) is held. Furthermore, if $f(\cdot, t)$ is Lipschitz on $K$ then, its solution is unique.*

This proof is based on the proof given in [9], and is given to guarantee that it also works in our context.

*Proof.* If $F$ is defined by

$$F(t) = \begin{cases} 0, & t < s; \\ \int_s^t m(r)dr, & s \leq t \leq b. \end{cases} \tag{A.3}$$

Then, it is clear that $F$ is continuous nondecreasing, and $F(s) = 0$. Therefore, there exists a positive constant $\eta > 0$, such that

$$(z_0 + F(t)e^{i\theta}, t) \in \mathbf{R}, \text{ for every } \ t \in [s, s+\eta], \ \text{ and } \ \theta \in [0, 2\pi]. \tag{A.4}$$

Let us define the approximation $\phi_k$, for $k = 1, 2, 3, \ldots$, by

$$\phi_k(t) = \begin{cases} z_0, & s \leq t \leq s + \frac{\eta}{k}; \\ z_0 + \int_s^{t-\eta/k} f(\phi_k(r), r)dr, & s + \frac{\eta}{k} < t \leq s + \eta. \end{cases} \tag{A.5}$$

Note that for all $k \in \mathbb{N}$, the function $\phi_k : [s, s+\eta] \longrightarrow K$ is continuous on $[s, s+\eta]$. Furthermore, if $t \in \left[s+\frac{\eta}{k}, s+\frac{2\eta}{k}\right]$ then, $\phi_k(t)-z_0 = \int_s^{t-\eta/k} f(\phi_k(r),r)dr$. Thus,

$$|\phi_k(t)-z_0| \leq \int_s^{t-\eta/k} m(r)dr = F(t-\eta/k).$$

In a similar way, for $1 < l < k$, we obtain

$$|\phi_k(t)-z_0| \leq F(t-\eta/k), \quad t \in \left[s+\frac{l\eta}{k}, s+\frac{(l+1)\eta}{k}\right],$$

for all $k \in \mathbb{N}$. Therefore, by induction we obtain that $\phi_k$ satisfies

$$\begin{aligned} &\phi_k(t) = z_0, && s \leq t \leq s+\tfrac{\eta}{k}; \\ &|\phi_k(t)-z_0| \leq F(t-\eta/k), && s+\tfrac{\eta}{k} < t \leq s+\eta. \end{aligned} \tag{A.6}$$

Now, if $t_1, t_2 \in [s, s+\eta]$, without lose of generality assume $t_1 < t_2$, and $k \in \mathbb{N}$ then, we have three options

1. $t_1, t_2 \leq s+\frac{\eta}{k}$,

2. $t_1 \leq s+\frac{\eta}{k} \leq t_2$,

3. $t_1, t_2 \geq s+\frac{\eta}{k}$.

It is not difficult to see that in all previous cases we have

$$|\phi_k(t_1)-\phi_k(t_2)| \leq \left|F\left(t_1-\frac{\eta}{k}\right) - F\left(t_2-\frac{\eta}{k}\right)\right|.$$

Since $F$ is continuous on $[s, s+\eta]$, it is uniformly continuous there. This implies that the family $\{\phi_k\}_{k\in\mathbb{N}}$ is an equicontinuous on $[s, s+\eta]$. Also, Equation (A.6) implies that the family $\{\phi_k\}_{k\in\mathbb{N}}$ is uniformly bounded on $[s, s+\eta]$. By Ascoli-Arzelà theorem there exists a subsequence $\{\phi_{k_j}\}_{j\in\mathbb{N}}$, which converge uniformly on $[s, s+\eta]$ to a continuous function $\phi$, as $j \longrightarrow +\infty$.

From condition (A.2), we have

$$|f(\phi_{k_j}(t),t)| \leq m(t), \quad t \in [s, s+\eta].$$

Since $f(\cdot,t)$ is continuous for fixed $t$ then,

$$f(\phi_{k_j}(t),t) \longrightarrow f(\phi(t),t), \quad \text{as} \;\; j \longrightarrow +\infty,$$

for every $t \in [s, s+\eta]$. Thus, the Dominated convergence theorem implies that

$$\lim_{j \longrightarrow +\infty} \int_s^t f(\phi_{k_j}(r),r)dr = \int_s^t f(\phi(t),t)dr,$$

for every $t \in [s, s+\eta]$. But,

$$\phi_{k_j}(t) = z_0 + \int_s^t f(\phi_k(r),r)dr - \int_{t-\eta/k}^t f(\phi_k(r),r)dr,$$

where $\int_{t-\eta/k}^t f(\phi_k(r),r)dr \longrightarrow 0$ as $j \longrightarrow +\infty$. Therefore, letting $j \longrightarrow +\infty$, we obtain

$$\phi(t) = z_0 + \int_s^t f(\phi(t),t)dr, \tag{A.7}$$

which is equivalent to our goal, with $I_{\mathbf{R}}(z_0,s) = s + \sup\{\eta : \text{(A.4) holds }\}$.

On the other hand, if $f(\cdot,t)$ is Lipschitz on $K$, and $u, v$ are two solutions of (A.1) in $[s, I_{\mathbf{R}}(z_0,s))$ then, Equation (A.7) implies

$$\begin{aligned} |u(t) - v(t)| &\leq \int_s^t |f(u(r),r) - f(u(r),r)|dr \\ &\leq \int_s^t C_K(r)|u(r) - v(r)|dr. \end{aligned}$$

The well known Gronwall inequality implies that $|u(t) - v(t)| = 0$. Thus, $u \equiv v$, i.e., the solution is unique. □

**Remark A.1.** Note that $\eta > 0$ given in the proof, actually, depends on $\mathbf{R}$ and therefore $I(z_0,s)$. That is why we have written $I_{\mathbf{R}}(z_0,s)$.

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