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Robust utility maximization for Lévy processes: Penalization and Solvability

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P R E S E N T A :
Leonel Ramón Pérez Hernández

DIRECTOR DE TESIS:
Dr. Daniel Hernández Hernández

S I N O D A L E S :
Dr. Rolando Cavazos Cadena (UAAAN)
Dra. Begoña Fernández Fernández (UNAM)
Dr. Onésimo Hernández Lerma (CINVESTAV)
Dr. Victor Rivero Mercado (CIMAT)

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A mis padres María Hernández y Leonel Pérez†

A mi esposa Beatriz Adriana

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Introduction

The conceptualization of the preferences of an economic agent is one of the cornerstone in the economic thought, and might be the central one. With only two axioms: Asymmetry, $x \succ y \Rightarrow y \not\succ x$, and negative transitivity, $x \succ y$ and z arbitrary $\Rightarrow (x \succ z) \vee (z \succ y)$, it is possible to describe the preferences of an economic agent, producing as a result, under mild assumptions, the existence of a numerical representation of the preferences. This means that $x \succ y \Leftrightarrow u(x) > u(y)$, for some real valued function u. In addition the inclusion of the independence or substitution axiom

$$x \succ y \Rightarrow \alpha x + (1-\alpha) z \succ \alpha y + (1-\alpha) z, \ \forall \alpha \in (0,1] \text{ and } \forall z,$$

as well as the Archimedean axiom

$$x \succ y \succ z \Rightarrow \exists \ \alpha, \beta \in (0,1) \ \text{ with } \alpha x + (1-\alpha) z \succ y \succ \beta x + (1-\beta) z,$$

are enough to guarantee the existence of an affine numerical representation of the preferences. Moreover, when the set of possible choices is the space of probability measures, under suitable conditions the representation occurs by means of a Von Neuman - Morgenstern representation

$$u(\mu) = \int U(x) \, d\mu(x) \, .$$

The paradigm of expected utility became one of the pillars in economics during the last century. Starting from an expected utility problem of the form

$$\mathbb{E}_{\mathbb{Q}} [U(X)] \rightarrow \max, \tag{1}$$

Harry Markowitz [24] derived in the early 50s, for the first time, a quantitative solution in form of his celebrated mean-variance analysis [25], and confronted the academic world with the ubiquitous trade-off between profit and risk in a financial market. Markowitz [26, pg. 287] also contributed to the real and substantial connection of utility and measure of risk. The theory of measures of risk was developed at an axiomatic level with the introduction of coherent risk measures by Artzner et. al. in [1] and [2], which was extended by Föllmer and Schied in [6] and [7] with the introduction of convex measures of risk.

It is common to refer to (1) as the Merton-problem, because a solution to (1) in the context of a continuous time Markovian market model was established in [28] and [29] using stochastic control methods. Harrison and Pliska accomplished in [13] and [14] the connection to stochastic calculus (initiated by Bachelier at the beginning of the last century), what led to the continuous time investment-consumption problems, widely studied in the second half of the last century.

It is merit of Pliska [31] to provide the martingale and duality approach, which is still one of the more influential ideas to solve the maximization expected utility problem. For the application of control method in the solution of the dual problem for utility maximization for an incomplete market model see [3] and [4]. Kramkov and Schachermayer [19] and [20] made very important contributions in this context, for utility functions defined in the positive halfline. There, the authors tackle the problem (1) in a dynamic setting for a fixed finite time horizon T

$$u_{\mathbb{Q}}(x) := \sup_{X \in \mathcal{X}(x)} \left\{ \mathbb{E}_{\mathbb{Q}} \left[U \left(X_T \right) \right] \right\},$$

over a set of admissible wealth processes $\mathcal{X}(x)$, which is explained later, in a very general semimartingale market model. In [19] a convex dual approach is used, considering the convex conjugate V of the function $-U(-x)$, and the dual value function is given by

$$v_{\mathbb{Q}}(y) := \inf_{Y \in \mathcal{Y}_{\mathbb{Q}}(y)} \left\{ \mathbb{E}_{\mathbb{Q}} \left[V(Y_T) \right] \right\},$$

where $\mathcal{Y}_{\mathbb{Q}}(y)$ is a class of processes defined later.

The choice of the market measure $\mathbb{Q}$ (model uncertainty or ambiguity) has risen many empirical studies, and has also motivated (beside some incongruous paradox) a reexamination of the axiomatic foundations of the theory of choice under uncertainty. Gilboa and Schmeidler [10] gave a significant step in this direction, introducing the "certainty-independence" axiom, what led to robust utility functionals

$$X \longrightarrow \inf_{\mathbb{Q} \in \mathcal{Q}} \left\{ \mathbb{E}_{\mathbb{Q}} \left[U(X) \right] \right\},$$

where the set of "prior" models $\mathcal{Q}$ is assumed to be a convex set of probability contents on the measurable space $(\Omega, \mathcal{F})$ (i.e. finite additive set functions $\mathbb{Q} : \mathcal{F} \rightarrow [0,1]$ with $\mathbb{Q}(\Omega) = 1$). The corresponding robust utility maximization problem

$$\inf_{\mathbb{Q} \in \mathcal{Q}} \left\{ \mathbb{E}_{\mathbb{Q}} \left[U(X) \right] \right\} \rightarrow \max, \tag{2}$$

has being studied by several authors. See [32], [11], [35], [9] and [12] and references therein.

A natural observation is that the worst case approach in (2) does not discriminate among all possible models in $\mathcal{Q}$, what again is reflected in inconsistencies in the axiomatic system proposed in [10]. Maccheroni, Marinacci and Rustichini [23] proposed a relaxed axiom system, which led to utility functionals

$$X \longrightarrow \inf_{\mathbb{Q} \in \mathcal{Q}} \left\{ \mathbb{E}_{\mathbb{Q}} \left[U(X) \right] + \vartheta(\mathbb{Q}) \right\},$$

where the penalty function ϑ assigns a weight $\vartheta(\mathbb{Q})$ to each model $\mathbb{Q} \in \mathcal{Q}$. Schied [34] developed the corresponding dual theory for utility functions defined in the positive halfline and utility functionals of the form

$$u(x) := \sup_{X \in \mathcal{X}(x)} \inf_{\mathbb{Q} \in \mathcal{Q}} \left\{ \mathbb{E}_{\mathbb{Q}} \left[U(X_T) \right] + \vartheta(\mathbb{Q}) \right\}.$$

Outline of the thesis.

The outline and description of the main contributions of this thesis is as follows: In Section 1 we propose the probability space on which our processes will be defined, and describe the class of absolutely continuous probabilities with respect to a reference probability measure $\mathbb{P}$. There, we also recall and develop some results from stochastic calculus. In the development of this work it was necessary to analyze the convergence of the quadratic variation of the densities, and the result is presented in Lemma 3.

Samuelson [33] seems to be the first to propose a geometric Brownian motion as a model for the prices of the underlying assets in a market; it is often referred (wrongly) as the Black & Scholes model, whom do not have the need of another merit. This idea led to the, almost ubiquitous, exponential semimartingales models. We use one of them to introduce the market model in Section 2, which need not to have independent increments but include certain Lévy exponential models, and has been used to study some problems closed to our, see for instance [27] and [30]. We also give in this section a characterization of the equivalent local martingale measures for the proposed model. This contribution extends to our setting a result of Kunita [22] for Lévy exponential models.

Section 3 includes results on static risk measures, which do not require the underlying market model. We provide necessary and sufficient conditions for a penalization function ϑ, concentrated in a convex subset of the class of absolutely continuous probability measures with respect to $\mathbb{P}$, to be the minimal penalty function of the associated convex measure of risk ρ. Also, in this section we propose a family of penalty functions which are minimal for the convex measures of risk generated by them. Both of them, Theorem 5 and Theorem 6, are the main contributions of this section.

Once we have introduced necessary conditions for the penalization and the corresponding convex measure of risk ρ, which are relevant to develop the duality theory for the maximization of a penalized robust expected utility problem as in Schied [34], we address in Section 4 the relationship between the choice of a penalty function and the existence of a solution to the dual problem. For the power and the logarithmic utility functions we provide, in each case, thresholds for the family of penalty functions, which guarantee the existence of solutions to the optimal allocation problem. These results are new and their proof is based on Theorem 6 in Section 3. We finish this section with a representation of the dual problem, given in Theorem 13, in terms of certain coefficients for an arbitrary utility function. To end the thesis, we have collected in the Appendixes all those calculations which regularly do not find a place in a journal publication, but we believe that are useful to complement the main contributions.

1 Preliminaries from stochastic calculus

Within a probability space which supports a semimartingale with the weak predictable representation property, there is a representation of the density processes of the absolutely continuous probability measures by means of two coefficients. Roughly speaking, this means that the "dimension" of the linear space of local martingales is two. Throughout these coefficients we can represent every local martingale as a combination of two components, namely an stochastic integral with respect to the continuous part of the semimartingale and an integral with respect to its compensated jump measure. This is of course the case for local martingales, and with more reason this observation about the dimensionality holds for the martingales associated with the corresponding densities processes. In this section we review those concepts of stochastic calculus that are relevant to understand this representation properties, and prove some kind of continuity property, not reported yet in the literature, for the quadratic variation of a sequence of densities converging in L^1.

1.1 Fundamentals of Lévy and semimartingales processes

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. We say that $L := \{L_t\}_{t\in\mathbb{R}_+}$ is a Lévy process for this probability space if it is an adapted cádlág process with independent stationary increments starting at zero. The filtration considered is $\mathbb{F} := \left\{\mathcal{F}_t^{\mathbb{P}}(L)\right\}_{t\in\mathbb{R}_+}$, the completion of its natural filtration, i.e. $\mathcal{F}_t^{\mathbb{P}}(L) := \sigma\{L_s : s \leq t\} \vee \mathcal{N}$ where $\mathcal{N}$ is the σ-algebra generated by all $\mathbb{P}$-null sets. The jump measure of L is denoted by $\mu : \Omega \times (\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R}_0)) \to \mathbb{N}$ where $\mathbb{R}_0 := \mathbb{R} \setminus \{0\}$. The dual predictable projection of this measure, also known as its Lévy system, satisfies the relation $\mu^{\mathcal{P}}(dt, dx) = dt \times \nu(dx)$, where $\nu(\cdot) := \mathbb{E}[\mu([0,1] \times \cdot)]$ is the, so called, intensity or Lévy measure of L.

The Lévy-Itô decomposition of L is given by

$$L_t = bt + W_t + \int\limits_{[0,t]\times\{0<|x|\leq 1\}} x\left\{\mu\left(ds,dx\right) - \nu\left(dx\right)ds\right\} + \int\limits_{[0,t]\times\{|x|>1\}} x\mu\left(ds,dx\right). \tag{1.1}$$

It implies that $L^c = W$ is the Wiener process, and hence $\left[L^c\right]_t = t$, where $(\cdot)^c$ and $[\,\cdot\,]$ denote the continuous martingale part and the process of quadratic variation of any semimartingale, respectively. For the predictable quadratic variation we use the notation $\langle\,\cdot\,\rangle$.

Even though most of the thesis deals with Lévy processes, we need to introduce some notation from the theory of semimartingales, and present some results needed in the next sections. Denote by $\mathcal{V}$ the set of cádlág, adapted processes with finite variation, and let $\mathcal{V}^+ \subset \mathcal{V}$ be the subset of non-decreasing processes in $\mathcal{V}$ starting at zero.

Let $\mathcal{A} \subset \mathcal{V}$ be the class of processes with integrable variation, i.e. $A \in \mathcal{A}$ if and only if $\bigvee_0^\infty A \in L^1\left(\mathbb{P}\right)$, where $\bigvee_0^t A$ denotes the variation of A over the finite interval $[0,t]$. The subset $\mathcal{A}^+ \subset \mathcal{A}$ represents those processes which are also increasing i.e. with non-negative right-continuous increasing trajectories. Furthermore, $\mathcal{A}_{loc}$ (resp. $\mathcal{A}^+_{loc}$) is the collection of adapted processes with locally integrable variation (resp. adapted locally integrable increasing processes). For a cádlág process X we denote by $X_- := (X_{t-})$ the left hand limit process, where $X_{0-} := X_0$ by convention, and by $\triangle X = (\triangle X_t)$ the jump process $\triangle X_t := X_t - X_{t-}$.

Given an adapted cádlág semimartingale U, the jump measure and its dual predictable projection (or compensator) are denoted by $\mu_U\left([0,t]\times A\right) := \sum_{s\leq t} \mathbf{1}_A\left(\triangle U_s\right)$ and $\mu_U^{\mathcal{P}}$, respectively. Further, we denote by $\mathcal{P} \subset \mathcal{F}\otimes\mathcal{B}\left(\mathbb{R}_+\right)$ the predictable σ-algebra and by $\widetilde{\mathcal{P}} := \mathcal{P}\otimes\mathcal{B}\left(\mathbb{R}_0\right).$ With some abuse of notation, we write $\theta_1 \in \widetilde{\mathcal{P}}$ when the function $\theta_1 : \Omega\times\mathbb{R}_+\times\mathbb{R}_0 \to \mathbb{R}$ is $\widetilde{\mathcal{P}}$-measurable and $\theta \in \mathcal{P}$ for predictable processes.

Let

$$\mathcal{L}(U^c) := \left\{\theta \in \mathcal{P} : \exists \{\tau_n\}_{n\in\mathbb{N}} \text{ sequence of stopping times with } \tau_n \uparrow \infty \right.$$
$$\left. \text{and } \mathbb{E}\left[\int_0^{\tau_n} \theta^2 d\left[U^c\right]\right] < \infty \ \forall n \in \mathbb{N}\right\} \tag{1.2}$$

be the class of predictable processes $\theta \in \mathcal{P}$ integrable with respect to U^c in the sense of local martingale, and by

$$\Lambda(U^c) := \left\{\int \theta_0 dU^c : \theta_0 \in \mathcal{L}(U^c)\right\}$$

the linear space of processes which admit a representation as the stochastic integral with respect to U^c. For an integer valued random measure $\widetilde{\mu}$ we denote by $\mathcal{G}(\widetilde{\mu})$ the class of $\widetilde{\mathcal{P}}$-measurable processes $\theta_1 : \Omega \times \mathbb{R}_+ \times \mathbb{R}_0 \rightarrow \mathbb{R}$ satisfying the following conditions:

(*i*) $\theta_1 \in \widetilde{\mathcal{P}}$,

(*ii*) $\int_{\mathbb{R}_0} |\theta_1(t,x)| \, \widetilde{\mu}^{\mathcal{P}}(\{t\}, dx) < \infty \ \forall t > 0$,

(*iii*) The process
$$\left\{\sqrt{\sum_{s\le t}\left\{\int_{\mathbb{R}_0}\theta_1(s,x)\,\widetilde{\mu}(\{s\},dx) - \int_{\mathbb{R}_0}\theta_1(s,x)\,\widetilde{\mu}^{\mathcal{P}}(\{s\},dx)\right\}^2}\right\}_{t\in\mathbb{R}_+} \in \mathcal{A}^+_{loc}.$$

The set $\mathcal{G}(\widetilde{\mu})$ represents the domain of the functional $\theta_1 \rightarrow \int \theta_1 d\left(\widetilde{\mu} - \widetilde{\mu}^{\mathcal{P}}\right)$. We use the notation $\int \theta_1 d\left(\widetilde{\mu} - \widetilde{\mu}^{\mathcal{P}}\right)$ to write the value of this functional in θ_1. It is important to point out that this integral functional is not, in general, the integral with respect to the difference of two measures. In the process of this thesis we need several properties of this functional which we could not find in the literature, and were included in the Appendix A, as well as a general construction of this functional. For a detailed exposition on these topics see He, Wang and Yan [15] or Jacod and Shiryaev [18], which are our basic references.

In particular, for the Lévy process L with jump measure μ,

$$\mathcal{G}(\mu) \equiv \left\{ \theta_1 \in \widetilde{\mathcal{P}} : \left\{ \sqrt{\sum_{s\leq t} \{\theta_1(s, \triangle L_s)\}^2 \mathbf{1}_{\mathbb{R}_0}(\triangle L_s)} \right\}_{t\in\mathbb{R}_+} \in \mathcal{A}_{loc}^+ \right\}, \tag{1.3}$$

since $\mu^{\mathcal{P}}(\{t\} \times A) = 0$, for any Borel set A of $\mathbb{R}_0$.

We say that the semimartingale U has the *weak property of predictable representation* when

$$\mathcal{M}_{loc,0} = \Lambda(U^c) + \left\{ \int \theta_1 d\left(\mu_U - \mu_U^{\mathcal{P}}\right) : \theta_1 \in \mathcal{G}(\mu_U) \right\}, \tag{1.4}$$

where the previous sum is the linear sum of the vector spaces, and $\mathcal{M}_{loc,0}$ is the linear space of local martingales starting at zero.

The integral representation of a semimartingale U asserts that

$$U_t = U_0 + \alpha_t^U + U_t^c + \int\limits_{[0,t]\times\{0<|x|\leq 1\}} x\left\{\mu_U(ds,dx) - \mu_U^{\mathcal{P}}(dx,ds)\right\} + \int\limits_{[0,t]\times\{|x|>1\}} x\mu_U(ds,dx), \tag{1.5}$$

where α_t^U is a predictable process with finite variation and $\alpha_0^U = 0$. Taking $\beta_t^U := [U^c]_t$ we define $\left(\alpha^U, \beta^U, \mu_U^{\mathcal{P}}\right)$ as the *predictable characteristics* (*predictable triplet*, *local characteristics*) of the semimartingale U.

1.2 Density processes

Given an absolutely continuous probability measure $\mathbb{Q} \ll \mathbb{P}$ in a filtered probability space, where a semimartingale with the weak predictable representation property is defined, the structure of the density process has been studied extensively by several authors; see Theorem 14.41 in He, Wang and Yan [15] or Theorem III.5.19 in Jacod and Shiryaev [18].

It is well known that the Lévy-processes satisfy the weak property of predictable representation when the completed natural filtration is considered. In the following lemma we present the characterization of the density processes for the case of these processes .

Lemma 1 *Given an absolutely continuous probability measure $\mathbb{Q} \ll \mathbb{P}$, there exist coefficients $\theta_0 \in \mathcal{L}(W)$ and $\theta_1 \in \mathcal{G}(\mu)$ such that*

$$\frac{d\mathbb{Q}_t}{d\mathbb{P}_t} = \mathcal{E}\left(Z^{\theta}\right)(t),$$

where

$$Z_t^{\theta} := \int_{]0,t]} \theta_0 dW + \int_{]0,t]\times\mathbb{R}_0} \theta_1(s,x)\left(\mu(ds,dx) - ds\ \nu(dx)\right), \tag{1.6}$$

and $\mathcal{E}$ represents the Doleans-Dade exponential of a semimartingale. The coefficients θ_0 and θ_1 are unique, $\mathbb{P}$-a.s. and $\mu_{\mathbb{P}}^{\mathcal{P}}(ds,dx)$-a.s., respectively.

Proof. Let $\mathbb{Q}$ be an arbitrary but fixed absolutely continuous probability measure with respect to $\mathbb{P}$. Define the cádlág density process $D_t := \mathbb{E}_{\mathbb{P}}\left[\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|\mathcal{F}_t\right]$. Using the fact that Lévy processes satisfy the weak property of predictable representation, and taking $\tau_n := \inf\left\{t \geq 0 : D_t \leq \frac{1}{n}\right\}$, for each local martingale $\left\{\int_0^t (D_{s-})^{-1} dD_s\right\}_{t\in\mathbb{R}_+}^{\tau_n}$, it follows that there exist processes $\theta_0^{(n)} \in \mathcal{L}(W)$ and $\theta_1^{(n)} \in \mathcal{G}(\mu)$ such that

$$\int_0^{t\wedge\tau_n} \frac{1}{D_{s-}} dD_s = \int_0^t \theta_0^{(n)}(s)\, dW_s + \int_{]0,t]\times\mathbb{R}_0} \theta_1^{(n)}(s,x)\left(\mu(ds,dx) - ds\ \nu(dx)\right) =: Z_{\theta}^{(n)}(t).$$

The previous identity yields that $D^{\tau_n} = \mathcal{E}\left(Z_{\theta}^{(n)}\right)$. Observe that $\mathbb{Q}\left[\inf_{t\geq 0} D_t = 0\right] = 0$, and hence $\tau_n \uparrow \infty$ $\mathbb{Q}$-a.s.. In view of the uniqueness of the solution to the stochastic integral equation we have the existence $\mathbb{Q}$-a.s. of the process Z^{θ}, as defined in the statement of the theorem, with $\left(Z^{\theta}\right)_t^{\tau_n} = Z_{\theta}^{(n)}(t)$ $\mathbb{Q}$-a.s. for all $n \in \mathbb{N}$. The uniqueness of the Radon Nikodym density yields that the representation holds also $\mathbb{P}$-a.s.. In order to prove the uniqueness of the coefficients θ_0 and θ_1, assume that the representation is satisfied by two vectors $\theta := (\theta_0, \theta_1)$ and $\widehat{\theta} := \left(\widehat{\theta}_0, \widehat{\theta}_1\right)$. Since two purely discontinuous local martingales with the same jumps are equal, it follows that

$$\int_{]0,t]\times\mathbb{R}_0} \theta_1(s,x)\left(\mu(ds,dx) - ds\ \nu(dx)\right) = \int_{]0,t]\times\mathbb{R}_0} \widehat{\theta}_1(s,x)\left(\mu(ds,dx) - ds\ \nu(dx)\right),$$

and thus

$$\int_{]0,t]} \theta_0(s)\, dW_s = \int_{]0,t]} \widehat{\theta}_0(s)\, dW_s.$$

Then,

$$0 = \left[\int \left\{\widehat{\theta}_0(s) - \theta_0(s)\right\} dW_s\right]_t = \int_{]0,t]} \left\{\widehat{\theta}_0(s) - \theta_0(s)\right\}^2 ds$$

and hence $\widehat{\theta}_0(s) = \theta_0(s)\ \ \forall s\ \mathbb{P}\text{-}a.s.$.

On the other hand,

$$\begin{aligned} 0 &= \left\langle \int \left\{\widehat{\theta}_1(s,x) - \theta_1(s,x)\right\}\left(\mu(ds,dx) - ds\ \nu(dx)\right)\right\rangle_t \\ &= \int_{]0,t]\times\mathbb{R}_0} \left\{\widehat{\theta}_1(s,x) - \theta_1(s,x)\right\}^2 \nu(dx)\, ds, \end{aligned}$$

implies that $\theta_1(s,x) = \widehat{\theta}_1(s,x)\quad \mu_{\mathbb{P}}^{\mathcal{P}}(ds,dx)$-a.s. . ■

For $\mathbb{Q} \ll \mathbb{P}$ the function $\theta_1(\omega,t,x)$ described in Lemma 1 determines the density of the predictable projection $\mu_{\mathbb{Q}}^{\mathcal{P}}(dt,dx)$ with respect to $\mu_{\mathbb{P}}^{\mathcal{P}}(dt,dx)$ (see He,Wang and Yan [15] or Jacod and Shiryaev [18]). More precisely, for $B \in (\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R}_0))$ we have

$$\mu_{\mathbb{Q}}^{\mathcal{P}}(\omega,B) = \int_B (1 + \theta_1(\omega,t,x))\, \mu_{\mathbb{P}}^{\mathcal{P}}(dt,dx)\,. \tag{1.7}$$

In what follows we restrict ourself to the time interval $[0,T]$, for some $T > 0$ fixed, and we take $\mathcal{F} = \mathcal{F}_T$. Let $\mathcal{Q}(\Omega,\mathcal{F})$ be the family of probability measures on the measurable space $(\Omega,\mathcal{F})$. We denote by $\mathcal{Q}_{\ll}(\mathbb{P})$ the subclass of absolutely continuous probability measure with respect to $\mathbb{P}$ and by $\mathcal{Q}_{\approx}(\mathbb{P})$ the subclass of equivalent probability measure. Of course,

$\mathcal{Q}_{\approx}(\mathbb{P}) \subset \mathcal{Q}_{\ll}(\mathbb{P}) \subset \mathcal{Q}(\Omega, \mathcal{F})$.

The corresponding classes of density processes associated to $\mathcal{Q}_{\ll}(\mathbb{P})$ and $\mathcal{Q}_{\approx}(\mathbb{P})$ are denoted by $\mathcal{D}_{\ll}(\mathbb{P})$ and $\mathcal{D}_{\approx}(\mathbb{P})$, respectively. For instance, in the former case

$$\mathcal{D}_{\ll}(\mathbb{P}) := \left\{ D = \{D_t\}_{t\in[0,T]} : \exists \mathbb{Q} \in \mathcal{Q}_{\ll}(\mathbb{P}) \text{ with } D_t = \left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} \right\}, \tag{1.8}$$

and the processes in this set are of the form

$$\begin{aligned} D_t = \quad & \exp\left\{ \int_{]0,t]} \theta_0 dW + \int_{]0,t]\times\mathbb{R}_0} \theta_1(s,x)\left(\mu(ds,dx) - \nu(dx)\,ds\right) - \tfrac{1}{2}\int_{]0,t]} (\theta_0)^2\, ds \right\} \times \\ & \times \exp\left\{ \int_{]0,t]\times\mathbb{R}_0} \{\ln(1+\theta_1(s,x)) - \theta_1(s,x)\}\,\mu(ds,dx) \right\} \end{aligned} \tag{1.9}$$

for $\theta_0 \in \mathcal{L}(W)$ and $\theta_1 \in \mathcal{G}(\mu)$.

The set $\mathcal{D}_{\ll}(\mathbb{P})$ is characterized as follow.

Corollary 2 *D belongs to $\mathcal{D}_{\ll}(\mathbb{P})$ if and only if there are $\theta_0 \in \mathcal{L}(W)$ and $\theta_1 \in \mathcal{G}(\mu)$ with $\theta_1 \geq -1$ such that $D_t = \mathcal{E}(Z^\theta)(t)$ $\mathbb{P}$-a.s. $\forall t \in [0,T]$ and $\mathbb{E}_{\mathbb{P}}[\mathcal{E}(Z^\theta)(t)] = 1$ $\forall t \geq 0$, where $Z^\theta(t)$ is defined by* (1.6).

Proof. The necessity follows from Lemma 1. Conversely, let $\theta_0 \in \mathcal{L}(W)$ and $\theta_1 \in \mathcal{G}(\mu)$ be arbitrarily chosen. Since $D_t = \int D_{s-} dZ_s^\theta \in \mathcal{M}_{loc}$ is a nonnegative local martingale, it is a supermartingale, with constant expectation from our assumptions. Therefore, it is a martingale, and hence the density process of an absolutely continuous probability measure.

■

The following lemma is interesting by itself to understand the continuity properties of the quadratic variation for a given convergent sequence of densities. It will play a central role in the proof of the lower semicontinuity of the penalization function introduced in the next sections. Observe that the assertion of this lemma is valid in a general filtered probability space and not only for the completed natural filtration of the Lévy process introduced above.

Lemma 3 *Let* $\left\{\mathbb{Q}^{(n)}\right\}_{n\in\mathbb{N}}$ *be a sequence in* $\mathcal{Q}_{\ll}(\mathbb{P})$*, with* $D_T^{(n)} := \left.\frac{d\mathbb{Q}^{(n)}}{d\mathbb{P}}\right|_{\mathcal{F}_T}$ *converging to* $D_T := \left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_T}$ *in* $L^1(\mathbb{P})$*. For the corresponding density processes* $D_t^{(n)} := \mathbb{E}_{\mathbb{P}}\left[D_T^{(n)} \left|\mathcal{F}_t\right.\right]$ *and* $D_t := \mathbb{E}_{\mathbb{P}}\left[D_T \left|\mathcal{F}_t\right.\right]$*, for* $t\in[0,T]$*, we have*

$$\left[D^{(n)}-D\right]_T \xrightarrow{\mathbb{P}} 0.$$

Proof. Since we can write $\left[D^{(n)}-D\right]_T = \left[D^{(n)}-D\right]_{T-} + \triangle\left[D^{(n)}-D\right]_T$, we will prove separately that $\left[D^{(n)}-D\right]_{T-} \xrightarrow{\mathbb{P}} 0$ and $\triangle\left[D^{(n)}-D\right]_T \xrightarrow{\mathbb{P}} 0$ hold. The proof of the convergence of $\triangle\left[D^{(n)}-D\right]_T \xrightarrow{\mathbb{P}} 0$ is equivalent to verify that

$$\lim_{n\to\infty} d\left(\left[D^n-D\right]_T, \left[D^{(n)}-D\right]_{T-}\right) = 0,$$

where $d(X,Y) := \inf\{\varepsilon>0 : \mathbb{P}\left[|X-Y|>\varepsilon\right]\le\varepsilon\}$ is the Ky Fan metric. Since

$$d\left(\left[D^n-D\right]_T, \left[D^{(n)}-D\right]_{T-}\right) = \inf\left\{\varepsilon>0 : \mathbb{P}\left[\left|\triangle\left(D^{(n)}-D\right)_T\right| > \sqrt{\varepsilon}\right] \le \varepsilon\right\},$$

the conclusion follows observing that the maximal inequality for supermartingales yields

$$\mathbb{P}\left[\left|\triangle\left(D^{(n)}-D\right)_T\right| > \sqrt{\varepsilon}\right] \le \mathbb{P}\left[\sup_{t\le T}\left|D_t^{(n)}-D_t\right| > \frac{\sqrt{\varepsilon}}{2}\right] \le \frac{2}{\sqrt{\varepsilon}}\mathbb{E}\left[\left|D_T^{(n)}-D_T\right|\right].$$

In order to prove that $\left[D^{(n)}-D\right]_{T-} \xrightarrow{\mathbb{P}} 0$, we show first that there is a double index sequence $\{\tau_k^n\}_{k\in\mathbb{N}}$, with $\left[D^{(n)}-D\right]_{\tau_k^n} \xrightarrow[n\to\infty]{\mathbb{P}} 0$ for all $k\in\mathbb{N}$. From the L^1 convergence of $D_T^{(n)}$ to D_T, we have that $\{D_T^{(n)}\}_{n\in\mathbb{N}} \cup \{D_T\}$ is uniformly integrable, which is equivalent to the existence of a convex and increasing function $G:[0,+\infty)\to[0,+\infty)$ such that

$$(i) \quad \lim_{x\to\infty}\frac{G(x)}{x} = \infty,$$

and

$$(ii) \quad \sup_{n\in\mathbb{N}} \mathbb{E}\left[G\left(D_T^{(n)}\right)\right] \vee \mathbb{E}\left[G\left(D_T\right)\right] < \infty.$$

Now, define the stopping times

$$\tau_k^n := \inf\left\{u > 0 : \sup_{t\le u}\left|D_t^{(n)} - D_t\right| \ge k\right\} \wedge T.$$

Observe that the estimation $\sup_{n\in\mathbb{N}} \mathbb{E}\left[G\left(D_{\tau_k^n}^{(n)}\right)\right] \le \sup_{n\in\mathbb{N}} \mathbb{E}\left[G\left(D_T^{(n)}\right)\right]$ implies the uniform integrability of $\left\{D_{\tau_k^n}^{(n)}\right\}_{n\in\mathbb{N}}$ for each k fixed. The same argument yields the uniform integrability of $\left\{D_{\tau_k^n}\right\}_{n\in\mathbb{N}}$ for all $k \in \mathbb{N}$, and hence $\left\{\sup_{t\le\tau_k^n}\left|D_t^{(n)} - D_t\right|\right\}_{n\in\mathbb{N}}$ is uniformly integrable and converge in L^1 to 0.

The maximal inequality for supermartingales

$$\begin{aligned}
\mathbb{P}\left[\sup_{t\le\tau_k^n}\left|D_t^{(n)} - D_t\right| \ge \varepsilon\right] &\le \mathbb{P}\left[\sup_{t\in[0,T]}\left|D_t^{(n)} - D_t\right| \ge \varepsilon\right] \\
&\le \frac{1}{\varepsilon}\left\{\sup_{t\in[0,T]} \mathbb{E}\left[\left|D_t^{(n)} - D_t\right|\right]\right\} \\
&\le \frac{1}{\varepsilon}\mathbb{E}\left[\left|D_T^{(n)} - D_T\right|\right] \longrightarrow 0,
\end{aligned}$$

and Davis' inequality guarantees that, for some constant c,

$$\mathbb{E}\left[\sqrt{\left[D^{(n)} - D\right]_{\tau_k^n}}\right] \le \frac{1}{c}\mathbb{E}\left[\sup_{t\le\tau_k^n}\left|D_t^{(n)} - D_t\right|\right] \xrightarrow[n\to\infty]{L^1} 0 \quad \forall k \in \mathbb{N},$$

and hence $\left[D^{(n)} - D\right]_{\tau_k^n} \xrightarrow[n\to\infty]{\mathbb{P}} 0$ for all $k \in \mathbb{N}$.

Finally, to prove that $\left[D^{(n)} - D\right]_{T-} \xrightarrow{\mathbb{P}} 0$ we assume that it is not true, and then $\left[D^{(n)} - D\right]_{T-} \overset{\mathbb{P}}{\nrightarrow} 0$ implies that there exist $\varepsilon > 0$ and $\{n_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$ with

$$d\left(\left[D^{(n_k)} - D\right]_{T-}, 0\right) \ge \varepsilon$$

for all $k \in \mathbb{N}$. We shall denote the subsequence as the original sequence, trying to keep the notation as simple as possible. Next, a subsequence $\{n_i\}_{i\in\mathbb{N}} \subset \mathbb{N}$ is chosen, with the property that $d\left(\left[D^{(n_i)} - D\right]_{\tau_k^{n_i}}, 0\right) < \frac{1}{k}$ for all $i \geq k$. Since

$$\lim_{k\to\infty} \left[D^{(n_i)} - D\right]_{\tau_k^{n_i}} = \left[D^{(n_i)} - D\right]_{T-} \quad \mathbb{P} - a.s.,$$

we can find some $k(n_i) \geq i$ such that

$$d\left(\left[D^{(n_i)} - D\right]_{\tau_{k(n_i)}^{n_i}}, \left[D^{(n_i)} - D\right]_{T-}\right) < \frac{1}{k}.$$

Then, using the estimation

$$\mathbb{P}\left[\left|\left[D^{(n_k)} - D\right]_{\tau_{k(n_k)}^{n_k}} - \left[D^{(n_k)} - D\right]_{\tau_k^{n_k}}\right| > \varepsilon\right] \leq \mathbb{P}\left[\left\{\sup_{t\leq T}\left|D_t^{(n_k)} - D_t\right| \geq k\right\}\right],$$

it follows that

$$d\left(\left[D^{(n_k)} - D\right]_{\tau_{k(n_k)}^{n_k}}, \left[D^{(n_k)} - D\right]_{\tau_k^{n_k}}\right) \underset{k\to\infty}{\longrightarrow} 0,$$

which yields a contradiction with $\varepsilon \leq d\left(\left[D^{(n_k)} - D\right]_{T-}, 0\right)$. Thus, $\left[D^{(n)} - D\right]_{T-} \xrightarrow{\mathbb{P}} 0$. ■

2 The market model

In this section, we introduce the market model considered in this dissertation. It is based on the generalization of the classical geometric Brownian setting, but in this case the coefficients are not constant and jumps are included in the model through an exogenous stochastic process. One of the most debatable feature about an stochastic process used for modelling stock market prices is the issue about the independent increments. A remarkable property of the proposed model is the fact that it does not need to have independent increments. It includes also a certain subclase of exponential Lévy models. We conclude the section with a characterization of the set of local equivalent martingale measures.

2.1 General description and martingale measures

First, consider the stochastic process Y_t with dynamics given by

$$Y_t := \int_{]0,t]} \alpha_s ds + \int_{]0,t]} \beta_s dW_s + \int_{]0,t]\times\mathbb{R}_0} \gamma(s,x)\left(\mu(ds,dx) - \nu(dx)\ ds\right), \tag{2.1}$$

where the processes α, β are cádlág, with $\beta \in \mathcal{L}(W)$ and $\gamma \in \mathcal{G}(\mu)$. Throughout we assume that the coefficients α, β and γ fulfill the following conditions:

(A 1) $\int_{]0,t]} (\alpha_s)^2 ds < \infty \quad \forall t \in \mathbb{R}_+ \quad \mathbb{P}\text{-}a.s.$.

(A 2) $0 < c \leq |\beta_t| \quad \forall t \in \mathbb{R}_+ \quad \mathbb{P}\text{-}a.s.$.

(A 3) $\int_0^T \left(\frac{\alpha_u}{\beta_u}\right)^2 du \in \mathfrak{M}_b$ i.e. it is a bounded random variable.

(A 4) $\gamma(t, \triangle L_t) \times \mathbf{1}_{\mathbb{R}_0}(\triangle L_t) \geq -1 \quad \forall t \in \mathbb{R}_+ \quad \mathbb{P}\text{-}a.s.$.

(A 5) $\{\gamma(t, \triangle L_t)\mathbf{1}_{\mathbb{R}_0}(\triangle L_t)\}_{t\in\mathbb{R}_+}$ is a locally bounded process.

The market model consists of two assets, one of them is the numéraire, having a strictly positive price. The dynamics of the other risky asset will be modeled as a function of the process Y_t defined above. More specifically, since we will be interested in the problem of robust utility maximization, the discounted capital process can be written in terms of the wealth invested in this asset, and hence the problem can be written using only the dynamics of the discounted price of this asset. For this reason, throughout we will be concentrated in the dynamics of this price.

The discounted price process S is determined by the process Y as its Doleans-Dade exponential

$$S_t = S_0 \mathcal{E}\left(Y_t\right). \tag{2.2}$$

The condition $(A\ 4)$ ensures that the price process is non-negative. This process is an exponential semimartingale, as it would be the case for an arbitrary semimartingale Y, if and only if the following two conditions are fulfilled:

$$\begin{aligned} &(i) \quad S = S\mathbf{1}_{[0,\tau]},\text{ for } \tau := \inf\left\{t > 0 : S_t = 0 \text{ or } S_{t-} = 0\right\}. \\ &(ii) \quad \tfrac{1}{S_{t-}}\mathbf{1}_{[S_{t-}\neq 0]} \text{ is integrable w.r.t. } S. \end{aligned} \tag{2.3}$$

The first property is conceptually very appropriate when we are interested in modelling the dynamics of a price process. Recall that a stochastically continuous semimartingale has independent increments if and only if its predictable triplet is non-random. Therefore in general, the price process S is not a Lévy exponential model, because $\left[Y^c\right]_t = \int_0^t \left(\beta_u\right)^2 du$ need not to be deterministic. However, observe that the model (2.2) includes the Lévy exponential for Lévy processes with $\triangle L_t \geq -1$.

For the model (2.2) the price process can be written explicitly as

$$\begin{aligned} S_t = \ & S_0 \exp\left\{ \int\limits_{]0,t]} \alpha_s ds + \int\limits_{]0,t]} \beta_s dW_s + \int\limits_{]0,t]\times\mathbb{R}_0} \gamma\left(s,x\right)\left(\mu\left(ds,dx\right) - \nu\left(dx\right)\ ds\right) - \tfrac{1}{2}\int\limits_{]0,t]} \left(\beta_s\right)^2 ds \right\} \\ & \times \exp\left\{ \int\limits_{]0,t]\times\mathbb{R}_0} \left\{\ln\left(1+\gamma\left(s,x\right)\right) - \gamma\left(s,x\right)\right\}\mu\left(ds,dx\right) \right\} \end{aligned} \tag{2.4}$$

Observe that $(A\ 5)$ is a necesary and sufficient condition for S to be a locally bounded process.

The predictable cádlág process $\{\pi_t\}_{t\in\mathbb{R}_+}$, satisfying the integrability condition $\int_0^t (\pi_s)^2 \, ds < \infty$ $\mathbb{P}$-a.s. for all $t \in \mathbb{R}_+$, shall denote the proportion of wealth at time t invested in the risky asset S. For an initial capital x, the discounted wealth $X_t^{x,\pi}$ associated with a self-financing investment strategy (x, π) fulfills the equation

$$X_t^{x,\pi} = x + \int_0^t \frac{X_{u-}^{x,\pi} \pi_u}{S_{u-}} \mathbf{1}_{[S_{u-} \neq 0]} dS_u. \tag{2.5}$$

We say that a self-financing strategy (x, π) is *admissible* if the wealth process satisfies $X_t^{x,\pi} > 0$ for all $t > 0$. The class of admissible wealth processes with initial wealth less than or equal to x is denoted by $\mathcal{X}(x)$.

The next result characterizes the class of *equivalent local martingale measures* defined as follows.

$$\mathcal{Q}_{elmm} := \{\mathbb{Q} \in \mathcal{Q}_{\approx}(\mathbb{P}) : \mathcal{X}(1) \subset \mathcal{M}_{loc}(\mathbb{Q})\} = \{\mathbb{Q} \in \mathcal{Q}_{\approx}(\mathbb{P}) : S \in \mathcal{M}_{loc}(\mathbb{Q})\}. \tag{2.6}$$

For details on the former equality see Appendix B. The class of density processes associated with $\mathcal{Q}_{elmm}$ is denoted by $\mathcal{D}_{elmm}(\mathbb{P})$. Kunita [22] gave conditions on the parameters (θ_0, θ_1) of a measure $\mathbb{Q} \in \mathcal{Q}_{\approx}$ in order that it is a local martingale measure for a Lévy exponential model i.e. when $S = \mathcal{E}(L)$. Observe that in this case $\mathcal{Q}_{elmm}(S) = \mathcal{Q}_{elmm}(L)$. Next proposition gives conditions on the parameters (θ_0, θ_1) under which an equivalent measure is a local martingale measure.

Proposition 4 *Given $\mathbb{Q} \in \mathcal{Q}_{\approx}$, let $\theta_0 \in \mathcal{L}(W)$ and $\theta_1 \in \mathcal{G}(\mu)$ be the corresponding processes describing the density processes found in Lemma 1. Then, the following equivalence holds:*

$$\mathbb{Q} \in \mathcal{Q}_{elmm} \Longleftrightarrow \alpha_t + \beta_t \theta_0(t) + \int_{\mathbb{R}_0} \gamma(t,x)\, \theta_1(t,x)\, \nu(dx) = 0 \ \forall t \geq 0 \quad \mathbb{P}\text{-}a.s. \tag{2.7}$$

Proof. Let $\mathbb{Q} \in \mathcal{Q}_{\approx}$ be an equivalent probability measure with density process given by $D_t := \mathbb{E}\left[d\mathbb{Q}/d\mathbb{P}|\,\mathcal{F}_t\right] = \mathcal{E}\left(Z^{\theta}\right)_t$, where we have used Lemma 1. Then, we have that

$$S \in \mathcal{M}^1_{loc}(\mathbb{Q}) \Longleftrightarrow SD \in \mathcal{M}^1_{loc}(\mathbb{P}).$$

Since $\gamma, \theta_1 \in \mathcal{G}(\mu)$ it follows that $\int \gamma(s,x)\,\theta_1(s,x)\,\mu(ds,dx) \in \mathcal{A}_{loc}$, and then $\gamma\theta_1 \in \mathcal{G}(\mu)$ *as* well as $\int \gamma\theta_1 d\left\{\mu - \mu^{\mathcal{P}}\right\} = \int \gamma\theta_1 d\mu - \int \gamma\theta_1 d\mu^{\mathcal{P}}$; see Proposition 17 in Appendix A for the details of this fact. Therefore,

$$\left[Y, Z^{\theta}\right]_t = \int_0^t \beta_s\theta_0 ds + \int_{]0,t]\times\mathbb{R}_0} \gamma\theta_1 d\left\{\mu - \mu^{\mathcal{P}}\right\} + \int_{]0,t]\times\mathbb{R}_0} \gamma\theta_1 d\mu^{\mathcal{P}}.$$

Now, we write

$$S_tD_t = S_0\mathcal{E}(Y)_t\,\mathcal{E}\left(Z^{\theta}\right)_t = S_0\mathcal{E}\left(Y + Z^{\theta} + \left[Y, Z^{\theta}\right]\right)_t,$$

and making some rearrangements we have that

$$\begin{aligned} & S_tD_t \\ = \;& S_0 + \int S_{u-}D_{u-}d\left\{Y + Z^{\theta} + \left[Y, Z^{\theta}\right]\right\}_u \\ = \;& S_0 + \int S_{u-}D_{u-}d\left\{\int(\beta+\theta_0)\,dW + \int(\gamma+\theta_1+\gamma\theta_1)\,d\left\{\mu-\mu^{\mathcal{P}}\right\}\right\}_u \\ & + \int S_{u-}D_{u-}d\left\{\int\left(\alpha_s + \beta_s\theta_0(s) + \int\gamma\theta_1\nu(dx)\right)ds\right\}_u. \end{aligned}$$

On the other hand, observe that

$$\int S_{u-}D_{u-}d\left\{\int(\beta+\theta_0)\,dW + \int(\gamma+\theta_1+\gamma\theta_1)\,d\left\{\mu-\mu^{\mathcal{P}}\right\}\right\}_u$$

belongs to the set of local martingales $\mathcal{M}_{loc}$, and

$$\int S_{u-}D_{u-}d\left\{\int\left(\alpha_s+\beta_s\theta_0\left(s\right)+\int\gamma\theta_1\nu\left(dx\right)\right)ds\right\}$$

is a finite variation continuous process in $\mathcal{V}^c$. To verify this claim, observe first that $(A\ 1)$ implies that $\int_0^t \alpha_s ds\in\mathcal{V}$. Further, for $\beta_s,\theta_0\in\mathcal{L}\left(W\right)$ we know that $\int_{[0,t]}\left\{\beta_s\right\}^2 ds<\infty$ $\mathbb{P}$-a.s., and $\int_{[0,t]}\left\{\theta_0\left(s\right)\right\}^2 ds<\infty$ $\mathbb{P}$-a.s. and from the Rogers-Hölder inequality

$$\int_0^t \left|\beta_s\right|\left|\theta_0\left(s\right)\right|ds\le\left(\int_0^t\left(\beta_s\right)^2 ds\right)^{\frac{1}{2}}\left(\int_0^t\left(\theta_0\left(s\right)\right)^2 ds\right)^{\frac{1}{2}}<\infty.$$

Then, the finite variation of $\int_0^t \beta_s\theta_0 ds$ is due to the absolutely integrability of the integrand, i.e. $\int_0^t \beta_s\theta_0 ds\in\mathcal{V}$. Since $\int\gamma\left(s,x\right)\theta_1\left(s,x\right)\mu\left(ds,dx\right)\in\mathcal{A}_{loc}$, it follows that

$$\int_{[0,t]\times\mathbb{R}_0}\gamma\left(s,x\right)\theta_1\left(s,x\right)\nu\left(dx\right)ds\in\mathcal{V}\ \mathbb{P}-a.s.\forall t\in\mathbb{R}_+.$$

See Appendix A for details. Summarizing,

$$\int_0^t \alpha_s ds+\int_0^t \beta_s\theta_0 ds+\int_{]0,t]\times\mathbb{R}_0}\gamma\theta_1\nu\left(dx\right)ds\in\mathcal{V}.$$

The equivalence (2.7) follows now observing that a predictable local martingale with locally integrable variation is constant. ■

3 Minimal penalty function of risk measures concentrated in $\mathcal{Q}_{\ll}(\mathbb{P})$.

In contrast with the first two sections, this one is not dedicated to the study of the structure of the market, based on Lévy processes. In this section we only need a probability space; except for the last result, where we need a probability space where a semimartingale with the weak predictable representation property is defined. We shall deal with the question of characterizing penalty functions that are minimal for the corresponding static risk measure. We begin establishing necessary and sufficient conditions for the penalty function for being minimal, which do not require any assumption on the probability space, and then we propose a family of penalty functions which are minimal for the generated risk measures.

3.1 Minimal penalty functions

Given a measurable space $(\Omega, \mathcal{F})$, we say that a set function $\mathbb{Q} : \mathcal{F} \to [0,1]$ is a *probability content* if it is finite additive and $\mathbb{Q}(\Omega) = 1$. The set of *probability contents* on this measurable space is denoted by $\mathcal{Q}_{cont}$.

From the general theory of static convex measure of risk, we know that any map $\psi : \mathcal{Q}_{cont} \to \mathbb{R} \cup \{+\infty\}$, with $\inf_{\mathbb{Q}\in\mathcal{Q}_{cont}} \psi(\mathbb{Q}) > -\infty$, induces a static convex measure of risk as a mapping $\rho : \mathfrak{M}_b \to \mathbb{R}$ given by

$$\rho(X) := \sup_{\mathbb{Q}\in\mathcal{Q}_{cont}} \left\{ \mathbb{E}_{\mathbb{Q}}[-X] - \psi(\mathbb{Q}) \right\}.$$

Here $\mathfrak{M}$ denotes the class of measurable functions and $\mathfrak{M}_b$ the subclass of bounded measurable functions. In [5], Föllmer and Schied proved that any convex measure of risk is essentially of this form. A convex measure of risk ρ on the space of bounded functions

$\mathfrak{M}_b(\Omega, \mathcal{F})$ has the representation

$$\rho(X) = \sup_{\mathbb{Q} \in \mathcal{Q}_{cont}} \left\{ \mathbb{E}_{\mathbb{Q}} [-X] - \psi_\rho^* (\mathbb{Q}) \right\}, \tag{3.1}$$

where

$$\psi_\rho^* (\mathbb{Q}) := \sup_{X \in \mathcal{A}\rho} \mathbb{E}_{\mathbb{Q}} [-X], \tag{3.2}$$

and $\mathcal{A}_\rho := \{X \in \mathfrak{M}_b : \rho(X) \leq 0\}$ is the *acceptance set* of ρ.

The penalty ψ_ρ^* is called the *minimal penalty function* associated to ρ because, for any other penalty function ψ fulfilling (3.1), $\psi(\mathbb{Q}) \geq \psi_\rho^*(\mathbb{Q})$, for all $\mathbb{Q} \in \mathcal{Q}_{cont}$. Furthermore, for the minimal penalty function, the next biduality relation is satisfied

$$\psi_\rho^* (\mathbb{Q}) = \sup_{X \in \mathfrak{M}_b(\Omega, \mathcal{F})} \{\mathbb{E}_{\mathbb{Q}} [-X] - \rho(X)\}, \quad \forall \mathbb{Q} \in \mathcal{Q}_{cont}. \tag{3.3}$$

Among the measures of risk, the class of them that are concentrated on the set of probability measures $\mathcal{Q} \subset \mathcal{Q}_{cont}$ are of special interest. Recall that a functional $I : E \subset \mathbb{R}^\Omega \to \mathbb{R}$ is *sequentially continuous from below (above)* when $\{X_n\}_{n \in \mathbb{N}} \uparrow X \Rightarrow \lim_{n\to\infty} I(X_n) = I(X)$ (respectively $\{X_n\}_{n \in \mathbb{N}} \downarrow X \Rightarrow \lim_{n\to\infty} I(X_n) = I(X)$). Föllmer and Schied [5] proved that any sequentially continuous from below convex measure of risk is concentrated on the set $\mathcal{Q}$. Later, Krätschmer [21, Prop. 3 pg 601] established that the sequential continuity from below is not only a sufficient but also a necessary condition in order to have a representation, by means of the minimal penalty function in terms of probability measures. The minimality property of the penalty function turns out to be quite relevant. This is the case, for instance, in the study of robust portfolio optimization problems. The following theorem characterizes in a precise way necessary and sufficient conditions for a penalty function to be the minimal penalty function of a convex measure of risk concentrated in $\mathcal{K} \subset \mathcal{Q}_{\ll}(\mathbb{P})$. When dealing with a set of measures $\mathcal{K} \subset \mathcal{Q}_{\ll}(\mathbb{P})$ we shall refer to some topological concepts, meaning that we are considering the corresponding set of densities and the strong topology in $L^1(\mathbb{P})$. Recall that in a locally convex space a convex set $\mathcal{K}$ is weakly closed if and only if $\mathcal{K}$ is closed in the original topology.

Theorem 5 *Let $\psi : \mathcal{K} \subset \mathcal{Q}_{\ll}(\mathbb{P}) \to \mathbb{R} \cup \{+\infty\}$ be a function with*

$$-\infty < \inf_{\mathbb{Q} \in \mathcal{Q}_{cont}} \psi(\mathbb{Q}) < \infty,$$

defining the extension $\psi(\mathbb{Q}) := \infty$ for each $\mathbb{Q} \in \mathcal{Q}_{cont} \setminus \mathcal{K}$ and $\mathcal{K}$ a convex closed set. Also, define the function Ψ, with domain in L^1, as

$$\Psi(D) := \begin{cases} \psi(\mathbb{Q}), & \text{if } D = d\mathbb{Q}/d\mathbb{P} \text{ for } \mathbb{Q} \in \mathcal{K} \\ \infty, & \text{otherwise.} \end{cases}$$

Then, the convex measure of risk $\rho(X) := \sup_{\mathbb{Q} \in \mathcal{Q}_{cont}} \{\mathbb{E}_{\mathbb{Q}}[-X] - \psi(\mathbb{Q})\}$ associated with ψ has the minimal penalty function ψ (i.e. $\psi = \psi_\rho^*$) if and only if Ψ is proper, convex and lower semicontinuous with respect to the (strong) L^1-topology or, equivalently, with respect to the weak topology $\sigma(L^1, L^\infty)$.

Proof. If $\psi(\mathbb{Q}) = \psi_\rho^*(\mathbb{Q}) = \sup_{X \in \mathfrak{M}_b(\Omega,\mathcal{F})} \left\{\int D^{\mathbb{Q}}(-X)\, d\mathbb{P} - \rho(X)\right\}$, we have that $\Psi(Z) = \sup_{X \in \mathfrak{M}_b(\Omega,\mathcal{F})} \left\{\int Z(-X)\, d\mathbb{P} - \rho(X)\right\}$ is the supremum of a family of convex lower semicontinuous functions with respect to the topology $\sigma(L^1, L^\infty)$, and $\Psi(Z)$ preserves both properties. In order to show that the condition is also sufficient, observe that from Theorem 4.31 in Föllmer and Schied [8] we have

$$\sup_{\mathbb{Q} \in \mathcal{K}} \{\mathbb{E}_{\mathbb{Q}}[-X] - \psi(\mathbb{Q})\} \equiv \rho(X) = \sup_{\mathbb{Q} \in \mathcal{Q}_{\ll}(\mathbb{P})} \left\{\mathbb{E}_{\mathbb{Q}}[-X] - \psi_\rho^*(\mathbb{Q})\right\}.$$

Therefore, we only need to show that $\psi_\rho^*(\mathbb{Q}) = \psi(\mathbb{Q})\ \forall \mathbb{Q} \in \mathcal{Q}_{\ll}(\mathbb{P})$.

However, observing that the Fenchel - Legendre transform $\Psi^*(U) = \rho(-U)$ and, considering the weak*-topology $\sigma(L^\infty(\mathbb{P}), L^1(\mathbb{P}))$, it follows that for $D = d\mathbb{Q}/d\mathbb{P}$ we have $\psi(\mathbb{Q}) = \Psi(D) = \Psi^{**}(D) = \psi_\rho^*(\mathbb{Q})$. $\blacksquare$

3.2 Penalty functions for densities

Now, we shall introduce a family of penalizations functions for the density processes described in Section 1, for the absolutely continuous measures $\mathbb{Q} \in \mathcal{Q}_{\ll}(\mathbb{P})$.

Let h, h_0 and h_1 be $\mathbb{R}_+$-valued convex functions defined in $\mathbb{R}$, with $0 = h(0) = h_0(0) = h_1(0)$, and h increasing. Define the penalty function

$$\begin{aligned} \vartheta(\mathbb{Q}) \quad : \quad &= \mathbb{E}_{\mathbb{Q}}\left[\int_0^T h\left(h_0(\theta_0(t)) + \int_{\mathbb{R}_0} h_1(\theta_1(t,x))\,\nu(dx)\right)dt\right]\mathbf{1}_{\mathcal{Q}_{\ll}}(\mathbb{Q}) \\ &+\infty \times \mathbf{1}_{\mathcal{Q}_{cont}\setminus\mathcal{Q}_{\ll}}(\mathbb{Q}), \end{aligned} \tag{3.4}$$

where θ_0, θ_1 are the processes associated to $\mathbb{Q}$ from Lemma 1. Further, define the convex measure of risk

$$\rho(X) := \sup_{\mathbb{Q}\in\mathcal{Q}_{\ll}(\mathbb{P})} \{\mathbb{E}_{\mathbb{Q}}[-X] - \vartheta(\mathbb{Q})\}. \tag{3.5}$$

Notice that ρ is a normalized and sensitive measure of risk .

For all the classes of probability measures introduced so far we define the subclasses of measures with finite penalization. Denote by $\mathcal{Q}^\vartheta$, $\mathcal{Q}^\vartheta_{\ll}(\mathbb{P})$ and $\mathcal{Q}^\vartheta_{\approx}(\mathbb{P})$ the analogous subclasses, i.e.

$$\mathcal{Q}^\vartheta := \{\mathbb{Q}\in\mathcal{Q} : \vartheta(\mathbb{Q}) < \infty\}, \ \mathcal{Q}^\vartheta_{\ll}(\mathbb{P}) := \mathcal{Q}^\vartheta \cap \mathcal{Q}_{\ll}(\mathbb{P}) \text{ and } \mathcal{Q}^\vartheta_{\approx}(\mathbb{P}) := \mathcal{Q}^\vartheta \cap \mathcal{Q}_{\approx}(\mathbb{P}). \tag{3.6}$$

It can be verified easily that $\mathcal{Q}^\vartheta_{\approx}(\mathbb{P}) \neq \varnothing$.

The next theorem establishes the minimality of the penalty function introduced above for the risk measure ρ. Its proof is based on the sufficient conditions given in Proposition 5.

Theorem 6 *The penalty function ϑ defined in* (3.4) *is the minimal penalty function of the convex risk measure ρ given by* (3.5)*.*

Proof. From Theorem 5, we need to show that the penalization ϑ is proper, convex and that the corresponding identification, defined as $\Theta(D) := \vartheta(\mathbb{Q})$ if $D = d\mathbb{Q}/d\mathbb{P} \in \mathcal{D}_{\ll}(\mathbb{P})$ and $\Theta(Z) := \infty$ on $L^1 \setminus \mathcal{D}_{\ll}(\mathbb{P})$, is lower semicontinuous with respect to the strong topology.

First, observe that the function ϑ is proper, since $\vartheta(\mathbb{P}) = 0$. To verify the convexity of ϑ, choose $\mathbb{Q}, \widetilde{\mathbb{Q}} \in \mathcal{Q}^{\vartheta}_{\ll}$ and define $\mathbb{Q}^{\lambda} := \lambda \mathbb{Q} + (1-\lambda)\widetilde{\mathbb{Q}}$, for $\lambda \in [0,1]$. Notice that the corresponding density process can be written as $D^{\lambda} := \dfrac{d\mathbb{Q}^{\lambda}}{d\mathbb{P}} = \lambda D + (1-\lambda)\widetilde{D}$ $\mathbb{P}$-a.s. .

Now, from Lemma 1, let (θ_0, θ_1) and $(\widetilde{\theta}_0, \widetilde{\theta}_1)$ be the processes associated to $\mathbb{Q}$ and $\widetilde{\mathbb{Q}}$, respectively. Defining $\tau_n^{\lambda} := \inf\left\{t \geq 0 : D_t^{\lambda} \leq \frac{1}{n}\right\}$, from the weak predictable representation property of the local martingale defined below, we have that

$$\int_0^{t\wedge\tau_n^{\lambda}} \left(D_{s-}^{\lambda}\right)^{-1} dD_s^{\lambda} = \int_0^{t\wedge\tau_n^{\lambda}} \theta_0^{\lambda}(s)\, dW_s + \int_{\left[0,t\wedge\tau_n^{\lambda}\right]\times\mathbb{R}_0} \theta_1^{\lambda}(s,x)\, d\left(\mu - \mu_{\mathbb{P}}^{\mathcal{P}}\right),$$

where

$$\theta_0^{\lambda}(s) := \frac{\lambda D_{s-}\theta_0(s) + (1-\lambda)\widetilde{D}_{s-}\widetilde{\theta}_0(s)}{\left(\lambda D_{s-} + (1-\lambda)\widetilde{D}_{s-}\right)},$$

and

$$\theta_1^{\lambda}(s,x) := \frac{\lambda D_{s-}\theta_1(s,x) + (1-\lambda)\widetilde{D}_{s-}\widetilde{\theta}_1(s,x)}{\left(\lambda D_{s-} + (1-\lambda)\widetilde{D}_{s-}\right)}.$$

The identification of $\theta_0^{\lambda}(s)$ and $\theta_1^{\lambda}(s,x)$ is possible thanks to the uniqueness of the representation in Lemma 1 . The convexity follows now from the convexity of h, h_0 and h_1, using the fact that any convex function is continuous in the interior of its domain. More specifically,

$$\begin{aligned}
\vartheta\left(\mathbb{Q}^{\lambda}\right) \leq\ & \mathbb{E}_{\mathbb{Q}^{\lambda}}\left[\int\limits_{[0,T]} \frac{\lambda D_s}{\left(\lambda D_s+(1-\lambda)\widetilde{D}_s\right)} h\left(h_0\left(\theta_0\left(s\right)\right)+\int\limits_{\mathbb{R}_0} h_1\left(\theta_1\left(s,x\right)\right)\nu\left(dx\right)\right)ds\right] \\
& +\mathbb{E}_{\mathbb{Q}^{\lambda}}\left[\int\limits_{[0,T]} \frac{(1-\lambda)\widetilde{D}_s}{\left(\lambda D_s+(1-\lambda)\widetilde{D}_s\right)} h\left(h_0\left(\widetilde{\theta}_0\left(s\right)\right)+\int\limits_{\mathbb{R}_0} h_1(\widetilde{\theta}_1\left(s,x\right))\nu\left(dx\right)\right)ds\right] \\
=\ & \int\limits_{[0,T]}\int\limits_{\Omega} \frac{\lambda D_s}{\left(\lambda D_s+\left(1-\lambda\right)\widetilde{D}_s\right)} h\left(h_0\left(\theta_0\left(s\right)\right)+\int\limits_{\mathbb{R}_0} h_1\left(\theta_1\left(s,x\right)\right)\nu\left(dx\right)\right) \\
& \qquad \times\mathbb{E}_{\mathbb{P}}[\lambda D_T+\left(1-\lambda\right)\widetilde{D}_T\left|\mathcal{F}_s\right.]\mathbf{1}_{\left\{\lambda D_s+(1-\lambda)\widetilde{D}_s>0\right\}}d\mathbb{P}ds \\
& +\int\limits_{[0,T]}\int\limits_{\Omega} \frac{\left(1-\lambda\right)\widetilde{D}_s}{\left(\lambda D_s+\left(1-\lambda\right)\widetilde{D}_s\right)} h\left(h_0\left(\widetilde{\theta}_0\left(s\right)\right)+\int\limits_{\mathbb{R}_0} h_1(\widetilde{\theta}_1\left(s,x\right))\nu\left(dx\right)\right) \\
& \qquad \times\mathbb{E}_{\mathbb{P}}[\lambda D_T+\left(1-\lambda\right)\widetilde{D}_T\left|\mathcal{F}_s\right.]\mathbf{1}_{\left\{\lambda D_s+(1-\lambda)\widetilde{D}_s>0\right\}}d\mathbb{P}ds \\
=\ & \lambda\vartheta\left(\mathbb{Q}\right)+\left(1-\lambda\right)\vartheta\left(\widetilde{\mathbb{Q}}\right).
\end{aligned}$$

It remains to prove the lower semicontinuity of Θ. As pointed out earlier, it is enough to consider a sequence of densities $D_T^{(n)} := \frac{d\mathbb{Q}^{(n)}}{d\mathbb{P}} \in \mathcal{D}_{\ll}\left(\mathbb{P}\right)$ converging in $L^1\left(\mathbb{P}\right)$ to $D_T := \frac{d\mathbb{Q}}{d\mathbb{P}}$. Denote the corresponding density processes by $D^{(n)}$ and D, respectively. In Lemma 3 was verified the convergence in probability to zero of the quadratic variation process

$$\begin{aligned}
\left[D^{(n)}-D\right]_T =\ & \int\limits_0^T \left\{D_{s-}^{(n)}\theta_0^{(n)}\left(s\right)-D_{s-}\theta_0\left(s\right)\right\}^2 ds \\
& +\int\limits_{[0,T]\times\mathbb{R}_0} \left\{D_{s-}^{(n)}\theta_1^{(n)}\left(s,x\right)-D_{s-}\theta_1\left(s,x\right)\right\}^2 d\mu\left(ds,dx\right).
\end{aligned}$$

This implies that

$$\left.\begin{array}{c}\int_0^T \left\{D_{s-}^{(n)}\theta_0^{(n)}(s) - D_{s-}\theta_0(s)\right\}^2 ds \xrightarrow{\mathbb{P}} 0, \\ \text{and} \\ \int\limits_{[0,T]\times\mathbb{R}_0} \left\{D_{s-}^{(n)}\theta_1^{(n)}(s,x) - D_{s-}\theta_1(s,x)\right\}^2 d\mu(ds,dx) \xrightarrow{\mathbb{P}} 0.\end{array}\right\} \tag{3.7}$$

Then, for an arbitrary but fixed subsequence, there exists a sub-subsequence such that $\mathbb{P}$-a.s.

$$\left\{D_{s-}^{(n)}\theta_0^{(n)}(s) - D_{s-}\theta_0(s)\right\}^2 \xrightarrow{L^1(\lambda)} 0$$

and

$$\left\{D_{s-}^{(n)}\theta_1^{(n)}(s,x) - D_{s-}\theta_1(s,x)\right\}^2 \xrightarrow{L^1(\mu)} 0,$$

where for simplicity we have denoted the sub-subsequence as the original sequence. Now, we claim that for any subsequence there is a sub-subsequence such that

$$\left\{\begin{array}{l} D_{s-}^{(n)}\theta_0^{(n)}(s) \xrightarrow{\lambda\times\mathbb{P}\text{-a.s.}} D_{s-}\theta_0(s), \\ \\ D_{s-}^{(n)}\theta_1^{(n)}(s,x) \xrightarrow{\mu\times\mathbb{P}\text{-a.s.}} D_{s-}\theta_1(s,x). \end{array}\right. \tag{3.8}$$

We present first the arguments for the proof of the second assertion in (3.8). Assuming the opposite, there exists $C \in \mathcal{B}([0,T]) \otimes \mathcal{B}(\mathbb{R}_0) \otimes \mathcal{F}_T$, with $\mu \times \mathbb{P}[C] > 0$, and such that for each $(s,x,\omega) \in C$

$$\lim_{n\to\infty} \left\{D_{s-}^{(n)}\theta_1^{(n)}(s,x) - D_{s-}\theta_1(s,x)\right\}^2 = c \neq 0,$$

or the limit does not exist.

Let $C(\omega) := \{(t,x) \in [0,T] \times \mathbb{R}_0 : (t,x,\omega) \in C\}$ be the ω-section of C. Observe that

$B := \{\omega \in \Omega : \mu[C(\omega)] > 0\}$ has positive probability: $\mathbb{P}[B] > 0$.

From (3.7), any arbitrary but fixed subsequence has a sub-subsequence converging $\mathbb{P}$-a.s. . Denoting such a sub-subsequence simply by n, we can fix $\omega \in B$ with

$$\begin{aligned} &\int_{C(\omega)} \left\{ D_{s-}^{(n)} \theta_1^{(n)}(s,x) - D_{s-}\theta_1(s,x) \right\}^2 d\mu(s,x) \\ \leq &\int_{[0,T]\times\mathbb{R}_0} \left\{ D_{s-}^{(n)} \theta_1^{(n)}(s,x) - D_{s-}\theta_1(s,x) \right\}^2 d\mu(s,x) \underset{n\to\infty}{\longrightarrow} 0, \end{aligned}$$

and hence $\left\{ D_{s-}^{(n)} \theta_1^{(n)}(s,x) - D_{s-}\theta_1(s,x) \right\}^2$ converges in μ-measure to 0 on $C(\omega)$. Again, for any subsequence there is a sub-subsequence converging μ-a.s. to 0. Furthermore, for an arbitrary but fixed $(s,x) \in C(\omega)$, when the limit does not exist

$$\begin{aligned} a &:= \liminf_{n\to\infty} \left\{ D_{s-}^{(n)} \theta_1^{(n)}(s,x) - D_{s-}\theta_1(s,x) \right\}^2 \\ &\neq \limsup_{n\to\infty} \left\{ D_{s-}^{(n)} \theta_1^{(n)}(s,x) - D_{s-}\theta_1(s,x) \right\}^2 =: b, \end{aligned}$$

and we can choose converging subsequences $n(i)$ and $n(j)$ with

$$\begin{aligned} \lim_{i\to\infty} \left\{ D_{s-}^{n(i)} \theta_1^{n(i)}(s,x) - D_{s-}\theta_1(s,x) \right\}^2 &= a \\ \lim_{j\to\infty} \left\{ D_{s-}^{n(j)} \theta_1^{n(j)}(s,x) - D_{s-}\theta_1(s,x) \right\}^2 &= b. \end{aligned}$$

From the above argument, there are sub-subsequences $n(i(k))$ and $n(j(k))$ such that

$$\begin{aligned} a &= \lim_{k\to\infty} \left\{ D_{s-}^{n(i(k))} \theta_1^{n(i(k))}(s,x) - D_{s-}\theta_1(s,x) \right\}^2 = 0 \\ b &= \lim_{k\to\infty} \left\{ D_{s-}^{n(j(k))} \theta_1^{n(j(k))}(s,x) - D_{s-}\theta_1(s,x) \right\}^2 = 0, \end{aligned}$$

which is clearly a contradiction.

For the case when

$$\lim_{n\to\infty}\left\{D_{s-}^{(n)}\theta_1^{(n)}(s,x)-D_{s-}\theta_1(s,x)\right\}^2=c\neq 0,$$

the same argument can be used, and get a subsequence converging to 0, having a contradiction again. Therefore, the second part of our claim in (3.8) holds.

Since $D_{s-}^{(n)}\theta_1^{(n)}(s,x)\,,\ D_{s-}\theta_1(s,x)\in\mathcal{G}(\mu)$, we have, in particular, that $D_{s-}^{(n)}\theta_1^{(n)}(s,x)\in\widetilde{\mathcal{P}}$ and $D_{s-}\theta_1(s,x)\in\widetilde{\mathcal{P}}$ and hence $C\in\widetilde{\mathcal{P}}$. From the definition of the predictable projection it follows that

$$\begin{aligned} 0 &= \mu\times\mathbb{P}[C]=\int_{\Omega}\int_{[0,T]\times\mathbb{R}_0}\mathbf{1}_C(s,\omega)\,d\mu d\mathbb{P}=\int_{\Omega}\int_{[0,T]\times\mathbb{R}_0}\mathbf{1}_C(s,\omega)\,d\mu_{\mathbb{P}}^{\mathcal{P}}d\mathbb{P}\\ &=\int_{\Omega}\int_{\mathbb{R}_0}\int_{[0,T]}\mathbf{1}_C(s,\omega)\,dsd\nu d\mathbb{P}=\lambda\times\nu\times\mathbb{P}[C]\,, \end{aligned}$$

and thus

$$D_{s-}^{(n)}\theta_1^{(n)}(s,x)\overset{\lambda\times\nu\times\mathbb{P}\text{-a.s.}}{\longrightarrow}D_{s-}\theta_1(s,x)\,.$$

Since $\int_{\Omega\times[0,T]}\left|D_{t-}^{(n)}-D_{t-}\right|dt\times d\mathbb{P}=\int_{\Omega\times[0,T]}\left|D_{t}^{(n)}-D_{t}\right|dt\times d\mathbb{P}\longrightarrow 0$, we have that $\left\{D_{t-}^{(n)}\right\}_{t\in[0,T]}\overset{L^1(\lambda\times\mathbb{P})}{\longrightarrow}\{D_{t-}\}_{t\in[0,T]}$ and $\left\{D_{t}^{(n)}\right\}_{t\in[0,T]}\overset{L^1(\lambda\times\mathbb{P})}{\longrightarrow}\{D_{t}\}_{t\in[0,T]}$. Then, for an arbitrary but fixed subsequence $\{n_k\}_{k\in\mathbb{N}}\subset\mathbb{N}$, there is a sub-subsequence $\{n_{k_i}\}_{i\in\mathbb{N}}\subset\mathbb{N}$ such that

$$\begin{array}{ccc} D_{t-}^{\left(n_{k_i}\right)}\theta_1^{\left(n_{k_i}\right)}(t,x) & \overset{\lambda\times\nu\times\mathbb{P}\text{-a.s.}}{\longrightarrow} & D_{t-}\theta_1(t,x)\,,\\ D_{t-}^{\left(n_{k_i}\right)} & \overset{\lambda\times\mathbb{P}\text{-a.s.}}{\longrightarrow} & D_{t-},\\ D_{t}^{\left(n_{k_i}\right)} & \overset{\lambda\times\mathbb{P}\text{-a.s.}}{\longrightarrow} & D_{t}. \end{array}$$

Furthermore, $\mathbb{Q} \ll \mathbb{P}$ implies that $\lambda \times \nu \times \mathbb{Q} \ll \lambda \times \nu \times \mathbb{P}$, and then

$$\begin{array}{ccc} D_{t-}^{\left(n_{k_i}\right)} \theta_1^{\left(n_{k_i}\right)} (t,x) & \overset{\lambda \times \nu \times \mathbb{Q}\text{-a.s.}}{\longrightarrow} & D_{t-}\theta_1 (t,x), \\ D_{t-}^{\left(n_{k_i}\right)} & \overset{\lambda \times \nu \times \mathbb{Q}\text{-a.s.}}{\longrightarrow} & D_{t-}, \end{array}$$

and

$$D_t^{\left(n_{k_i}\right)} \overset{\lambda \times \nu \times \mathbb{Q}\text{-a.s.}}{\longrightarrow} D_t. \tag{3.9}$$

Finally, noting that $\inf D_t > 0$ $\mathbb{Q}$-a.s.

$$\theta_1^{\left(n_{k_i}\right)} (t,x) \overset{\lambda \times \nu \times \mathbb{Q}\text{-a.s.}}{\longrightarrow} \theta_1 (t,x). \tag{3.10}$$

The first assertion in (3.8) can be proved using essentially the same kind of ideas used above for the proof of the second part, concluding that

$$\left\{ D_t^{\left(n_{k_i}\right)} \right\}_{t \in [0,T]} \overset{\lambda \times \mathbb{Q}\text{-a.s.}}{\longrightarrow} \{D_t\}_{t\in[0,T]} \tag{3.11}$$

and

$$\left\{ \theta_0^{\left(n_{k_i}\right)} (t) \right\}_{t \in [0,T]} \overset{\lambda \times \mathbb{Q}\text{-a.s.}}{\longrightarrow} \{\theta_0 (t)\}_{t\in[0,T]}. \tag{3.12}$$

We are now ready to finish the proof of the theorem, observing that

$$\liminf_{n\to\infty} \vartheta \left(\mathbb{Q}^{(n)}\right) = \liminf_{n\to\infty} \int_{\Omega\times[0,T]} \left\{ h\left(h_0 \left(\theta_0^{(n)}(t)\right) + \int_{\mathbb{R}_0} h_1 \left(\theta_1^{(n)}(t,x)\right) \nu(dx) \right) \right\} \frac{D_t^{(n)}}{D_t} d(\lambda \times \mathbb{Q}).$$

Let $\{n_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$ be a subsequence for which the limit inferior is realized. Using (3.9), (3.10), (3.11), and (3.12) we can pass to a sub-subsequence $\{n_{k_i}\}_{i\in\mathbb{N}} \subset \mathbb{N}$ and, from the continuity of $h,\ h_0$ and h_1, it follows

$$
\begin{aligned}
&\liminf_{n\to\infty}\vartheta\left(\mathbb{Q}^{(n)}\right)\\
\geq\ &\int\limits_{\Omega\times[0,T]}\liminf_{i\to\infty}\left(\left\{h\left(h_0\left(\theta_0^{\left(n_{k_i}\right)}(t)\right)+\int_{\mathbb{R}_0}h_1\left(\theta_1^{\left(n_{k_i}\right)}(t,x)\right)\nu(dx)\right)\right\}\frac{D_t^{\left(n_{k_i}\right)}}{D_t}\right)d(\lambda\times\mathbb{Q})\\
\geq\ &\int\limits_{\Omega\times[0,T]}h\left(h_0(\theta_0(t))+\int_{\mathbb{R}_0}h_1(\theta_1(t,x))\,\nu(dx)\right)d(\lambda\times\mathbb{Q})\\
=\ &\vartheta(\mathbb{Q}).
\end{aligned}
$$

■

Remark 7 *The assertion of Theorem 6 remains valid if in the proposed family of penalties* (3.4) *we add a coefficient* $\delta(t,x):\mathbb{R}_+\times\mathbb{R}_0\to\mathbb{R}_+$ *in the integral with respect to the intensity measure* $\nu(dx)$, *that is*

$$
\begin{aligned}
\vartheta(\mathbb{Q}) :=\ &\mathbb{E}_{\mathbb{Q}}\left[\int_0^T h\left(h_0(\theta_0(t))+\int_{\mathbb{R}_0}\delta(t,x)\,h_1(\theta_1(t,x))\,\nu(dx)\right)dt\right]\mathbf{1}_{\mathcal{Q}_{\ll}}(\mathbb{Q})\\
&+\infty\times\mathbf{1}_{\mathcal{Q}_{cont}\setminus\mathcal{Q}_{\ll}}(\mathbb{Q}).
\end{aligned}
$$

4 Robust utility maximization

In this section we review, first, the dual approach in the robust and classical settings for the expected utility, and then establish the connection between penalty functions and the existence of solutions to the penalized robust expected utility problem. In the last subsection we formulate the dual problem in terms of control processes for an arbitrary utility function.

4.1 Formulation of the dual problem

Within the market model introduced in the Section 2, consider payoffs described by $\mathcal{F}_T$-measurable random variables, and an economic agent with preferences, characterized by a numerical representation taking into account his risk preferences and model uncertainty, of the form introduced by Maccheroni, Marinacci and Rustichini in [23]. These numerical representations have the form

$$X \longrightarrow \inf_{\mathbb{Q} \in \mathcal{Q}_{\ll}} \left\{ \mathbb{E}_{\mathbb{Q}} \left[U \left(X \right) \right] + \vartheta \left(\mathbb{Q} \right) \right\},$$

where the utility function $U : (0, \infty) \longrightarrow \mathbb{R}$ is strictly increasing, strictly concave, continuously differentiable, and satisfies the Inada conditions (i.e. $U' (0+) = +\infty$ and $U' (\infty-) = 0$). The log-utility $U(x) = \log(x)$ and the power utility $U(x) = \frac{1}{q} x^q$, with $q \in (-\infty, 1) \setminus \{0\}$, satisfy those properties, and are in the group of utility functions that more attention have received in the literature.

Given the restrictions on the set of admissible strategies, we shall be interested only on strictly positive payoffs , according with the definition of the set $\mathcal{X}(x)$. To guarantee that the $\mathbb{Q}$-expectation is well defined, we extend the operator $\mathbb{E}_{\mathbb{Q}} [U(\cdot)]$ to $\mathcal{L}^0$, as in Schied [34, pg 111], by

$$\mathbb{E}_{\mathbb{Q}} [X] := \sup_{n \in \mathbb{N}} \mathbb{E}_{\mathbb{Q}} [X \wedge n] = \lim_{n \to \infty} \mathbb{E}_{\mathbb{Q}} [X \wedge n] \quad X \in \mathcal{L}^0 (\Omega, \mathcal{F}). \tag{4.13}$$

The goal of the economic agent, with an initial capital $x > 0$, will be to maximize the penalized expected utility from a terminal wealth in the worst case model. This means that the agent seeks to solve the robust expected utility problem associated with value function

$$u(x) := \sup_{X \in \mathcal{X}(x)} \inf_{\mathbb{Q} \in \mathcal{Q}^{\vartheta}_{\ll}} \left\{ \mathbb{E}_{\mathbb{Q}} [U(X_T)] + \vartheta(\mathbb{Q}) \right\}. \tag{4.14}$$

In the non-robust setting a probability measure $\mathbb{Q} \in \mathcal{Q}$ is fixed a priori as the market measure, and the primal problem (4.14) is reduced to

$$u_{\mathbb{Q}}(x) := \sup_{X \in \mathcal{X}(x)} \left\{ \mathbb{E}_{\mathbb{Q}} \left[U(X_T) \right] \right\}. \tag{4.15}$$

Kramkov and Schachermayer [19] and [20] studied this problem in a very general semimartingale setting. Their analysis was based on the dual formulation, which had been successfully used for other models. The basic idea is to pass to the convex conjugate V (also known as the Legendre-Frechet transformation) of the function $-U(-x)$, defined by

$$V(y) = \sup_{x>0} \{U(x) - xy\}, \qquad y > 0. \tag{4.16}$$

From the conditions imposed to the utility function U, we have that the conjugate function V is continuously differentiable, decreasing, and strictly convex, satisfying: $V'(0+) = -\infty$, $V'(\infty) = 0$, $V(0+) = U(\infty)$, $V(\infty) = U(0+)$. Further, the biconjugate of U is again U itself; in other words the bidual relationship holds

$$U(x) = \inf_{y>0} \{V(y) + xy\}, \qquad x > 0.$$

This approach has been very powerful and has had a tremendous impact in the development of Mathematical Finance and, in general, in the theory of convex optimization; somehow we could say that it became classical.

Kramkov and Schachermayer [19] formulated the dual problem in the non-robust setting in terms of the value function

$$v_{\mathbb{Q}}(y) := \inf_{Y \in \mathcal{Y}_{\mathbb{Q}}(y)} \left\{ \mathbb{E}_{\mathbb{Q}} \left[V(Y_T) \right] \right\}, \tag{4.17}$$

where

$$\mathcal{Y}_{\mathbb{Q}}(y) := \quad \{Y \geq 0 : \ Y_0 = y, YX \ \mathbb{Q}\text{-supermartingale} \ \forall X \in \mathcal{X}(1)\}. \tag{4.18}$$

Observe that any $Y \in \mathcal{Y}_{\mathbb{Q}}(y)$ is a $\mathbb{Q}$-supermartingale, since $X \equiv 1 \in \mathcal{X}(1)$. When the utility function U has asymptotic elasticity strictly less than one,

$$\limsup_{x \to \infty} \frac{xU'(x)}{U(x)} < 1,$$

it was proved in Kramkov and Schachermayer [19] that:

(i) There is always a unique solution for all $x > 0$, i.e. there exists a unique $\widehat{X} \in \mathcal{X}(x)$ such that $u_{\mathbb{Q}}(x) = \mathbb{E}_{\mathbb{Q}}\left[U\left(\widehat{X}_T\right)\right]$.

(ii) The value function $u_{\mathbb{Q}}(x)$ is a utility function i.e. strictly increasing, strictly concave, continuously differentiable and satisfies the Inada conditions ($u'(0+) = +\infty$ and $u'(\infty-) = 0$).

(iii) The dual problem satisfies $v_{\mathbb{Q}}(y) < \infty, \ \forall y > 0$, and it can be restricted to the class of equivalent local martingale measures $\mathcal{Q}_{elmm}(\mathbb{Q})$,

$$v_{\mathbb{Q}}(y) = \inf_{\widetilde{\mathbb{Q}} \in \mathcal{Q}_{elmm}(\mathbb{Q})} \left\{\mathbb{E}_{\mathbb{Q}}\left[V\left(yd\widetilde{\mathbb{Q}}/d\mathbb{Q}\right)\right]\right\}.$$

The previous assertions (i) - (iii) hold when the classical problem (4.15) is finite for at least some $x > 0$, and the non-arbitrage condition $\mathcal{Q}_{elmm}(\mathbb{Q}) \neq \varnothing$ as well as the Inada conditions for U are satisfied. Clearly, the asymptotic elasticity hypothesis involves only the utility function U and hence such condition is independent of the financial market.

In a more recent contribution Kramkov and Schachermayer [20] proved that a necessary and sufficient condition for (i) - (iii) to hold, is that the *dual function is finite*. Moreover, the authors showed that the following assertions are equivalent

$$\begin{aligned} v_{\mathbb{Q}}(y) &< \infty \text{ for all } y>0 \\ \lim_{x\to\infty}\frac{u_{\mathbb{Q}}(x)}{x} &= 0 \\ \inf_{\widetilde{\mathbb{Q}}\in\mathcal{Q}_{elmm}(\mathbb{Q})} \mathbb{E}_{\mathbb{Q}}\left[V\left(yd\widetilde{\mathbb{Q}}/d\mathbb{Q}\right)\right] &< \infty \text{ for all } y>0. \end{aligned} \tag{4.19}$$

When any of these conditions is satisfied, it can be concluded that:

(iv) $u_{\mathbb{Q}}(x) < \infty$, for all $x > 0$.

(v) The primal and dual problems have optimal solutions, $\widehat{X} \in \mathcal{X}(x)$ and $\widehat{Y} \in \mathcal{Y}_{\mathbb{Q}}(y)$ respectively, and are unique. Moreover, for $y = u'_{\mathbb{Q}}(x)$ it follows that

$$U'\left(\widehat{X}_T(x)\right) = \widehat{Y}_T(y).$$

(vi) The primal and dual value functions, $u_{\mathbb{Q}}(x)$ and $v_{\mathbb{Q}}(y)$ respectively, are conjugate

$$\begin{aligned} u_{\mathbb{Q}}(x) &= \inf_{y>0}\{v_{\mathbb{Q}}(y) + xy\}, \\ v_{\mathbb{Q}}(y) &= \sup_{x>0}\{u_{\mathbb{Q}}(x) - xy\}. \end{aligned}$$

The extension of the above results to the robust setting (4.14) was first studied by Schied [34]. The corresponding dual value function was defined by

$$v(y) := \inf_{\mathbb{Q}\in\mathcal{Q}^{\vartheta}_{\ll}}\{v_{\mathbb{Q}}(y) + \vartheta(\mathbb{Q})\}. \tag{4.20}$$

In this robust setting the necessary and sufficient condition given in 4.19 is transformed into

$$v_{\mathbb{Q}}(y) < \infty \quad \text{for all } \mathbb{Q} \in \mathcal{Q}_{\approx}^{\vartheta} \quad \text{and} \quad y > 0. \tag{4.21}$$

Remark 8 *When the conjugate convex function V is bounded from above it follows immediately that the penalized robust utility maximization problem (4.14) has a solution for any proper penalty function ϑ. This is the case, for instance, of the power utility function $U(x) := \frac{1}{q}x^q$, for $q \in (-\infty, 0)$, where the convex conjugate is the nonpositive function given by $V(x) = \frac{1}{p}x^{-p}$ with $p := \frac{q}{1-q}$.*

Let ϑ be a penalty function bounded from below, which corresponds to the minimal penalty function of a convex risk measure (3.1) such that the normalizing and sensitivity conditions hold. Assuming condition (4.21), the following assertions hold for the robust problem (4.14).

(vii) The robust value function $u(x)$ is strictly concave and takes only finite values.

(viii) The "minimax property" is satisfied

$$\sup_{X \in \mathcal{X}(x)} \inf_{\mathbb{Q} \in \mathcal{Q}_{\ll}^{\vartheta}} \{\mathbb{E}_{\mathbb{Q}}[U(X_T)] + \vartheta(\mathbb{Q})\} = \inf_{\mathbb{Q} \in \mathcal{Q}_{\ll}^{\vartheta}} \sup_{X \in \mathcal{X}(x)} \{\mathbb{E}_{\mathbb{Q}}[U(X_T)] + \vartheta(\mathbb{Q})\};$$

in other words,

$$u(x) = \inf_{\mathbb{Q} \in \mathcal{Q}_{\ll}^{\vartheta}} \{u_{\mathbb{Q}}(x) + \vartheta(\mathbb{Q})\}.$$

(ix) u and v are conjugate

$$u(x) = \inf_{y>0} (v(y) + xy) \qquad \text{and} \qquad v(y) = \sup_{x>0} (u(x) - xy).$$

(x) v is convex, continuously differentiable, and take only finite values.

(xi) The dual problem (4.20) has an optimal solution. That is, there exist $\mathbb{Q}^* \in \mathcal{Q}_{\ll}^{\vartheta}$ and $Y^* \in \mathcal{Y}_{\mathbb{Q}^*}(y)$ such that

$$\mathbb{E}_{\mathbb{Q}^*}\left[V\left(Y_T^*\right)\right] + \vartheta\left(\mathbb{Q}^*\right) = \inf_{\mathbb{Q}\in\mathcal{Q}_{\ll}^{\vartheta}} \left\{ \inf_{Y\in\mathcal{Y}_{\mathbb{Q}}(y)} \left\{\mathbb{E}_{\mathbb{Q}}\left[V\left(Y_T\right)\right]\right\} + \vartheta\left(\mathbb{Q}\right) \right\},$$

which is maximal in the sense that any other solution $(\mathbb{Q}, Y)$ satisfies $\mathbb{Q} \ll \mathbb{Q}^*$ and $Y_T = Y_T^*$ $\mathbb{Q}$-a.s. .

(xii) For each $x > 0$ there exists an optimal solution $X^* \in \mathcal{X}(x)$ to the robust problem (4.14). Furthermore, let $y > 0$, such that $v'(y) = -x$, and $(\mathbb{Q}^*, Y^*)$ be a solution to the dual problem (4.20). Then $(\mathbb{Q}^*, X^*)$ with

$$X_T^* := -V'\left(Y_T^*\right),$$

is a saddlepoint for the robust problem

$$u(x) = \mathbb{E}_{\mathbb{Q}^*}\left[U\left(X_T^*\right)\right] + \vartheta\left(\mathbb{Q}^*\right) = \inf_{\mathbb{Q}\in\mathcal{Q}_{\ll}^{\vartheta}} \sup_{X\in\mathcal{X}(x)} \left\{\mathbb{E}_{\mathbb{Q}}\left[U\left(X_T\right)\right] + \vartheta\left(\mathbb{Q}\right)\right\}.$$

4.2 Penalties and solvability

Let us now introduce the class

$$\mathcal{C} := \left\{ \mathcal{E}\left(Z^{\xi}\right) : \begin{array}{l} \xi := \left(\xi^{(0)}, \xi^{(1)}\right), \ \xi^{(0)} \in \mathcal{L}(W), \ \xi^{(1)} \in \mathcal{G}(\mu), \text{ with} \\ \alpha_t + \beta_t \xi_t^{(0)} + \int_{\mathbb{R}_0} \gamma(t,x)\, \xi^{(1)}(t,x)\, \nu(dx) = 0 \text{ Lebesgue } \forall t \end{array} \right\}, \tag{4.22}$$

with Z^{ξ} as in (1.6). Observe that $\mathcal{D}_{elmm}(\mathbb{P}) \subset \mathcal{C} \subset \mathcal{Y}_{\mathbb{P}}(1)$. Details about this relation will be presented in Appendix B. It should be pointed out that this relation between these three sets plays a crucial role in the formulation of the dual problem, even in the non-robust case.

Theorem 9 *Let $U(x) := \frac{1}{q}x^q$ be the power utility function, with $q \in (-\infty, 1) \setminus \{0\}$. For the functions $h,\ h_0,\ h_1 : \mathbb{R} \to \mathbb{R}$ such that $h(x) \geq \exp(\kappa_1 x^2) - 1$, with $\kappa_1 := 1 \vee 3(2p^2 + p)T$, $h_0(x) \geq |x|$, and $h_1(x) \geq \frac{|x|}{c}$, define the penalty function*

$$\vartheta_{x^q}(\mathbb{Q}) := \mathbb{E}_{\mathbb{Q}}\left[\int_0^T h\left(h_0(\theta_0(t)) + \int_{\mathbb{R}_0} |\gamma(t,x)|\, h_1(\theta_1(t,x))\,\nu(dx)\right) dt\right].$$

Then, the robust utility maximization problem (4.14) *has an optimal solution.*

Proof. The penalty function ϑ_{x^q} is bounded from below, and by Theorem 6 and Remark 7 it is the minimal penalty function of the (normalized and sensitive) convex measure of risk defined in (3.1). Therefore, we only need to prove that condition (4.21) holds.

(1) In Lemma 4.2, Schied [34] establishes that for $\mathbb{Q} \in \mathcal{Q}_{\ll}$, with density process Z, the next equivalence holds

$$Y \in \mathcal{Y}_{\mathbb{Q}}(y) \Leftrightarrow YZ \in \mathcal{Y}_{\mathbb{P}}(y).$$

Therefore, for $\mathbb{Q} \in \mathcal{Q}_{\ll}^{\vartheta_{x^q}}$, with coefficient $\theta = (\theta_0, \theta_1)$, it follows that

$$\begin{aligned} v_{\mathbb{Q}}(y) &\equiv \inf_{Y \in \mathcal{Y}_{\mathbb{Q}}(y)} \{\mathbb{E}_{\mathbb{Q}}[V(Y_T)]\} \\ &= \inf_{Y \in \mathcal{Y}_{\mathbb{P}}(1)} \left\{\mathbb{E}_{\mathbb{Q}}\left[V\left(y\frac{Y_T}{D_T^{\mathbb{Q}}}\right)\right]\right\} \leq \inf_{\xi \in \mathcal{C}} \left\{\mathbb{E}_{\mathbb{Q}}\left[V\left(y\frac{\mathcal{E}(Z^{\xi})_T}{\mathcal{E}(Z^{\theta})_T}\right)\right]\right\}. \end{aligned}$$

(2) Denote by

$$\varepsilon_t := \alpha_t + \beta_t \theta_0(t) + \int_{\mathbb{R}_0} \gamma(t,x)\,\theta_1(t,x)\,\nu(dx)$$

the process described in the definition of the class $\mathcal{C}$.

When ε_t is identically zero for all $t > 0$, Proposition 4 implies that $\mathbb{Q} \in \mathcal{Q}_{elmm}$. However, for $\mathbb{Q} \in \mathcal{Q}_{elmm}$ the constant process $Y \equiv y$ belongs to $\mathcal{Y}_{\mathbb{Q}}(y)$, and it follows that $v_{\mathbb{Q}}(y) < \infty$, for all $y > 0$. In this case the proof is concluded.

If ε is not identically zero, consider $\xi_t^{(0)} := \theta_0(t) - \frac{\varepsilon_t}{\beta_t}$ and $\xi^{(1)} := \theta_1$. Since

$$\infty > \vartheta_{x^q}(\mathbb{Q}) \geq \mathbb{E}_{\mathbb{Q}}\left[\int_0^T \left(\frac{1}{|\beta_t|}\int_{\mathbb{R}_0} |\gamma(t,x)\,\theta_1(t,x)|\,\nu(dx)\right)^2 dt\right],$$

it follows that $\left\{\frac{1}{|\beta_t|}\int_{\mathbb{R}_0}\gamma(t,x)\,\theta_1(t,x)\,\nu(dx)\right\}_{t\in[0,T]} \in \mathcal{L}(W')$ for W' a $\mathbb{Q}$-Wiener process.

Using Girsanov, we obtain $\frac{\mathcal{E}(Z^\xi)_T}{\mathcal{E}(Z^\theta)_T} = \exp\left\{\int_{]0,T]}\left(-\frac{\varepsilon_t}{\beta_t}\right)dW' - \frac{1}{2}\int_{]0,T]}\left(\frac{\varepsilon_t}{\beta_t}\right)^2 dt\right\}$.

(3) The Hölder inequality yields

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}}\left[V\left(y\frac{\mathcal{E}(Z^\xi)_T}{\mathcal{E}(Z^\theta)_T}\right)\right] = \frac{1}{p}y^{-p}\mathbb{E}_{\mathbb{Q}}\left[\exp\left\{p\int_{]0,T]}\left(\frac{\varepsilon_t}{\beta_t}\right)dW' + \frac{p}{2}\int_{]0,T]}\left(\frac{\varepsilon_t}{\beta_t}\right)^2 dt\right\}\right] \\
&\leq \frac{1}{p}y^{-p}\mathbb{E}_{\mathbb{Q}}\left[\exp\left\{2p\int_{]0,T]}\left(\frac{\varepsilon_t}{\beta_t}\right)dW' - \frac{4p^2}{2}\int_{]0,T]}\left(\frac{\varepsilon_t}{\beta_t}\right)^2 dt\right\}\right]^{\frac{1}{2}} \\
&\times \mathbb{E}_{\mathbb{Q}}\left[\exp\left\{\left(\frac{4p^2}{2}+p\right)\int_{]0,T]}\left(\frac{\varepsilon_t}{\beta_t}\right)^2 dt\right\}\right]^{\frac{1}{2}}.
\end{aligned}$$

On the other hand, the process

$$\exp\left\{2p\int_{]0,T]}\left(\frac{\varepsilon_t}{\beta_t}\right)dW' - \frac{4p^2}{2}\int_{]0,T]}\left(\frac{\varepsilon_t}{\beta_t}\right)^2 dt\right\} \in \mathcal{M}_{loc}(\mathbb{Q})$$

is a local $\mathbb{Q}$-martingale and, since it is positive, is a supermartingale. Hence,

$$\mathbb{E}_{\mathbb{Q}}\left[\exp\left\{2p\int_{]0,T]}\left(\frac{\varepsilon_t}{\beta_t}\right)dW' - \frac{4p^2}{2}\int_{]0,T]}\left(\frac{\varepsilon_t}{\beta_t}\right)^2 dt\right\}\right] \leq 1.$$

Finally, observe that for $\mathbb{Q} \in \mathcal{Q}_{\ll}^{\vartheta_{x^q}}$, using that it has finite penalization $\vartheta_{x^q}(\mathbb{Q}) < \infty$ and Jensen's inequality, we have

$$\begin{aligned} \infty &> \mathbb{E}_{\mathbb{Q}}\left[\exp\left\{\frac{\kappa_1}{T}\int_0^T\left(h_0(\theta_0(t)) + \int_{\mathbb{R}_0}|\gamma(t,x)|\,h_1(\theta_1(t,x))\,\nu(dx)\right)^2 dt\right\}\right] \\ &\geq \mathbb{E}_{\mathbb{Q}}\left[\exp\left\{3\left(2p^2+p\right)\int_0^T\left(|\theta_0(t)| + \frac{1}{c}\int_{\mathbb{R}_0}|\gamma(t,x)|\,|\theta_1(t,x)|\,\nu(dx)\right)^2 dt\right\}\right]. \end{aligned}$$

■

The next theorem establishes a sufficient condition for the existence of a solution to the robust utility maximization problem (4.14) for an arbitrary utility function.

Theorem 10 *Suppose that a utility function $\widetilde{U}$ is bounded above by a power utility U, with penalty function ϑ_{x^q} associated to U in Theorem 9. Then the robust utility maximization problem* (4.14) *for $\widetilde{U}$ with penalty ϑ_{x^q} has an optimal solution.*

Proof. Since $U(x) := \frac{1}{q}x^q \geq \widetilde{U}(x)$ for all $x > 0$, for some $q \in (-\infty, 1) \setminus \{0\}$ the corresponding convex conjugate functions satisfy $V(y) \geq \widetilde{V}(y)$ for each $y > 0$. As it was pointed out in Remark 8, we can restrict ourself to the positive part $\widetilde{V}^+(y)$. From Proposition 9, we can fix some $Y \in \mathcal{Y}_{\mathbb{Q}}(y)$ such that $\mathbb{E}_{\mathbb{Q}}[V(Y_T)] < \infty$ for any $\mathbb{Q} \in \mathcal{Q}_{\approx}^{\vartheta_{x^q}}$ and $y > 0$, arbitrary, but fixed. Furthermore, the inequality $V(y) \geq \widetilde{V}(y)$ implies that their inverse functions satisfy $(V^+)^{(-1)}(n) \geq \left(\widetilde{V}^+\right)^{(-1)}(n)$ for all $n \in \mathbb{N}$, and hence

$$\sum_{n=1}^{\infty}\mathbb{Q}\left[Y_T \leq \left(\widetilde{V}^+\right)^{(-1)}(n)\right] \leq \sum_{n=1}^{\infty}\mathbb{Q}\left[Y_T \leq \left(V^+\right)^{(-1)}(n)\right] < \infty.$$

The moments Lemma ($\mathbb{E}_{\mathbb{Q}}[X] < \infty \Leftrightarrow \sum_{n=1}^{\infty}\mathbb{Q}[|X| \geq n] < \infty$) yields $\mathbb{E}_{\mathbb{Q}}\left[\widetilde{V}^+(Y_T)\right] < \infty$, and the assertion follows. ■

Example 11 *The logarithm utility function satisfies the conditions of Theorem 10. However, this case will be studied more deeply in Section 4.3, since the techniques involve interesting arguments related to the relative entropy.*

From the proof of Theorem 10 it is clear that the behavior of the convex conjugate function in a neighborhood of zero is fundamental. From this observation we conclude the following.

Corollary 12 *Let U be a utility function with convex conjugate V, and let ϑ be a penalization function such that the robust utility maximization problem (4.14) has a solution. For a utility function $\widetilde{U}$ such that their convex conjugate function $\widetilde{V}$ is majorized in an ε-neighborhood of zero by V, the corresponding utility maximization problem (4.14) has a solution.*

Theorem 13 *For a utility function U with asymptotic elasticity strictly less than one, satisfying condition* (4.21)*, the dual value function can be written as*

$$v(y) = \inf_{\mathbb{Q}\in\mathcal{Q}_{\ll}} \left\{ \inf_{\xi\in\mathcal{C}} \left\{ \mathbb{E}_{\mathbb{Q}} \left[V\left(y\frac{\mathcal{E}(Z^{\xi})_T}{D_T^{\mathbb{Q}}} \right) \right] \right\} + \vartheta(\mathbb{Q}) \right\}. \tag{4.23}$$

Proof. Condition (4.21), together with Lemma 4.4 in [34] and Theorem 2.2 (iv) in [19], imply the following identity

$$v(y) = \inf_{\mathbb{Q}\in\mathcal{Q}_{\approx}^{\vartheta}} \left\{ \inf_{\widetilde{\mathbb{Q}}\in\mathcal{Q}_{elmm}(\mathbb{Q})} \left\{ \mathbb{E}_{\mathbb{Q}} \left[V\left(yd\widetilde{\mathbb{Q}}/d\mathbb{Q} \right) \right] \right\} + \vartheta(\mathbb{Q}) \right\}.$$

Since $\mathcal{D}_{elmm}(\mathbb{P}) \subset \mathcal{C}$, we get

$$v(y) \geq \inf_{\mathbb{Q}\in\mathcal{Q}_{\ll}} \left\{ \inf_{\xi\in\mathcal{C}} \left\{ \mathbb{E}_{\mathbb{Q}} \left[V\left(y\frac{\mathcal{E}(Z^{\xi})_T}{D_T^{\mathbb{Q}}} \right) \right] \right\} + \vartheta(\mathbb{Q}) \right\}.$$

Finally, using that $\mathcal{C} \subset \mathcal{Y}_{\mathbb{P}}(1)$ the reverse inequality holds, and the result follows. ■

4.3 The logarithmic utility case

As it was pointed out above in Example 11, the existence of a solution to the dual problem for the logarithmic utility function $U(x) = \log(x)$ can be read from the results presented in the previous subsection. However, the nature of the optimization problem arising in the case of a logarithmic utility deserves a deeper study. Let h, h_0 and h_1 be $\mathbb{R}_+$-valued convex functions defined on $\mathbb{R}$, such that $0 = h(0) = h_0(0) = h_1(0)$, and the following growth conditions hold.

$$
\begin{aligned}
h(x) &\geq x, \\
h_0(x) &\geq \frac{1}{2}x^2, \\
h_1(x) &\geq \{|x| \vee x\ln(1+x)\}\mathbf{1}_{(-1,0)}(x) + \{|x| \vee (1+x)\ln(1+x)\}\mathbf{1}_{\mathbb{R}_+}(x).
\end{aligned}
$$

Now, define the penalization function

$$
\begin{aligned}
\vartheta_{\log}(\mathbb{Q}) := \; & \mathbb{E}_{\mathbb{Q}}\left[\int_0^T h\left(h_0(\theta_0(t)) + \int_{\mathbb{R}_0} h_1(\theta_1(t,x))\,\nu(dx)\right)dt\right]\mathbf{1}_{\mathcal{Q}_{\ll}}(\mathbb{Q}) \\
& + \infty \times \mathbf{1}_{\mathcal{Q}_{cont}\setminus\mathcal{Q}_{\ll}}(\mathbb{Q}).
\end{aligned}
\tag{4.24}
$$

Remark 14 *Notice that when $\mathbb{Q} \in \mathcal{Q}_{\ll}^{\vartheta_{\log}}(\mathbb{P})$ has a finite penalization, we obtain following the $\mathbb{Q}$-integrability conditions:*

(14.i) $$\int_{[0,T]\times\mathbb{R}_0} \theta_1(t,x)\,\mu_{\mathbb{P}}^{\mathcal{P}}(dt,dx) \in \mathcal{L}^1(\mathbb{Q}).$$

(14.ii) $$\int_{[0,T]\times\mathbb{R}_0} \{1+\theta_1(t,x)\}\ln(1+\theta_1(t,x))\,\mu_{\mathbb{P}}^{\mathcal{P}}(dt,dx) \in \mathcal{L}^1(\mathbb{Q}).$$

(14.iii) $$\int_{[0,T]\times\mathbb{R}_0} \ln(1+\theta_1(s,x))\,\mu(ds,dx) \in \mathcal{L}^1(\mathbb{Q}).$$

For $\mathbb{Q} \in \mathcal{Q}_{\ll}(\mathbb{P})$, the relative entropy function is defined as

$$H(\mathbb{Q}|\mathbb{P}) := \mathbb{E}\left[D_T^{\mathbb{Q}} \log\left(D_T^{\mathbb{Q}}\right)\right].$$

Lemma 15 *Given* $\mathbb{Q} \in \mathcal{Q}_{\approx}^{\vartheta_{\log}}(\mathbb{P})$, *it follows that*

$$H(\mathbb{Q}|\mathbb{P}) \leq \vartheta_{\log}(\mathbb{Q}).$$

Proof. For $\mathbb{Q} \in \mathcal{Q}_{\approx}^{\vartheta_{\log}}(\mathbb{P})$, Remark 14 implies that

$$\begin{aligned} H(\mathbb{Q}|\mathbb{P}) &= \mathbb{E}_{\mathbb{Q}}\left[\frac{1}{2}\int_0^T (\theta_0)^2 ds + \int_{]0,T]\times\mathbb{R}_0} \ln(1+\theta_1(s,x))\,\mu(ds,dx) - \int_0^T\int_{\mathbb{R}_0} \theta_1(s,x)\,\nu(dx)\,ds\right] \\ &\leq \mathbb{E}_{\mathbb{Q}}\left[\int_0^T \left\{\frac{1}{2}(\theta_0)^2 ds + \int_{\mathbb{R}_0} \{\ln(1+\theta_1(s,x))\}\,\theta_1(s,x)\,\nu(dx)\right\} ds\right] \\ &\leq \vartheta_{\log}(\mathbb{Q}). \end{aligned}$$

■

Lemma 16 *Let* $U(x) = \log(x)$ *and* $\vartheta_{\log}$ *be as in* (4.24). *Then the robust utility maximization problem* (4.14) *has an optimal solution.*

Proof. Let $\mathbb{Q} \in \mathcal{Q}_{\ll}$ be fixed. Then

$$v_{\mathbb{Q}}(y) \leq \inf_{\xi\in\mathcal{C}}\left\{\mathbb{E}\left[D_T^{\mathbb{Q}} \log\left(\frac{D_T^{\mathbb{Q}}}{\mathcal{E}(Z^{\xi})_T}\right) - \log(y) - 1\right]\right\}.$$

Also, Proposition 4 yields for $\widetilde{\xi} \in \mathcal{C}$, with $\widetilde{\xi}^{(0)} := -\dfrac{\alpha_s}{\beta_s}$ and $\widetilde{\xi}^{(1)} := 0$, that $\widetilde{\mathbb{Q}} \in \mathcal{Q}_{elmm}^{\vartheta_{\log}}$, where $d\widetilde{\mathbb{Q}} \backslash d\mathbb{P} = D_T^{\widetilde{\xi}} := \mathcal{E}\left(Z^{\widetilde{\xi}}\right)_T$. Further, from Lemma 15 we conclude that

$$\mathbb{E}\left[D_T^{\mathbb{Q}} \log\left(\frac{D_T^{\mathbb{Q}}}{D_T^{\widetilde{\xi}}}\right)\right] = H\left(\mathbb{Q}\,|\mathbb{P}\right) + \mathbb{E}_{\mathbb{Q}}\left[\int_0^T \frac{\alpha_s}{\beta_s}\theta_s^{(0)} ds + \frac{1}{2}\int_0^T \left(\frac{\alpha_s}{\beta_s}\right)^2 ds\right] < \infty$$

and the claim follows. ■

A Appendix: The integral $\int\limits_{[0,t]\times\mathbb{R}_0} \theta_1(t,x)\, d\{\widetilde{\mu} - \widetilde{\mu}^{\mathcal{P}}\}$ for Lévy processes

In this appendix we present a definition of the integral $\int\limits_{[0,t]\times\mathbb{R}_0} \theta_1(t,x)\, d\left\{\widetilde{\mu} - \widetilde{\mu}^{\mathcal{P}}\right\}$, and study some of its properties.

The $\sigma-$ algebra on $\Omega\times\mathbb{R}$, generated by all cádlág adapted processes, is called the *optional* *σ-algebra.* A stochastic process X is called *optional,* if it is measurable with respect to the optional $\sigma-$algebra.

An optional process is said to be *thin* if the stochastic set $[\![X \neq 0]\!] := \{(\omega,t) : X_t(\omega) \neq 0\} \subset \Omega\times\mathbb{R}_+$ is a thin set, i.e. $[\![X \neq 0]\!] = \bigcup_{n=1}^{\infty} [\![\tau_n]\!] \equiv \bigcup_{n=1}^{\infty} \{(\omega,t) : \tau_n(\omega) = t\}$ for a sequence of stopping times τ_n. For a thin process X, a natural question is if it corresponds to the jump process $\triangle M$ of a local martingale $M \in \mathcal{M}_{loc}$. This is true under the following conditions, which are necessary and sufficient.

(i) The predictable projection vanishes, ${}^{\mathcal{P}}X = 0$.

(ii) $\sqrt{\sum_{s\leq t} X_s^2} \in \mathcal{A}_{loc}^{+}$.

Let $\widetilde{\mu}$ be an integer valued random measure with dual predictable projection $\widetilde{\mu}^{\mathcal{P}}$. The measure $\widetilde{\mu}^{\mathcal{P}}$ is by definition a predictable measure, meaning that for $\theta_1 \in \widetilde{\mathcal{P}}$, such that $\int \theta_1 d\widetilde{\mu}^{\mathcal{P}}$ exists, then $\left\{\int_{[0,t]\times\mathbb{R}_0} \theta_1 d\widetilde{\mu}^{\mathcal{P}}\right\}_{t\in\mathbb{R}_+} \in \mathcal{P}$ is a predictable process. Furthermore, given a non-negative function $\theta_1 \in \widetilde{\mathcal{P}}^{+}$ and a predictable time τ, it follows

$$\int_{\mathbb{R}_0} \theta_1(\tau,x)\, \widetilde{\mu}^{\mathcal{P}}(\{\tau\}, dx)\, \mathbf{1}_{[\tau<\infty]} = \mathbb{E}\left[\int_{\mathbb{R}_0} \theta_1(\tau,x)\, \widetilde{\mu}(\{\tau\}, dx)\, \mathbf{1}_{[\tau<\infty]}\, |\mathcal{F}_{\tau-}\right] \ a.s.$$

For a process $\theta_1 \in \widetilde{\mathcal{P}}$ with $\int_{\mathbb{R}_0} |\theta_1(t,x)|\, \widetilde{\mu}^{\mathcal{P}}(\{t\}, dx) < \infty$ for $t \geq 0$, and a predictable time

τ, considering the positive and negative parts of θ_1, we have that

$$\left(\int_{\mathbb{R}_0} \theta_1(\tau,x)\,\widetilde{\mu}^{\mathcal{P}}(\{\tau\},dx)\right)\mathbf{1}_{[\tau<\infty]} = \mathbb{E}\left[\int_{\mathbb{R}_0} \theta_1(\tau,x)\,\widetilde{\mu}(\{\tau\},dx)\,\mathbf{1}_{[\tau<\infty]}\,|\mathcal{F}_{\tau-}\right] \; a.s.$$

Since

$$\left\{\int_{\mathbb{R}_0} \theta_1(t,x)\,\widetilde{\mu}^{\mathcal{P}}(\{t\},dx)\right\}_{t\in\mathbb{R}_+} \in \mathcal{P},$$

the predictable projection of the process $\left\{\int_{\mathbb{R}_0} \theta_1(t,x)\,\widetilde{\mu}(\{t\},dx)\right\}_t$ is then

$$^{\mathcal{P}}\left(\int_{\mathbb{R}_0} \theta_1(t,x)\,\widetilde{\mu}(\{t\},dx)\right) = \int_{\mathbb{R}_0} \theta_1(t,x)\,\widetilde{\mu}^{\mathcal{P}}(\{t\},dx),$$

and hence

$$^{\mathcal{P}}\left(\int_{\mathbb{R}_0} \theta_1(t,x)\,\widetilde{\mu}(\{t\},dx) - \int_{\mathbb{R}_0} \theta_1(t,x)\,\widetilde{\mu}^{\mathcal{P}}(\{t\},dx)\right) = 0.$$

Therefore, for each element of the class $\mathcal{G}(\widetilde{\mu})$ of functions $\theta_1 : \Omega\times\mathbb{R}_+\times\mathbb{R}_0 \to \mathbb{R}$ such that

- (i) $\theta_1 \in \widetilde{\mathcal{P}}$.
- (ii) $\int_{\mathbb{R}_0} |\theta_1(t,x)|\,\widetilde{\mu}^{\mathcal{P}}(\{t\},dx) < \infty \; \forall t > 0$.
- (iii) The process

$$\left\{\sqrt{\sum_{s\le t}\left\{\int_{\mathbb{R}_0} \theta_1(s,x)\,\widetilde{\mu}(\{s\},dx) - \int_{\mathbb{R}_0} \theta_1(s,x)\,\widetilde{\mu}^{\mathcal{P}}(\{s\},dx)\right\}^2}\right\}_{t\in\mathbb{R}_+} \in \mathcal{A}^+_{loc},$$

there is a unique purely discontinuous local martingale $M \in \mathcal{M}^d_{loc}$ with the property

$$\triangle M_t = \int_{\mathbb{R}_0} \theta_1(t,x)\,\widetilde{\mu}(\{t\},dx) - \int_{\mathbb{R}_0} \theta_1(t,x)\,\widetilde{\mu}^{\mathcal{P}}(\{t\},dx) \quad \text{for every } t \ge 0.$$

The process M is called the *stochastic integral* of θ_1 with respect to $\widetilde{\mu}-\widetilde{\mu}^{\mathcal{P}}$ and we write

$$M_t = \int_{[0,t]\times\mathbb{R}_0} \theta_1(t,x)\, d\left\{\widetilde{\mu}-\widetilde{\mu}^{\mathcal{P}}\right\}.$$

For a process $A \in \mathcal{A}_{loc}$ we have that the dual predictable projection $A^{\mathcal{P}}$ is the unique predictable process with finite variation (and thus with locally integrable variation) such that $A - A^{\mathcal{P}}$ is a local martingale with locally integrable variation $A - A^{\mathcal{P}} \in \mathcal{M}_{loc,0} \cap \mathcal{A}_{loc}$. For $\theta_1 \in \widetilde{\mathcal{P}}$ assume now that the process $\int_{[0,t]\times\mathbb{R}_0} \theta_1(t,x)\, d\mu \in \mathcal{A}_{loc}$, then

$$\begin{aligned}
&(i) \quad \left\{\int_{[0,t]\times\mathbb{R}_0} \theta_1(t,x)\,\widetilde{\mu}(dt,dx)\right\}^{\mathcal{P}} = \int_{[0,t]\times\mathbb{R}_0} \theta_1(t,x)\,\widetilde{\mu}^{\mathcal{P}}(dt,dx),\\
&(ii) \quad \int_{[0,t]\times\mathbb{R}_0} \theta_1(t,x)\,\widetilde{\mu}^{\mathcal{P}}(dt,dx) \in \mathcal{A}_{loc}.\\
&(iii) \quad M := \left\{\int_{[0,t]\times\mathbb{R}_0} \theta_1(t,x)\,\widetilde{\mu}(dt,dx) - \int_{[0,t]\times\mathbb{R}_0} \theta_1(t,x)\,\widetilde{\mu}^{\mathcal{P}}(dt,dx)\right\}_{t\in\mathbb{R}_+} \in \mathcal{M}_{loc,0} \cap \mathcal{A}_{loc}.
\end{aligned}$$

Clearly $\triangle M_t = \int_{\mathbb{R}_0} \theta_1(t,x)\,\widetilde{\mu}(\{t\},dx) - \int_{\mathbb{R}_0} \theta_1(t,x)\,\widetilde{\mu}^{\mathcal{P}}(\{t\},dx)$ and $\theta_1 \in \mathcal{G}(\widetilde{\mu})$. Since $M \in \mathcal{M}^d_{loc,0}$ is a purely discontinuous local martingale it follows that

$$\int_{[0,t]\times\mathbb{R}_0} \theta_1(t,x)\, d\left\{\widetilde{\mu}-\widetilde{\mu}^{\mathcal{P}}\right\} = \int_{[0,t]\times\mathbb{R}_0} \theta_1(t,x)\,\widetilde{\mu}(dt,dx) - \int_{[0,t]\times\mathbb{R}_0} \theta_1(t,x)\,\widetilde{\mu}^{\mathcal{P}}(dt,dx).$$

As it was mentioned before, for a Lévy process L with jump measure μ the domain of integration $\mathcal{G}(\mu)$ has a simpler structure, due to the fact that $\mu^{\mathcal{P}}(\{t\},dx) = 0$.

Proposition 17 *Let L be a Lévy process and consider $\mathcal{G}(\mu)$ for the jump measure μ. Then, the following assertion hold.*

$$\theta_1, \theta_1' \in \mathcal{G}(\mu) \Longrightarrow \int_{[0,t]\times\mathbb{R}_0} |\theta_1\theta_1'|\, d\mu \in \mathcal{A}^+_{loc},$$

and thus in particular $\theta_1 \times \theta_1' \in \mathcal{G}(\mu)$ and $\int \theta_1(t,x)\, d\left\{\widetilde{\mu}-\widetilde{\mu}^{\mathcal{P}}\right\} = \int \theta_1(t,x)\,\widetilde{\mu}(dt,dx) -$

$\int \theta_1(t,x)\, \widetilde{\mu}^{\mathcal{P}}(dt,dx)$.

Proof. For $\theta_1, \theta_1' \in \mathcal{G}(\mu)$, we have

$$\begin{aligned}
& \sum_{s\leq t} \left|\theta_1(s,\triangle L_s)\,\theta_1'(s,\triangle L_s)\right| \mathbf{1}_{\mathbb{R}_0}(\triangle L_s) \\
\leq\ & \sum_{s\leq t} \left(\theta_1(s,\triangle L_s)\right)^2 \mathbf{1}_{\{|\theta_1(s,\triangle L_s)|\geq|\theta_1'(s,\triangle L_s)|\}} \mathbf{1}_{\mathbb{R}_0}(\triangle L_s) \\
& + \sum_{s\leq t} \left(\theta_1'(s,\triangle L_s)\right)^2 \mathbf{1}_{\{|\theta_1(s,\triangle L_s)|<|\theta_1'(s,\triangle L_s)|\}} \mathbf{1}_{\mathbb{R}_0}(\triangle L_s) \\
\leq\ & \sum_{s\leq t} \left(\theta_1(s,\triangle L_s)\right)^2 \mathbf{1}_{\mathbb{R}_0}(\triangle L_s) + \sum_{s\leq t} \left(\theta_1'(s,\triangle L_s)\right)^2 \mathbf{1}_{\mathbb{R}_0}(\triangle L_s).
\end{aligned}$$

Let $\left\{\tau_n^{(1)}\right\}_{n\in\mathbb{N}}$ and $\left\{\tau_n^{(2)}\right\}_{n\in\mathbb{N}}$ be localizing sequences of $\theta_1 \in \mathcal{G}(\mu)$ and $\theta_1' \in \mathcal{G}(\mu)$ respectivily and define $\tau_n := \tau_n^{(1)} \wedge \tau_n^{(2)}$. The claim follows from the observation

$$\begin{aligned}
\sum_{s\leq\tau_n} \left|\theta_1(s,\triangle L_s)\,\theta_1'(s,\triangle L_s)\right| \mathbf{1}_{\mathbb{R}_0}(\triangle L_s) \ \leq\ & \sqrt{\sum_{s\leq\tau_n^{(1)}} \left(\theta_1(s,\triangle L_s)\right)^2 \mathbf{1}_{\mathbb{R}_0}(\triangle L_s)} \\
& + \sqrt{\sum_{s\leq\tau_n^{(2)}} \left(\theta_1'(s,\triangle L_s)\right)^2 \mathbf{1}_{\mathbb{R}_0}(\triangle L_s)}
\end{aligned}$$

■

B Appendix: Results on equivalent local martingale measures

In this appendix we shall present some fundamental results on equivalent (local) martingale measures, which were important in the development of this thesis. We shall prove that

$$\mathcal{D}_{elmm}(\mathbb{P}) \subset \mathcal{C} \subset \mathcal{Y}_{\mathbb{P}}(1), \text{ and} \tag{B.25}$$

$$\{\mathbb{Q} \in \mathcal{Q}_{\approx}(\mathbb{P}) : \mathcal{X}(1) \subset \mathcal{M}_{loc}(\mathbb{Q})\} = \{\mathbb{Q} \in \mathcal{Q}_{\approx}(\mathbb{P}) : S \in \mathcal{M}_{loc}(\mathbb{Q})\} \tag{B.26}$$

First, recall that the classes of sets $\mathcal{D}_{elmm}(\mathbb{P})$ and $\mathcal{Y}_{\mathbb{P}}(1)$ were defined in (2.6) and (4.18), respectively. As it was mentioned in Section 4, the original formulation of the dual problem of the utility maximization problem was based in these sets. However, in our case, we introduced the set $\mathcal{C}$, defined below, since it allow us to take advantage of the Lévy structure of the price process.

Let

$$\mathcal{C} := \Big\{\mathcal{E}\left(Z^{\xi}\right) : \xi^{(0)} \in \mathcal{L}(W),\ \xi^{(1)} \in \mathcal{G}(\mu), \\ \alpha_t + \beta_t \xi_t^{(0)} + \int_{\mathbb{R}_0} \gamma(t,x)\, \xi^{(1)}(t,x)\, \nu(dx) = 0 \text{ Lebesgue } \forall t\Big\}.$$

Now, we prove the first claim $(B.25)$. The inclusion $\mathcal{D}_{elmm}(\mathbb{P}) \subset \mathcal{C}$ follows directly from Proposition 4, and then we shall prove only that $\mathcal{C} \subset \mathcal{Y}_{\mathbb{P}}(1)$. First, observe that $U_t = 1 + \int_0^t U_{u-} dZ_u^{\xi}$ is a $\mathbb{P}$-supermartingale, with $U_0 = 1$. Then, we only need to show that, given $X \in \mathcal{X}(1)$, the process UX is a $\mathbb{P}$-supermartingale. Since

$$UX = \mathcal{E}\left(Z^{\xi}\right) \mathcal{E}\left(\int \pi_u dY_u\right) = \mathcal{E}\left(Z^{\xi} + \int \pi_u dY_u + \left[Z^{\xi}, \int \pi_u dY_u\right]\right)$$

the supermartingale property will follow if we prove that

$$Z^{\xi}+\int \pi_u dY_u+\left[Z^{\xi},\int \pi_u dY_u\right]\in\mathcal{M}_{loc}(\mathbb{P}) \text{ is a } \mathbb{P}\text{-local martingale.}$$

For the last term $\left[Z^{\xi},\int \pi_u dY_u\right]=\int \pi_u d\left[Z^{\xi},Y\right]_u$, notice that

$$\begin{aligned}&\left[Z^{\xi},Y\right]_u\\ =&\int\limits_{[0,u]}\xi_s^{(0)}\beta_s ds+\left[\int\xi^{(1)}(s,x)\left(\mu(ds,dx)-ds\ \nu(dx)\right),\int\gamma(s,x)\left(\mu(ds,dx)-\nu(dx)\ ds\right)\right]_u\end{aligned}$$

and

$$\begin{aligned}&\left[\int\xi^{(1)}(s,x)\left(\mu(ds,dx)-ds\ \nu(dx)\right),\int\gamma(s,x)\left(\mu(ds,dx)-\nu(dx)\ ds\right)\right]_u\\ &=\sum_{s\le u}\triangle\left(\int\xi^{(1)}(s,x)\left(\mu(ds,dx)-ds\ \nu(dx)\right)\right)\triangle\left(\int\gamma(s,x)\left(\mu(ds,dx)-\nu(dx)\ ds\right)\right)\\ &=\sum_{s\le u}\int\limits_{\mathbb{R}_0}\xi^{(1)}(s,x)\,\mu(\{s\},dx)\int\limits_{\mathbb{R}_0}\gamma(s,x)\,\mu(\{s\},dx)\\ &=\sum_{s\le u}\xi^{(1)}(s,\triangle L_s)\,\gamma(s,\triangle L_s)\,\mathbf{1}_{\mathbb{R}_0}(\triangle L_s)\\ &=\int\limits_{[0,u]\times\mathbb{R}_0}\xi^{(1)}(s,x)\,\gamma(s,x)\,\mu(ds,dx).\end{aligned}$$

Hence,

$$\begin{aligned}\left[Z^{\xi},\int\pi_u dY_u\right]_t&=\int\pi_u d\left[Z^{\xi},Y\right]_u\\ &=\int\pi_u d\left\{\int\xi_s^{(0)}\beta_s ds+\int_{[0,u]\times\mathbb{R}_0}\xi^{(1)}(s,x)\,\gamma(s,x)\,\mu(ds,dx)\right\}\\ &=\int\pi_s\xi_s^{(0)}\beta_s ds+\int_{[0,t]\times\mathbb{R}_0}\pi_s\xi^{(1)}(s,x)\,\gamma(s,x)\,\mu(ds,dx).\end{aligned}$$

Therefore, the process $Z^{\xi}+\int\pi_u dY_u+\left[Z^{\xi},\int\pi_u dY_u\right]$ is a local martingale if and only if

$$\int_0^t\pi_s\alpha_s ds+\int_0^t\pi_s\beta_s\xi_s^{(0)}ds+\int_{[0,t]\times\mathbb{R}_0}\pi_s\xi^{(1)}(s,x)\,\gamma(s,x)\,\mu(ds,dx)\in\mathcal{M}_{loc}$$

is a local martingale. Also, note that

$$\begin{aligned}
&\int_0^t \pi_s\alpha_s ds + \int_0^t \pi_s\beta_s\xi_s^{(0)} ds + \int_{[0,t]\times\mathbb{R}_0} \pi_s\xi^{(1)}(s,x)\,\gamma(s,x)\,\mu(ds,dx) \\
= &\int_0^t \pi_s\alpha_s ds + \int_0^t \pi_s\beta_s\xi_s^{(0)} ds + \int_{[0,t]\times\mathbb{R}_0} \pi_s\xi^{(1)}(s,x)\,\gamma(s,x)\,\mu_{\mathbb{P}}^{\mathcal{P}}(ds,dx) \\
&+ \int_{[0,t]\times\mathbb{R}_0} \pi_s\xi^{(1)}(s,x)\,\gamma(s,x)\left(\mu(ds,dx) - \mu_{\mathbb{P}}^{\mathcal{P}}(ds,dx)\right).
\end{aligned}$$

Since $\int_{[0,t]\times\mathbb{R}_0} \pi_s\xi^{(1)}(s,x)\,\gamma(s,x)\left(\mu(ds,dx) - \mu_{\mathbb{P}}^{\mathcal{P}}(ds,dx)\right) \in \mathcal{M}_{loc}^d$ is a discontinuous local martingale, we need to show that

$$\int_0^t \pi_s\alpha_s ds + \int_0^t \pi_s\beta_s\xi_s^{(0)} ds + \int_{[0,t]\times\mathbb{R}_0} \pi_s\xi^{(1)}(s,x)\,\gamma(s,x)\,\mu_{\mathbb{P}}^{\mathcal{P}}(ds,dx) \in \mathcal{M}_{loc}$$

is a local martingale, but this follows immediately from the definition of $\mathcal{C}$.

Now we turn our attention to the claim $(B.26)$. Recall that an equivalent probability measure $\mathbb{Q} \in \mathcal{Q}_{\approx}(\mathbb{P})$ is called an *equivalent local martingale measure* if any wealth process $X \in \mathcal{X}(1)$ is a local $\mathbb{Q}$-martingale (Kramkov & Schachermayer 1999 Def. 2.1 p. 906), and from the admissibility condition ($X_t \geq 0\ \forall t$) it is in fact a supermartingale). The equivalence (**??**) has been mentioned in several papers, but we elaborate on their proof since it is based on fine properties of the stochastic integral, which are usually left as an exercise. Let us start with the following remark on stochastic integration before the proof itself.

Remark 18 *1. If a predictable process H is locally bounded, this process is integrable with respect to all semimartingales X.*

2. If H is an unbounded process, then the process is X-integrable if and only if the sequence $\int \left(H1_{\{|H|\leq n\}}\right) dX$ converges in the semimartingale topology. Moreover, in this case, the limit of the sequence equals $\int HdX$.

3. An X-integrable process H is called an admissible integrand if there exists a constant *a such that*

$$a+\int_{[0,t]} HdX \geq 0 \quad t>0.$$

4. *It is possible to show that a stochastic integral with respect to a local martingale is not a local martingale. However, if $M \in \mathcal{M}_{loc}$ is a local martingale and H is an admissible integrand for M, then $\int HdM \in \mathcal{M}_{loc}$.*

Now, we present an equivalent formulation of set $\mathcal{Q}_{elmm}(\mathbb{P})$ of equivalent local martingale measures.

Theorem 19 (*a*) *For a bounded price process S, it holds that*

$$\mathbb{Q} \in \mathcal{Q}_{elmm}(\mathbb{P}) \Longleftrightarrow S \in \mathcal{M}(\mathbb{Q}) \text{ is a } \mathbb{Q}\text{-martingale.}$$

(*b*) For a locally bounded price process S, it holds that

$$\mathbb{Q} \in \mathcal{Q}_{elmm}(\mathbb{P}) \Longleftrightarrow S \in \mathcal{M}_{loc}(\mathbb{Q}) \text{ is a local } \mathbb{Q}\text{-martingale.}$$

Proof. We only present the proof of part (a), since the same arguments can be applied for part (b)

We start with the necessity. First, observe that conceptually the price process $\{S_t\}_{t\in\mathbb{R}_+}$ has to be non-negative, and therefore we have for the predictable process $H := \frac{1}{S_0}\mathbf{1}_{[0,T]}(t)$ that

$$X_t = 1+\int_{[0,t]} H_u dS_u = \frac{S_t}{S_0} \geq 0, \quad \forall t \geq 0.$$

and thus H is an admissible integrand with respect to S. Given $\mathbb{Q} \in \mathcal{Q}_{elmm}(\mathbb{P})$ we have that $X_t = S_t/S_0 \in \mathcal{M}_{loc}(\mathbb{Q})$ is a local $\mathbb{Q}$-martingale, and hence $\{S_t\}_{t\in\mathbb{R}_+}$ is also a local $\mathbb{Q}-$martingale. Recall now that a process Z is of class D on $I \subset \mathbb{R}_+$, when $\{Z_\tau : \tau \in \mathcal{T}(I)\}$ is uniformly integrable, where $\mathcal{T}(I)$ is the class of stopping times with values in I. Furthermore,

Z is of clase DL when Z is of class D, for any $I = [0, \alpha]$ $\forall \alpha > 0$. Clearly, when S is bounded it belongs to the class DL, and hence a $\mathbb{Q}$-martingale, i.e. $S \in \mathcal{M}(\mathbb{Q})$.

For the sufficiency, take $S \in \mathcal{M}(\mathbb{Q})$ a $\mathbb{Q}$-martingale and H a predictable S-integrable process. Then, the admissible condition for strategies $1 + \int_{[0,t]} H_u dS_u > 0$ yields from the Remark 18 that $X_t = 1 + \int_{[0,t]} H_u dS_u \in \mathcal{M}_{loc}(\mathbb{Q})$ is a local $\mathbb{Q}$-martingale, and therefore $\mathbb{Q} \in \mathcal{Q}_{elmm}(\mathbb{P})$. ■

C Index of Notation

$\mathcal{A}$	adapted process with integrable variation	pg 9
$\mathcal{A}^+$	adapted, non-negative, non-decreasing, integrable process	pg 9
$\mathcal{A}_\rho$	set of admissible positions for risk measure ρ	pg 24
$\mathcal{C}$	set of control	pg 39
$\mathcal{D}_{\ll}(\mathbb{P}), \mathcal{D}_{\approx}(\mathbb{P})$	class of density process for $\mathcal{Q}_{\ll}(\mathbb{P})$ and $\mathcal{Q}_{\approx}(\mathbb{P})$	pg 14
$\mathcal{D}_{elmm}(\mathbb{P})$	class of density process for $\mathcal{Q}_{elmm}(\mathbb{P})$	pg 20
$\mathcal{M}, \mathcal{M}^c, \mathcal{M}^d, \mathcal{M}_\infty$	spaces of martingales, continous, purely discontinous, and uniformly integrables martingales	
$\mathcal{M}_{loc}, \mathcal{M}_{loc}^c, \mathcal{M}_{loc}^d$	localized martingale spaces	pg 11
$\mathfrak{M}, \mathfrak{M}_b$	measurable and bounded measurable functions respectively	pg 33
$\mathcal{P} \subset \mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+)$	the predictable σ-algebra	
$\widetilde{\mathcal{P}} := \mathcal{P} \otimes \mathcal{B}(\mathbb{R}_0)$		
$\mathcal{Q}(\Omega, \mathcal{F})$	set of probability measures on the measurable space $(\Omega, \mathcal{F})$	pg 14
$\mathcal{Q}_{\ll}(\mathbb{P})$	class of absolutely continuous probability measure w.r.t. $\mathbb{P}$	pg 14
$\mathcal{Q}_{\ll}^{\vartheta}(\mathbb{P})$	elements of $\mathcal{Q}_{\ll}(\mathbb{P})$ with a finite penalization $\vartheta(\mathbb{Q}) < \infty$	pg 26
$\mathcal{Q}_{\approx}(\mathbb{P})$	class of equivalent probability measure w.r.t. $\mathbb{P}$	pg 14
$\mathcal{Q}_{\approx}^{\vartheta}(\mathbb{P})$	elements of $\mathcal{Q}_{\approx}(\mathbb{P})$ with a finite penalization $\vartheta(\mathbb{Q}) < \infty$	pg 26
$\mathcal{Q}_{cont}(\Omega, \mathcal{F})$	class probability contents on $(\Omega, \mathcal{F})$	pg 23
$\mathcal{Q}_{elmm}(\mathbb{P})$	class of equivalent local martingale measures w.r.t. $\mathbb{P}$	pg 20
$u, u_{\mathbb{Q}}$	value function for robust and classical problem respectively	pg 34
$\mathcal{V}$	cádlág, adapted processes with finite variation	pg 9
$\mathcal{V}^+$	cádlág, adapted, non-decreasing starting at zero	pg 9
$X^{x,\pi}$	wealth process	pg 20
$\mathcal{X}(x)$	admissible wealth processes with initial capital x	pg 20

Y	exogeneous process	pg 18
$\mathcal{Y}_{\mathbb{Q}}(y)$		pg 36
Z^{θ}		pg 12
α, β, γ	coeficients of Y	pg 18
κ_1, κ_2	coeficients in coercivity conditions	pg 39
μ	jump measure	pg 8
ν	intensity measure	pg 8
π	portfolio	pg 20
θ_0, θ_1	coeficients in the density representation	pg 12
ρ	risk measure	pg 26
ϑ	penalty function	pg 26
ξ	control processes	pg 39
$\widetilde{\Omega}$	$\widetilde{\Omega} := \Omega \times \mathbb{R}_+ \times \mathbb{R}_0$	

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