Centro de Investigación en Matemáticas

PhD Thesis

Properties and Applications of Motives associated to Fibrations

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Introduction

The concept of the category of mixed motives arises as the conjectural categorical framework for the universal cohomology theory of algebraic varieties. Although the category of mixed motives has yet to be constructed, many of its desired properties have been described. We provide a partial list of such properties:

1. For each scheme S, we should have the category $\mathcal{MM}_S$ of mixed motives over S, which is an abelian tensor category.
2. For each S we have a functor natural in S
$$M_S : (\mathbf{Sm}/S)^{op} \to \mathcal{MM}_S$$
where $\mathbf{Sm}/S$ is the category of smooth S-schemes; $M_S(X)$ is the mixed motive of X.
3. There are external products
$$M_S(X) \otimes M_S(Y) \to M_S(X \times_S Y)$$
which are isomorphisms, called the Künneth isomorphisms.

An interesting question arising at this point is if we can obtain a motivic analog of the Leray-Hirsch theorem (see for example [Hat02, Theorem 4.D.1]). The statement in the algebraic topology context is as follows: let $\pi : Y \to X$ be a fibre bundle with fibre Z. Assume that for each p the vector space $H^p(Z, \mathbb{Q})$ of singular cohomology has finite dimension m_p and that we can find classes

$$T_{p,1}, \cdots, T_{p,m_p} \in H^p(Y, \mathbb{Q})$$

that restrict, on each fibre Z to a basis of the cohomology in degree p. Then we have an isomorphism of $H^*(X, \mathbb{Q})$-modules

$$H^*(Y, \mathbb{Q}) \cong H^*(X, \mathbb{Q}) \otimes H^*(Z, \mathbb{Q}).$$

So in the motivic framework we ask if we can find an isomorphism

$$M_S(Y) \cong M_S(X) \otimes M_S(Z)$$

for a fibration $\pi : Y \to X$ with fibre Z, and the conditions that guarantee such an isomorphism.

Given the conjectural nature of the category $\mathcal{MM}_S$, it can be hard to answer this question in full generality, so we consider a special case.

If instead of considering the category $\mathbf{Sm}/S$ we consider the category $V(k)$ of smooth and projective k-schemes (where k is an algebraically closed field with characteristic zero), the category $\mathcal{M}_k$ of *pure* motives over k can be constructed (here we use the notation $\mathcal{M}_k$ instead of $\mathcal{M}_{\mathrm{Spec}(k)}$). Moreover, since the question we ask deals only with motives associated to objects of $V(k)$, we can restrict ourselves to work with a smaller category containing the image of the functor

$$M_k : V(k) \to \mathcal{M}_k,$$

namely with the category $\mathfrak{M}_k$ of *effective* motives over k.

In order to construct the category $\mathfrak{M}_k$, we have to take into account some considerations. Grothendieck's definition of motive involves replacing the category $V(k)$ with a category with the same objects, but whose morphisms are correspondences, modulo a suitable equivalence relation. Depending on the relation chosen, one gets rather different theories.

It is usual to take numerical or homological equivalence, obtaining motive categories denoted by $\mathfrak{M}_k^{num}$ and respectively by $\mathfrak{M}_k^{hom}$, but in this work we will consider the category of motives for rational equivalence, denoted by $\mathfrak{M}_k^{\mathrm{Rat}}$, and usually refered to as Chow motives. One reason to do this is that rational equivalence is the finest adequate equivalence on cycles (for example, to obtain the Chow moving lemma, see Lemma 1.2.2), so that $\mathfrak{M}_k^{\mathrm{Rat}}$ is in some sense universal.

In this context, denote by

$$h : V(k) \to \mathfrak{M}_k^{\mathrm{Rat}}$$

the functor assigning to each variety its Chow motive. Then we ask whether or not we can give an isomorphism

$$h(Y) \cong h(X) \otimes h(Z), \tag{1}$$

for a fibration $\pi : Y \to X$ with fibre Z.

As a reference on this topic we have the work of Guillet and Soulé [GS96] in which they obtain the isomorphism (1) for a fibration locally trivial in the Zariski topology of the base space (see also [DL98] and [GN02]). In this work, we provide the isomorphism (1) in the case when the fibre has Chow groups satisfying certain pairing conditions (see Theorem 4.2.4) by using more elementary techniques. As an example of the fibrations covered by our assumptions we have the projective, grassmannian and flag bundles associated to a given vector bundle E over a variety X. Our approach provides additional information about the fibration, starting with the fact that, in order to prove Theorem 4.2.4, we first give an explicit description of the Chow ring of such a fibration as a module over the Chow ring of the base space (see Theorem 3.2.4). Moreover, we made explicit the isomorphism (1), identifying in the process several strata in the motive $h(Y)$, each stratum corresponding to certain decomposition of the diagonal of the fibre (see Lemma 4.2.2 and the Remark following Theorem 4.2.4).

As a future application, we hope the additional information just described let us answer a part of the conjectures proposed by Murre in [Mur93].

The organization of this work is as follows. In Chapter 1 we recall the basic notions related to intersection theory and Chow rings. In Chapter 2, we recall from [Man68] the construction of the category $\mathfrak{M}_k^{\mathrm{Rat}}$. In Chapter 3 we establish the isomorphism (3.2) for the Chow ring of the fibration and in the last chapter we prove the isomorphism (4.9) on the category $\mathfrak{M}_k^{\mathrm{Rat}}$.

Chapter 1

Preliminaries

1.1 Global intersection theory

Let k be an algebraically closed field with characteristic zero. For a *variety* over k we always mean a reduced and irreducible algebraic scheme of finite type over k. A point will always be a closed point.

In this section we consider the category $V(k)$ of smooth projective k-schemes, with morphisms given by the usual morphisms between schemes.

Definition 1.1.1. *Let Λ be a commutative ring. A* global intersection theory *on $V(k)$ (with coefficients in Λ) consists of the following data:*

a) A contravariant functor assigning to each variety X a Λ-algebra $C(X)$. For any morphism $f : X \to Y$ the corresponding morphism of Λ-algebras is denoted by $f^* : C(Y) \to C(X)$. The identity of the ring is denoted by the element $1_X \in C(X)$.

b) A covariant functor from $V(k)$ to the category of Λ-modules, assigning to each $X \in \mathrm{Obj}(V(k))$ the element $C(X)$ considered as a Λ-module. The morphism of Λ-modules $C(X) \to C(Y)$ corresponding to the morphism $f : X \to Y$ is denoted by f_*.

c) For any $X,\ Y \in V(k)$ we give a Λ-algebra morphism

$$C(X) \otimes_\Lambda C(Y) \to C(X \times Y).$$

The image of the element $x \otimes y$ under this mapping will be denoted by $x \times y$.

d) For an irreducible element $X \in V(k)$ we have an augmentation morphism $C(X) \to \Lambda$.

The data described should satisfy certain axioms, we list the most used in this work in what follows, for the complete list refer to [Gro58]:

Multiplication Axiom. *Let $X \in V(k)$, and let $\delta_X : X \to X \times X$ be the diagonal morphism. Then the composition morphism of Λ-algebras*

$$C(X) \otimes_\Lambda C(X) \longrightarrow C(X \times X) \xrightarrow{\delta_X^*} C(X)$$

coincides with the morphism of multiplication:

$$\delta_X^*(x \times y) = xy. \tag{1.1}$$

Projection Formula. *Let $f : X \to Y$ be a morphism, and take $x \in C(X)$, $y \in C(Y)$. Then*

$$f_*(xf^*(y)) = f_*(x)y. \tag{1.2}$$

In the following sections we will recall the global intersection theory for projective varieties given by the Chow ring of a variety.

1.2 Chow Rings

Let X be an algebraic variety over a field k. A *cycle of codimension* r (r*-cycle*) on X is an element of the free abelian group generated by the closed subvarieties of X of codimension r, we denote this group by $Z^r(X)$. If Z is a closed subscheme of codimension r let $Y_1, \ldots, Y_t$ be those irreducible components of Z which have codimension r, and define the *cycle associated* to Z to be:

$$Z := \sum_{i=1}^{t} n_i Y_i,$$

where n_i is the length of the local ring $\mathcal{O}_{\xi_i,Z}$ of the generic point ξ_i of Y_i on Z.

We proceed to describe the *rational equivalence* for r-cycles. For any $(r-1)$-codimensional subvariety $W \subset X$ and any $\varphi \in K(W)^*$ (where $K(W)$ denotes the function field of the variety W) define a r-cycle $\operatorname{div}(\varphi) \in Z^r(X)$ by

$$\operatorname{div}(\varphi) := \sum \operatorname{ord}_V(\varphi)V$$

the sum over all codimension one subvarieties V of W; where ord_V is the order function on $K(W)^*$ defined by the local ring $\mathcal{O}_{V,W}$.

A r-cycle α is *rationally equivalent to zero*, written $\alpha \sim 0$, if there are a finite number of $(r-1)$-codimensional subvarieties W_i of X, and $\varphi_i \in K(W_i)^*$, such that

$$\alpha = \sum \operatorname{div}(\varphi_i).$$

Since $\operatorname{div}(\varphi^{-1}) = -\operatorname{div}(\varphi)$, the r-cycles rationally equivalent to zero form a subgroup $\operatorname{Rat}^r(X)$ of $Z^r(X)$. The r^{th} *Chow group* of X is the factor group

$$CH^r(X) := Z^r(X)/\operatorname{Rat}^r(X);$$

we will denote by $[V]$ the class in $CH^r(X)$ of a variety V.

Observe that $Z^r(X)$ and consequently $CH^r(X)$ are equal to zero if $r > \dim(X)$. Define $Z^*(X)$ (respectively $CH^*(X)$) to be the direct sum of the groups $Z^r(X)$ (respectively $CH^r(X)$) for $r = 0, \dots, \dim(X)$. A *cycle* (respectively *cycle class*) is an element of $Z^*(X)$ (respectively $CH^*(X)$). If α is a cycle in $Z^*(X)$, then $\alpha = (\alpha_r)_r$ with $\alpha_r \in Z^r(X)$, we call α_r the *components* of α.

Fulton proves in [Ful98, p. 21] that we can calculate the Chow groups of a variety X in terms of the Chow groups of a closed subscheme and its complement, namely we have the following.

Theorem 1.2.1. *Let W be a closed subscheme of a scheme X, and let $U = X \backslash W$. Let $j : W \to X$, $i : U \to X$ be the inclusions, and set $m = \operatorname{codim}_X(W)$. Then the sequence*

$$CH^{k-m}(W) \xrightarrow{j_*} CH^k(X) \xrightarrow{i^*} CH^k(U) \longrightarrow 0$$

is exact for all k.

As a special case consider $k < m$. Then we have $CH^k(X) \cong CH^k(U)$. In particular, if X is a variety and W is a proper subvariety then $CH^0(U) \cong CH^0(X)$.

We can endow $CH^*(X)$ with a $\mathbb{Z}$-algebra structure, in order to do this we need the following lemma, the reader can find proofs in [Che58] and [Rob72].

Lemma 1.2.2. ***(Chow's moving lemma)***. *Let $X \in V(k)$ and let Y, Z be two cycles on X. Then there is a cycle Z' on X, rationally equivalent to Z, such that Y and Z' intersect properly. Furthermore, if Z'' is another such cycle, then $Y \cap Z'$ and $Y \cap Z''$ are rationally equivalent.* □

With the notation of Lemma 1.2.2 define the multiplication of cycle classes in $CH^*(X)$ as:

$$[Y][Z] := [Y \cap Z'].$$

With respect to this multiplication the identity is given by the class $[X]$, we will denote this class by 1_X.

1.3 $CH^*(\cdot)$ as a global intersection theory

In this section we will sketch why $CH^*(\cdot)$ can be regarded as a global intersection theory on $V(k)$ with coefficients in $\mathbb{Z}$. We start by recalling the definitions of the functors involved.

Push-forward of Cycle Classes. Let $f : X \to Y$ be a morphism of varieties, let $V \subset X$ be a subvariety and let $W := \overline{f(V)}$. Define the *degree* of V over W by:

$$\deg(V/W) := \begin{cases} [K(V):K(W)] & if \quad \dim V = \dim W \\ 0 & if \quad \dim V > \dim W. \end{cases}$$

The push-forward of a variety V is given by the formula

$$f_*(V) := \deg(V/W)W$$

which extends by linearity to a morphism of cycles

$$f_* : Z^r(X) \to Z^{r-n}(Y),$$

where $n = \dim(X) - \dim(Y)$.

We have the following Theorem, see [Ful98, p. 11].

Theorem 1.3.1. *If $f : X \to Y$ is a proper morphism of varieties then the homomorphism $f_* : Z^r(X) \to Z^{r-n}(Y)$ induces a well defined homomorphism $f_* : CH^r(X) \to CH^{r-n}(Y)$.*

Flat Pull-back of Cycle Classes. Let $f : X \to Y$ be a flat morphism. For any subvariety $V \subset Y$, let $f^*(V)$ be the cycle associated to the inverse image scheme $f^{-1}(V)$. This can be extended by linearity to a *pull-back* morphism of r-cycles

$$f^* : Z^r(Y) \to Z^r(X) \ .$$

The following theorem is needed, among other things, to demonstrate that we can use the previous morphism to induce one between the corresponding Chow groups. See [Ful98, p. 18].

Theorem 1.3.2. *(Base Change) Let X, Y, X', Y' be smooth varieties and let*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

be a fibre square, with g flat and f proper. Then g' is flat, f' is proper and for all $\alpha \in Z^*(X)$,

$$(f'_* \circ g'^*)(\alpha) = (g^* \circ f_*)(\alpha)$$

in $Z^*(Y')$.

We also have (see for example [Ful98, p. 19]).

Theorem 1.3.3. *Let $f : X \to Y$ be a flat morphism and $\alpha \in Z^r(Y)$ be such that $\alpha \sim 0$. Then $f^*(\alpha) \sim 0$, and therefore we have a well defined morphism*

$$f^* : CH^r(Y) \to CH^r(X).$$

Sketching the proof of some properties. Let ${}_{\mathbb{Z}}\mathfrak{Mod}$ denote the category of left $\mathbb{Z}$-modules and consider the assignations given by

$$\begin{array}{ccc} V(k) & \longrightarrow & {}_{\mathbb{Z}}\mathfrak{Mod} \quad . \\ \\ X & \longmapsto & CH^*(X) \\ \\ f & \longmapsto & f_* \end{array}$$

In order to verify that this is a functor, it is enough to show that

$$(f \circ g)_* = f_* \circ g_* \tag{1.3}$$

for any two morphisms of varieties $f : Y \to Z$ and $g : X \to Y$. Now, let $V \subset X$ be a subvariety and let $V' = g(V)$, $V'' = f(V')$. Then we have the equality

$$\deg(V/V'') = \deg(V/V') \deg(V'/V'');$$

this follows from the multiplicative formula of the degree of a tower of field extensions (see Proposition 1.20 in [Mor96]). Now, keeping the notation for V, V' and V'' we have that

$$\begin{aligned} (f_* \circ g_*)([V]) &= f_*(\deg(V/V')[V']) \\ &= \deg(V/V') \deg(V'/V'')[V''] \\ &= \deg(V/V'')[V''] \\ &= (f \circ g)_*([V]), \end{aligned}$$

therefore equality (1.3) follows from the fact that the classes [V] where V is a subvariety of X generate the group $CH^*(X)$.

Now we will see that the pull-back of morphisms can be used to define a contravariant functor from the category $V(k)$ to the category of $\mathbb{Z}$-algebras (denoted by ${}_{\mathbb{Z}}\mathfrak{Alg}$). Consider the assignations

$$\begin{array}{ccc} V(k)^{op} & \longrightarrow & {}_{\mathbb{Z}}\mathfrak{Alg} \\ \\ X & \longmapsto & CH^*(X) \\ \\ f & \longmapsto & f^* \end{array}$$

We will show that

$$(f \circ g)^* = g^* \circ f^*. \tag{1.4}$$

for any two morphisms $f : Y \to Z$, $g : X \to Y$.

If $V \subset Z$ is a subvariety then

$$(f \circ g)^*([V]) = [(f \circ g)^{-1}(V)] = [g^{-1}(f^{-1}(V))] = g^*([f^{-1}(V)]) = (g^* \circ f^*)([V]).$$

Again, identity (1.4) follows from the fact the cycle classes $[V]$ where V is a subvariety of Z generate $CH^*(Z)$.

The verification of the conditions asked for this pair of functors to define a global intersection theory requires several results, so we provide references to their proofs, formula (1.1) as well as the projection formula (1.2) can be found in Remark 8.3 and Theorem 8.3.(c) in [Ful98, p. 140] respectively.

Finally, the augmentation morphism ϵ for an irreducible variety X is such that

$$\epsilon(CH^i(X)) = 0 \ for \ i > 0,$$

and

$$\begin{aligned} \epsilon : CH^0(X) &\longrightarrow \mathbb{Z} \\ d \cdot 1_X &\longrightarrow d. \end{aligned}$$

Chapter 2

Correspondences and Motives

In this chapter we consider a global intersection theory (see Definition 1.1.1) with coefficients in a ring Λ and develop the tools needed for the construction of the category of effective motives.

2.1 The Category of Correspondences

Let k be an algebraically closed field with characteristic zero, and let C be a global intersection theory on the category of smooth projective varieties $V(k)$.

Definition 2.1.1. *A C-correspondence between the varieties X, $Y \in V(k)$ is any element of the ring $C(X \times Y)$.*

Given correspondences $f \in C(X \times Y)$ and $g \in C(Y \times Z)$, the composition of f and g is given by the formula

$$g \circ f := p_{13*}\big(p_{12}^*(f) \cdot p_{23}^*(g)\big) \in C(X \times Z),$$

where the morphism p_{ij} is the projection in the ij-factor:

$$\begin{tikzcd} & X \times Y \times Z \arrow[dl, "p_{12}"'] \arrow[dd, "p_{13}"] \arrow[dr, "p_{23}"] & \\ X \times Y & & Y \times Z \\ & X \times Z & \end{tikzcd}.$$

If the choice of the intersection theory C is clear, we will refer to the C-correspondences simply as correspondences.

Lemma 2.1.2.

a) *Let* $\Delta_X = \delta_{X*}(1_X) \in C(X \times X)$ *be the diagonal class. Then for any correspondences* $f \in C(X \times Y)$, $g \in C(Y \times X)$ *we have*

$$f \circ \Delta_X = f, \quad \Delta_X \circ g = g.$$

b) *The composition of correspondences is associative.*

Proof.

a) We will prove the equality

$$f \circ \Delta_X = f.$$

We have

$$p_{12}^*(\Delta_X) = \Delta_X \times 1_Y = (\delta_X \times id_Y)_*(1_{X\times Y}) \ .$$

Also, we have the commutative diagram

$$\begin{tikzcd}
& X \times Y \\
X \times Y \arrow[ru, bend left, "id_{X\times Y}"] \arrow[r, "\delta_X \times id_Y"] \arrow[rd, bend right, "id_{X\times Y}"'] & X \times X \times Y \arrow[u, "p_{13}"'] \arrow[d, "p_{23}"] \\
& X \times Y \ .
\end{tikzcd}$$

So, by the projection formula

$$\begin{aligned}
\Big((\delta_X \times id_Y)_*(1_{X\times Y})\Big) \cdot p_{23}^*(f) &= (\delta_X \times id_Y)_*\Big(1_{X\times Y} \cdot \Big((\delta_X \times id_Y)^*(p_{23}^*(f))\Big)\Big) \\
&= (\delta_X \times id_Y)_*((id_{X\times Y})^*(f)) \\
&= (\delta_X \times id_Y)_*(f).
\end{aligned}$$

The morphisms involved in the composition $f \circ \Delta_X$ can be displayed in this diagram:

$$\begin{tikzcd}
& C(X \times X \times Y) \arrow[dd, "p_{13*}"'] & \\
\Delta_X \in C(X \times X) \arrow[ur, "p_{12}^*"] & & f \in C(X \times Y) \arrow[ul, "p_{23}^*"'] \\
& C(X \times Y) & .
\end{tikzcd}$$

Now, by definition

$$\begin{aligned} f \circ \Delta_X &= p_{13*}\Big(p_{12}^*(\Delta_X) \cdot p_{23}^*(f)\Big) \\ &= p_{13*}((\delta_X \times id_Y)_*(f)) \\ &= (id_{X\times Y})_*(f) \\ &= id_{C(X\times Y)}(f) \\ &= f\ . \end{aligned}$$

The proof of the equality $\Delta_X \circ g = g$ is similar.

***b*)** The verification of this part can be done by a direct calculation of the compositions involved. □

Now, we have at our disposal the ingredients that are needed in the following definition.

Definition 2.1.3. *The category of C-correspondences, denoted by $CV(k)$, is defined by the following data:*

a) $\mathrm{Obj}\, CV(k) = \mathrm{Obj}\, V(k)$.

b) For any two objects $X,\ Y$ of $CV(k)$,

$$\mathrm{Hom}_{CV(k)}(X, Y) := C(X \times Y).$$

c) The composition of morphisms in $CV(k)$ is given by the composition of correspondences.

For each morphism of $V(k)$ we have a morphism going in the opposite direction in the category of correspondences. In order to see this, let $\varphi : Y \to X$ be a morphism of $V(k)$. Following Manin's convention (see [Man68, p. 446]) we will define its *graph* as the morphism

$$\Gamma_\varphi := (\varphi \times id_Y) \circ \delta_Y :\ Y \xrightarrow{\delta_Y} Y \times Y \xrightarrow{\varphi \times id_Y} X \times Y.$$

Observe that this lets us define a push-forward morphism

$$\Gamma_{\varphi *} : C(Y) \to C(X \times Y);$$

in this way, if we denote by 1_Y the identity of the ring $C(Y)$ we can define the following.

Definition 2.1.4. *If $\varphi : Y \to X$ is a morphism of $V(k)$, define*

$$c(\varphi) := \Gamma_{\varphi *}(1_Y) \in \mathrm{Hom}_{CV(k)}(X, Y).$$

Theorem 2.1.5. *The assignation*

$$X \rightsquigarrow X, \qquad \varphi \rightsquigarrow c(\varphi)$$

defines a contravariant functor from $V(k)$ to $CV(k)$.

Proof. We see that if we have a sequence of morphisms

$$Z \xrightarrow{\varphi} Y \xrightarrow{\psi} X$$

in $V(k)$, then $c(\psi \circ \varphi) = c(\varphi) \circ c(\psi)$. We have the following commutative diagram

$$\begin{tikzcd} Z \arrow[r, "\Gamma_\varphi"] \arrow[drr, "\Gamma_{\psi\circ\varphi}"'] & Y \times Z \arrow[r, "\Gamma_\psi \times id_Z"] & X \times Y \times Z \arrow[d, "p_{13}"] \\ & & X \times Z \end{tikzcd}$$

from which it follows

$$c(\psi \circ \varphi) = p_{13*}\left((\Gamma_\psi \times id_Z)_*(\Gamma_{\varphi*}(1_Z))\right).$$

On the other hand, by definition we have

$$c(\varphi) \circ c(\psi) = p_{13*}\big((1_X \times \Gamma_{\varphi*}(1_Z)) \cdot (\Gamma_{\psi*}(1_Y) \times 1_Z)\big),$$

therefore it is be enough to see that

$$(1_X \times \Gamma_{\varphi*}(1_Z)) \cdot (\Gamma_{\psi*}(1_Y) \times 1_Z) = (\Gamma_\psi \times id_Z)_*(\Gamma_{\varphi*}(1_Z)) \ .$$

By the projection formula:

$$\Big(1_X \times \Gamma_{\varphi*}(1_Z)\Big) \cdot \Big((\Gamma_\psi \times id_Z)_*(1_{Y\times Z})\Big) = (\Gamma_\psi \times id_Z)_*\Big((\Gamma_\psi \times id_Z)^*\big(1_X \times \Gamma_{\varphi*}(1_Z)\big)\Big),$$

but

$$(\Gamma_\psi \times id_Z)^*\big(1_X \times \Gamma_{\varphi*}(1_Z)\big) = (\Gamma_\psi \times id_Z)^*\big(p_{23}^*(\Gamma_{\varphi*}(1_Z))\big),$$

and since $p_{23} \circ (\Gamma_\psi \times id_Z) = id_{Y\times Z}$ we have that

$$(\Gamma_\psi \times id_Z)^*\big(p_{23}^*(\Gamma_{\varphi*}(1_Z))\big) = \Gamma_{\varphi*}(1_Z) \ .$$

Therefore,

$$\begin{aligned} \big(1_X \times \Gamma_{\varphi*}(1_Z)\big) \cdot \big(\Gamma_{\psi*}(1_Y) \times 1_Z\big) &= \Big(1_X \times \Gamma_{\varphi*}(1_Z)\Big) \cdot \Big((\Gamma_\psi \times id_Z)_*(1_{Y\times Z})\Big) \\ &= (\Gamma_\psi \times id_Z)_*(\Gamma_{\varphi*}(1_Z)). \ \square \end{aligned}$$

This result lets us replace the category $V(k)$ with an additive category. To be more precise, the category $CV(k)$ is Λ-additive: all the groups of morphisms

$\mathrm{Hom}_{CV(k)}(X,Y)$ are Λ-modules and the composition is Λ-linear with respect to all of its arguments.

For any correspondence $f \in \mathrm{Hom}_{CV(k)}(X,Y)$ we will denote by f^t the image of f under the morphism permuting the factors

$$C(X \times Y) \to C(Y \times X).$$

We have $(f \circ g)^t = g^t \circ f^t$, in this way we see that $CV(k)$ is equivalent to its opposite category.

Moreover, we have the following:

a) $X \oplus Y := X \amalg Y$, with morphisms given by $c(i_X)^t,\ c(i_Y)^t$ where

$$X \xrightarrow{i_X} X \amalg Y$$

$$Y \xrightarrow{i_Y} X \amalg Y$$

are the canonical inclusions.

b) Let $f_i \in \mathrm{Hom}_{CV(k)}(X_i, Y_i)$, $i = 1, 2$. Define

$$f_1 \oplus f_2 := c(i_{Y_1})^t \circ f_1 \circ c(i_{X_1}) + c(i_{Y_2})^t \circ f_2 \circ c(i_{X_2}) \in \mathrm{Hom}_{CV(k)}(X_1 \oplus X_2, Y_1 \oplus Y_2),$$

c) $X \otimes Y := X \times Y$.

d) Let $f_i \in \mathrm{Hom}_{CV(k)}(X_i, Y_i)$, $i = 1, 2$. Define

$$f_1 \otimes f_2 := s_{23*}(p_{13}^*(f_1) p_{34}^*(f_2)) \in \mathrm{Hom}_{CV(k)}(X_1 \otimes X_2, Y_1 \otimes Y_2),$$

where s_{23} is the morphism interchanging the second and third factors:

$$s_{23} : X_1 \times Y_1 \times X_2 \times Y_2 \to X_1 \times X_2 \times Y_1 \times Y_2.$$

From the definition we have the following equality:

$$(f_1 \otimes f_2) \circ (g_1 \otimes g_2) = (f_1 \circ g_1) \otimes (f_2 \circ g_2).$$

2.2 Functorial Properties of Correspondences.

A reference for the properties listed in this section can be found in [Man68].

Let $T,\ X \in \mathrm{Obj}(CV(k))$ and let $X(T) := \mathrm{Hom}_{CV(k)}(T, X)$ (in some literature $X(T)$ is denoted by $h_X(T)$).

If $f \in \mathrm{Hom}_{CV(k)}(Y, X)$ and $T \in \mathrm{Obj}(CV(k))$, we can define the natural homomorphism of modules of T-points $f_T : Y(T) \to X(T)$ by setting $f_T(g) := f \circ g$ for each element $g \in Y(T)$.

It is useful to consider the following special case. We put $T = e = \mathrm{Spec}(k)$, and $g \in Y(T) = C(e \times Y) = C(Y)$. Then $f_e(g) = p_{2*}(fp_1^*(g))$, where $p_1,\ p_2$ are the projections given in the following diagram

$$\begin{tikzcd} & Y \times X \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ Y & & X. \end{tikzcd}$$

We can construct another morphism between the same target and source as f_T. In order to do this observe that every correspondence $f \in \mathrm{Hom}_{CV(k)}(Y, X)$ defines a correspondence

$$\Delta_T \otimes f \in \mathrm{Hom}_{CV(k)}(T \otimes Y, T \otimes X),$$

and a morphism of Λ-modules

$$(\Delta_T \otimes f)_e : (T \otimes Y)(e) \to (T \otimes X)(e),$$

where $(T \otimes Y)(e) = C(T \times Y) = Y(T)$ and $(T \otimes X)(e) = C(T \times X) = X(T)$.

We show that both morphisms coincide.

Lemma 2.2.1. *$f_T = (\Delta_T \otimes f)_e$.*

Proof. Let $g \in Y(T)$ and consider the commutative diagram

$$\begin{tikzcd} & & T \times Y & & \\ Y \times X & T \times Y \times X \arrow[l, "p_{23}"'] \arrow[ur, "p_{12}"] \arrow[dr, "p_{13}"'] \arrow[r, "\delta_T \times id_{Y \times X}"'] & T \times T \times Y \times X \arrow[r, "s_{23}"] & T \times Y \times T \times X \arrow[ul, "p'_{12}"'] \arrow[dl, "p_{34}"] \\ & & T \times X & & \end{tikzcd}.$$

By definition

$$f_T(g) = f \circ g = p_{13*}(p_{12}^*(g) \cdot p_{23}^*(f)),$$
$$(\Delta_T \otimes f)_e(g) = p_{34*}\left((\Delta_T \otimes f) \cdot p_{12}^{\prime *}(g)\right).$$

By applying the projection formula, we see that

$$\begin{aligned} & \left(s_{23*}(\delta_T \times id_{Y\times X})_*(1_X \times f)\right) \cdot p_{12}^{\prime *}(g) \\ = \quad & s_{23*}(\delta_T \times id_{Y\times X})_*\left((1_X \times f) \cdot \left((\delta_T \times id_{Y\times X})^* s_{23}^* p_{12}^{\prime *}(g)\right)\right) \end{aligned}$$

and since

$$p_{34}s_{23}(\delta_T \times id_{Y\times X}) = p_{13}, \quad p'_{12}s_{23}(\delta_T \times id_{Y\times X}) = p_{12},$$

we see that

$$p_{34*}\left((\Delta_T \otimes f)\cdot p_{12}'^{*}(g)\right) = p_{34*}\Bigg(\left(s_{23*}(\delta_T \times id_{Y\times X})_*(1_X \times f)\right)\cdot p_{12}'^{*}(g)\Bigg)$$

$$= p_{34*}s_{23*}(\delta_T \times id_{Y\times X})_*\Bigg((1_X \times f)\cdot\left((\delta_T \times id_{Y\times X})^* s_{23}^* p_{12}'^{*}(g)\right)\Bigg)$$

$$= p_{13*}\left((1_X \times f)\cdot p_{12}^*(g)\right) = p_{13*}\left(p_{12}^*(g)\cdot p_{23}^*(f)\right),$$

which proves the desired equality. □

Definition 2.2.2. *We say that the homomorphism $h : C(X) \to C(Y)$ of Λ-modules is represented by the correspondence $f \in \mathrm{Hom}_{CV(k)}(X,Y)$ if $h = f_e$.*

We show that, given a morphism on $V(k)$ $\varphi : X \to Y$, the homomorphisms φ^* and φ_* can be represented by correspondences. The homomorphism $m_\alpha : C(X) \to C(X)$ of multiplication by any element α of the ring $C(X)$ can be represented by the correspondence $c_\alpha := \delta_{X*}(\alpha)$.

Lemma 2.2.3. *Let $\varphi : X \to Y$ be a morphism on $V(k)$, $\alpha \in C(X)$. Then*

a) $c(\varphi)_e = \varphi^*$,

b) $c(\varphi)_e^t = \varphi_*$,

c) $(c_\alpha)_e = m_\alpha$.

Proof. Consider the projections

$$\begin{tikzcd} & Y \times X \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ Y & & X. \end{tikzcd}$$

and observe that we have the identities:

$$p_1\Gamma_\varphi = \varphi \quad \textit{and} \quad p_2\Gamma_\varphi = id_X.$$

Now, if $y \in C(Y)$ we have that

$$c(\varphi)_e(y) = p_{2*}\left(\Gamma_{\varphi*}(1_X)p_1^*(y)\right) = p_{2*}\Gamma_{\varphi*}\left(1_X\Gamma_\varphi^* p_1^*(y)\right) = \varphi^*(y).$$

In a similar way, if $x \in C(X)$ then

$$c(\varphi)_e^t(x) = p_{2*}\left(\Gamma_{\varphi*}(1_X)^t p_1^*(x)\right) = p_{1*}\Gamma_{\varphi*}(1_X\Gamma_\varphi^* p_2^*(x)) = \varphi_*(x).$$

Finally, if $x \in C(X)$, since $p_2\delta_X = p_1\delta_X = id_X$ we have that

$$(c_\alpha)_e(x) = p_{2*}(\delta_{X*}(\alpha)p_1^*(x)) = p_{2*}\delta_{X*}(\alpha\delta_X^* p_1^*(x)) = m_\alpha(x),$$

so we have shown all the desired equalities. □

In order to improve the previous Lemma, we need to establish the following identities.

Lemma 2.2.4. *Let $\varphi : X \to Y$ be a morphism of varieties. Then for any variety T, and every element $\alpha \in C(X)$ we have the following equalities*

a) $\Delta_T \otimes c(\varphi) = c(id_T \times \varphi)$,

b) $\Delta_T \otimes c(\varphi)^t = c(id_T \times \varphi)^t$,

c) $\Delta_T \otimes c_\alpha = c_{1_T \times \alpha}$.

Proof. We only prove the first equality since all of them can be easily derived from the definitions. We have a commutative diagram

$$\begin{tikzcd} T \times X \arrow[r, "\delta_T \times \varphi"] \arrow[dr, "\Gamma_{id_T \times \varphi}"'] & T \times T \times Y \times X \arrow[d, "s_{23}"] \\ & T \times Y \times T \times X \end{tikzcd}$$

from which we obtain the following equalities

$$c(id_T \times \varphi) = \Gamma_{(id_T \times \varphi)*}(1_{T \times X}) = (s_{23}(\delta_T \times \Gamma_\varphi))_*(1_{T \times X})$$

$$= s_{23*}(\delta_{T*}(1_T) \times \Gamma_{\varphi*}(1_X)) = s_{23*}(\Delta_T \times c(\varphi)) = \Delta_T \otimes c(\varphi).$$

The remaining identities can be derived in a similar way. □

From Lemmas 2.2.1, 2.2.3 and 2.2.4 we have the following corollary.

Corollary 2.2.5. *For any morphism $\varphi : X \to Y$, any $T \in \mathrm{Obj}(V(k))$ and any element $\alpha \in C(X)$ we have*

a) $c(\varphi)_T = (id_T \times \varphi)^*$,

b) $c(\varphi)^t_T = (id_T \times \varphi)_*$,

c) $(c_\alpha)_T = m_{1_T \times \alpha}$.

Proof. For equality **a)** observe that

$$\begin{aligned} c(\varphi)_T &= (\Delta_T \otimes c(\varphi))_e && \textit{Lemma } 2.2.1 \\ &= c(id_T \times \varphi)_e && \textit{Lemma } 2.2.4 \\ &= (id_T \times \varphi)^* && \textit{Lemma } 2.2.3 \end{aligned}$$

The remaining cases can be obtained analogously. □

Now, we recall some useful facts from category theory, see for example [Mac98]. Let $\mathfrak{C}$ be a small category and let $X \in \mathrm{Obj}(\mathfrak{C})$. Denote by $\mathfrak{Sets}$ the category of sets. We can construct a contravariant functor:

$$h_X : \mathfrak{C} \longrightarrow \mathfrak{Sets}$$

$$T \longmapsto \mathrm{Mor}_{\mathfrak{C}}(T, X)$$

$$f \longmapsto h_X(f)$$

where $h_X(f)(g) = g \circ f$. Yoneda's Lemma (see [Mac98, p. 61]) relates an arbitrary covariant functor $F : \mathfrak{C} \to \mathfrak{Sets}$ with a functor h_X in the following way.

Theorem 2.2.6. ***Yoneda's Lemma****. Let $\mathfrak{C}$ be a small category and assume that $F : \mathfrak{C} \to \mathfrak{Sets}$ is a contravariant functor. Let X be an object of $\mathfrak{C}$ and let* $\mathrm{Nat}(h_X, F)$ *be the set of natural transformations from h_X to F. Then there is a bijection*

$$\mathrm{Nat}(h_X, F) \longrightarrow F(X)$$

$$\Phi \longmapsto \Phi_X(id_X).$$

The inverse assignation is given by

$$F(X) \longrightarrow \mathrm{Nat}(h_X, F)$$

$$a \longmapsto \Phi_a$$

where $\Phi_{a,T}(g) = F(g)(a)$ for any $T \in \mathrm{Obj}(\mathfrak{C})$ and any $g \in h_X(T)$. □

As a consequence of Yoneda's Lemma consider the case when the functor F is a functor of points for some $Y \in \mathrm{Obj}(\mathfrak{C})$. Then there is a bijection

$$\mathrm{Mor}_{\mathfrak{C}}(X, Y) = h_Y(X) \cong \mathrm{Nat}(h_X, h_Y).$$

We rewrite this bijection in the context of the category of correspondences. The assignation

$$\mathrm{Hom}_{CV(k)}(X, Y) \longrightarrow \mathrm{Nat}(h_X, h_Y)$$

$$f \longmapsto \Phi_f$$

is such that

$$\Phi_{f,T}(g) = h_Y(g)(f) = f \circ g = f_T(g).$$

This has the following consequence.

Corollary 2.2.7. *Let $\mathcal{D}$ be a diagram of objects and morphisms from the category $CV(k)$. Furthermore, let I be*

$$I = \sum_{i=1}^{r} a_i f_i,$$

where $a_i \in \Lambda$ and f_i are some correspondences between the objects of the diagram $\mathcal{D}$. For $T \in \mathrm{Obj}(V(k))$, let I_T be

$$I_T = \sum_{i=1}^{r} a_i (f_i)_T.$$

Then $I = 0$ if and only if $I_T = 0$ for all $T \in \mathrm{Obj}(V(k))$. □

Suppose we have a diagram $\mathcal{D}$ of objects and morphisms of the category $V(k)$, and let J be

$$J = \sum_{i=1}^{r} a_i F_i,$$

where $a_i \in \Lambda$ and every homomorphism F_i is a composition of a finite number of homomorphisms of the form φ^*, φ_*, m_α for $\alpha \in C(X)$, $X \in \mathrm{Obj}(\mathcal{D})$, $\varphi \in \mathrm{Mor}(\mathcal{D})$.

For any $T \in \mathrm{Obj}(V(k))$ we denote by $T \times J$ the identity obtained from J by changing all the objects X by $T \times X$, all the morphisms φ by $id_T \times \varphi$ and all the morphisms m_α by $m_{1_T \times \alpha}$.

In a similar way, denote by $c(J)$ the identity obtained from J by changing all the morphisms φ^* by $c(\varphi)$, all the morphisms φ_* by $c(\varphi)^t$ and all the morphisms m_α by c_α.

The following result will be used exhaustively in Chapter 4.

Theorem 2.2.8. ***Manin's Identity Principle*** *([Man68, p.450]). Let J be as before. The following two assertions are equivalent.*

a) *$T \times J = 0$ for all $T \in \mathrm{Obj}(V(k))$.*

b) *$c(J) = 0$.*

Proof.

$$T \times J = 0 \;\forall\, T \quad \overset{Cor.2.2.5}{\Longleftrightarrow} \quad c(J)_T = 0 \;\forall\, T \quad \overset{Cor.2.2.7}{\Longleftrightarrow} \quad c(J) = 0. \square$$

2.3 Graded Correspondences.

In this section we will assume that the intersection theory C being considered is a functor taking its values in the category of commutative and positively graded Λ-algebras, moreover, for any $X \in \mathrm{Obj}(V(k))$ we suppose $C^i(X) = 0$ for $i > \dim X$. We will impose the following conditions for the grading:

a) The morphisms φ^* are homogeneous with degree zero.

b) If $X,\ Y \in \mathrm{Obj}(V(k))$ have pure dimension n and m respectively then for any morphism $\varphi : X \to Y$ the morphism of graded rings $\varphi_* : C(X) \to C(Y)$ is homogeneous of degree $m - n$.

c) $C(X) \otimes_\Lambda C(Y) \to C(X \times Y)$ is a homogeneous morphism of degree zero.

d) For any irreducible variety X the augmentation morphism $\epsilon : C(X) \to \Lambda$ sends $C^i(X)$, $i \geq 1$, to zero and induces an isomorphism between $C^0(X)$ and Λ.

We will show that the groups of morphisms in the category of C-correspondences also have a natural grading.

For each $X,\ Y \in \mathrm{Obj}(CV(k))$, with X having pure dimension n, define

$$\mathrm{Hom}^i(X,Y) := C^{i+n}(X \times Y).$$

We will call the elements of $\mathrm{Hom}^i(X,Y)$ *homogeneous correspondences of degree i*. In the general case we decompose X as a union of equidimensional components and define the degree corresponding to each component. We will assume both X and Y irreducibles. Observe that the degree of a correspondence can be negative.

Lemma 2.3.1. *The degree of a composition of correspondences equals the sum of the degrees of the correspondences involved.*

Proof. Let $n = \dim X$, $m = \dim Y$, $f \in \mathrm{Hom}^i(X,Y)$, $g \in \mathrm{Hom}^j(Y,Z)$. We have

$$p_{12}^* C^{i+n}(X \times Y) \subset C^{i+n}(X \times Y \times Z)$$
$$p_{23}^* C^{j+m}(Y \times Z) \subset C^{j+m}(X \times Y \times Z)$$

In this way $p_{12}^*(f)p_{23}^*(g) \in C^{i+j+n+m}(X \times Y \times Z)$. Moreover,

$$\begin{aligned} p_{13*}C^{i+j+n+m}(X \times Y \times Z) &\subset C^{i+j+n+m+\dim X \times Z - \dim X \times Y \times Z}(X \times Z) \\ &= C^{i+j+n}(X \times Z) \\ &= \mathrm{Hom}^{i+j}(X,Z)\ . \end{aligned}$$

Therefore, $g \circ f = p_{13*}(p_{12}^*(f)p_{23}^*(g)) \in \mathrm{Hom}^{i+j}(X,Z)$. □

Examples.

1. Let $\varphi : X \to Y$ be a morphism, where $X,\ Y \in \mathrm{Obj}(V(k))$. Then the degree of $c(\varphi)$ is zero and the degree of $c(\varphi)^t$ is $m - n$. In fact, $c(\varphi) = \Gamma_{\varphi*}(1_X)$ and $\Gamma_{\varphi*}C^0(X) \subset C^{0+(m+n)-n}(Y \times X) = \mathrm{Hom}^0(Y,X)$. On the other hand, $c(\varphi)^t = s_*(c(\varphi))$, where $s : Y \times X \to X \times Y$ is the morphism permuting the factors; in this way, $s_*C^m(Y \times X) \subset C^{m+(n+m)-(m+n)}(X \times Y) = C^{(m-n)+n}(X \times Y) = \mathrm{Hom}^{m-n}(X,Y)$.

2. Let $x \in C^i(X)$. Then the degree of $c_x = \delta_{X*}(x)$ is i. In particular, $\Delta_X = \delta_{X*}(1_X)$ is of degree zero.

Since the identity correspondence has degree zero, we can consider a new category $CV^0(k)$, whose objects are the objects of $CV(k)$ and the morphisms are given by the correspondences of degree zero.

As well as $CV(k)$, the category $CV^0(k)$ is not abelian, for example, there are projectors which do not correspond to a decomposition of an object as a direct sum. Karoubi introduced, in the context of Banach categories, the idea of formally adding the kernels and images of all projectors. Grothendieck adapted this construction to the category of correspondences, obtaining the category of effective motives. We give a brief explanation of Karoubi's construction in the next section.

2.4 Pseudo-abelian Categories.

The concepts recalled in this section can be found in §5 of [Man68].

Definition 2.4.1. *An additive category $\mathfrak{D}$ is said to be* pseudo-abelian *if it satisfies the following condition:*

(P) *For any projector $p \in \mathrm{Hom}(X, X)$, $X \in \mathrm{Obj}\,\mathfrak{D}$, there exists a kernel $\mathrm{Ker}\,p$, and the canonical morphism $\mathrm{Ker}(p) \oplus \mathrm{Ker}(id_X - p) \to X$ is an isomorphism.*

Definition 2.4.2. *Let $\mathfrak{D}$ be an additive category. Its **pseudo-abelian completion** (also named **Karoubi envelope**) is the category $\tilde{\mathfrak{D}}$ defined by the following data: $\mathrm{Obj}\,\tilde{\mathfrak{D}}$ consists of pairs (X, p), where $X \in \mathrm{Obj}\,\mathfrak{D}$ and $p \in \mathrm{Hom}_{\mathfrak{D}}(X, X)$ is an arbitrary projector,*

$$\mathrm{Hom}_{\tilde{\mathfrak{D}}}((X,p),(Y,q)) = \{f \in \mathrm{Hom}_{\mathfrak{D}}(X,Y) \ : \ fp = qf\}/\{f \ : \ fp = qf = 0\}.$$

The composition of morphisms in $\tilde{\mathfrak{D}}$ is induced by the composition in $\mathfrak{D}$. We will denote by $\tilde{f}$ the class of the morphism f in $\mathrm{Hom}_{\tilde{\mathfrak{D}}}((X,p),(Y,q))$.

The name given to $\tilde{\mathfrak{D}}$ is justified by the following Lemma.

Lemma 2.4.3.

a) The category $\tilde{\mathfrak{D}}$ is pseudo-abelian.

b) The assignation $X \rightsquigarrow \tilde{X} = (X, id)$, $f \rightsquigarrow \tilde{f}$ can be extended in a unique way to a functor $G : \mathfrak{D} \to \tilde{\mathfrak{D}}$, which is fully-faithful and possesses the following universal property:

For each additive functor $F : \mathfrak{D} \to \mathfrak{E}$, where $\mathfrak{E}$ is a pseudo-abelian category, there exists an additive functor $\tilde{F} : \tilde{\mathfrak{D}} \to \mathfrak{E}$ such that the functors F and $\tilde{F}G$ are equivalent. □

2.5 Chow Motives

In this section we define the category $\mathfrak{M}_k^{\mathrm{Rat}}$ of effective Chow motives and we study some of its basic properties.

Let C be a graded intersection theory on $V(k)$. The *category of effective C-motives*, denoted by $\mathfrak{M}_k^C$, is the pseudo-abelian completion of the category $CV^0(k)$ of C-correspondences with degree zero.

Remark . We can restrict ourselves to consider correspondences with degree zero since all of the correspondences Δ_X, where X is a variety and $c(\varphi)$, where φ is a morphism of varieties are of degree zero. It may seem that we are losing information by doing this, but we can restore the correspondences of degree greater than zero by tensorizing with powers of the motive $(\mathbb{P}^1, 1_{\mathbb{P}^1} \times e)$, where e is a point in $\mathbb{P}^1$; for further details on this procedure see §8 in [Man68].

An object of $\mathfrak{M}_k^C$ will be called *C-motive*; if C is the global intersection theory given by $CH^*(\cdot)$, we will refer to the objects of $\mathfrak{M}_k^C$ as *Chow motives* and the category $\mathfrak{M}_k^C$ will be denoted by $\mathfrak{M}_k^{\mathrm{Rat}}$. In particular, for a variety $X \in \mathrm{Obj}(V(k))$, the object $h(X) := (X, \Delta_X) \in \mathrm{Obj}(\mathfrak{M}_k^C)$ is called the *motive associated to X*. Each C-motive has a representative of the form (X, p), where $X \in \mathrm{Obj}(CV(k)) = \mathrm{Obj}(V(k))$ and $p \in \mathrm{Hom}^0_{CV(k)}(X, X)$, $p^2 = p$.

We have a functor

$$h : V(k)^{op} \longrightarrow \mathfrak{M}_k^C$$

$$X \longmapsto (X, \Delta_X)$$

$$\varphi \longmapsto \widetilde{c(\varphi)}.$$

Remark . We define the direct sum of motives as well as tensor products:

a) $(X, p) \oplus (Y, q) := (X \oplus Y, p \oplus q)$

b) $(X, p) \otimes (Y, q) := (X \otimes Y, p \otimes q)$.

Now, suppose we have morphisms $p_i : (X, \Delta_X) \to (X, \Delta_X)$ for $i = 1, ..., m$ such that

$$\sum_{i=1}^{m} p_i = \Delta_X, \tag{2.1}$$

and

$$p_i \circ p_j = \delta_{i,j} p_i, \tag{2.2}$$

where $\delta_{i,j}$ is a Dirac delta.

Observe that the property **(P)** of Section 2.4 implies that

$$(X, \Delta_X) \cong \bigoplus_{i=1}^{m} (X, p_i), \tag{2.3}$$

therefore, we can obtain a decomposition of the motive $h(X)$, provided we can find a finite collection of morphisms $(p_{i,j})_{i=1}^{m}$ satisfying (2.1) and (2.2).

Chapter 3

The Chow Ring of a Locally Trivial Fibration

In this chapter we calculate the Chow ring of a locally trivial fibration with fibres isomorphic to a variety having a cell decomposition.

3.1 A Duality Theorem

Definition 3.1.1. *Let Z be a smooth projective variety with dimension n. We say that Z satisfies the* ***Chow pairing conditions*** *if for each p such that $0 \leq p \leq n$ we can find cycle classes $\tau_{p,1}, ..., \tau_{p,m_p} \in CH^p(Z)$ such that*

1. $CH^n(Z) \cong \mathbb{Z}\tau_{n,1}$.
2. *For $p < n$, $CH^p(Z)$ is a free $\mathbb{Z}$-module with finite rank*

$$CH^p(Z) \cong \bigoplus_{i=1}^{m_p} \mathbb{Z}\tau_{p,i}.$$

3. *For each $p < n$, we can give a perfect pairing*

$$CH^p(Z) \times CH^{n-p}(Z) \to CH^n(Z) \cong \mathbb{Z}\tau_{n,1}$$

satisfying

$$\tau_{p,i} \cap \tau_{n-p,j} = \begin{cases} \tau_{n,1} & if \quad i = j \\ 0 & if \quad i \neq j. \end{cases}$$

Definition 3.1.2. *Let Z be a smooth projective variety. We say that Z has a* ***Chow stratification*** *if*

1. *Z has a cellular decomposition*

$$Z = Z_d \supset Z_{d-1} \supset \cdots \supset Z_0 \supset Z_{-1} = \emptyset$$

by closed subvarieties such that each $Z_i - Z_{i-1}$ is a disjoint union of schemes $U_{i,j}$ isomorphic to affine spaces $\mathbb{A}^{d_{i,j}}$.

2. *Z satisfies the Chow pairing conditions by taking the cycles appearing in Definition 3.1.1 as $\tau_{i,j} := \overline{U}_{i,j}$.*

Remark . (See Example 1.9.1 in [Ful98, p. 23]). If a variety Z has a cellular decomposition then $CH^*(Z)$ is finitely generated as a $\mathbb{Z}$-module by the cycle classes $\overline{U}_{i,j}$; therefore the variety Z will have a Chow stratification if it satisfies conditions 1 and 3 in Definition 3.1.1.

Let $\pi : Y \to X$ be a locally trivial fibration with π being a proper morphism and fibres isomorphic to a n-dimensional variety Z having a Chow stratification. We will calculate the Chow ring of such a fibration; in order to do this we will construct a basis of $CH^*(Y)$ as a $CH^*(X)$-module, using the basis of $CH^*(Z)$.

Remark . For the rest of this work, by saying that $\pi : Y \to X$ is a locally trivial fibration we mean that π is also a proper morphism with fibres isomorphic to a n-dimensional variety Z.

Let U be an open subset of X for which Y becomes trivial, and $W := X \setminus U$. Let $i : U \to X$, $j : W \to X$, $\imath : Y|_U \to Y$ and $\jmath : Y|_W \to Y$ be the inclusions, and denote by π_U (resp. π_W) the restriction of π to $Y|_U$ (resp. $Y|_W$). Let $\eta : Y|_U \to Z$ be the morphism induced by the projection of $U \times Z$ on the second factor.

Then for each p with $0 \le p \le n$ we have the following diagram

$$\begin{tikzcd}
 & & CH^p(Z) \arrow[d, "\eta^*"] & \\
CH^{p-m}(Y|_W) \arrow[r, "\jmath_*"] \arrow[d, "\pi_{W*}"] & CH^p(Y) \arrow[r, "\imath^*"] \arrow[d, "\pi_*"] & CH^p(Y|_U) \arrow[r] \arrow[d, "\pi_{U*}"] & 0 \\
CH^{p-n-m}(W) \arrow[r, "j_*"'] & CH^{p-n}(X) \arrow[r, "i^*"'] & CH^{p-n}(U) \arrow[r] & 0
\end{tikzcd} \tag{3.1}$$

where m denotes the codimension of W in X; the rows are exact by Theorem 1.2.1, the left square commute by the functoriality of the push-forward and the right square commute by Theorem 1.3.2.

Using this diagram we then define elements $T_{p,i} \in CH^p(Y)$ such that $\imath^* T_{p,i} = \eta^* \tau_{p,i}$. We have the following theorem, which provides a generalization of Proposition 14.6.3. in [Ful98, p. 267] .

Theorem 3.1.3. *(Duality Theorem). Let $\pi : Y \to X$ be a locally trivial fibration and suppose the fibre Z satisfies the Chow pairing conditions. Then for any p, q satisfying $p + q \leq n$ and any $\alpha \in CH^*(X)$:*

$$\pi_*(\pi^*(\alpha) \cap T_{p,i} \cap T_{q,j}) = \begin{cases} \alpha & if\ (q,j) = (n-p, i) \\ 0 & otherwise \end{cases}$$

Proof. By the projection formula

$$\pi_*(\pi^*(\alpha) \cap T_{p,i} \cap T_{q,j}) = \alpha \cap \pi_*(T_{p,i} \cap T_{q,j}),$$

so it is enough to calculate $\pi_*(T_{p,i} \cap T_{q,j})$. Observe that

$$\pi_*(T_{p,i} \cap T_{q,j}) \in CH^{p+q-n}(X),$$

and therefore $\pi_*(T_{p,i} \cap T_{q,j}) = 0$ if $p + q < n$. From now on we will suppose $q = n - p$. In this case, by (3.1) we see that $CH^0(X) \cong CH^0(U)$.

Since the right square in (3.1) commutes

$$i^*\pi_*(T_{p,i} \cap T_{q,j}) = \pi_{U*}\imath^*(T_{p,i} \cap T_{q,j}).$$

Then

$$\imath^*(T_{p,i} \cap T_{q,j}) = \imath^*(T_{p,i}) \cap \imath^*(T_{q,j}) = \eta^*(\tau_{p,i}) \cap \eta^*(\tau_{q,j}) = \eta^*(\tau_{p,i} \cap \tau_{q,j}).$$

Now, if $j = i$ then $\tau_{p,i} \cap \tau_{q,j} = \tau_{n,1}$ and then

$$\imath^*(T_{p,i} \cap T_{q,j}) = \eta^*(\tau_{n,1}) = 1_U \times \tau_{n,1},$$

but $\tau_{n,1}$ is the class of a point in Z, therefore we have

$$i^*\pi_*(T_{p,i} \cap T_{q,j}) = \pi_{U*}(1_U \times \tau_{n,1}) = 1_U\ .$$

So, being i^* injective for $CH^0(X)$, we have $\pi_*(T_{p,i} \cap T_{q,j}) = 1_X$ if $j = i$, and therefore in this case

$$\pi_*(\pi^*(\alpha) \cap T_{p,i} \cap T_{q,j}) = \alpha\ .$$

Now, suppose $j \neq i$, so we have $\tau_{p,i} \cap \tau_{q,j} = 0$ and then

$$\imath^*(T_{p,i} \cap T_{q,j}) = \eta^*(0) = 0,$$

as a consequence

$$i^*\pi_*(T_{p,i} \cap T_{q,j}) = \pi_{U*}\imath^*(T_{p,i} \cap T_{q,j}) = 0$$

and since i^* is injective for $CH^0(X)$

$$\pi_*(T_{p,i} \cap T_{q,j}) = 0$$

from which we conclude

$$\pi_*(\pi^*(\alpha) \cap T_{p,i} \cap T_{q,j}) = 0$$

for $j \neq i$. □

3.2 The Chow ring of a locally trivial fibration

As a consequence of the Duality Theorem of the last section we have the following.

Corollary 3.2.1. *Let $\pi : Y \to X$ be a locally trivial fibration as in Theorem 3.1.3. Then the morphism of groups*

$$\begin{aligned} \varphi : \bigoplus_{i=0}^{p}\bigoplus_{j=1}^{m_i} CH^{p-i}(X)\otimes \mathbb{Z}\tau_{i,j} &\to CH^p(Y) \\ (\alpha_{i,j}\otimes\tau_{i,j})_{i,j} &\mapsto \sum_{i=0}^{p}\sum_{j=1}^{m_i}\pi^*(\alpha_{i,j})\cap T_{i,j} \end{aligned}$$

is injective.

Proof. Let $(\alpha_{i,j}\otimes\tau_{i,j})_{i,j}\in\ker\varphi$, so it satisfies the equation

$$\sum_{i,j}\pi^*(\alpha_{i,j})\cap T_{i,j}=0.$$

Suppose we have $(\alpha_{i,j}\otimes\tau_{i,j})_{i,j}\neq 0$ and let (k,l) be the (lexicographically) greatest index such that $\alpha_{k,l}\neq 0$. Multiplying the last equality by $T_{n-k,l}$ and then applying π_* we obtain

$$0=\pi_*\left(\sum\pi^*(\alpha_{i,j})\cap T_{i,j}\cap T_{n-k,l}\right)=\sum\pi_*(\pi^*(\alpha_{i,j})\cap T_{i,j}\cap T_{n-k,l})=\alpha_{k,l}$$

which yields a contradiction. Therefore, $\ker\varphi=0$ and φ is injective. □

Remark . The group

$$\bigoplus_{i=0}^{p}\bigoplus_{j=1}^{m_i} CH^{p-i}(X)\otimes\mathbb{Z}\tau_{i,j}$$

is isomorphic to the p-graded part of the graded ring $CH^*(X)\otimes CH^*(Z)$. In fact

$$\begin{aligned} \bigoplus_{i=0}^{p} CH^{p-i}(X)\otimes CH^i(Z) &\cong \bigoplus_{i=0}^{p} CH^{p-i}(X)\otimes\left(\bigoplus_{j=1}^{m_i}\mathbb{Z}\tau_{i,j}\right) \\ &\cong \bigoplus_{i=0}^{p}\bigoplus_{j=1}^{m_i} CH^{p-i}(X)\otimes\mathbb{Z}\tau_{i,j}. \end{aligned}$$

In this way, Corollary 3.2.1 can be restated as follows.

Corollary 3.2.2. *Let $\pi : Y \to X$ be a fibration as in Corollary 3.2.1. Then $CH^*(X)\otimes CH^*(Z)$ is a $CH^*(X)$-submodule of $CH^*(Y)$.* □

Now, we center our attention on deciding when the morphism defined in Corollary 3.2.1 is surjective. In order to answer this question we require an additional condition, namely we require a Chow stratification in the fibre.

Lemma 3.2.3. *If Z has a Chow stratification, then for any variety X we have*

$$CH^*(X \times Z) \cong CH^*(X) \otimes CH^*(Z).$$

Proof. Notice that, in this case, the morphism from Corollary 3.2.1 can be written as

$$\begin{array}{rcl} \varphi : \displaystyle\bigoplus_{i=0}^{p} CH^{p-i}(X) \otimes CH^i(Z) & \to & CH^p(X \times Z) \\ (\alpha_i \otimes \beta_i)_i & \mapsto & \displaystyle\sum_{i=0}^{p} \pi_X^*(\alpha_i) \cap \pi_Z^*(\beta_i) \end{array}$$

where π_X, π_Z are the projections from $X \times Z$ to X and Z respectively, and therefore this morphism is injective. Moreover, we have the equality

$$\sum_i \pi_X^*(\alpha_i) \cap \pi_Z^*(\beta_i) = \sum_i \alpha_i \times \beta_i \ .$$

In this way, proceeding as in Example 1.10.2 from [Ful98, p. 25] we obtain the surjectivity. □

To conclude this chapter, we present the following Theorem.

Theorem 3.2.4. *Let $\pi : Y \to X$ be a locally trivial fibration and suppose that the fibre Z has a Chow stratification. Then we have an isomorphism of $CH^*(X)$-modules*

$$CH^*(Y) \cong CH^*(X) \otimes CH^*(Z). \tag{3.2}$$

Moreover, the isomorphism is induced by the morphism described in Corollary 3.2.1.

Proof. We only have to show that the morphism defined in Corollary 3.2.1 is surjective. In order to do this, we proceed by induction on the dimension of the base space X.

For $\dim X = 0$ the result is trivial.

Now, suppose $\dim X > 0$, let $U \subset X$ be an open set such that Y becomes trivial on U, and set $W = X \setminus U$, $m = \operatorname{codim}_X(W)$. We have a diagram

$$\begin{tikzcd}
\bigoplus_{i=0}^{p-m} \bigoplus_{j=1}^{m_i} CH^{p-m-i}(W) \otimes \mathbb{Z}\tau_{i,j} \arrow[r, "g'"] \arrow[d, "h'"] & CH^{p-m}(Y|_W) \arrow[r] \arrow[d, "f'"] & 0 \\
\bigoplus_{i=0}^{p} \bigoplus_{j=1}^{m_i} CH^{p-i}(X) \otimes \mathbb{Z}\tau_{i,j} \arrow[r, "\varphi"] \arrow[d, "h"] & CH^p(Y) \arrow[d, "f"] & \\
\bigoplus_{i=0}^{p} CH^{p-i}(U) \otimes CH^i(Z) \arrow[r, "g"] \arrow[d] & CH^p(U \times Z) \arrow[r] \arrow[d] & 0 \\
0 & 0 &
\end{tikzcd} \tag{3.3}$$

where the first row is exact by induction hypothesis since $\dim W < \dim X$, the last row is given by Lemma 3.2.3 and the left column is obtained factor by factor by tensorizing the corresponding exact sequences obtained from the localization theorem.

Choose an element $\beta \in CH^p(Y)$ and set $\alpha_1 := f(\beta)$. Define elements α_2, α_3 satisfying $g(\alpha_2) = \alpha_1$ and $h(\alpha_3) = \alpha_2$. Then

$$f(\varphi(\alpha_3)) = g(h(\alpha_3)) = g(\alpha_2) = \alpha_1 = f(\beta)$$

and therefore $\beta - \varphi(\alpha_3) \in \operatorname{Ker} f = \operatorname{Im} f'$, so we can write $\beta - \varphi(\alpha_3) = f'(\alpha_4)$ for some α_4. Finally define α_5 as an element satisfying that $g'(\alpha_5) = \alpha_4$. Then

$$\varphi(\alpha_3 + h'(\alpha_5)) = \varphi(\alpha_3) + f'(g'(\alpha_5)) = \beta$$

and we have that φ is surjective. $\square$

Chapter 4

The Chow Motive of a Locally Trivial Fibration

In this chapter we calculate the Chow Motive of a locally trivial fibration of the form described in the last chapter. In order to do this we follow the ideas given by Manin in [Man68] to calculate the Chow motive of a projective bundle associated to a given vector bundle, and Köck in [Köc91] who generalized the construction of Manin for grassmannian bundles.

4.1 The decomposition of the diagonal

We keep the notation introduced in Chapter 3. We will define correspondences $p_{i,j} \in \mathrm{Hom}(Y, Y)$ using the isomorphism described in Corollary 3.2.1. In order to do this, consider the sets

$$W_{i,j} = \{(i, l) | j < l \leq m_i\} \cup \{(k, l) | k > i,\ 1 \leq l \leq m_k\}.$$

Observe that we have an ordering:

$$W_{1,1} \supset W_{1,2} \supset \cdots \supset W_{1,m_1} \supset W_{2,1} \supset \cdots \supset W_{n,m_n-1} \supset \emptyset$$

so we can use this to define the correspondences $p_{i,j}$ by a downward induction, starting with

$$p_{n,m_n} = c_{T_{n,m_n}} \circ c(\pi) \circ c(\pi)^t \circ c_{T_{0,m_n}}$$

and in the general case by writing

$$p_{i,j} = c_{T_{i,j}} \circ c(\pi) \circ c(\pi)^t \circ c_{T_{n-i,j}} \circ \left(\Delta_Y - \sum_{(k,l) \in W_{i,j}} p_{k,l} \right) . \tag{4.1}$$

We will show that the correspondences just defined have degree zero and satisfy the identities:

$$\sum_{i,j} p_{i,j} = \Delta_Y$$

$$p_{i,j} \circ p_{k,l} = \delta_{(i,j)}^{(k,l)} p_{i,j}$$

where $\delta_{(i,j)}^{(k,l)}$ denotes a Dirac delta.

Let $e := \mathrm{Spec}(k)$. First, we have to study the morphisms

$$(p_{i,j})_e : CH^*(Y) \to CH^*(Y) \ .$$

Now, since $(p_{i,j})_e(CH^p(Y)) \subset CH^p(Y)$ we can restrict our calculations to each graded part of the Chow ring of Y.

From Theorem 3.2.4, any element from $CH^p(Y)$ can be written as

$$\sum_{r=0}^{p} \sum_{s=1}^{m_r} \pi^*(\alpha_{r,s}) \cap T_{r,s}$$

for some $\alpha_{r,s} \in CH^{p-r}(X)$. We have the following lemma.

Lemma 4.1.1.

$$(p_{i,j})_e \left(\sum_{r=0}^{p} \sum_{s=1}^{m_r} \pi^*(\alpha_{r,s}) \cap T_{r,s} \right) = \begin{cases} \pi^*(\alpha_{i,j}) \cap T_{i,j} & if \quad i \leq p \\ 0 & if \quad i > p \end{cases} \quad \square$$

Assume Lemma 4.1.1. Then

$$\sum_{i=0}^{n} \sum_{j=1}^{m_i} (p_{i,j})_e \left(\sum_{r=0}^{p} \sum_{s=1}^{m_r} \pi^*(\alpha_{r,s}) \cap T_{r,s} \right) = \sum_{i=0}^{p} \sum_{j=1}^{m_i} \pi^*(\alpha_{i,j}) \cap T_{i,j}.$$

So, we have that $\sum_{i,j} (p_{i,j})_e|_{CH^p(Y)} = id_{CH^p(Y)}$, and therefore

$$\sum_{i,j} (p_{i,j})_e = id_{CH^*(Y)} = (\Delta_Y)_e \ . \tag{4.2}$$

Another consequence of Lemma 4.1.1, is that

$$(p_{i,j})_e \circ (p_{k,l})_e = \delta_{(i,j)}^{(k,l)} (p_{i,j})_e \ . \tag{4.3}$$

In fact:

$$(p_{i,j})_e \circ (p_{k,l})_e \left(\sum_{r=0}^{p} \sum_{s=1}^{m_r} \pi^*(\alpha_{r,s}) \cap T_{r,s} \right) = (p_{i,j})_e \left(\pi^*(\alpha_{k,l}) \cap T_{k,l} \right)$$

but

$$(p_{i,j})_e\left(\pi^*(\alpha_{k,l})\cap T_{k,l}\right)=\begin{cases}\pi^*(\alpha_{i,j})\cap T_{i,j} & if \quad (i,j)=(k,l)\\ 0 & if \quad (i,j)\neq(k,l)\end{cases}$$

which gives us the desired result.

Now, we give the proof of Lemma 4.1.1.

Proof of Lemma 4.1.1

We use a downward induction. First, observe that

$$\begin{aligned}
&(p_{n,m_n})_e\left(\sum_{r=0}^{p}\sum_{s=1}^{m_r}\pi^*(\alpha_{r,s})\cap T_{r,s}\right)\\
= \;& m_{T_{n,m_n}}\left(\pi^*\left(\pi_*\left(\sum_{r=0}^{p}\sum_{s=1}^{m_r}\pi^*(\alpha_{r,s})\cap T_{r,s}\cap T_{0,m_n}\right)\right)\right)\\
= \;& \sum_{r=0}^{p}\sum_{s=1}^{m_r} m_{T_{n,m_n}}\left(\pi^*\left(\pi_*\left(\pi^*(\alpha_{r,s})\cap T_{r,s}\cap T_{0,m_n}\right)\right)\right)
\end{aligned}$$

Now, by using Theorem 3.1.3, last expression becomes

$$m_{T_{n,m_n}}(\pi^*(\alpha_{n,m_n}))=\pi^*(\alpha_{n,m_n})\cap T_{n,m_n},$$

so we have verified the Lemma in this case.

Now, in order to prove the general case, consider the sets

$$M_{i,j}:=\{(k,l)\mid k<i\}\cup\{(i,l)\mid l\leq j\}.$$

Then, by applying the induction hypothesis:

$$\begin{aligned}
&\left((\Delta_Y)_e-\sum_{(k,l)\in W_{i,j}}(p_{k,l})_e\right)\left(\sum_{r=0}^{p}\sum_{s=1}^{m_r}\pi^*(\alpha_{r,s})\cap T_{r,s}\right)\\
= \;& \sum_{r=0}^{p}\sum_{s=1}^{m_r}\pi^*(\alpha_{r,s})\cap T_{r,s}-\sum_{(k,l)\in W_{i,j}}\pi^*(\alpha_{k,l})\cap T_{k,l}\\
= \;& \sum_{(r,s)\in M_{i,j}}\pi^*(\alpha_{r,s})\cap T_{r,s}
\end{aligned}$$

and so

$$
\begin{aligned}
&(p_{i,j})_e\left(\sum_{r=0}^{p}\sum_{s=1}^{m_r}\pi^*(\alpha_{r,s})\cap T_{r,s}\right)\\
= \quad & m_{T_{i,j}}\left(\pi^*\left(\pi_*\left(\sum_{(r,s)\in M_{i,j}}\pi^*(\alpha_{r,s})\cap T_{r,s}\cap T_{n-i,j}\right)\right)\right)\\
= \quad & \sum_{(r,s)\in M_{i,j}} m_{T_{i,j}}\left(\pi^*\left(\pi_*\left(\pi^*(\alpha_{r,s})\cap T_{r,s}\cap T_{n-i,j}\right)\right)\right).
\end{aligned}
$$

If $(r,s)\in M_{i,j}$ then $r\leq i$, and therefore $r+n-i\leq n$; then applying Theorem 3.1.3 we obtain

$$
\pi_*\left(\pi^*(\alpha_{r,s})\cap T_{r,s}\cap T_{n-i,j}\right)=\begin{cases}\alpha_{i,j} & if\ (r,s)=(i,j)\\ 0 & otherwise\end{cases}.
$$

In this way we can write

$$
(p_{i,j})_e\left(\sum_{r=0}^{p}\sum_{s=1}^{m_r}\pi^*(\alpha_{r,s})\cap T_{r,s}\right)=m_{T_{i,j}}(\pi^*(\alpha_{i,j}))=\pi^*(\alpha_{i,j})\cap T_{i,j}
$$

as desired. □

The next result gives us a decomposition of the diagonal Δ_Y of the fibration into a sum of pairwise orthogonal projectors.

Theorem 4.1.2. *Let $p_{i,j}$ be the correspondences defined before. Then we have the following:*

1. *The correspondences $p_{i,j}$ are of degree zero.*
2. $\sum_{i,j} p_{i,j}=\Delta_Y$
3. $p_{i,j}\circ p_{k,l}=\delta_{(i,j)}^{(k,l)}p_{i,j}$

Proof. The first affirmation is clear from the definition of the correspondences $p_{i,j}$. In order to prove the remaining assertions we will use Manin's Identity Principle 2.2.8.

Define morphisms $\rho_{i,j}:CH^*(Y)\to CH^*(Y)$ by a downward induction:

$$
\rho_{i,j}:=m_{T_{i,j}}\circ\pi^*\circ\pi_*\circ m_{T_{n-i,j}}\circ\left(id_{CH^*(Y)}-\sum_{(k,l)\in W_{i,j}}\rho_{k,l}\right),
$$

and for a variety T, denote by $\rho_{i,j}^T$ the morphism:

$$
\rho_{i,j}^T:=m_{1_T\times T_{i,j}}\circ(id_T\times\pi)^*\circ(id_T\times\pi)_*\circ m_{1_T\times T_{n-i,j}}\circ\left(id_{CH^*(T\times Y)}-\sum_{(k,l)\in W_{i,j}}\rho_{k,l}^T\right).
$$

Manin's Identity Principle asserts that identities 2 and 3 of Theorem 4.1.2 hold if and only if the identities

$$\sum_{i,j} \rho_{i,j}^T = id_{CH^*(T\times Y)} \tag{4.4}$$

and

$$\rho_{i,j}^T \circ \rho_{k,l}^T = \delta_{(i,j)}^{(k,l)} \rho_{i,j}^T \tag{4.5}$$

hold for every variety T.

Now if

$$p_{i,j}^T = c_{1_T \times T_{i,j}} \circ c(id_T \times \pi) \circ c(id_T \times \pi)^t \circ c_{1_T \times T_{n-i,j}} \circ \left(\Delta_{T\times Y} - \sum_{(k,l)\in W_{i,j}} p_{k,l}^T \right)$$

then

$$(p_{i,j}^T)_e = \rho_{i,j}^T.$$

Equation (4.2) shows that, if e denotes the variety consisting of a single point, then we have

$$\sum_{i,j} \rho_{i,j} = \sum_{i,j} (p_{i,j})_e = (\Delta_Y)_e = id_{CH^*(Y)},$$

and, by equation (4.3)

$$\rho_{i,j} \circ \rho_{k,l} = (p_{i,j})_e \circ (p_{k,l})_e = \delta_{(i,j)}^{(k,l)} (p_{i,j})_e = \delta_{(i,j)}^{(k,l)} \rho_{i,j}.$$

But these identities were proved for a locally trivial fibration $\pi : Y \to X$ satisfying the hypothesis of Theorem 3.2.4, and the locally trivial fibration

$$id_T \times \pi : T \times Y \to T \times X$$

satisfy such hypotheses. In order to see this, we only have to show that the elements $1_T \times T_{i,j}$ generate the Chow ring $CH^*(T\times Y)$ as a $CH^*(T\times X)$-module. Since $(T\times Y)|_{T\times U} = T \times Y|_U \cong (T\times U)\times Z$, by (3.1) we have the diagram

$$\begin{tikzcd}
 & & CH^p(Z) \arrow[d, "q_Z^*"] \arrow[dd, bend left, "p_Z^*"] & \\
 & & CH^p(T\times Z) \arrow[d, "(id_T\times\eta)^*"] & \\
CH^{p-m}(T\times Y|_W) \arrow[r, "(id_T\times j)_*"] \arrow[d, "(id_T\times\pi_W)_*"] & CH^p(T\times Y) \arrow[r, "(id_T\times\imath)^*"] \arrow[d, "(id_T\times\pi)_*"] & CH^p(T\times Y|_U) \arrow[r] \arrow[d, "(id_T\times\pi_U)_*"] & 0 \\
CH^{p-n-m}(T\times W) \arrow[r, "(id_T\times j)_*"'] & CH^{p-n}(T\times X) \arrow[r, "(id_T\times i)^*"'] & CH^{p-n}(T\times U) \arrow[r] & 0,
\end{tikzcd}$$

where $p_Z : T \times Y|_U \to Z$ is the morphism induced by the projection on Z and $q_Z \circ (id_T \times \eta) = p_Z$. Therefore, in order to apply Theorem 3.2.4 we need to show that

$$(id_T \times \imath)^*(1_T \times T_{i,j}) = p_Z^*(\tau_{i,j}),$$

but

$$\begin{aligned}(id_T \times \imath)^*(1_T \times T_{i,j}) &= id_T^*(1_T) \times \imath^*(T_{i,j}) \\ &= id_T^*(1_T) \times \eta^*(\tau_{i,j}) \\ &= (id_T \times \eta)^*(1_T \times \tau_{i,j}) \\ &= (id_T \times \eta)^*(q_Z^*(\tau_{i,j})) \\ &= p_Z^*(\tau_{i,j})\end{aligned}$$

as desired. □

4.2 The Main Theorem

In this section we prove the main theorem of this work and some lemmas needed to prove it. In this chapter by an isomorphism of motives we mean an isomorphism between the additive structures of the motives involved. We start by calculating some factors appearing in the decomposition that we will give later for the motive $h(Y)$.

Lemma 4.2.1. *Let $\pi : Y \to X$ and $\pi' : Y' \to X$ be two locally trivial fibrations satisfying the hypothesis of Theorem 3.2.4. Then we have an isomorphism of motives*

$$h(Y) \cong h(Y').$$

Proof. Denote by $T_{i,j}$ (respectively $T'_{i,j}$) the generators of $CH^*(Y)$ (respectively $CH^*(Y')$) as a $CH^*(X)$-module.

By Theorem 4.1.2 and (2.3):

$$h(Y) = (Y, \Delta_Y) = \left(Y, \sum_{i,j} p_{i,j} \right) \cong \bigoplus_{i,j} (Y, p_{i,j})$$

and

$$h(Y') \cong \bigoplus_{i,j} (Y', p'_{i,j}),$$

where $p_{i,j},\ p'_{i,j}$ are defined as in (4.1), so it is enough to show that the factors appearing in these decompositions are isomorphic.

In order to do this, define morphisms of motives $h_{i,j} \in \mathrm{Hom}_{\mathfrak{M}_k}((Y, p_{i,j}), (Y', p'_{i,j}))$ by the formula:

$$h_{i,j} := c_{T'_{i,j}} \circ c(\pi') \circ c(\pi)^t \circ c_{T_{n-i,j}} \circ \left(\Delta_Y - \sum_{(k,l) \in W_{i,j}} p_{k,l} \right) ;$$

analogously define the morphisms $h'_{i,j} \in \mathrm{Hom}_{\mathfrak{M}_k}((Y', p'_{i,j}), (Y, p_{i,j}))$ by:

$$h'_{i,j} := c_{T_{i,j}} \circ c(\pi) \circ c(\pi')^t \circ c_{T'_{n-i,j}} \circ \left(\Delta_{Y'} - \sum_{(k,l) \in W_{i,j}} p'_{k,l} \right) .$$

At this point we would have to show the commutativity of the diagrams

$$\begin{tikzcd} Y \arrow[r, "h_{i,j}"] \arrow[d, "p_{i,j}"'] & Y' \arrow[d, "p'_{i,j}"] \\ Y \arrow[r, "h_{i,j}"'] & Y' \end{tikzcd} \qquad \begin{tikzcd} Y' \arrow[r, "h'_{i,j}"] \arrow[d, "p'_{i,j}"'] & Y \arrow[d, "p_{i,j}"] \\ Y' \arrow[r, "h'_{i,j}"'] & Y \end{tikzcd}$$

but this is a consequence of Manin's Identity Principle 2.2.8, since we have both

$$(h_{i,j})_e \left(\sum_{r=0}^{p} \sum_{s=1}^{m_r} \pi^*(\alpha_{r,s}) \cap T_{r,s} \right) = \pi'^*(\alpha_{i,j}) \cap T'_{i,j}$$

and a similar equation holding for $(h'_{i,j})_e$.

By following this procedure we can also obtain the identities

$$h'_{i,j} \circ h_{i,j} = \Delta_Y \mod p_{i,j}, \quad h_{i,j} \circ h'_{i,j} = \Delta_{Y'} \mod p'_{i,j}$$

which expose both $h_{i,j}$ and $h'_{i,j}$ as the desired isomorphisms. □

Lemma 4.2.2. *Let $\pi : Y \to X$ be a locally trivial fibration satisfying the hypothesis of Theorem 3.2.4. Then we have an isomorphism of motives*

$$(Y, p_{i,j}) \cong h(X) \otimes (Z, p_{i,j,Z}),$$

where the projectors $p_{i,j,Z} \in \mathrm{Hom}_{\mathfrak{M}_k}(Z, Z)$ are defined by the formula

$$p_{i,j,Z} := \tau_{n-i,j} \times \tau_{i,j}.$$

Proof. By Lemma 4.2.1 we have that

$$(Y, p_{i,j}) \cong (X \times Z, q_{i,j})$$

where the projectors $q_{i,j}$ are the ones defined for the trivial fibration

$$X \times Z \xrightarrow{\rho} X$$

by using the formula (4.1).

We will show that

$$(X \times Z, q_{i,j}) = (X \times Z, \Delta_X \otimes p_{i,j,Z}) .$$

We start by observing that the correspondences $p_{i,j,Z}$ are pairwise orthogonal projectors in the category of motives. Clearly, the correspondences $p_{i,j,Z}$ have degree zero. Now,

$$\begin{aligned} p_{i,j,Z} \circ p_{k,l,Z} &= \pi_{13*}\left(\pi_{12}^*(\tau_{n-i,j} \times \tau_{i,j}) \cap \pi_{23}^*(\tau_{n-k,l} \times \tau_{k,l})\right) \\ &= \pi_{13*}(\tau_{n-i,j} \times (\tau_{i,j} \cap \tau_{n-k,l}) \times \tau_{k,l}) \ . \end{aligned}$$

Suppose $\tau_{i,j} \cap \tau_{n-k,l} \neq 0$. Then

$$\pi_{13}(\tau_{n-i,j} \times (\tau_{i,j} \cap \tau_{n-k,l}) \times \tau_{k,l}) = \tau_{n-i,j} \times \tau_{k,l} \ .$$

Since

$$\dim(\tau_{n-i,j} \times (\tau_{i,j} \cap \tau_{n-k,l}) \times \tau_{k,l}) = n,$$

$$\dim(\tau_{n-i,j} \times \tau_{k,l}) = n + i - k$$

we see that they have the same dimension if and only if $k = i$. Therefore,

$$\pi_{13*}(\tau_{n-i,j} \times (\tau_{i,j} \cap \tau_{n-k,l}) \times \tau_{k,l}) = \begin{cases} \tau_{n-i,j} \times \tau_{i,l} & if \quad k = i \\ 0 & if \quad k \neq i \end{cases}$$

but since we are assuming $\tau_{i,j} \cap \tau_{n-i,l} \neq 0$, we have that $l = j$, so

$$\pi_{13*}(\tau_{n-i,j} \times (\tau_{i,j} \cap \tau_{n-k,l}) \times \tau_{k,l}) = \delta_{(i,j)}^{(k,l)} \tau_{n-i,j} \times \tau_{i,j}$$

and therefore

$$p_{i,j,Z} \circ p_{k,l,Z} = \delta_{(i,j)}^{(k,l)} p_{i,j,Z} \ .$$

Now consider the case when $\tau_{i,j} \cap \tau_{n-k,l} = 0$. Then $(i,j) \neq (k,l)$, otherwise we would have $\tau_{i,j} \cap \tau_{n-k,l} = e$; therefore $\delta_{(i,j)}^{(k,l)} = 0$. Another consequence of $\tau_{i,j} \cap \tau_{n-k,l} = 0$ is that, by Proposition 1.10 in [Ful98, p. 24] we have that

$$\tau_{n-i,j} \times (\tau_{i,j} \cap \tau_{n-k,l}) \times \tau_{k,l} = 0.$$

So in this case we also have the equality

$$p_{i,j,Z} \circ p_{k,l,Z} = \delta_{(i,j)}^{(k,l)} p_{i,j,Z} \ .$$

Therefore, the correspondences $p_{i,j,Z}$ define mutually orthogonal projectors on Z. Now we proceed to verify that the projectors $q_{i,j}$ and $\Delta_X \otimes p_{i,j,Z}$ coincide.

By Lemma 4.1.1 we have that

$$(q_{i,j})_e \left(\sum_{r=0}^{p} \sum_{s=1}^{m_r} \rho^*(\alpha_{r,s}) \cap 1_X \times \tau_{r,s} \right) = \rho^*(\alpha_{i,j}) \cap 1_X \times \tau_{i,j} \ . \qquad (4.6)$$

On the other hand,

$$\begin{aligned}(\Delta_X \otimes p_{i,j,Z})_e(\rho^*(\alpha_{r,s}) \cap 1_X \times \tau_{r,s}) &= (p_{i,j,Z})_X(\rho^*(\alpha_{r,s}) \cap 1_X \times \tau_{r,s}) \\ &= p_{i,j,Z} \circ (\rho^*(\alpha_{r,s}) \cap 1_X \times \tau_{r,s}) \\ &= p_{13*}(\alpha_{r,s} \times (\tau_{r,s} \cap \tau_{n-i,j}) \times \tau_{i,j}) \\ &= \begin{cases} \alpha_{i,j} \times \tau_{i,j} & if \quad (r,s) = (i,j) \\ 0 & if \quad (r,s) \neq (i,j) \end{cases} \\ &= \delta^{(k,l)}_{(i,j)} \rho^*(\alpha_{i,j}) \cap 1_X \times \tau_{i,j} \ .\end{aligned}$$

Therefore

$$\begin{aligned}&(\Delta_X \otimes p_{i,j,Z})_e \left(\sum_{r=0}^{p} \sum_{s=1}^{m_r} \rho^*(\alpha_{r,s}) \cap 1_X \times \tau_{r,s} \right) \\ = \quad & \rho^*(\alpha_{i,j}) \cap 1_X \times \tau_{i,j} \ .\end{aligned} \tag{4.7}$$

In this way, by (4.6) and (4.7) we obtain

$$(q_{i,j})_e = (\Delta_X \otimes p_{i,j,Z})_e$$

and by applying Manin's Identity Principle 2.2.8 we obtain

$$q_{i,j} = \Delta_X \otimes p_{i,j,Z} \ . \tag{4.8}$$

To conclude the proof of this lemma observe that

$$\begin{aligned}(Y, p_{i,j}) &\cong (X \times Z, q_{i,j}) \\ &= (X \times Z, \Delta_X \otimes p_{i,j,Z}) \quad by\ (4.8) \\ &\cong (X, \Delta_X) \otimes (Z, p_{i,j,Z}) \quad by\ definition\ of\ tensor\ product. \square\end{aligned}$$

Lemma 4.2.3. *Under the hypothesis of Lemma 4.2.2 we have that*

$$h(Z) \cong \bigoplus_{i,j} (Z, p_{i,j,Z}) \ .$$

Proof. We have already shown that the morphisms $p_{i,j,Z}$ induce pairwise orthogonal projectors. So, our proof will be finished if we can show that

$$\sum_{i,j} p_{i,j,Z} = \Delta_Z \ .$$

The elements of $CH^*(Z)$ can be written as

$$\sum_{r=0}^{n} \sum_{s=1}^{m_r} n_{r,s} \tau_{r,s}$$

for some $n_{r,s} \in \mathbb{Z}$. Observe that

$$(p_{i,j,Z})_e \left(\sum_{r=0}^{n} \sum_{s=1}^{m_r} n_{r,s} \tau_{r,s} \right) = p_{i,j,Z} \circ \left(\sum_{r=0}^{n} \sum_{s=1}^{m_r} n_{r,s} \tau_{r,s} \right) = \sum_{r=0}^{n} \sum_{s=1}^{m_r} n_{r,s} p_{i,j,Z} \circ \tau_{r,s}$$

besides,

$$\begin{aligned} p_{i,j,Z} \circ \tau_{r,s} &= p_{2*}((\tau_{n-i,j} \times \tau_{i,j}) \cap (\tau_{r,s} \times 1_Z)) \\ &= p_{2*}((\tau_{n-i,j} \cap \tau_{r,s}) \times \tau_{i,j}) \\ &= \begin{cases} \tau_{i,j} & if \quad (r,s) = (i,j) \\ 0 & if \quad (r,s) \neq (i,j) \end{cases} \\ &= \delta_{(i,j)}^{(r,s)} \tau_{i,j} \ . \end{aligned}$$

In this way we obtain that

$$(p_{i,j,Z})_e \left(\sum_{r=0}^{n} \sum_{s=1}^{m_r} n_s \tau_{r,s} \right) = n_{i,j} \tau_{i,j}$$

and therefore

$$\sum_{i,j} (p_{i,j,Z})_e = id_{CH^*(Z)} = (\Delta_Z)_e$$

and by using Manin's Identity Principle 2.2.8 we obtain the desired result. □

Now, we have at our disposal all the tools needed to prove the main theorem.

Theorem 4.2.4. *Let $\pi : Y \to X$ be a locally trivial fibration and suppose that the fibre Z has a Chow stratification. Then*

$$h(Y) \cong h(X) \otimes h(Z). \tag{4.9}$$

Proof. We have that

$$\begin{aligned} h(Y) = (Y, \Delta_Y) &\cong \bigoplus_{i,j} (Y, p_{i,j}) \\ &\cong \bigoplus_{i,j} (h(X) \otimes (Z, p_{i,j,Z})) \\ &\cong h(X) \otimes \left(\bigoplus_{i,j} (Z, p_{i,j,Z}) \right) \\ &\cong h(X) \otimes h(Z). \end{aligned}$$

□

Remark . Although it may seem at first glance that we have lost all the information of the fibration by calculating its motive, we have detected several strata in the motive $h(Y)$ given by the motives $(Y, q_{i,j})$, each of these is related to a projector appearing in the decomposition of the diagonal Δ_Z. Moreover, we have seen that if $\pi' : Y' \to X$ is another fibration as in Lemma 4.2.1 the isomorphism between $h(Y)$ and $h(Y')$ has to be constructed stratum by stratum, involving in each step the fibrations π and π', as well as the generators of the Chow groups of the fibre. As a final observation, recall that all the isomorphisms we have calculated in this chapter relate the additive structures of the motives involved, nothing has been said about the multiplicative structures of the motives; this can be an interesting question to answer in a future work.

4.3 Applications

In this section we present some applications of Theorems 3.2.4 and 4.2.4. To be more specific we present some examples of varieties having a Chow stratification.

We remind the reader that for a locally trivial fibration $\pi : Y \to X$ we mean that π is also a proper morphism with fibres isomorphic to a n-dimensional variety Z. We have the following corollary.

Corollary 4.3.1. *Let $\pi : Y \to X$ be a locally trivial fibration with fibre Z isomorphic to a quotient of a linear algebraic group by a parabolic subgroup (in particular, if Z is isomorphic to a variety of complete flags) then*

$$CH^*(Y) \cong CH^*(X) \otimes CH^*(Z)$$

and

$$h(Y) \cong h(X) \otimes h(Z).$$

If the fibre is isomorphic to a smooth projective toric variety then

$$CH^*(Y, \mathbb{Q}) \cong CH^*(X, \mathbb{Q}) \otimes CH^*(Z, \mathbb{Q}),$$

where $CH^*(\cdot, \mathbb{Q}) := CH^*(\cdot) \otimes_{\mathbb{Z}} \mathbb{Q}$. Moreover,

$$h_{\mathbb{Q}}(Y) \cong h_{\mathbb{Q}}(X) \otimes h_{\mathbb{Q}}(Z);$$

here $h_{\mathbb{Q}}$ denotes the motive that we obtain by considering the intersection theory given by the Chow rings $CH^*(\cdot, \mathbb{Q})$. $\square$

This is a direct consequence of Theorem 3.2.4, Theorem 4.2.4, and the fact that each of the varieties Z proposed in Corollary 4.3.1 has a Chow stratification, as we explain in the following paragraphs.

Complete flags. We start by considering the case of the variety of complete flags. A *complete flag* in n-dimensional projective space $\mathbb{P}^n$ is a sequence

$$F: \; F_0 \subset F_1 \subset \cdots \subset F_{n-1} \subset \mathbb{P}^n$$

of linear subspaces F_i ($\dim F_i = i$).

The locus of all complete flags F in $\mathbb{P}^n$ is called the *variety of complete flags*, and is denoted by $F(n+1)$.

Ehresmann in [Ehr34] studied, among other things, the topological properties of flag manifolds. He found that a basis of algebraic cells may be obtained for the Chow groups of such varieties by considering condition symbols. A condition symbol for flags F of $\mathbb{P}^n$ may be written in the form of a triangular matrix

$$\sigma = \begin{bmatrix} a_{00} & 0 & \cdots & 0 \\ a_{10} & a_{11} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1,0} & a_{n-1,1} & \cdots & a_{n-1,n-1} \end{bmatrix}$$

where the elements a_{ij} are non-negative integers, $0 \leq a_{ij} \leq n$, and

$$a_{i0} < a_{i1} < \cdots < a_{ii} \quad (1 \leq i \leq n-1).$$

The symbol is said to be irreducible if every element of each row except the last appears in the next row, either below or to the right. We can assign to each symbol a subvariety of F(n+1) in the following way: Fix a nest of spaces (the same nest is used for all values of i)

$$S_0 \subset S_1 \subset \cdots \subset S_{n-1} \subset \mathbb{P}^n, \quad \dim S_j = j,$$

then the variety determined by σ is

$$Y_\sigma := \{F \in F(n+1) \ : \ \dim(F_i \cap S_{a_{ij}}) \geq j\}.$$

The subvariety of $F(n+1)$ determined by an irreducible symbol is called a fundamental variety.

The fundamental varieties provide a subdivision of $F(n+1)$ into algebraic cells giving a basis for its Chow groups.

In [Mon59] Monk shows that we have the perfect pairing required for the basis just described. Therefore, a flag variety has a Chow stratification.

Quotients of linear algebraic groups. Another example that generalizes the variety of complete flags is the following. Let k be a field and let G be a k-split reductive linear algebraic group defined over k. Fix a maximal k-split k-torus T in G and a Borel subgroup B of G containing T. We consider the set of simple B-positive roots and let S be the corresponding set of reflections in the Weyl group W. Furthermore, we fix a subset θ of S and let W_θ be the subgroup of W generated by θ. Let $P_\theta := BW_\theta B$ be the corresponding parabolic subgroup of G and let Y be the projective smooth k-variety G/P. Let W^θ be the following subset of W

$$W^\theta := \{w \in W \ : \ l(ws) = l(w) + 1 \ for \ all \ s \in \theta\},$$

where $l : W \to \mathbb{N}_0$ is the length function relative to the system S of generators of W.

For any $w \in W^\theta$ let Y_w° (respectively Y_w) denote the cell BwP/P (respectively the closure of BwP/P in Y). Then we have that (see for example [Köc91]):

$$Y = \coprod_{w \in W^\theta} Y_w^\circ,$$

and for any $w \in W^\theta$ the cell Y_w° is k-isomorphic to the affine space $\mathbb{A}_k^{l(w)}$ of dimension $l(w)$. Moreover,

$$CH^*(Y) \cong \bigoplus_{w \in W^\theta} \mathbb{Z}[Y_w]$$

and we have the orthogonality relations

$$[Y_w] \cdot [Y_{w'}] = \delta_{w, w_0 w' v_0}[e],$$

for any $w,\ w' \in W^\theta$ with $l(w) + l(w') \leq \dim Y = l(w_0) - l(v_0)$ where w_0, respectively v_0 are the elements of maximal length in W, respectively W_θ. Therefore, Y has a Chow stratification.

Toric varieties. For the results stated in this part, we refer the reader to [Ful93]. A *toric* variety is a normal variety X that contains a torus $T \cong (\mathbb{C}^*)^n$ as a dense open subset, together with an action $T \times X \to X$ of T on X that extends the natural action of T on itself.

Consider a lattice $N \cong \mathbb{Z}^n$. A strongly convex rational polyhedral σ in $N_\mathbb{R} = N \otimes_\mathbb{Z} \mathbb{R}$ is a cone with apex at the origin, generated by a finite number of vectors; rational means that it is generated by vectors in N and strong convexity means that it contains no line through the origin; if there is no risk of confusion we call such a cone simply a cone in N. A fan Δ is a collection of cones in N satisfying the conditions analogous to those for a simplicial complex: every face of a cone in Δ is also a cone in Δ, and the intersection of two cones in Δ is a face of each.

A toric variety can be constructed from a lattice N and a fan Δ in N. In order to see this let $M = \mathrm{Hom}(N, \mathbb{Z})$, and for a cone σ in Δ define

$$S_\sigma = \{u \in M \ : \ \langle u, v\rangle \geq 0 \quad \forall v \in \sigma\}.$$

This semigroup is finitely generated, so its group algebra $\mathbb{C}[S_\sigma]$ is a finitely generated commutative $\mathbb{C}$-algebra. Such an algebra corresponds to the affine variety

$$U_\sigma = \mathrm{Spec}(\mathbb{C}[S_\sigma]).$$

If τ is a face of sigma, then $S_\sigma \subset S_\tau$ and therefore $\mathbb{C}[S_\sigma]$ is a subalgebra of $\mathbb{C}[S_\tau]$, in this way we get a morphism $U_\tau \to U_\sigma$, moreover U_τ can be regarded as a principal open subset of U_σ. With these identifications, these affine varieties fit together to form an algebraic variety which we denote by $X(\Delta)$, and it can be seen that $X(\Delta)$ is a toric variety, in fact its corresponding torus is given by

$$T = T_N = \mathrm{Spec}(\mathbb{C}[M]) = \mathrm{Hom}(M, \mathbb{C}^*) = N \otimes_{\mathbb{Z}} \mathbb{C}^*.$$

For each cone τ in Δ can be constructed an orbit of $X(\Delta)$ by the action of the torus $T = T_N$ which we will denote by $\mathcal{O}_\tau$, its closure will be denoted by $V(\tau)$.

For a smooth projective toric variety $Z = X(\Delta)$, it can be shown that

$$CH^p(Z, \mathbb{Q}) := CH^p(Z) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \bigoplus_{i=1}^{m_p} \mathbb{Q}[V(\tau_{p,i})]$$

for some cones $\tau_{p,1}, ..., \tau_{p,m_p}$ in Δ having codimension p. Moreover, the closed sets $V(\tau_{p,i})$ come from the closures of the cells of a cellular decomposition for $X(\Delta)$.

For these varieties we have that the Chow groups and the cohomology groups coincide, therefore by Poincaré duality we obtain perfect pairings

$$CH^p(X(\Delta), \mathbb{Q}) \otimes CH^{n-p}(X(\Delta), \mathbb{Q}) \to \mathbb{Q}$$

where $n = \dim X(\Delta)$. Therefore, if we relax the requirements of Definitions 3.1.1 and 3.1.2 in order to consider the groups $CH^*(\cdot, \mathbb{Q})$ instead of the groups $CH^*(\cdot)$ a smooth projective toric variety has a Chow stratification.

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