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“BRANCHED COVERINGS OF SEIFERT MANIFOLDS”

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Introduction

A Seifert manifold $M$ is a 3-manifold which is a disjoint union of circles (fibers). Seifert manifolds $M$ were defined and classified (up to fiber preserving homeomorphisms) by H. Seifert [Se] according to a Seifert symbol associated to $M$. Because of the fact that Seifert manifolds are classified, they play a useful role in the Theory of 3–manifolds. Since the invention of Seifert manifolds in the 30’s, an interesting problem is to understand the branched coverings $\varphi : \tilde{M} \to M$ when $M$ is a closed Seifert manifold.

Let $M$ be a closed Seifert manifold and suppose $\varphi : \tilde{M} \to M$ is a covering of $M$ branched along fibers, that is, the branching of $\varphi$ is a finite union of fibers of $M$. It is known that $\tilde{M}$ is also a Seifert manifold [G-H]. In [Se], H. Seifert also found the Seifert symbol for the orientation double covering of $M$. More recently, V. Núñez and E. Ramírez-Losada [N-RL] compute the Seifert symbol for $\tilde{M}$ when $M$ is orientable and $\varphi : \tilde{M} \to M$ satisfies some properties. But in general, if $\varphi : \tilde{M} \to M$ is a covering of a Seifert manifold $M$ branched along fibers, the Seifert Symbol for $\tilde{M}$ is unknown. Therefore a basic problem is to determine the Seifert symbol of $\tilde{M}$ in terms of $\varphi$ and the Seifert symbol of $M$. In this work we solve the above problem (Theorem 2.3.8 and Theorem 2.3.15).

On the other hand, Heegaard genera for almost all Seifert manifolds are known. M. Boileau and H. Zieschang [B-Z] computed the Heegaard genera for almost all orientable Seifert manifolds and V. Núñez [Nu] computed the Heegaard genera for almost all non-orientable Seifert manifolds. In both cases, orientable or non-orientable, the Heegaard genus of $M$ is expressed in terms of the Seifert symbol of $M$. 

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Let $M$ be a Seifert manifold with infinite fundamental group. Suppose $\varphi : \tilde{M} \to M$ is a covering of $M$ branched along fibers. If we know the Heegaard genus of $M$, $h(M)$, and we compute the Seifert symbol of $\tilde{M}$, we can compare the Heegaard genus of $\tilde{M}$, $h(\tilde{M})$, with $h(M)$. What one can “reasonable” expect is that $h(\tilde{M}) \geq h(M)$, but we find families of manifolds $M$, with infinite fundamental group, having a covering $\tilde{M}$ such that $h(\tilde{M}) < h(M)$ (Corollary 3.2.4 and Corollary 3.2.5). This implies (translating into fundamental group) that there are infinite families of infinite groups $G$ associated to 3-manifolds that have a subgroup $H < G$ of finite index with an unexpected and surprising property: $\text{rank}(H) < \text{rank}(G)$.

In Chapter 1, we deal with basic topics to be used along this work. The basic topics to consider are: Topology of manifolds, Heegaard splittings and Branched coverings. In the last section of Chapter 1, we write a list of Theorems that we will be needed later.

Let $M$ be a Seifert manifold and $\varphi : \tilde{M} \to M$ a branched covering space of $M$. Suppose $\tilde{M}$ is connected. In chapter 2, we prove that there are coverings $\psi : \tilde{M} \to M'$ and $\zeta : M' \to M$ branched along fibers such that the following diagram commutes

$$
\begin{array}{ccc}
\tilde{M} & \xrightarrow{\psi} & M' \\
\downarrow{\varphi} & & \downarrow{\zeta} \\
M & & \\
\end{array}
$$

and if $\omega_\psi$ and $\omega_\zeta$ are the representations associated to $\psi$ and $\zeta$, respectively, we have that $\omega_\psi(h') = \varepsilon_n$ and $\omega_\zeta(h) = (1)$, where $(1)$ is the identity permutation in $S_n$ and $\varepsilon_n$ is the standard $n$-cycle $(1, 2, \ldots, n)$, and $h$ and $h'$ are regular fibers of $M$ and $M'$, respectively. Thus we reduce the study of coverings of $M$ to coverings $\varphi : \tilde{M} \to M$, such that $\omega_\varphi$, the representation associated to $\varphi$, sends a regular fiber $h$ of $M$ into the identity permutation or into the $n$-cycle $(1, \ldots, n)$. In both cases, $\omega(h) = (1)$ or $\omega(h) = \varepsilon_n$, we calculate the Seifert symbol of $\tilde{M}$. 


In chapter 3, given a $\varphi : \tilde{M} \to M$ covering of $M$ branched along fibers such that $\omega_\varphi$, the representation associated to $\varphi$, sends a regular fiber $h$ of $M$ into the identity permutation or into the $n$-cycle $(1, \ldots, n)$, we apply the theory in Chapter 2 to compare the Heegaard genus of $\tilde{M}$, $h(\tilde{M})$, with the Heegaard genus of $M$, $h(M)$. The genus $h(\tilde{M})$ is computed in terms of $\omega_\varphi$ and the Seifert symbol of $M$. We show that there are Seifert manifolds of $M$ and coverings $\tilde{M}$ such that $h(\tilde{M}) < h(M)$. 
Chapter 1

Preliminaries

This chapter is a brief review about facts in low-dimensional topology.

1.1 3-manifolds and Heegaard genus

Definition 1.1.1 Let $M$ be a Hausdorff topological space. We say $M$ is an $n$-manifold if and only if each element $x$ of $M$ has a neighborhood homeomorphic to $\mathbb{R}^n$ or $\mathbb{R}^n_+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_i \geq 0, \forall i = 1, \ldots, n\}$.

If $M$ is an $n$-manifold and there is a point in $M$ having no neighborhood homeomorphic to $\mathbb{R}^n$, we say that $M$ is an $n$-manifold with boundary and we call this point a boundary point. The set of boundary points is called the boundary of $M$ and we denote it by $\partial M$. The space $M - \partial M$ is called the interior of $M$ and it is denoted by $M^o$. An $n$-manifold $M$ is a closed manifold if it is compact and $\partial M = \emptyset$.

Definition 1.1.2 A 3-manifold $M$ is irreducible if every 2–sphere $S^2$ in $M$ bounds a 3-ball.

Definition 1.1.3 A disk $D^2$ in a 3-manifold with boundary $M$ is said to be properly embedded if $D^2 \cap \partial M = \partial D^2$. 

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Definition 1.1.4 Let $V$ be an orientable irreducible compact and connected 3-manifold with non-empty boundary. If there exist $k$ properly embedded pairwise disjoint 2-disks $D_j$ such that $\bigcup_{j=1}^{k} D_j$ splits $V$ into a 3-ball, we say that $V$ is a handlebody of genus $k$.

Handlebody

Note that the boundary of $V$ is a closed, connected and orientable surface of genus $k$.

Heegaard’s theorem 1.1.1 Let $M$ be a connected closed and orientable 3–manifold. Then $M$ is union of two handlebodies of genus $g$, for some $g \geq 0$.

Proof.
It is well-known that $M$ is triangulable [Mo]. Let $K$ be a triangulation for $M$. Define $V_1$ to be a regular neighborhood of the 1-skeleton of $K$ and $V_2$ to be $M - V_1$. \hfill \Box

Definition 1.1.5 Let $M$ be a connected, closed 3-manifold and let $F \subset M$ be a closed, connected and orientable surface. If $F$ splits $M$ into two handlebodies, then $(M, F)$ is a Heegaard splitting of $M$.

Definition 1.1.6 The genus of a Heegaard splitting is the genus of the surface $F$, and the Heegaard genus of $M$, $h(M)$, is the smallest integer $h$ such that $M$ has a Heegaard splitting of genus $h$.

Example 1.1.1 $h(S^3) = 0$

1.2 Branched coverings

Definition 1.2.1 Let $X$ and $\tilde{X}$ be two path-connected topological spaces. A surjective map $f : \tilde{X} \to X$ is a covering space map if and only if for every $x \in X$ there exists a neighborhood $V_x$ of $x$ satisfying the following properties:
1.2. BRANCHED COVERINGS

(a) \( f^{-1}(V_x) = \bigcup_{\alpha \in J} \tilde{V}_\alpha \), with \( \tilde{V}_\alpha \cap \tilde{V}_\beta = \emptyset \) if \( \alpha \neq \beta \) and

(b) \( f \mid : \tilde{V}_\alpha \to V_x \) is a homeomorphism, for all \( \alpha \in J \).

If \( |J| = n \) is a natural number, then \( f \) is a finite covering space and we say that \( f \) is a covering of \( n \)-sheets or that \( f \) is an \( n \)-fold covering.

Let \( \Omega \) be a set of \( n \) elements; we write \( S_n = S(\Omega) \) for the symmetric group on the \( n \) elements of \( \Omega \). When no confusion arises about the set \( \Omega \), we only write \( S_n \).

Let \( \tilde{N} \) and \( N \) be \( n \)-manifolds. Suppose \( f : \tilde{N} \to N \) is a map. We say that \( f \) is a proper map if \( f^{-1}(\partial N) = \partial \tilde{N} \). The map \( f \) is finite-to-one if \( f^{-1}(x) \) is finite, for all \( x \in N \).

**Definition 1.2.2** A proper map \( f : \tilde{N} \to N \) between two \( m \)-manifolds is called a branched covering if it is finite-to-one and open.

Usually one can check if an open map \( f \) between manifolds is a branched covering by finding a minimal subcomplex \( B \) of \( N \) of codimension two such that \( f \mid : \tilde{N} - f^{-1}(B) \to N - B \) is a finite covering space [Fo].

The subcomplex \( B \) is called the branch set of \( f \) and \( f^{-1}(B) \) is called the singular set of \( f \). In our examples the set \( B \) is always a submanifold.

If \( f \mid (\tilde{N} - f^{-1}(B)) \) is an \( n \)-fold covering, we say that \( f \) is a branched covering of \( n \)-sheets or that \( f \) is an \( n \)-fold branched covering.

Note that a finite covering space map (unbranched) between manifolds is a branched covering with \( B = \emptyset \).

**Remark 1.2.1** The following facts about coverings and branched coverings are known:
(a) An $n$-fold covering space $\eta: \tilde{X} \to X$ determines and is determined by a homomorphism $\omega_f : \pi_1(X) \to S_n$, where $S_n$ is the symmetric group on $n$ symbols. This homomorphism $\omega$ is called a representation of $\pi_1(X)$. Also $\tilde{X}$ is connected if and only if $\omega$ is transitive.

Let $\varphi: \tilde{X} \to X$ be a branched covering and let $B$ be the branch set of $\varphi$.

(b) The covering $\varphi| : \tilde{X} - \varphi^{-1}(B) \to X - B$ determines the branched covering $\varphi$ through a Fox compactification [Fo].

(c) By (a) and (b), a branched covering determines and is determined by a representation $\omega_f : \pi_1(N - \text{Branch set of } f) \to S_n$

(d) If $X$ is orientable, $\tilde{X}$ is also orientable [B-E], for if $w_1(X)$ is the first Stiefel-Whitney class of $X$ then $\varphi^*w_1(X) = w_1(\tilde{X})$, where $\varphi^* : H^1(M, \mathbb{Z}_2) \to H^1(\tilde{M}, \mathbb{Z}_2)$ is the homomorphism induced by $\varphi: \tilde{X} \to X$ in the cohomology groups.

1.3 Some preliminary Theorems

If $M$ is 3–manifold, let $w_1(M) : H_1(M) \to \mathbb{Z}_2$ be a homomorphism such that if $\alpha \subset M$ is an orientation preserving curve then $w_1(\alpha) = 1$, and if $\alpha$ is orientation reversing then $w_1(\alpha) = -1$.

The homomorphism $w_1(M)$ is the first Stiefel-Whitney class of $M$. If $\varphi: \tilde{M} \to M$ is a branched covering of $M$, it is proved in [B-E] that $w_1(\tilde{M}) = \varphi^*(w_1(M))$, where $\varphi^* : H^1(M, \mathbb{Z}_2) \to H^1(\tilde{M}, \mathbb{Z}_2)$ is the homomorphism induced by $\varphi$ in the cohomology groups.

We write $PD : H^1(M, \mathbb{Z}_2) \to H_2(M, \mathbb{Z}_2)$ for the Poincaré duality isomorphism associated to the 3-manifold $M$.

**Definition 1.3.1** Let $M$ be a non-orientable 3-manifold and $F \subset M$ be an orientable surface. We call $F$ a Stiefel-Whitney surface for $M$ if and only if $F$ is connected and $[F] = PDw_1(M) \in H_2(M; \mathbb{Z}_2)$. 
Assume $M$ is a manifold. Let $\beta : H^i(M, \mathbb{Z}_2) \to H^{i+1}(M, \mathbb{Z})$ denote the Bockstein homomorphism associated to the short exact sequence of coefficients

$$0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_2 \to 0.$$ 

**Lemma 1.3.1 [B-E]** Let $M$ be a non-orientable 3-manifold. Then $\beta w_1(M) = 0$ if and only if there exists $S \subset M$ a two-sided Stiefel-Whitney surface for $M$.

Let $M = (Xx, g, \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r)$ be a Seifert manifold, where $Xx$ is a symbol in \{Oo, On, No, NnI, NnII, NnIII\} (See Chapter 3). Write $e_0(M) = \sum \beta_i/\alpha_i$ and, $\lambda(M) = \text{lcm}\{\alpha_1, \ldots, \alpha_r\} e_0(M)$, where $\text{lcm}\{\alpha_1, \ldots, \alpha_r\}$ denotes the least common multiple of $\alpha_1, \ldots, \alpha_r$. Notice that $\lambda(M)$ is an integer number.

**Theorem 1.3.1 [Nu]** If $M$ is a non-orientable Seifert manifold with orbit projection $p : M \to F$, then $\beta w_1(M) \neq 0$ if and only if either $M \in NnII$ or $M \in NnI$, $g(F)$ is odd and $\lambda(M)$ is even.

**Theorem 1.3.2 [Nu]** Let $M$ be a non-orientable Seifert manifold. Then there exists a fibered torus $T \subset M$, where fibered means that $T$ is a union of fibers of $M$, such that $T$ is a Stiefel-Whitney surface for $M$. In the following cases $T$ is two-sided in $M$:

(i) $M \in (No, g)$.

(ii) $M \in (NnI, 2g)$.

(iii) $M \in (NnIII, g)$.

And in the following cases $T$ is one-sided in $M$:

(iv) $M \in (NnI, 2g + 1)$.

(v) $M \in (NnII, g)$.

**Theorem 1.3.3 [Nu]** Let $M$ be a non-orientable Seifert manifold and $T$ be a fibered torus in $M$. 

• Suppose $M \in (NnI, 2g + 1)$ or $M \in (NnII, g)$. If $T \subset M$ is a two-sided fibered torus, then $M - T$ is non-orientable;

• Assume $M \in (No, g)$ or $M \in (NnI, 2g)$ or $M \in (NnIII, g)$. If $T \subset M$ is an one-sided fibered torus, then $M - T$ is non-orientable.
Chapter 2

Coverings of Seifert manifolds

2.1 Coverings and bundles

Recall that if $\Omega$ is a set of $n$ elements, then $S_n = S(\Omega)$ denotes the symmetric group on the $n$ elements of $\Omega$.

The identity permutation of $S_n$ is the permutation that fix all the elements of $\Omega$. We denote the identity permutation of $S_n$ by $(1)$.

Let $\sigma \in S_n$, the order of $\sigma$, denoted by $\text{order}(\sigma)$, is the smallest natural number $n$ such that $\sigma^n = (1)$.

Let $\sigma \in S_n$, the order of $\sigma$, denoted by $\text{order}(\sigma)$, is the smallest natural number $n$ such that $\sigma^n = (1)$.

A cycle $\rho = (a_1, \ldots, a_s)$ in $S_n = S(\Omega)$ is the permutation that fixes the elements in $\Omega$ different from $a_i$, for all $i = 1, \ldots, s$, it sends the element $a_i \in \Omega$ into $a_{i+1}$, for each $i = 1, \ldots, s-1$, and sends the element $a_s$ into $a_1$. One can verify easily that if $\rho = (a_1, \ldots, a_s)$ then $\text{order}(\rho) = s$. Throughout this work the standard $n$-cycle of $S_n$ is the permutation $(1, 2, \ldots, n) \in S_n$ and it will be denoted by $\varepsilon_n$.

Recall that if $\sigma$ is a permutation in $S_n$ then $\sigma$ can be represented as a product of disjoint cycles. Throughout this work all permutations in $S_n$ will be represented as a product of disjoint cycles.
Definition 2.1.1 Suppose \( m, n \in \mathbb{N} - \{1\} \) and \( H \leq S_{mn} = S(\Omega) \), where \( \Omega \) is a set of \( m, n \)-elements; then we say that \( H \) is \( m, n \)-imprimitive if there are \( \Delta_1, \ldots, \Delta_n \subset \Omega \) such that:

(a) \( \Omega = \bigsqcup_{i=1}^n \Delta_i \), where \( \bigsqcup \) denotes the disjoint union.

(b) \( \#\Delta_i = m \), for all \( i = 1, \ldots, n \).

(c) The elements of \( H \) leave the sets \( \Delta_i \) invariant, that is \( \sigma(\Delta_i) = \Delta_j \), for each \( i \) and \( \sigma \) and for some \( j \in \{1, \ldots, n\} \).

The sets \( \Delta_1, \ldots, \Delta_n \) are called sets of \( m, n \)-imprimitivity for \( H \).

Note that if \( H \) is \( m, n \)-imprimitive then \( H \leq S_{mn} \).

Given \( x \in \Omega \), the stabilizer of \( x \) is the subgroup \( St(x) = \{ \sigma \in S(\Omega) | \sigma(x) = x \} \leq S(\Omega) \).

Let \( H \) be \( m, n \)-imprimitive. The quotient \( \Delta_1 \sqcup \ldots \sqcup \Delta_n / \{\Delta_1, \ldots, \Delta_n\} \) which sends all symbols of \( \Delta_i \) into the symbol \( \Delta_i \) for each \( i \), induces a “quotient homomorphism” \( q : H \to S_n = S(\{\Delta_1, \ldots, \Delta_n\}) \). If \( H_1 = q^{-1}(St(\Delta_1)) \), then the “restriction homomorphism” \( \gamma : H_1 \to S_m = S(\Delta_1) \) such that \( \gamma(\sigma) = \sigma|\Delta_1 \), is a group homomorphism.

Lemma 2.1.1 Let \( \varphi : X \to Y \) be an \( mn \)-fold covering space and let \( \omega : \pi_1(Y) \to S_{mn} \) be the associated representation; write \( H = Im(\omega) \). Then \( H \) is \( m, n \)-imprimitive if and only if \( \varphi \) factors through an \( m \)-fold covering \( \psi : X \to Z \) and an \( n \)-fold covering \( \zeta : Z \to Y \).

Proof.

If \( H \) is \( m, n \)-imprimitive, then there exists sets of \( m, n \)-imprimitivity, \( \Delta_1, \ldots, \Delta_n \), for \( H \). Consider the representation

\[
\omega \zeta : \pi_1(Y) \xrightarrow{\omega} H \xrightarrow{q} S_n = S(\{\Delta_1, \ldots, \Delta_n\}),
\]
where \( q \) is the quotient homomorphism determined by \( \Delta_1, \ldots, \Delta_n \). Let \( \zeta : Z \to Y \) be the \( n \)-fold covering associated to \( \omega \zeta \); then \( Z \) is a topological space such that \( \pi_1(Z) \cong (q \circ \omega)^{-1}(St(\Delta_1)) \). Notice that \( \omega^{-1}(St(1)) \subset (q \circ \omega)^{-1}(St(\Delta_1)) \) by definition of \( q \). Therefore there is an \( m \)-fold covering \( \psi : X \to Z \) such that \( \zeta \circ \psi = \varphi \).

Notice that the representation associated to \( \psi \) is

\[
\omega_\psi : \pi_1(Z) \cong (q \circ \omega)^{-1}(St(\Delta_1)) \xrightarrow{\omega} q^{-1}(St(\Delta_1)) \xrightarrow{\sim} S_m = S(\Delta_1),
\]

where \( \gamma \) is the restriction homomorphism determined by \( \Delta_1, \ldots, \Delta_n \).

Now suppose there are \( \psi : X \to Z \) and \( \zeta : Z \to Y \) covering spaces of \( m \)-sheets and \( n \)-sheets, respectively, such that \( \varphi = \psi \circ \zeta \). Let \( y_0 \in Y \). Then \( \zeta^{-1}(y_0) = \{z_1, \ldots, z_n\} \) and

\[
\varphi^{-1}(y_0) = \{x_{1,1}, \ldots, x_{1,m}, x_{2,1}, \ldots, x_{2,m}, \ldots, x_{n,1}, \ldots, x_{n,m}\}.
\]

By renumbering the points, if necessary, we can suppose that \( \psi(x_{i,j}) = z_i \), for \( 1 \leq i \leq n \) and for \( 1 \leq j \leq m \). Define \( \Delta_i = \{x_{i,1}, \ldots, x_{i,m}\} \), for each \( i \in \{1, \ldots, n\} \). Using the Path Lifting Theorem for covering spaces, it is clear that the \( \Delta_i \)'s are sets of \( m, n \)-imprimitivity. \( \square \)

Suppose \( N \) is an \( n \)-manifold and \( \varphi : \tilde{N} \to N \) is an \( m \)-fold covering of \( N \). Let \( \omega : \pi_1(N) \to S_m \) be the representation determined by \( \varphi \) and \( \theta : H_1(N) \to \mathbb{Z}_2 \) be a homomorphism. Note that we can consider the homomorphism \( \theta \circ p_{ab} : \pi_1(N) \to \mathbb{Z}_2 \), where \( p_{ab} : \pi_1(N) \to H_1(N) \) is the abelianization quotient. Since \( p_{ab}[x]_{\pi_1} = [x]_{H_1} \), for all \( [x] \in \pi_1(N) \), throughout this work we also write \( \theta \) to denote the homomorphism \( \theta \circ p_{ab} \).

If \( \varphi_\theta : N_{\theta} \to N \) is the 2-fold covering associated to \( \theta \), define \( \tilde{\theta} = \varphi^*(\theta) \), where \( \varphi^* : H^1(N; \mathbb{Z}_2) \to H^1(\tilde{N}; \mathbb{Z}_2) \) is the cohomology induced homomorphism. Notice that \( \tilde{\theta} \) can be regarded as an element of \( H^1(\tilde{N}; \mathbb{Z}_2) \), that is \( \tilde{\theta} : H_1(N) \to \mathbb{Z}_2 \) is a homomorphism.

Note that if \( \theta \) is non-trivial, then \( \theta \) is an epimorphism (i.e. \( \theta \) is a transitive representation). Consequently \( \pi_1(N_{\theta}) \cong Ker(\theta) \), for \( \varphi_\theta \) is regular and thus \( Ker(\theta) = \theta^{-1}(St(1)) \).
CHAPTER 2. COVERINGS OF SEIFERT MANIFOLDS

Remark 2.1.1 If $\theta$ is trivial, then $\tilde{\theta}$ is trivial.

Proof.
In this case $N_\theta = N \sqcup N$, where $\sqcup$ denotes the disjoint union. Suppose $\tilde{\alpha} \in H_1(\tilde{N})$, then $\tilde{\theta}(\tilde{\alpha}) = \theta(\varphi_*(\tilde{\alpha})) = (1)$. □

Remark 2.1.2 If $\theta$ is non-trivial, then $\tilde{\theta}$ is trivial if and only if there exists a $m/2$-fold covering $\psi : \tilde{N} \to N_\theta$ such that $\psi \circ \varphi_\theta = \varphi$.

Proof.
Let us suppose that $\tilde{\theta}$ is trivial; then $\tilde{\theta}(\tilde{\alpha}) = \theta(\varphi_*(\tilde{\alpha})) = (1)$, for all $\tilde{\alpha} \in H_1(\tilde{N})$. Therefore $\varphi_*(H_1(\tilde{N})) \subset \text{Ker}(\theta)$ and there is a $m/2$-fold covering $\psi : \tilde{N} \to N_\theta$ satisfying that $\psi \circ \varphi_\theta = \varphi$.

On the other hand, if there exists a covering $\psi : \tilde{N} \to N_\theta$ such that $\psi \circ \varphi_\theta = \varphi$, then $\varphi_*(H_1(\tilde{N})) \subset \text{Ker}(\theta)$ and thus $\tilde{\theta}$ is trivial. □

Theorem 2.1.1 Assume $N$ is an $n-$manifold and $\varphi : \tilde{N} \to N$ is an $m-$fold covering of $N$. Let $\omega : \pi_1(N) \to S_m$ be the representation determined by $\varphi$ and $\theta : H_1(N) \to \mathbb{Z}_2$ be a homomorphism. Let $\tilde{\theta} = \varphi^*(\theta)$. Suppose that $\theta$ is non-trivial.

Then $\tilde{\theta}$ is trivial if and only if $\text{Im}(\omega)$ is $m/2$-2-imprimitive and there are sets of $m/2$, 2-imprimitivity for $\text{Im}(\omega)$, $\Delta_1$ and $\Delta_2$, such that the quotient homomorphism $q : \text{Im}(\omega) \to S_2$ satisfies that $q \circ \omega = \theta$.

Proof.
If $\tilde{\theta}$ is trivial, by Remark 2.1.2 there exists an $m/2$-fold covering $\psi : \tilde{N} \to N_\theta$ such that $\psi \circ \varphi_\theta = \varphi$. Then, by Lemma 2.1.1, there exist $\Delta_1$ and $\Delta_2$ sets of $m/2$, 2-imprimitivity for $\text{Im}(\omega)$ such that the representation induced by $\varphi_\theta$ is $q \circ \omega : \pi_1(N) \to S_2$. Therefore $q \circ \omega = \theta$.

On the other hand, if there are sets of $m/2$, 2-imprimitivity for $\text{Im}(\omega)$, $\Delta_1$ and $\Delta_2$, such that $q \circ \omega = \theta$, then by Lemma 2.1.1 there is a covering $\psi : \tilde{N} \to N_\theta$ of $m/2$-sheets such that $\varphi = \psi \circ \varphi_\theta$. Thus, by Remark 2.1.2, $\tilde{\theta}$ is trivial. □
**Definition 2.1.2** Let $N$ be a connected $m$–manifold and let $n \in \mathbb{N}$. Assume $\omega : \pi_1(N) \to S_n$ is a transitive representation and $\theta \in H^1(N, \mathbb{Z}_2)$. We say that $\omega$ trivializes the bundle of $\theta$ if and only if $\text{Im}(\omega)$ is $\frac{m}{2}$–imprimitive and there are sets of $\frac{m}{2}$–imprimitivity for $\text{Im}(\omega)$, $\Delta_1$ and $\Delta_2$, such that the quotient homomorphism $q : \text{Im}(\omega) \to S_2$ satisfies that $q \circ \omega = \theta$.

When a permutation in an imprimitive subgroup contains an odd order cycle, computations are somewhat eased as it is shown in the following example.

**Example 2.1.1** Consider the permutations $a = (1, 2, 3)(4, 5, 6)$ and $b = (1, 4)(2, 5)(3, 6)$ in $S_6$. Let $H = \langle a, b \rangle$ be the subgroup in $S_6$ generated by the permutations $a$ and $b$. It can be seen that $H$ is $3, 2$–imprimitive. Let us calculate a system of $3, 2$–imprimitivity for $H$. There exist sets of $3, 2$–imprimitivity, $\Delta_1$ and $\Delta_2$ for $H$. Note that $a \cdot \Delta_1$ must be equal to $\Delta_1$ or $\Delta_2$ because $\Delta_1$ is a set of $3, 2$–imprimitivity. Assume $1 \in \Delta_1$.

If $a \cdot \Delta_1 = \Delta_1$, then $2, 3 \in \Delta_1$ for $a(1) = 2$ and $a(2) = 3$; thus $\{1, 2, 3\} \subset \Delta_1$ and we get $\Delta_1 = \{1, 2, 3\}$ because $\# \Delta_1 = 3$.

Note that $a \cdot \Delta_1 = \Delta_2$ cannot happen. If $a \cdot \Delta_1 = \Delta_2$, then $2 \in \Delta_2$ for $1 \in \Delta_1$ and $a(1) = 2$. Of course $3$ should belong to $\Delta_2$ because $a(3) = 1$; otherwise, if $3 \in \Delta_1$ we have $1 \in \Delta_2$. But $3 \in \Delta_2$ implies that $a \cdot \Delta_2 = \Delta_2$ for $a(2) = 3$ and $2, 3 \in \Delta_2$. Thus $1 \in \Delta_2$ since $a(3) = 1$ and this contradicts our assumption that $1 \in \Delta_1$.

Therefore $\Delta_1 = \{1, 2, 3\}$ and $\Delta_2 = \{4, 5, 6\}$ are the only sets of $3, 2$–imprimitivity for $H$. One can see easily that if $q : H \to S_2$ is the quotient homomorphism associated to $\Delta_1$ and $\Delta_2$, then $q(a)$ is the identity in $S_2 = S(\{\Delta_1, \Delta_2\})$ and $q(b) = (\Delta_1, \Delta_2) \in S(\{\Delta_1, \Delta_2\})$. □

In general, we obtain the following corollary.

**Corollary 2.1.1** Assume $N$ is an $n$–manifold and $\varphi : \tilde{N} \to N$ is an $m$–fold covering of $N$. Let $\omega : \pi_1(N) \to S_m$ be the representation determined by $\varphi$ and $\theta : H_1(N) \to \mathbb{Z}_2$ be a
homomorphism. Let \( \tilde{\theta} = \varphi^*(\theta) \). Suppose that \( v_j \) is a generator for \( \pi_1(N) \) such that in the disjoint cycle decomposition of \( \omega(v_j) \) there is a cycle \( (a_{j,1}, \ldots, a_{j,k}) \) of odd order and \( \theta(v_j) = (1,2) \).

Then \( \tilde{\theta} \) is non-trivial.

Proof.

Assume that \( \tilde{\theta} \) is trivial. Then there are sets \( \Delta_1 \) and \( \Delta_2 \) of \( m^2 \) \( 2 \)-imprimitive for \( \text{Im}(\omega) \). Since \( (a_{j,1} \cdots a_{j,k}) \) has odd order and \( \omega(v_j) \) must leave the sets \( \Delta_1 \) and \( \Delta_2 \) invariant, it follows that \( \{a_{j,1}, \ldots, a_{j,k}\} \subset \Delta_1 \) or \( \{a_{j,1}, \ldots, a_{j,k}\} \subset \Delta_2 \). Without loss of generality, we suppose that \( \{a_{j,1}, \ldots, a_{j,k}\} \subset \Delta_1 \), thus \( (q \circ \omega(v_j))(\Delta_1) = \Delta_1 \) and \( q \circ \omega \neq \theta \). Therefore \( \tilde{\theta} \) is non-trivial. \( \square \)

Let \( N \) be a manifold and let \( \theta \) be equal to \( w_1(N) \), the first Stiefel-Whitney class of \( N \), and recall that if \( \varphi : \tilde{N} \to N \) is a covering space then \( w_1(\tilde{N}) = \varphi^*(w_1(N)) \). Then we can apply the previous theorem to get the following corollary.

**Corollary 2.1.2** Suppose that \( N \) is a non-orientable manifold and consider a transitive representation \( \omega : \pi_1(N) \to S_m \). Let \( \varphi : \tilde{N} \to N \) be the covering space associated to \( \omega \) and \( w_1(N) \) be the first Stiefel-Whitney class of \( N \).

Then \( \tilde{N} \) is orientable if and only if \( \text{Im}(\omega) \) trivializes the bundle of \( w_1(N) \).

**Remark 2.1.3** Let \( F \) be a non-orientable surface of genus \( k \) and let \( \{v_j\}_{j=1}^k \) be a basis for \( \pi_1(F) \) such that \( v_j \) is an orientation reversing loop, for all \( j \in \{1, \ldots, k\} \). Suppose that \( n \geq 2 \), \( \varphi : \tilde{F} \to F \) is a covering space and let \( \omega : \pi_1(F) \to S_n \) be the representation associated to \( \varphi \). By Corollary (2.1.1) and Corollary (2.1.2)

1. If the order of a cycle of \( \omega(v_m) \) is odd, for some \( m \in \{1, \ldots, k\} \), then \( \tilde{F} \) is non-orientable.
2. If \( n \) is an odd number, \( \tilde{F} \) is non-orientable.
3. Suppose that all the cycles of \( w(v_j) \) have even order (therefore \( n \) is an even number), for each \( j = 1, \ldots, k \); then \( \tilde{F} \) is orientable if and only if \( \text{Im}(\omega) \) trivializes the bundle of \( w_1(F) \).
2.2 Seifert manifolds

Let $\alpha$ and $\beta$ be coprime integers numbers and $\alpha \geq 1$; Suppose $r : D^2 \to D^2$ is the rotation defined by $r(x) = xe^{2\pi i (\alpha/\beta)}$. Then the fibered solid torus $T(\beta/\alpha)$ is the quotient space $D^2 \times I/(x, 0) \sim (r(x), 1)$, where $I = [0, 1]$.

The fibers of $T(\beta/\alpha)$ are the images of the intervals $\{x\} \times I$ (under the identification). Note that almost all fiber in $T(\beta/\alpha)$ is the union of the images of $\beta$ intervals; the only exception is the core of $T(\beta/\alpha)$ because this fiber is the image of just the interval from $\{0\} \times I$.

Suppose $T(\beta/\alpha)$ and $T(\beta'/\alpha')$ are fibered solid tori. A fiber preserving homeomorphism $f$ of $T(\beta/\alpha)$ and $T(\beta'/\alpha')$ is a homeomorphism $f : T(\beta/\alpha) \to T(\beta'/\alpha')$ that sends each fiber of $T(\beta/\alpha)$ onto a fiber of $T(\beta'/\alpha')$.

Definition 2.2.1 A Seifert manifold $M$ is a connected closed 3-manifold that can be decomposed into disjoint circles called fibers of $M$, such that for every fiber $h$ there exist a neighborhood $V_h$, and coprime integer numbers $\alpha \geq 1$ and $\beta$, and a fiber preserving homeomorphism $f : V_h \to T(\beta/\alpha)$ such that $f(h)$ is the core of $T(\beta/\alpha)$.

If $\alpha \geq 2$, the core of $V_h$ is called an exceptional fiber of multiplicity $\alpha$ of $M$, otherwise it is a regular fiber of $M$.

Note that by collapsing each fiber into a point we get a well-defined quotient $p : M \to F$, where $F$ is a closed surface of genus $g$; $F$ is orientable or non-orientable. This quotient is called the orbit quotient of $M$ or the orbit projection of $M$, and $F$ is called the orbit surface of $M$. Since each fiber $h$ in $M$ has a neighborhood $V_h$ homeomorphic to a fibered solid torus, one can show that $\text{int}(\{p(V_h)\})$ is a basis for the topology of $F$, where $\text{int}$ denotes the interior of a topological space. The image of a regular fiber is a regular point and the image of an exceptional fiber is an exceptional point.

Given a triangulation $T$ of $F$ it is possible to construct a system of neighborhoods of fibers
of $M$, where each neighborhood is homeomorphic to a fibered solid torus and projects onto a triangle of $F$. Also we can pick $T$, in such way, that every triangle contains at most one exceptional point. We will consider only triangulations of $F$ with this property.

Assume $F$ is triangulated by $T$. Let $x_1, y_1 \in F$ and suppose there is a triangle $T_1$ which misses exceptional points and such that $x_1, y_1 \in T_1$. Let $c_1 \subset T_1$ be a path joining $x_1$ and $y_1$. Let us fix an orientation of $p^{-1}(x_1)$. Since $p^{-1}(x)$ and $p^{-1}(y)$ are fibers of the fibered solid torus $p^{-1}(T_1)$, we can induce an orientation on the fiber $p^{-1}(y_1)$ by translating the fiber $p^{-1}(x)$ along the path $c_1$ and we say that $p^{-1}(y)$ has the orientation induced by $p^{-1}(x)$ along $c_1$.

In general, let $x, y \in F$ and suppose there is a path $c$, connecting $x$ with $y$, which misses exceptional points, we may assume, refining $T$, if necessary, that there exists a finite number of $s$ triangles $T_i$ without exceptional points, where $i = 1, \ldots, s$, such that $c \subset \bigcup_{i=1}^{s} T_i$. Let $V_i$ be the solid torus determined by $T_i$, for all $i = 1, \ldots, s$. Note that we can also suppose that the set $c_i = c \cap T_i$ does not contain the vertices of $T_i$. If $p^{-1}(x)$ has an orientation then we can induce an orientation on the fiber $p^{-1}(y)$ by translating the orientation of $p^{-1}(x)$, triangle by triangle, along the curves $c_i$. Then if $x = y$ and the fiber $p^{-1}(x)$ is oriented we can follow the induced orientation of $p^{-1}(x)$ along loops $c$ based at $x$. Thus we have a homomorphism $e : \pi_1(F) \to \mathbb{Z}_2$ such that $e(c) = +1$, if $c$ preserves the orientation of the fiber when the fiber is translated along $c$; otherwise, if $c$ reverses the orientation of the fiber, $e(c) = -1$. This homomorphism is called the valuation homomorphism. Of course, it is enough to define $e$ in a basis for $\pi_1(F)$ or $H_1(F)$.

Since $M$ is compact, the number of exceptional fibers in a Seifert manifold is finite.

Seifert manifolds were classified by H. Seifert [Se] according to a Seifert symbol and six classes, depending on the orientability of $F$, the valuation homomorphism and the multiplicities of exceptional fibers. In order to state the classification in classes of Seifert manifolds we fix the following facts and notation.
Let \( \{ h_i \}_{i=1}^r \) be a set of disjoint fibers of \( M \) which contains all the exceptional fibers and some regular fibers. By refining \( T \), if necessary, each fiber \( h_i \) has a neighborhood \( V_i \) fiber preserving homeomorphic to a fibered solid torus such that \( V_i \cap V_j = \emptyset \), if \( i \neq j \). We will always consider this neighborhoods \( V_i \)'s to be pairwise disjoint. Let \( T(\beta_i/\alpha_i) \) be the fibered solid torus homeomorphic to \( V_i \), for all \( i = 1, \ldots, r \). Recall that \( \alpha_i \) and \( \beta_i \) are coprime numbers and \( \alpha_i \geq 1 \).

We always assume \( \alpha_i \) be greater than or equal to 1 and coprime with \( \beta_i \).

We write \( M_0 = M - \bigcup V_i \). It is very important to remark that each fiber of \( M_0 \) is a regular fiber of \( M \). Note that we have a quotient \( p|: M_0 \to F_0 \), where \( F_0 \) is a surface with boundary. The boundary of \( F_0 \) has \( r \) components, one for each component of \( \partial M_0 \). Let \( q_1, \ldots, q_r \) be the components of \( \partial F_0 \) and \( h \) be a fiber of \( M_0 \) (i.e. a regular fiber of \( M \) different from \( h_i \), for all \( i \)). It is very important to note that \( e(q_i) = +1 \) since \( q_i \) bounds a disk in \( F \).

Now the list of classes of Seifert manifolds is the following (we use the notations of the previous paragraphs).

**\((Oo)\)** \( M \) is orientable, the orbit surface \( F \) is orientable of genus \( g \) and \( e \) is the trivial homomorphism.

**The Seifert symbol** associated to this manifold is

\[
M = (Oo, g; \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r).
\]

If \( \{ v_i \}_{i=1}^{2g} \) is a basis for \( \pi_1(F) \), presentations for the fundamental groups of \( M \) and \( M_0 \) are the following:

\[
\pi_1(M) \cong \langle v_1, \ldots, v_{2g}, q_1, \ldots, q_r, h; [h, v_j] = 1, [h, q_i] = 1, \\
q_1 q_2 \cdots q_r = \prod_{j=1}^{g} [v_{2j-1}, v_{2j}], q_i^{\alpha_i} h^{\beta_i} = 1 \rangle.
\]
\[ \pi_1(M_0) \cong \langle v_1, \ldots, v_{2g}, q_1, \ldots, q_r, h; [h, v_j] = 1, [h, q_1] = 1, q_1q_2 \cdots q_r = \prod_{j=1}^{g} [v_{2j-1}, v_{2j}] \rangle. \]

(On) \( M \) is orientable, the orbit surface \( F \) of \( M \) is non-orientable of genus \( g \) and if \( \{v_1, \ldots, v_g\} \) is a basis for \( \pi_1(F) \) such that each \( v_j \) is orientation reversing then \( e(v_j) = -1 \), for \( j = 1, \ldots, g \).

The Seifert symbol associated to this manifold is

\[ M = (On, g; \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r). \]

Presentations for the fundamental groups of \( M \) and \( M_0 \) are

\[ \pi_1(M) \cong \langle v_1, \ldots, v_g, q_1, \ldots, q_r, h; v_j h v_j^{-1} = h^{-1}, [h, q_i] = 1, q_1q_2 \cdots q_r = \prod_{j=1}^{g} q_j^2 q_i^\alpha h^\beta = 1 \rangle. \]

\[ \pi_1(M_0) \cong \langle v_1, \ldots, v_g, q_1, \ldots, q_r, h; v_j h v_j^{-1} = h^{-1}, [h, q_i] = 1, q_1q_2 \cdots q_r = \prod_{j=1}^{g} v_j^2 \rangle. \]

(No) \( M \) is non-orientable, the orbit surface \( F \) is orientable of genus \( g \) and if \( \{v_j\} \) is a basis for \( \pi_1(F) \) then \( e(v_1) = -1 \) and \( e(v_j) = +1 \), for \( j \geq 2 \).

The Seifert symbol associated to this manifold is

\[ M = (No, g; \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r). \]

Fundamental groups of \( M \) and \( M_0 \) are isomorphic to the following presentations:

\[ \pi_1(M) \cong \langle v_1, \ldots, v_{2g}, q_1, \ldots, q_s, h; q_1q_2 \cdots q_r = \prod_{j=1}^{g} [v_{2j-1}, v_{2j}], [h, q_i] = 1, q_i^\alpha h^\beta = 1, v_1 h v_1^{-1} = h^{-1}, [v_j, h] = 1 \text{ for } j \geq 2 \rangle. \]
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\[ \pi_1(M_0) \cong \langle v_1, \ldots, v_{2g}, q_1, \ldots, q_s, h; q_1q_2 \cdots q_r = \prod_{j=1}^{g} [v_{2j-1}, v_{2j}], \]
\[ [h, q_i] = 1, v_1 hv_1^{-1} = h^{-1}, [v_j, h] = 1 \text{ for } j \geq 2 \rangle. \]

(NnI) \( M \) is non-orientable, the orbit surface \( F \) is non-orientable of genus \( g \) and the valuation is trivial.

**The Seifert symbol** for this class is

\[ M = (NnI, g; \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r). \]

In this case, if \( \{v_j\} \) is a basis for \( \pi_1(F) \) of orientation reversing curves, then presentations for the fundamental groups of \( M \) and \( M_0 \) are

\[ \pi_1(M) \cong \langle v_1, \ldots, v_g, q_1, \ldots, q_r, h; [v_j, h] = 1, [h, q_i] = 1, \]
\[ q_1q_2 \cdots q_r = \prod_{j=1}^{g} v_j^2, q_i^\alpha h^\beta = 1 \rangle. \]

\[ \pi_1(M_0) \cong \langle v_1, \ldots, v_g, q_1, \ldots, q_r, h; [v_j, h] = 1, [h, q_i] = 1, \]
\[ q_1q_2 \cdots q_r = \prod_{j=1}^{g} v_j^2 \rangle. \]

(NnII) \( M \) is non-orientable, the orbit surface \( F \) is non-orientable of genus \( g \geq 2 \) and if \( \{v_j\} \) is a orientation reversing basis for \( \pi_1(F) \), then \( e(v_1) = +1 \) and \( e(v_j) = -1, \) for all \( j \geq 2. \)

**The Seifert symbol** associated to this Seifert manifold is

\[ M = (NnII, g; \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r), \]

and, in this case, presentations for the fundamental groups of \( M \) and \( M_0 \) are

\[ \pi_1(M) \cong \langle v_1, \ldots, v_g, q_1, \ldots, q_r, h; [h, q_i] = 1, q_1q_2 \cdots q_r = \prod_{j=1}^{g} v_j^2, \]
\[ q_i^\alpha h^\beta = 1, [v_1, h] = 1, v_j hv_j^{-1} = h^{-1}, \text{ for each } j \geq 2 \rangle. \]
π₁(M₀) ≅ ⟨v₁, ..., v₉, q₁, ..., q₉, h; [h, qᵢ] = 1, q₁q₂ · · · q₉ = \prod_{j=1}^{9} v_j^2, [v₁, h] = 1, v_jhv_j^{-1} = h^{-1}, \text{ for each } j \geq 2⟩.

(NnIII) M is non-orientable, the orbit surface F is non-orientable of genus g ≥ 3 and if \{v_j\} is a orientation reversing basis for π₁(F), then e(v₁) = e(v₂) = +1 and e(v_j) = −1, for each j ≥ 2.

The Seifert symbol associated to this manifold is

\[ M = (NnIII, g; β₁/α₁, ..., β_r/α_r). \]

The fundamental groups of M and M₀ have the following presentations:

\[
\pi₁(M) \cong ⟨v₁, ..., v₉, q₁, ..., q₉, h; [h, qᵢ] = 1, q₁q₂ · · · q₉ = \prod_{j=1}^{9} v_j^2, q_i^{α_i}h^{β_i} = 1, [v₁, h] = 1, [v₂, h] = 1, v_jhv_j^{-1} = h^{-1}, \text{ for each } j \geq 3⟩.
\]

\[
\pi₁(M₀) \cong ⟨v₁, ..., v₉, q₁, ..., q₉, h; [h, qᵢ] = 1, q₁q₂ · · · q₉ = \prod_{j=1}^{9} v_j^2, [v₁, h] = 1, [v₂, h] = 1, v_jhv_j^{-1} = h^{-1}, \text{ for each } j \geq 3⟩.
\]

The set \{h, qᵢ, v_j\} is called a standard system of generators of π₁(M) and of π₁(M₀).

The Seifert Classification Theorem is:

**Theorem 2.2.1** [Se] Two Seifert symbols represent homeomorphic Seifert manifolds by a fiber preserving homeomorphism if and only if one of the symbols can be changed into the other by a finite sequence of the following moves:

1. Permute the ratios.
2. Add or delete 0/1.
3. Replace the pair \(β_i/α_i, β_j/α_j\) by \((β_i + kα_i)/α_i, (β_j − kα_j)/α_j\).
Definition 2.2.2 The rational number $e_0(M) = \sum_{i=1}^{r} \beta_i / \alpha_i$ is called the Euler number of $M$.

2.3 Coverings of Seifert manifolds branched along fibers

Definition 2.3.1 If $M$ is a Seifert manifold and $\varphi : \tilde{M} \rightarrow M$ is a branched covering space of $M$, we say $\varphi$ is branched along fibers if the branch set of $\varphi$ is a finite union of fibers of $M$.

Let $\{h_i\}_{i=1}^{r}$ be a set of fibers of $M$ which contains all the exceptional fibers of $M$ and a finite number of regular fibers of $M$. Recall each fiber has a fibered neighborhood $V_i$ fiber preserving homeomorphic to a fibered solid torus $T(\beta_i/\alpha_i)$, for $i = 1, \ldots, r$. Recall $M_0 = \overline{M - \bigcup_i V_i}$. Note that $M_0$ is equal to $M$ with all the exceptional fibers and some regular fibers drilled out.

Remember also that $q_i = p(\partial V_i)$, where $p : M \rightarrow F$ is the orbit projection.

A covering of $M$ branched along fibers is determined by a representation $\omega : \pi_1(M - \bigcup_i h_i) \rightarrow S_n$ and therefore by a representation $\omega : \pi_1(M_0) \rightarrow S_n$.

To describe a covering of $M$ branched along fibers our procedure is as follows:

- Let $M$ be a Seifert manifold and consider the subspace $M_0$.

- Consider a representation $\omega : \pi_1(M_0) \rightarrow S_n$. This determines a finite covering space $\varphi_0 : \tilde{M}_0 \rightarrow M_0$.

- Let $T_i = q_i \times h$, where $h$ is a fiber of $M_0$. Let $f_i : \partial V_i \rightarrow T_i$ be the glueing homeomorphisms.

Using $\varphi_0$, lift the homeomorphisms $f_i : \partial V_i \rightarrow T_i$ to glueing homeomorphisms $\tilde{f}_i : \tilde{V}_i \rightarrow \tilde{T}_i$, where $\tilde{T}_i \subset \varphi^{-1}(T_i)$ is a component.

- In this way we obtain a covering $\varphi : \tilde{M} \rightarrow M$ of $M$ branched along fibers.
Lemma 2.3.1 Suppose $M$ is a Seifert manifold and $\omega : \pi_1(M_0) \to S_n$ is a transitive representation. Assume $\omega(h) \neq (1)$ and $\omega(h) = \sigma_1 \cdots \sigma_k$, is the disjoint cycle decomposition of $\omega(h)$. Then $\text{order}(\sigma_1) = \text{order}(\sigma_2) = \cdots = \text{order}(\sigma_k)$.

Proof.

Note that the subgroup generated by $h$, denoted by $\langle h \rangle$, is a normal subgroup of $\pi_1(M_0)$; thus $\langle \omega(h) \rangle$ is normal in $\text{Im}(\omega)$. Let $\sigma_1 = (a_{1,1}, \ldots, a_{1,m})$; then $A = \{a_{1,1}, \ldots, a_{1,m}\}$ is an orbit of $\langle \omega(h) \rangle$.

Let $a_{s,1} \in \{1, \ldots, n\}$. We assume that $a_{s,1}$ appears non-trivially in the orbit of the cycle $\sigma_s$. Since $\omega$ is transitive there is an $\alpha \in \pi_1(M_0)$ such that $\omega(\alpha)(a_{1,1}) = a_{s,1}$. Let us write $\omega(\alpha)(A) = \{a_{s,1}, \ldots, a_{s,m}\}$.

Also
\[
\langle \omega(h) \rangle \omega(\alpha)(A) = (\langle \omega(h) \rangle \omega(\alpha))(A) = (\omega(\alpha) \langle \omega(h) \rangle)(A) \text{ since } \langle \omega(h) \rangle \text{ is normal,} = \omega(\alpha)(\langle \omega(h) \rangle)(A) = \omega(\alpha)(A) \text{ since } A \text{ is an orbit of } \langle \omega(h) \rangle.
\]

Thus $\{a_{s,1}, \ldots, a_{s,m}\}$ is an orbit of $\langle \omega(h) \rangle$ and $\sigma_s = (a_{s,1} \cdots a_{s,m})$. □

By mean of Lemma 2.1.1 we can prove the following theorem which is our main tool to study coverings of a Seifert manifold.

Theorem 2.3.1 Let $M$ be a Seifert manifold and assume that $\varphi : \tilde{M} \to M$ is an $n$-fold covering branched along fibers of $M$. Assume $\tilde{M}$ is connected. Then there are coverings $\psi : \tilde{M} \to M'$ and $\zeta : M' \to M$ branched along fibers such that the following diagram is commutative.
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Also if $\omega_\psi$ and $\omega_\zeta$ are the representations associated to $\psi$ and $\zeta$, respectively, we have that $\omega_\psi(h') = \varepsilon_m$ and $\omega_\zeta(h) = (1)$, where $(1)$ is the identity permutation of $S_k$, $\varepsilon_m = (1,2,\ldots,m)$ is the standard $m$–cycle, and $h$ and $h'$ are regular fibers of $M$ and $M'$, respectively.

Proof.

Since $\tilde{M}$ is connected then $\omega_\varphi$, the representation determined by $\varphi$, is transitive. If $\omega(h) = \sigma_1 \cdots \sigma_k$ is the disjoint cycle decomposition of $\omega(h)$ in the proof of the previous lemma we also proved that each cycle $\sigma_s = (a_{s,1} \cdots a_{s,m})$ of $\omega(h)$ gives us a set of $m,k$–imprimitivity for $\text{Im}(\omega)$, namely, $\Delta_s = \{a_{s,1},\ldots,a_{s,m}\}$.

The quotient homomorphism $q : \text{Im}(\omega) \to S(\{\Delta_1,\ldots,\Delta_k\})$ satisfies that $q(\omega(h))(\Delta_i) = \Delta_i$. Therefore $q \circ \omega(h) = (\Delta_1)$, the identity permutation in $S(\{\Delta_1,\ldots,\Delta_k\})$.

Also $\omega(h) \in H_1 = q^{-1}(\text{St}(\Delta_1))$ and $\gamma_1 : H_1 \to S_m = S(\Delta_1)$ sends $h$ into an $m$–cycle. □

Therefore in order to understand the connected coverings of a Seifert manifold $M$ branched along fibers, we only need to study representations that send a regular fiber $h$ of $M$ into the identity permutation and representations that send a regular fiber $h$ of $M$ into an standard $n$–cycle.

2.3.1 The case $\omega(h) = (1)$, the identity permutation

If $M = (Xx,g;\beta_1/\alpha_1,\ldots,\beta_r/\alpha_r)$, where $Xx$ is a symbol in $\{Oo,On,No,NnI,NnII,NnIII\}$, we will write $M_0$ for the manifold obtained from $M$ by drilling out the fibers corresponding to...
the ratios $\beta_1/\alpha_1, \ldots, \beta_r/\alpha_r$. Recall that some ratios $\beta_k/\alpha_k$ could be regular fibers of $M$.

In this section the set $\{h, q_i, v_j\}$ is a standard system of generators of $\pi_1(M_0)$ and $\omega : \pi_1(M_0) \to S_n$ is a transitive representation such that
\[
\omega(h) = (1), \\
\omega(q_i) = \sigma_{i,1}\cdots\sigma_{i,\ell_i}, \text{ for } i = 1, \ldots, r \text{ and} \\
\omega(v_j) = \rho_{j,1}\cdots\rho_{j,s_j},
\]
where $\sigma_{i,1}\cdots\sigma_{i,\ell_i}$ and $\rho_{j,1}\cdots\rho_{j,s_j}$ are the disjoint cycle decompositions of $\omega(q_i)$ and $\omega(v_j)$, respectively.

Let $\tilde{M}_0 = \varphi^{-1}(M_0)$.

**Lemma 2.3.2** Suppose that $M$ is a Seifert manifold with orbit projection $p : M \to F$ and assume $n \in \mathbb{N}$. Let $\omega : \pi_1(M_0) \to S_n$ be a representation defined by
\[
\omega(h) = (1), \\
\omega(q_i) = \sigma_{i,1}\cdots\sigma_{i,\ell_i}, \text{ for } i = 1, \ldots, r \text{ and} \\
\omega(v_j) = \rho_{j,1}\cdots\rho_{j,s_j},
\]
where $\sigma_{i,1}\cdots\sigma_{i,\ell_i}$ and $\rho_{j,1}\cdots\rho_{j,s_j}$ are the disjoint cycle decompositions of $\omega(q_i)$ and $\omega(v_j)$, respectively.

Let $\varphi : \tilde{M} \to M$ be the branched covering associated to $\omega$ and let $\tilde{p} : \tilde{M} \to G$ be the orbit projection of $\tilde{M}$. Assume $\tilde{g}$ is the genus of $G$.

i) Suppose $F$ is non-orientable. If $G$ is orientable, then
\[
\tilde{g} = 1 - \frac{n(2 - g) + \sum_{i=1}^{r} \ell_i - nr}{2};
\]
otherwise,
\[
\tilde{g} = n(g - 2) + 2 + nr - \sum_{i=1}^{r} \ell_i.
\]
ii) If $F$ is orientable, then $\tilde{g} = 1 + n(g - 1) + \frac{nr - \sum_{i=1}^{r} \ell_i}{2}$.

Proof.

This is essentially the Riemann-Hurwitz formula. Let $F_0$ be the orbit surface of $M_0$ and $G_0$ be the orbit surface of $\tilde{M}_0 = \varphi^{-1}(M_0)$. Note that $G$, the orbit surface of $\tilde{M}$, is obtained by capping off the boundaries of $G_0$ with discs.

It is easy to see that $\varphi^{-1}(h)$ has $n$-components, $\tilde{h}_1, \ldots, \tilde{h}_n$. Thus if $\tilde{x}, \tilde{y} \in \tilde{h}_t$, for some $t \in \{1, \ldots, n\}$, we have $\tilde{p}(\tilde{x}) = \tilde{p}(\tilde{y})$ and $p(\varphi(\tilde{x})) = p(\varphi(\tilde{y}))$; by the Universal Property of Quotients we have a covering of $n$-sheets $\overline{\varphi} : G_0 \to F_0$ such that the following diagram is commutative:

\[
\begin{array}{ccc}
\tilde{M}_0 & \xrightarrow{\varphi} & M_0 \\
\downarrow \tilde{p} & & \downarrow p \\
G_0 & \xrightarrow{\overline{\varphi}} & F_0
\end{array}
\]

The representation $\overline{\varphi} : \pi_1(F_0) \to S_n$ associated to $\overline{\varphi}$ is defined as

\[
\overline{\varphi}(q_i) = \sigma_{i,1} \cdots \sigma_{i,\ell_i}, \text{ for } i = 1, \ldots, r \text{ and } \\
\overline{\varphi}(v_j) = \rho_{j,1} \cdots \rho_{j,s_j}, \text{ for } j = 1, \ldots, g.
\]

That is $\overline{\varphi} = \varphi|G_0$. Since $\omega$ is transitive and $\omega(h) = (1)$, then $\tilde{F}_0 = \varphi^{-1}(F_0)$ is connected. It is easy to see that $\tilde{F}_0$ is a horizontal surface, then $\tilde{p}_| : \tilde{F}_0 \to G_0$ is a covering. Also we know that $\varphi| : \tilde{F}_0 \to F_0$ is a covering of $n$ sheets.

Then there exists a commutative diagram

\[
\begin{array}{ccc}
\tilde{F}_0 & \xrightarrow{\tilde{p}} & G_0 \\
\downarrow \varphi| & & \downarrow \overline{\varphi} \\
F_0 & \xrightarrow{\varphi} & F_0
\end{array}
\]
Thus \( \tilde{F}_0 \cong G_0 \). Let \( \tilde{F} \) be the closed surface obtained by filling in the boundaries of \( \tilde{F}_0 \) with discs, then \( \tilde{F} \cong G \) and there exists a covering \( \varphi : G \to F \) of \( F \). We also called this covering \( \varphi \) since this extends the covering \( \varphi : G_0 \to F_0 \), that is \( \varphi|G_0 = \varphi|G_0 \).

Since \( \tilde{F}_0 \) is a covering of \( n \) sheets of \( F_0 \), then \( \chi(\tilde{F}_0) = n \chi(F_0) \). Since \( \omega(q_i) = \sigma_{i,1} \cdots \sigma_{i,s_i} \), therefore \( \varphi^{-1}(q_i) \) has \( \ell_i \) components; thus \( \partial \tilde{F}_0 \) has \( \sum_{i=1}^{r} \ell_i \) components for \( \partial F_0 = \sqcup q_i \). Hence

\[
\chi(\tilde{F}) = n \chi(F_0) + \sum_{i=1}^{r} \ell_i \quad (2.1)
\]

i) Suppose \( F \) is non-orientable; then \( \chi(F_0) = 2 - g - r \) and Equation (2.1) has the following form

\[
\chi(\tilde{F}) = n(2 - g - r) + \sum_{i=1}^{r} \ell_i.
\]

If \( G \) is orientable, then \( G \) has Euler characteristic equal to \( 2 - 2\tilde{g} \) and

\[
\tilde{g} = 1 - \frac{n(2 - g) + \sum_{i=1}^{r} \ell_i}{2}.
\]

If \( G \) is non-orientable, we know that \( \chi(G) = 2 - \tilde{g} \). Therefore,

\[
\tilde{g} = n(g - 2) + 2 + nr - \sum_{i=1}^{r} \ell_i.
\]

ii) When \( F \) is orientable, \( G \) is also orientable. Since \( \chi(F_0) = 2 - 2g - r \) and \( \chi(G) = 2 - 2\tilde{g} \), by (2.1) we conclude

\[
\tilde{g} = 1 + n(g - 1) + \frac{nr - \sum_{i=1}^{r} \ell_i}{2}.
\]

□

Since \( M_0 \) is an \( S^1 \)-bundle over \( F \) and \( \omega(h) = (1) \), then \( \tilde{M}_0 \) is the pullback of \( M_0 \) by \( \varphi : G_0 \to F_0 \) and the following lemma follows.

**Lemma 2.3.3** If \( M \) is a Seifert manifold and \( \omega : \pi_1(M_0) \to S_n \) is a representation defined by

\[
\omega(h) = (1),
\omega(q_i) = \sigma_{i,1} \cdots \sigma_{i,\ell_i}, \text{ for } i = 1, \ldots, r \text{ and }
\omega(v_j) = \rho_{j,1} \cdots \rho_{j,s_j},
\]

then the following holds.
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where \( \sigma_1, \ldots, \sigma_{i,t_i} \) and \( \rho_1, \ldots, \rho_{j,s_j} \) are the disjoint cycle decompositions of \( \omega(q_i) \) and \( \omega(v_j) \), respectively. Let \( \varphi : \tilde{M} \to M \) be the covering determined by \( \omega \).

Then \( \tilde{e} = \varphi^*(e) \), where \( e \) and \( \tilde{e} \) are the valuations of \( M \) and \( \tilde{M} \), respectively.

**Lemma 2.3.4** Let \( M \) be a non-orientable Seifert manifold. Let \( F \) and \( G \) be the orbit surfaces of \( M \) and \( \tilde{M} \), respectively. Consider the orbit projections \( \tilde{p} : \tilde{M} \to G \) and \( p : M \to F \). Suppose \( \varphi : G \to F \) is the induced covering of orbit surfaces. Let \( F_0 \) and \( G_0 \) be the orbit surfaces of \( M_0 \) and \( \tilde{M}_0 = \varphi^{-1}(M_0) \), respectively. Recall that \( \varphi|G_0 = \varphi|G_0 \).

If \( v \) is a simple closed curve in \( F_0 \) and if \( \tilde{v} \subset G_0 \) is the component of \( \varphi^{-1}(v) \) corresponding to the cycle \( \rho = (a_1, \ldots, a_t) \) of \( \omega(v) \), then:

(a) \( \varphi| : \tilde{p}^{-1}(\tilde{v}) \to p^{-1}(v) \) is a \( t \)-fold covering space, where \( t = \text{order}(\rho) \).

(b) If \( e(v) = +1 \), then \( \tilde{e}(\tilde{v}) = +1 \).

(c) Suppose that \( e(v) = -1 \). Then \( \tilde{e}(\tilde{v}) = +1 \) if and only if \( \text{order}(\rho) \) is even.

**Proof.**

Note that \( p^{-1}(v) \) and \( \tilde{p}^{-1}(\tilde{v}) \) are \( S^1 \)-bundles over \( v \) and \( \tilde{v} \), respectively.

(a) It is easy to see that \( \varphi(\tilde{p}^{-1}(\tilde{v})) = p^{-1}(v) \) because \( \overline{\varphi}(\tilde{v}) = v \) and the following diagram commutes.

\[
\begin{array}{ccc}
\tilde{M}_0 & \xrightarrow{\varphi} & M_0 \\
\downarrow{\tilde{p}} & & \downarrow{p} \\
G_0 & \xrightarrow{\varphi|} & F_0
\end{array}
\]

Thus \( \varphi| : \tilde{p}^{-1}(\tilde{v}) \to p^{-1}(v) \) is a covering space and the representation associated to this covering is \( \omega' : \pi_1(p^{-1}(v)) \to S_t = S(\{a_1, \ldots, a_t\}) \) defined by

\[
\omega'(h) = (1) \quad \text{and} \quad \omega'(v) = \rho.
\]
(b) Since $p^{-1}(v)$ and $\tilde{p}^{-1}(\tilde{v})$ are $S^1$-bundles over $v$ and $\tilde{v}$, respectively, $\varphi : \tilde{p}^{-1}(\tilde{v}) \to p^{-1}(v)$ is a covering, $\varphi(\tilde{v}) = v$ and $e(v) = +1$ then by Remark (2.1.1) we get $\tilde{e}(\tilde{v}) = +1$.

(c) Note that $t$ odd implies $\tilde{e}(\tilde{v}) = -1$ (Corollary 2.1.1). Thus $\tilde{e}(\tilde{v}) = +1$ only if $t$ is even.

On the other hand, suppose $t$ even and let $\rho = (1 \cdots t)$. Define $\Delta_1 = \{a_1, a_3, \ldots, a_{t-1}\}$ and $\Delta_2 = \{a_2, a_4, \ldots, a_t\}$, then $q : Im(\omega') \to S_2 = S(\{\Delta_1, \Delta_2\})$ sends $v$ into $(\Delta_1, \Delta_2)$ and we have $q \circ \omega = e$. Therefore $\tilde{e}$ is trivial and $\tilde{e}(\tilde{v}) = +1$ (See Remark 2.1.1) \qed

Lemma 2.3.5 Suppose that $X$ and $X'$ are $n$-manifolds with boundary. Let $Y$ and $Y'$ be connected $n-1$ sub-manifolds of $\partial X$ and $\partial X'$, respectively. If $f : Y \to Y'$ is a homeomorphism, then $Z = X \sqcup X'/f$ is orientable if and only if $X$ and $X'$ are orientable.

Proof. Assume $O_z$ is an orientation of $Z$. Then $O_z|X$ and $O_z|X'$ are orientations for $X$ and $X'$, respectively.

Now, suppose $O$ and $O'$ are orientations of $X$ and $X'$, respectively.

- If $f$ is orientation reversing, it is clear that $O \cup O'$ is an orientation of $Z$.
- Is $f$ is orientation preserving, then $O \cup (-O')$ is an orientation for $Z$. \qed

Suppose $M$ is a Seifert manifold with orbit projection $p : M \to F$. Let $\omega : \pi_1(M_0) \to S_n$ be a representation such that

\[
\begin{align*}
\omega(h) &= (1), \\
\omega(q_i) &= \sigma_{i,1} \cdots \sigma_{i,\ell_i}, \text{ for } i = 1, \ldots, r \text{ and } \\
\omega(v_j) &= \rho_{j,1} \cdots \rho_{j,s_j},
\end{align*}
\]

where $\sigma_{i,1} \cdots \sigma_{i,\ell_i}$ and $\rho_{j,1} \cdots \rho_{j,s_j}$ are the disjoint cycle decompositions of $\omega(q_i)$ and $\omega(v_j)$, respectively, and $M_0$ is the Seifert manifold $M$ with the exceptional fibers drilled out and without
some singular fibers that appear in the Seifert symbol.

Assume \( \varphi : \tilde{M} \to M \) is the covering of \( M \) branched along fibers associated to \( \omega \). Let \( \bar{p} : \tilde{M} \to G \) be the orbit projection of \( \tilde{M} \). Write \( F_0 = p(M_0) \) and note that a presentation for \( \pi_1(F_0) \) is \( \langle v_1, \ldots, v_{2g}, q_1, \ldots, q_r : - \rangle \). Let \( \tilde{M}_0 = \varphi^{-1}(M_0) \) and \( G_0 \) be the orbit surface of \( \tilde{M}_0 \). Note that by filling in with discs the boundaries of \( G_0 \) we obtain the surface \( G \). Recall that there is a covering \( \varphi : G \to F \) such that \( \varphi| : G_0 \to F_0 \) is a covering of \( F_0 \) and \( \varphi|G_0 = \varphi|G_0 \).

In order to determine what class of Seifert manifold \( \tilde{M} \) belong to, we analyze two cases: \( M \) orientable and \( M \) non-orientable. By Lemma (2.3.5), to see if \( \tilde{M} \) and \( G \) are orientable we only need to determine the orientability of \( \tilde{M}_0 = \varphi^{-1}(M_0) \) and \( G_0 \).

(a) **The case \( M \) orientable.**

Assume \( M = (Oo, g; \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r) \) is an orientable Seifert manifold and assume that the orbit surface \( F \) of \( M \) is orientable of genus \( g \). Recall also that \( \alpha \geq 1 \) and \( \beta_i \) are coprime numbers. The numbers \( \beta_i/\alpha_i \) in the Seifert symbol are defined by a fibered torus \( T(\beta_i/\alpha_i) \) which is a fibered neighborhood of some fiber \( h_i \) of \( M \). All the exceptional fibers are contained in the set \([h_i]_{i=1}^{r}\). Recall that \( M_0 = \tilde{M} - \sqcup T(\beta_i/\alpha_i) \). Note that \( \partial M_0 = \sqcup_{i=1}^{r} T_i \), where \( T_i \) is a torus for \( i = 1, \ldots, r \) and \( \sqcup_{i=1}^{r} T_i \) denotes the disjoint union of the tori \( T_i \). Let \( q_i = p(T_i) \), where \( p : M \to F \) is the orbit projection of \( M \).

If \( \{v_j\}_{j=1}^{2g} \) is a basis for \( \pi_1(F) \), a presentation for the fundamental groups of \( M \) and \( M_0 \) are

\[
\pi_1(M) \cong \langle v_1, \ldots, v_{2g}, q_1, \ldots, q_r, h; [h, v_j] = 1, [h, q_i] = 1, \\
q_1q_2\cdots q_r = \prod_{j=1}^{g} [v_{2j-1}, v_{2j}], q_i^{\alpha_i}h^\beta_i = 1 \rangle.
\]

\[
\pi_1(M_0) \cong \langle v_1, \ldots, v_{2g}, q_1, \ldots, q_r, h; [h, v_j] = 1, [h, q_i] = 1, \\
q_1q_2\cdots q_r = \prod_{j=1}^{g} [v_{2j-1}, v_{2j}] \rangle.
\]
Theorem 2.3.2 Suppose that \( M = (O_0, g; \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r) \) and \( \omega : \pi_1(M_0) \to S_n \) is a transitive representation defined by

\[
\begin{align*}
\omega(h) &= (1), \\
\omega(q_i) &= \sigma_{i,1} \cdots \sigma_{i,\ell_i}, \text{ for } i = 1, \ldots, r \quad \text{and} \\
\omega(v_j) &= \rho_{j,1} \cdots \rho_{j,s_j}, \text{ for } j = 1, \ldots, 2g;
\end{align*}
\]

where \( \sigma_{i,1} \cdots \sigma_{i,\ell_i} \) and \( \rho_{j,1} \cdots \rho_{j,s_j} \) are the disjoint cycle decompositions of \( \omega(q_i) \) and \( \omega(v_j) \), respectively, and \( \{h, q_i, v_j\} \) is a standard system of generators of \( M_0 \). Assume that \( \varphi : \tilde{M} \to M \) is the covering branched along fibers associated to \( \omega \) and \( \tilde{\varphi} : \tilde{M} \to G \) is the orbit projection of \( \tilde{M} \).

Then \( \tilde{M} \in O_0 \), that is, \( M \) is orientable and \( G \) is orientable.

Proof.

Since \( M \) and \( F \) are orientable, then \( M_0 \) and \( F_0 \) are orientable. Thus the first Stiefel-Whitney classes of \( M_0 \) and \( F_0 \), \( w_1(M_0) \) and \( w_1(F_0) \), respectively, are trivial. Recall we have coverings \( \varphi| : \tilde{M}_0 \to M \) and \( \overline{\varphi}| : G_0 \to F_0 \), where \( \tilde{M}_0 = \varphi^{-1}(M_0) \) and \( G_0 \) is the orbit surface of \( \tilde{M}_0 \). Then \( \tilde{M}_0 \) and \( G_0 \) are orientable since \( w_1(M_0) \) and \( w_1(G_0) \) are trivial (Remark 2.1.1). Therefore \( \tilde{M} \) is orientable and \( G \) is orientable. \( \square \)

Let \( M = (O_n, g; \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r) \) be a Seifert manifold: \( M \) is orientable and the orbit surface \( F \) of \( M \) is non-orientable of genus \( g \). Again the numbers \( \beta_i/\alpha_i \) in the Seifert symbol are defined by a fibered torus \( T(\beta_i/\alpha_i) \) which is a neighborhood of some fiber \( h_i \) of \( M \). All exceptional fibers belong to the set \( \{h_i\}_{i=1}^r \). Consider the manifold with boundary \( M_0 = \overline{M} - \sqcup T(\beta_i/\alpha_i) \). Note that \( \partial M_0 = \sqcup_{i=1}^r T_i \), where \( T_i \) is a torus for \( i = 1, \ldots, r \). Let \( q_i = p(T_i) \), where \( p : M \to F \) is the orbit projection of \( M \).

If \( \{v_1, \ldots, v_g\} \) is a basis for \( \pi_1(F) \) such that each \( v_j \) is orientation reversing, then a presentation for the fundamental groups of \( M \) and \( M_0 \) are
Theorem 2.3.3 Let $M = (On, g; \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r)$. Suppose

$$\omega: \pi_1(M_0) \to S_n$$

is a representation such that

\[
\begin{align*}
\omega(h) &= (1), \\
\omega(q_i) &= \sigma_{i,1} \cdots \sigma_{i,\ell_i}, \text{ for } i = 1, \ldots, r \text{ and } \\
\omega(v_j) &= \rho_{j,1} \cdots \rho_{j,s_j}, \text{ for } j = 1, \ldots, g;
\end{align*}
\]

where $\sigma_{i,1} \cdots \sigma_{i,\ell_i}$ and $\rho_{j,1} \cdots \rho_{j,s_j}$ are the disjoint cycle decompositions of $\omega(q_i)$ and $\omega(v_j)$, respectively, and $\{h, q_i, v_j\}$ a standard system of generators of $\pi_1(M_0)$.

Assume $\varphi: \tilde{M} \to M$ is the covering of $M$ branched along fibers determined by $\omega$ and $\tilde{p}: \tilde{M} \to G$ is the orbit projection of $\tilde{M}$.

Then $\tilde{M} \in O_o$ ($\tilde{M}$ and $G$ are orientable) or $\tilde{M} \in On$ ($\tilde{M}$ is orientable and $G$ is non-orientable).

Also $\tilde{M} \in O_o$ if and only if $\omega|\pi_1(F_0)$ trivializes the bundle of $w_1(F_0)$, where $w_1(F_0)$ is the first Stiefel-Whitney class of $F_0$.

Proof.

Note that $M_0$ is orientable since $M$ is orientable. Then the first Stiefel-Whitney class of $M_0$, $w_1(M_0)$, is trivial. By Lemma 2.1.1, we have that the first Stiefel-Whitney class of $\tilde{M}_0 = \varphi^{-1}(M_0)$, $w_1(\tilde{M}_0)$, is trivial. Thus $\tilde{M}_0$ is orientable and we conclude $\tilde{M}$ is orientable.
We have only two classes of orientable Seifert manifolds, namely, $Oo$ and $On$. Therefore $\tilde{M} \in Oo$ or $\tilde{M} \in On$. By Corollary 2.1.2, the surface $G_0$ is orientable (and $\tilde{M} \in Oo$) if and only if $\omega|\pi_1(F_0)$ has sets of $\frac{n}{2}$ $2$–imprimitivity, $\Delta_1$ and $\Delta_2$, such that the quotient homomorphism $q: Im(\omega|\pi_1(F_0)) \to S_2$ satisfies that $q \circ \omega = w_1(F_0)$.

Example 2.3.1

Let $M = (On, 1; 1/2)$. Since $M \in On$, $M$ is orientable and the orbit surface of $M$, $F$, is non-orientable. The genus of $F$ is 1, that is, $F$ is a projective plane. Let $T(1/2)$ be the solid fibered torus homeomorphic (under a fiber preserving homeomorphism) to a neighborhood of the only exceptional fiber. The boundary of $M_0 = M - T(1/2)$ is a torus $T_1$. Let $q_1 = p(T_1)$, where $p: M \to F$ is the orbit projection of $M$. Let $v_1$ be the generator of $\pi_1(F)$ and let $h$ be a regular fiber of $M$.

Note that
\[
\pi_1(M_0) \cong \langle v_1, q_1, h : [h, q_1] = 1, v_1 h v_1^{-1} = h, q_1 = v_1^2 \rangle
\]
and
\[
\pi_1(M) \cong \langle v_1, q_1, h : [h, q_1] = 1, v_1 h v_1^{-1} = h^{-1}, q_1 = v_1^2, q_1^2 h = 1 \rangle
\]

- Consider the representation $\omega : \pi_1(M_0) \to S_2$ defined by
  \[
  \omega(h) = (1), \\
  \omega(q_1) = (1, 2) \text{ and } \\
  \omega(v_1) = (1).
  \]

Assume $\varphi : \tilde{M} \to M$ is the covering determined by $\omega$. Note that the only sets of $1, 2$–imprimitivity for $Im(\omega|\pi_1(F_0))$ are $\Delta_1 = \{1\}$ and $\Delta_2 = \{2\}$. It is clear that $q: Im(\omega|\pi_1(F_0)) \to S_2 = S(\{\Delta_1, \Delta_2\})$ holds the relation: $q(v_1) = (\Delta_1)$, the identity permutation in $S_2$. Thus $\tilde{M} \in On$ (Cf. Theorem 2.3.3).
• If we consider \( \omega : \pi_1(M_0) \to S_2 \) defined by

\[
\begin{align*}
\omega(h) &= (1), \\
\omega(q_1) &= (1,2) \text{ and} \\
\omega(v_1) &= (1,2),
\end{align*}
\]

then \( \tilde{M} \) is the 2-fold covering space of orientation and \( \tilde{M} \in Oo \) (Cf. Theorem 2.3.2).

(b) **The case \( M \) non-orientable.**

(i) The case \( M \in No \).

Assume \( M = (No,g; \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r) \). Recall that in this kind of Seifert manifolds \( M \) is non-orientable and the orbit surface \( F \) is orientable of genus \( g \); The numbers \( \beta_i/\alpha_i \) in the Seifert symbol are defined by a fibered torus \( T(\beta_\alpha) \) which is a fibered neighborhood of some fiber \( h_i \) of \( M \). The set of exceptional fibers is contained in the set \( \{ h_i \}_i \). Recall \( M_0 = \overline{M - \sqcup T(\beta_\alpha)} \). Note that \( \partial M_0 = \sqcup T_i \), where \( T_i \) is a torus for \( i = 1, \ldots, r \). Let \( q_i = p(T_i) \), where \( p : M \to F \) is the orbit projection of \( M \).

If \( h \) is a regular fiber and \( \{ v_j \}_i \) is a basis for \( \pi_1(F) \) then the valuation homomorphism \( e : \pi_1(M) \to S_n \) satisfies \( e(v_1) = -1 \) and \( e(v_j) = +1 \), for \( j \geq 2 \).

Fundamental groups of \( M \) and \( M_0 \) have the following presentations:

\[
\begin{align*}
\pi_1(M) &\cong \langle v_1, \ldots, v_{2g}, q_1, \ldots, q_s, h; q_1q_2 \cdots q_r = \prod_{j=1}^{g}[v_{2j-1}, v_{2j}], \\
&\quad [h, q_i] = 1, q_i^{\alpha_i} h^{\beta_i} = 1, v_1 h v_1^{-1} = h^{-1}, [v_j, h] = 1 \text{ for } j \geq 2 \rangle. \\
\pi_1(M_0) &\cong \langle v_1, \ldots, v_{2g}, q_1, \ldots, q_s, h; q_1q_2 \cdots q_r = \prod_{j=1}^{g}[v_{2j-1}, v_{2j}], \\
&\quad [h, q_i] = 1, v_1 h v_1^{-1} = h^{-1}, [v_j, h] = 1 \text{ for } j \geq 2 \rangle.
\end{align*}
\]
The orbit projection of $M_0$ is $p : M_0 \to F_0$, where $F_0 \subset F$ is a surface. If $e' : \pi_1(F_0) \to S_n$ is the valuation homomorphism in $M_0$ then $e' = i_\# \circ e$, where $e$ is the valuation homomorphism of $M$ and $i : M_0 \to M$ is the natural inclusion map.

**Theorem 2.3.4** Consider $M = (\mathbb{N}, g; \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r)$ and suppose \{v_1, \ldots, v_{2g}\} is a basis for the orbit surface $F$ of $M$. Assume that $\omega : \pi_1(M_0) \to S_n$ is a representation defined by

\[
\begin{align*}
\omega(h) &= (1), \\
\omega(q_i) &= \sigma_{i,1} \cdots \sigma_{i,\ell_i}, \text{ for } i = 1, \ldots, r \text{ and } \\
\omega(v_j) &= \rho_{j,1} \cdots \rho_{j,s_j}, \text{ for } j = 1, \ldots, 2g,
\end{align*}
\]

where $\sigma_{i,1} \cdots \sigma_{i,\ell_i}$ and $\rho_{j,1} \cdots \rho_{j,s_j}$ are the disjoint cycle decompositions of $\omega(q_i)$ and $\omega(v_j)$, respectively. Assume $\varphi : \tilde{M} \to M$ is the covering of $M$ branched along fibers determined by $\omega$ and $\tilde{p} : \tilde{M} \to G$ is the orbit projection of $\tilde{M}$. Let $e' : \pi_1(F_0) \to S_2$ be the valuation homomorphism of $M_0$.

Then $\tilde{M} \in Oo$ ($\tilde{M}$ and $G$ are orientable) or $\tilde{M} \in No$ ($\tilde{M}$ is non-orientable and $G$ is orientable). Furthermore $\tilde{M} \in Oo$ if and only if $\omega|_{\pi_1(F_0)}$ trivializes the bundle of $e'$.

**Proof.**

Recall $\tilde{M}_0 = \varphi^{-1}(M_0)$, $G_0 = G \cap \tilde{M}_0 = \varphi^{-1}(F_0)$. We have coverings $\varphi : \tilde{M}_0 \to M_0$ and $\varphi : G_0 \to F_0$. Since the first Stiefel-Whitney class of $F_0$, $w_1(F_0)$, is trivial then $w_1(G_0)$ is trivial (Remark 2.1.1). Therefore $\tilde{M} \in No$ or $\tilde{M} \in Oo$.

By Remark 1.2.1.(b), the valuation homomorphism $e : \pi_1(F) \to \mathbb{Z}_2 \cong S_2$ gives us a covering $\varphi_e : (F_e)_0 \to F_0$ of 2-sheets.

Let $e' : \pi_1(F_0) \to \mathbb{Z}_2 \cong S_2$ be the valuation homomorphism of $M_0$. According to Lemma 2.3.3 and Theorem 2.1.1, $e'$ is trivial if and only if $\omega|_{\pi_1(F_0)}$ trivializes the bundle of $e'$. In the class $No$ the valuation homomorphism is non-trivial. Therefore
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\[ \tilde{M} \in \text{Oo} \text{ if and only if } \omega|\pi_1(F_0) \text{ trivializes the bundle of } e'. \]

\[ \square \]

**Remark 2.3.1** Let \( M = (N_o, g; \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r) \) with orbit projection \( p : M \to F \). Suppose \( \{v_j\}_{j=1}^{2g} \) is a basis for \( \pi_1(F) \) and \( M_0 = M - \sqcup T(\beta_i/\alpha_i) \), where \( T(\beta_i/\alpha_i) \) is a fibered neighborhood of either a exceptional fiber or a regular fiber. Recall \( F_0 = F \cap M_0 \). Assume \( \varphi : M \to M \) is an \( n \)-fold covering of \( M \) branched along fibers, where \( \tilde{M} \) is connected. Let \( \omega : \pi_1(M_0) \to S_n \) be the transitive representation determined by \( \varphi \), and let \( h \) be a regular fiber of \( M \).

If \( \omega(h) = (1) \), the identity permutation in \( S_n \), a useful criterion to determine if \( \tilde{M} \in \text{No} \) or \( \tilde{M} \in \text{Oo} \) is the following:

1. If \( n \) is odd, then \( M \in \text{No} \)
2. If \( \omega(v_1) \) has a cycle of odd order then \( \tilde{M} \in \text{No} \)
3. If \( \text{Im}(\omega|\pi_1(F_0)) \) is not \( \frac{n}{2} \)-imprimitive then \( \tilde{M} \in \text{No} \).
4. If \( \text{Im}(\omega|\pi_1(F_0)) \) is \( \frac{n}{2} \)-imprimitive, then \( \tilde{M} \in \text{Oo} \) if and only if \( \omega|\pi_1(F_0) \) trivializes the bundle of \( e' \), where \( e' : \pi_1(F_0) \to \mathbb{Z}_2 \cong S_2 \) is the valuation homomorphism of \( M_0 \).

**Example 2.3.2**

Let \( M = (N_o, 1; 1/2) \). The manifold \( M \) is non-orientable and \( F \), the orbit surface of \( M \), is an orientable surface of genus 1. Note that \( M \) has exactly one exceptional fiber \( h' \). Then there exists a fibered neighborhood of \( h' \) homeomorphic to the solid fibered torus \( T(1/2) \). Consider \( M_0 = M - T(1/2) \) and \( \{v_1, v_2\} \) a basis for \( \pi_1(F) \). Note that \( \partial M_0 \) is a torus \( T_1 \). Let \( q_1 = p(T_1) \), where \( p : M \to F \) is the orbit projection of \( M \) and let \( h \) be a regular fiber of \( M \).

Presentations for the fundamental groups of \( M_0 \) and \( M \) are

\[ \pi_1(M_0) \cong \langle v_1, v_2, q_1, h : v_1h^{-1} = h^{-1}, [v_2, h] = 1, [h, q_1] = 1, q_1 = [v_1, v_2] \rangle \]
\( \pi_1(M_0) \cong \langle v_1, v_2, q_1, h : v_1 h v_1^{-1} = h^{-1}, [v_2, h] = 1, [h, q_1] = 1, q_1 = [v_1, v_2], q_1^2 h = 1 \rangle. \)

- Let \( \omega : \pi_1(M_0) \to S_4 \) be the representation defined by
  \[
  \omega(h) = (1), \quad \omega(v_1) = (1,2)(3,4), \quad \omega(v_2) = (1,3)(2,4), \text{ and } \omega(q_1) = (1).
  \]

Suppose \( \varphi : \tilde{M} \to M \) is the covering of \( M \) determined by \( \omega \).

Observe that \( \Delta_1 = \{1,3\} \) and \( \Delta_2 = \{2,4\} \) are sets of 2,2-imprimitivity for \( \text{Im} (\omega|\pi_1(F_0)) \) such that \( q : \text{Im}(\omega|\pi_1(F_0)) \to S(\{\Delta_1, \Delta_2\}) \) satisfies

\[
q(v_1) = (\Delta_1, \Delta_2) \quad q(v_2) = (\Delta_1), \text{ the identity permutation in } S(\{\Delta_1, \Delta_2\}), \text{ and } \quad q(q_1) = (\Delta_1).
\]

On the other hand,

\[
e(v_1) = (1,2) = -1 \quad e(v_2) = (1) = +1, \text{ and } \quad e(q_1) = (1) = +1.
\]

Therefore \( \tilde{M} \in Oo \) (Cf. Theorem 2.3.4).

- Suppose \( \omega : \pi_1(M_0) \to S_3 \) is the representation such that
  \[
  \omega(h) = (1), \quad \omega(v_1) = (1,2,3) \quad \omega(v_2) = (1,2,3) \text{ and } \omega(q_1) = (1).
  \]

Let \( \varphi : \tilde{M} \to M \) be the covering of \( M \) determined by \( \omega \). In this case \( \tilde{M} \in No \) because 3 is odd (Cf. Theorem 2.3.4).
(ii) The case $M \in NnI$. 
Suppose $M = (NnI, g; \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r)$. That is $M$ is non-orientable, the orbit surface $F$ is non-orientable of genus $g$ and the valuation is trivial. Consider $M_0 = M - T(\beta_i/\alpha_i)$, where $T(\beta_i/\alpha_i)$ is the solid fibered torus corresponding to the ratio $\beta_i/\alpha_i$. Note that $\partial M_0 = \sqcup_{i=1}^r T_i$, where $T_i$ is a torus for $i = 1, \ldots, r$. Let $F_0 = p(M_0)$ and $q_i = p(T_i)$, where $p : M \to F$ is the orbit projection of $M$. If $h$ is a regular fiber of $M$ and $\{v_j\}$ is a basis for $\pi_1(F)$ of orientation reversing curves, then presentations for the fundamental groups of $M$ and $M_0$ are:

$$\pi_1(M) \cong \langle v_1, \ldots, v_g, q_1, \ldots, q_r, h; [v_j, h] = 1, [h, q_i] = 1, q_1q_2 \cdots q_r = \prod_{j=1}^g v_j^2, q_i^{\alpha_i} h^{\beta_i} = 1 \rangle.$$ 

$$\pi_1(M_0) \cong \langle v_1, \ldots, v_g, q_1, \ldots, q_r, h; [v_j, h] = 1, [h, q_i] = 1, q_1q_2 \cdots q_r = \prod_{j=1}^g v_j^2 \rangle.$$ 

The valuation homomorphism of $M_0$, $e' : \pi_1(F_0) \to S_n$, also is trivial.

**Theorem 2.3.5** Let $M = (NnI, g; \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r)$ be a non-orientable Seifert manifold. Consider a representation $\omega : \pi_1(M_0) \to S_n$ defined by

$$\omega(h) = (1),$$
$$\omega(q_i) = \sigma_{i,1} \cdots \sigma_{i,\ell_i}, \text{ for } i = 1, \ldots, r \text{ and }$$
$$\omega(v_j) = \rho_{j,1} \cdots \rho_{j,s_j},$$

where $\sigma_{i,1} \cdots \sigma_{i,\ell_i}$ and $\rho_{j,1} \cdots \rho_{j,s_j}$ are the disjoint cycle decompositions of $\omega(q_i)$ and $\omega(v_j)$, respectively. Suppose $\varphi : \tilde{M} \to M$ is the covering associated to $\omega$. Let $\tilde{M} \to G$ be the orbit projection of $\tilde{M}$.

Then $\tilde{M} \in Oo$ or $\tilde{M} \in NnI$. Moreover, $\tilde{M} \in Oo$ if and only if $\omega|_{\pi_1(F_0)}$ trivializes the bundle of $w_1(F_0)$, where $w_1(F_0)$ is the first Stiefel-Whitney class of $F_0$. 
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Proof.
Recall \( \tilde{M} = \varphi^{-1}(M_0) \) and \( G_0 = \varphi^{-1}(F_0) \). Let \( \tilde{e} : \pi_1(G_0) \to S_2 \) be the valuation homomorphism of \( M_0 \). Since \( e \) is trivial we have \( \tilde{e} \) trivial by Lemma 2.3.3 and Remark 2.1.1. There are only two classes of Seifert manifolds having trivial valuation homomorphism, namely, \( \tilde{M} \in Oo \) or \( \tilde{M} \in NnI \). Therefore \( \tilde{M} \in Oo \) or \( \tilde{M} \in NnI \).

Since \( \varphi : G \to F \) is a covering, by Corollary (2.1.2), \( G_0 \) is orientable if and only if there are sets of \( \frac{n}{2} \), 2–imprimitivity, \( \Delta_1 \) and \( \Delta_2 \), such that \( q \circ (\omega|\pi_1(F_0)) = w_1(F_0) \). Therefore \( \tilde{M} \in Oo \) if and only if there are sets of \( \frac{n}{2} \), 2–imprimitivity, \( \Delta_1 \) and \( \Delta_2 \), such that \( q \circ (\omega|\pi_1(F_0)) = w_1(F_0) \). □

Example 2.3.3

Consider \( M = (NnI, 1; 1/2) \). Suppose \( p : M \to F \) is the orbit projection of \( M \). In this case, \( F \) is a non-orientable surface of genus 1. Note that \( M \) has exactly one exceptional fiber \( h' \). Then there exists a fibered neighborhood of \( h' \) homeomorphic to the solid fibered torus \( T(1/2) \). Consider \( M_0 = \overline{M - T(1/2)} \) and let \( \{v_1\} \) be a basis for \( \pi_1(F) \). Note that \( \partial M_0 \) is a torus \( T_1 \). Let \( F_0 = p(M_0) \) and \( q_1 = p(T_1) \), where \( p : M \to F \) is the orbit projection of \( M \) and let \( h \) be a regular fiber of \( M \).

Presentations for the fundamental groups of \( M_0 \) and \( M \) are the following:

\[
\pi_1(M_0) \cong \langle v_1, q_1, h : [v_1, h] = 1, [q_1, h] = 1, q_1 = v_1^2 \rangle
\]

and

\[
\pi_1(M_0) \cong \langle v_1, q_1, h : [v_1, h] = 1, [q_1, h] = 1, q_1 = v_1^2, q_1^2h = 1 \rangle.
\]

• Assume that \( \omega : \pi_1(M_0) \to S_3 \) is the representation such that

\[
\omega(h) = (1),
\omega(q_1) = (1, 3, 2) \quad \text{and} \quad \omega(v_1) = (1, 2, 3).
\]
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Let \( \varphi : \tilde{M} \to M \) be the covering determined by \( \omega \). Suppose \( G \) is the orbit surface of \( \tilde{M} \). Then \( G \) is non-orientable because \( n \) is odd. Therefore \( \tilde{M} \in NnI \) (Cf. Theorem 2.3.5)

\[
\begin{align*}
\bullet \quad & \text{If } \omega : \pi_1(M_0) \to S_4 \text{ is a representation defined by} \\
& \omega(h) = (1), \\
& \omega(q_1) = (1, 3)(2, 4) \text{ and} \\
& \omega(v_1) = (1, 2, 3, 4). \\
\end{align*}
\]

Suppose \( \varphi : \tilde{M} \to M \) be the covering associated to \( \omega \) and \( G \) is the orbit surface of \( \tilde{M} \).

Then \( \Delta_1 = \{1, 3\} \) and \( \Delta_2 = \{2, 4\} \) are sets of \( 2, 2 \)–imprimitivity for \( Im(\omega|\pi_1(F_0)) \), such that \( q(v_1) = (\Delta_1, \Delta_2) \) and \( q(q_1) = (\Delta_1) \), the identity permutation in \( S(\{\Delta_1, \Delta_2\}) \). Of course, \( w_1(F_0)(v_1) = (1, 2) \) and \( w_1(F_0)(q_1) = (1) \). Therefore \( \tilde{M} \in Oo \) (Cf. Theorem 2.3.5).

(iii) The case \( M \in NnII \).

Suppose \( M = (NnII, g; \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r) \) and \( p : M \to F \) is the orbit projection. Since \( M \in NnII \) then \( F \) is non-orientable. Assume that the genus of \( F \) is \( g \). Write \( M_0 = \tilde{M} - T(\beta_i/\alpha_i) \), where \( T(\beta_i/\alpha_i) \) is the solid fibered torus homeomorphic to a neighborhood of either a exceptional fiber or a singular fiber. Then \( \partial M_0 = \bigsqcup_{i=1}^r T_i \), where \( T_i \) is a torus for \( i = 1, \ldots, r \). Let \( F_0 = p(M_0) \) and \( q_i = p(T_i) \). If \( h \) is a regular fiber of \( M \) and \( \{v_j\}_{j=1}^g \) is a basis for \( \pi_1(F) \) of orientation reversing curves, then presentations for the fundamental groups of \( M \) and \( M_0 \) are:

\[
\pi_1(M) \cong \langle v_1, \ldots, v_g, q_1, \ldots, q_r, h; [h, q_i] = 1, q_1q_2\cdots q_r = \prod_{j=1}^g v_j^2, \quad q_i^a h^\beta_i = 1, [v_1, h] = 1, v_j hv_j^{-1} = h^{-1}, \text{ for each } j \geq 2 \rangle.
\]
\[ \pi_1(M_0) \cong \langle v_1, \ldots, v_g, q_1, \ldots, q_r, h; [h, q_i] = 1, q_1 q_2 \cdots q_r = \prod_{j=1}^{g} v_j^2, \]

\[ [v_1, h] = 1, v_j h v_j^{-1} = h^{-1}, \text{ for each } j \geq 2 \rangle. \]

**Lemma 2.3.6** Suppose that \( M = (NnII, g; \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r) \) and \( \omega : \pi_1(M_0) \to S_n \) is a representation such that

\[
\begin{align*}
\omega(h) &= (1), \\
\omega(q_i) &= \sigma_{i,1} \cdots \sigma_{i,\ell_i}, \text{ for } i = 1, \ldots, r \text{ and} \\
\omega(v_j) &= \rho_{j,1} \cdots \rho_{j,s_j}, \text{ for } j = 1, \ldots, g,
\end{align*}
\]

where \( \sigma_{i,1} \cdots \sigma_{i,\ell_i} \) and \( \rho_{j,1} \cdots \rho_{j,s_j} \) are the disjoint cycle decompositions of \( \omega(q_i) \) and \( \omega(v_j) \), respectively. Let \( \varphi : \tilde{M} \to M \) be the covering associated to \( \omega \) and let \( \tilde{p} : \tilde{M} \to G \) be the orbit projection of \( \tilde{M} \). Assume the valuation homomorphism \( e : \pi_1(F) \to \mathbb{Z}_2 \cong S_2 \) is non-trivial and \( \tilde{M} \) is non-orientable (i.e. \( M \in NnII \) or \( M \in NnIII \)).

1. If the number of cycles of \( \omega(v_1) \) having odd order is odd, then \( M \in NnII \).
2. If the number of cycles of \( \omega(v_1) \) having odd order is even, then \( M \in NnIII \).

**Proof.**

Note that \( v_1 \) is an orientation reversing curve in \( M_0 \) because \( v_1 \) is orientation reversing in \( F_0 \) and \( e(v_1) = +1 \). Then \( p^{-1}(v_1) \) is a 2-sided vertical torus \( T^2 \). Let \( \mathcal{N}(p^{-1}(v_1)) \) be an open regular neighborhood of \( p^{-1}(v_1) \). Then \( M - \mathcal{N}(p^{-1}(v_1)) \) is orientable for \( v_2, \ldots, v_g, q_1, \ldots, q_r \) and \( h \) are orientation preserving curves in \( M_0 \).

Let \( \tilde{v}_{1,j} \) be the components of \( \varphi^{-1}(v_1) \) corresponding to \( \rho_{1,j} \). Then \( \varphi^{-1}(T^2) = \sqcup_{j=1}^{s_1} (\tilde{v}_{1,j} \times S^1) \).

Suppose \( \mathcal{N}(\sqcup (\tilde{v}_{1,j} \times S^1)) \) is an open regular neighborhood of \( \sqcup (\tilde{v}_{1,j} \times S^1) \). It is clear that \( \tilde{M} - \mathcal{N}(\sqcup (\tilde{v}_{1,j} \times S^1)) \) is orientable because \( T^2 \) is a Stiefel-Whitney surface for \( M_0 \) (Theorem 1.3.2).
Let $PD : H^1(M, \mathbb{Z}_2) \to H_2(M, \mathbb{Z}_2)$ denote the Poincaré duality isomorphism associated to $M$.

Since $\varphi^*(w_1(M_0)) = w_1(\tilde{M}_0)$ then

$$PDw_1(\tilde{M}_0) = [\varphi^{-1}(T^2)]$$

$$= \bigcup_{j=1}^{s_1} (\tilde{v}_{1,j} \times S^1)$$

$$= [\tilde{v}_{1,1} \times S^1] + [\tilde{v}_{1,2} \times S^1] + \cdots + [\tilde{v}_{1,s_1} \times S^1],$$

where possibly some classes $[\tilde{v}_j \times S^1]$ are trivial. Since the cycles $\rho_{1,j}$ are disjoint and the homology groups are abelian, without loss of generality, we may assume that there is a $k \in \{1, \ldots, s_1\}$, such that $[T_j]$ is trivial for all $k < j \leq s_1$. Thus

$$PDw_1(\tilde{M}) = [\tilde{v}_{1,1} \times S^1] + [\tilde{v}_{1,2} \times S^1] + \cdots + [\tilde{v}_{1,k} \times S^1].$$

Of course, if $\rho_{1,j}$ has odd order then $1 \leq j \leq k$ since $\tilde{v}_{1,j}$ is the core of a Moebius strip contained in $G_0$ and this is a non-separating curve in $G_0$; consequently $\tilde{p}^{-1}(\tilde{v}_{1,j}) = \tilde{v}_{1,j} \times S^1$ is a non-separating surface in $\tilde{M}_0$ and the class $[\tilde{p}^{-1}(\tilde{v}_{1,j})]$ is non-trivial in $H_2(\tilde{M}_0)$.

Let $\tilde{v}$ be a simple closed curve in $G_0$ homologous to $\tilde{v}_{1,1} + \cdots + \tilde{v}_{1,k}$ and note that $PDw_1(\tilde{M}_0) = [\tilde{v} \times S^1]$; it means $\tilde{v} \times S^1$ is a Stiefel-Whitney surface for $\tilde{M}_0$ and for $\tilde{M}$. Thus $\tilde{v} \times S^1$ is a vertical torus which is a Stiefel-Whitney surface. Of course, $\tilde{v} \times S^1$ is one-sided in $M_0$ and $M$ if and only if $\tilde{v}$ is one sided in $F_0$. By Theorem (1.3.3), if the number of cycles of $\omega(v_1)$ having odd order is odd then $\tilde{M} \in N_{nII}$; Otherwise, $\tilde{M} \in N_{nIII}$. □

**Theorem 2.3.6** Assume that $M = (N_{nII}, g; \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r)$ and $n \in \mathbb{N}$. Consider a representation $\omega : \pi_1(M_0) \to S_n$ such that

$$\omega(h) = (1),$$

$$\omega(q_i) = \sigma_{i,1} \cdots \sigma_{i,\ell_i}, \text{ for } i = 1, \ldots, r \text{ and}$$

$$\omega(v_j) = \rho_{j,1} \cdots \rho_{j,s_j}, \text{ for } j = 1, \ldots, g,$$

where $\sigma_{i,1} \cdots \sigma_{i,\ell_i}$ and $\rho_{j,1} \cdots \rho_{j,s_j}$ are the disjoint cycle decompositions of $\omega(q_i)$ and $\omega(v_j)$, respectively. Let $\varphi : \tilde{M} \to M$ be the covering associated to $\omega$ and let $\tilde{p} : \tilde{M} \to \tilde{M}_0$. 


Let \( G \) be the orbit projection of \( \tilde{M} \). Let \( e' : \pi_1(F_0) \to S_n \) be the valuation homomorphism of \( M_0 \).

(a) Suppose that \( n \) is an odd number.

(1) If \( \omega(v_1) \) has an odd number of cycles of odd order, then \( \tilde{M} \in N_{nII} \).

(2) If \( \omega(v_1) \) has an even number of cycles of odd order, then \( \tilde{M} \in N_{nIII} \).

(b) Assume that \( n \) is an even number and that there exists \( v_j \), such that \( \omega(v_j) \) has at least a cycle of odd order.

(1) Suppose that the number of cycles of \( \omega(v_1) \) having odd order is a non-zero even number.

   If there exists \( k \neq 1 \) such that \( \omega(v_k) \) has a cycle of odd order then \( \tilde{M} \in N_{nIII} \).

   Otherwise, if for \( k \neq 1 \) each cycle of \( \omega(v_k) \) has even order, then \( \tilde{M} \in N_{nI} \) or \( \tilde{M} \in N_{nIII} \).

   Moreover \( \tilde{M} \in N_{nI} \) if and only if \( \omega|_{\pi_1(F_0)} \) trivializes the bundle of \( e' \).

(2) If every cycle of \( \omega(v_1) \) has even order, then \( \tilde{M} \in On \) or \( \tilde{M} \in N_{nIII} \).

Furthermore, \( \tilde{M} \in On \) if and only if \( \omega \) trivializes the bundle of \( w_1(M_0) \), where \( w_1(M_0) \) is the first Stiefel-Whitney class of \( M_0 \).

(c) If \( n \) is an even number and every cycle of \( \omega(v_j) \) has even order, for \( j = 1, \ldots, g \), then \( \tilde{M} \notin N_{nII} \). In this case it is possible \( \tilde{M} \in O_o \), or \( \tilde{M} \in On \), or \( \tilde{M} \in N_o \), or \( \tilde{M} \in N_{nI} \) or \( \tilde{M} \in N_{nIII} \).

Proof.

Suppose \( \{v_j\} \) is a basis of orientation reversing curves for \( \pi_1(F) \). The valuation homomorphism \( e : \pi_1(F) \to \mathbb{Z}_2 \cong S_2 \) is such that \( e(v_1) = +1 \) and \( e(v_j) = -1 \), for \( j \geq 2 \).

Recall we have \( e' : \pi_1(F_0) \to S_2 \), the valuation homomorphism of \( M_0 \), and \( w_1(F_0) : \pi_1(F_0) \to S_2 \), the first Stiefel-Whitney class of \( F_0 \), and \( w_1(M_0) : \pi_1(M_0) \to S_2 \), the first Stiefel-Whitney class of \( M_0 \). Let \( \tilde{e} \) be the valuation homomorphism of \( \tilde{M} \).
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(a) If \( n \) is an odd number. Corollary 2.1.1 applied to \( w_1(M_0) \) and to \( w_1(F_0) \) give us that \( w_1(\tilde{M}_0) \) and \( w_1(G_0) \) are non-trivial, where \( \tilde{M}_0 = \varphi^{-1}(M_0) \) and \( G_0 = G \cap \tilde{M}_0 = \varphi^{-1}(F_0) \). Therefore \( \tilde{M}_0 \) and \( G_0 \) are non-orientable. Then \( \tilde{M} \) and \( G \) are non-orientable. Applying Theorem 2.1.1 to the valuation homomorphism \( e \), we obtain that \( \tilde{e} \), the valuation homomorphism of \( \tilde{M} \), is non-trivial. Therefore \( \tilde{M} \in NnII \) or \( \tilde{M} \in NnIII \); The result follows from Lemma 2.3.6.

(b) Recall \( \{v_j\} \) is a basis of reversing orientation curves for \( \pi_1(F) \).

Since \( n \) is an even number and there exists \( v_j \) such that \( \omega(v_j) \) has at least one cycle of odd order, then the orbit surface \( G \) of \( \tilde{M} \) is non-orientable (Corollary 2.1.1).

(1) Note that \( \tilde{M} \) is non-orientable since Corollary (2.1.1) applied to \( \theta = w_1(M_0) \) gives us \( w_1(\tilde{M}_0) \) is non-trivial.

If there exists \( k \neq 1 \) such that \( v_k \) has a cycle of odd order, then the valuation homomorphism of \( \tilde{M} \), \( \tilde{e} \), is non-trivial by Corollary 2.1.1 applied to \( e \). Since the number of cycles of \( \omega(v_1) \) having odd order is even, by Lemma 2.3.6 we obtain \( \tilde{M} \in NnIII \).

If each cycle of \( \omega(v_k) \) has even order, for all \( k \neq 1 \), then \( \tilde{M} \in NnI \) or \( \tilde{M} \in NnIII \) and the result follows from Theorem (2.1.1).

(2) First note that \( G_0 \) is non-orientable and the valuation homomorphism of \( \tilde{M} \), \( \tilde{e} \), is non-trivial, by Corollary 2.1.2. Also, by Lemma 2.3.6, we conclude \( \tilde{M} \notin NnII \). Thus \( \tilde{M} \in On \) or \( \tilde{M} \in NnIII \). We can decide if \( \tilde{M} \in On \) applying Theorem (2.1.1) to \( \theta = w_1(M_0) \) as required.

(c) If \( n \) is an even number and every cycle of \( \omega(v_j) \) has even order, for all \( j = 1, \ldots, g \), then we have the following cases:

If \( \text{Im}(\omega|\pi_1(M_0)) \) and \( \text{Im}(\omega|\pi_1(F_0)) \) are not \( \frac{n}{2} \)-imprimitive, then \( w_1(\tilde{F}_0) \), \( w_1(\tilde{M}_0) \) and \( \tilde{e} \) are non-trivial by Theorem (2.1.1) applied to \( e \), to \( w_1(M_0) \) and
to $w_1(F_0)$. Therefore $\tilde{M}$ and $G$ are non-trivial. Since every cycle of $\omega(v_1)$ has even order and $e$ is non-trivial then $\tilde{M} \in NnIII$ by Lemma 2.3.6.

Assume $\text{Im}(\omega|\pi_1(M_0))$ is $\frac{n}{2},2$–imprimitive. If $w_1(\tilde{M}_0)$ is trivial we have that $\tilde{M} \in Oo$ or $\tilde{M} \in On$. If $w_1(\tilde{M}_0)$ is non-trivial, then $\tilde{M} \in No$, or $\tilde{M} \in NnI$, or $\tilde{M} \in NnIII$. Note that $\tilde{M} \notin NnII$ due to Lemma 2.3.6. □

(iv) The case $M \in NnIII$.

Let $M = (NnIII, g; \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r)$ and let $F$ be the non-orientable orbit surface of $M$. Assume that the genus of $F$ is $g$. Consider $M_0 = \overline{M - T(\beta_i/\alpha_i)}$, where $T(\beta_i/\alpha_i)$ is the solid fibered torus homeomorphic to a neighborhood of either an exceptional fiber or a singular fiber. Notice that $\partial M_0 = \sqcup_{i=1}^r T_i$, where $T_i$ is a torus for $i = 1, \ldots, r$. Let $F_0 = p(M_0)$ and $q_i = p(T_i)$. Let $h$ be a regular fiber of $M$ and $\{v_j\}_{j=1}^g$ be a basis for $\pi_1(F)$ of orientation reversing curves.

The fundamental groups of $M$ and $M_0$ have the following presentations:

$$
\pi_1(M) \cong \langle v_1, \ldots, v_g, q_1, \ldots, q_r, h; [h, q_i] = 1, q_1q_2 \cdots q_r = \prod_{j=1}^g v_j^2, q_i^{q_i}h^{\beta_i} = 1, [v_1, h] = 1, [v_2, h] = 1, v_jh v_j^{-1} = h^{-1}, \text{for each } j \geq 3 \rangle.
$$

$$
\pi_1(M_0) \cong \langle v_1, \ldots, v_g, q_1, \ldots, q_r, h; [h, q_i] = 1, q_1q_2 \cdots q_r = \prod_{j=1}^g v_j^2, [v_1, h] = 1, [v_2, h] = 1, v_jh v_j^{-1} = h^{-1}, \text{for each } j \geq 3 \rangle.
$$

If $e : \pi_1(M) \to \mathbb{Z}_2$ is the valuation homomorphism of $M$, then $e(v_1) = e(v_2) = +1$ and $e(v_j) = -1$ for $j \geq 3$.

Recall $\beta : H^i(M, \mathbb{Z}_2) \to H^{i+1}(M, \mathbb{Z})$ is the Bockstein homomorphism associated to
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the short exact sequence of coefficients

\[ 0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_2 \to 0. \]

Suppose that \( M \in NnIII \) and consider a branched covering \( \varphi : \tilde{M} \to M \), then \( \beta w_1(\tilde{M}) = 0 \) for \( \beta w_1(M) = 0 \) and \( \beta \) is natural with respect to continuous functions \( (\varphi_* \beta = \beta \varphi_*) \). Thus \( \tilde{M} \in Oo \) or \( \tilde{M} \in On \) or \( \tilde{M} \in No \) or \( \tilde{M} \in NnI \) or \( \tilde{M} \in NnIII \) by Theorem 1.3.1 (and \( \tilde{M} \in NnII \)).

**Theorem 2.3.7** Suppose \( M \in NnIII \) with \( p : M \to F \), the orbit projection of \( M \). Let \( n \in \mathbb{N} \). Assume \( \{v_j\} \) is a basis of reversing orientation curves for \( \pi_1(F) \).

Let \( \omega : \pi_1(M_0) \to S_n \) be a representation defined by

\[
\begin{align*}
\omega(h) &= (1), \\
\omega(q_i) &= \sigma_{i, 1} \cdots \sigma_{i, \ell_i}, \text{ for } i = 1, \ldots, r \text{ and} \\
\omega(v_j) &= \rho_{j, 1} \cdots \rho_{j, s_j}, \text{ for } j = 1, \ldots, g,
\end{align*}
\]

where \( \sigma_{i, 1} \cdots \sigma_{i, \ell_i} \) and \( \rho_{j, 1} \cdots \rho_{j, s_j} \) are the disjoint cycle decompositions of \( \omega(q_i) \) and \( \omega(v_j) \), respectively. Suppose \( \varphi : \tilde{M} \to M \) is the covering determined by \( \omega \) and \( \tilde{p} : \tilde{M} \to G \) is the orbit projection of \( \tilde{M} \). Let \( e' : \pi_1(F_0) \to S_2 \) be the evaluation of \( M_0 \).

(a) If \( n \) is an odd number, then \( \tilde{M} \in NnIII \).

(b) Suppose that \( n \) is an even number and there exists \( v_j \) such that \( \omega(v_j) \) has at least one cycle of odd order.

(i) If each cycle of \( \omega(v_1) \) and \( \omega(v_2) \) has even order, then \( \tilde{M} \in On \) or \( \tilde{M} \in NnIII \). Also, \( \tilde{M} \in On \) if and only if \( \omega \) trivializes the bundle of \( w_1(M_0) \), where \( w_1(M_0) \) is the first Stiefel-Whitney class of \( M_0 \).

(ii) If \( \omega(v_1) \) or \( \omega(v_2) \) have a cycle of odd order, then \( \tilde{M} \in NnI \) or \( \tilde{M} \in NnIII \).

(c) If \( n \) is an even number and each cycle of \( \omega(v_j) \) has even order, for all \( j = 1, \ldots, g \), then \( \tilde{M} \in Oo \) or \( \tilde{M} \in No \) or \( \tilde{M} \in NnI \) or \( \tilde{M} \in NnIII \).
Theorem 2.1.1. Thus $\tilde{M}$ and $G$ are non-orientable. Thus $\tilde{M} \in N_nIII$ for $\tilde{e}$ is non-trivial and $\beta(w_1(M)) = 0$.

Proof.

Let $\tilde{e}$ be the valuation homomorphism of $\tilde{M}$.

(a) If $n$ is an odd number, then $w_1(G_0)$ and $w_1(\tilde{M}_0)$ are non-trivial by Corollary 2.1.2; the homomorphism $\tilde{e}$ is also non-trivial by Theorem 2.1.1. Thus $\tilde{M}$ and $G$ are non-orientable. Thus $\tilde{M} \in N_nIII$ for $\tilde{e}$ is non-trivial and $\beta(w_1(M)) = 0$.

(b) Since there is one $\omega(v_j)$ having a cycle of odd order, then $w_1(G_0)$ is non-trivial because of Corollary (2.1.2). Thus $G$ is non-orientable. Recall $e(v_1) = e(v_2) = +1$ and $e(v_k) = -1$, for $k \geq 3$.

(i) Since $v_j \neq v_1$ and $v_j \neq v_2$, then $\tilde{e}$ is non-trivial due to Corollary 2.1.1. Therefore $\tilde{M} \in On$ or $\tilde{M} \in N_nIII$. By Theorem 2.1.1 applied to $w_1(M_0)$ we can decide when $\tilde{M} \in On$ as stated.

(ii) Suppose that $\omega(v_1)$ or $\omega(v_2)$ have a cycle of odd order. Note that $v_1$ and $v_2$ are orientation reversing curves in $M_0$ since they are 1-sided in $F_0$ and $e(v_1) = e(v_2) = +1$. By Corollary 2.1.1, $w_1(\tilde{M}_0)$ is non-trivial and we conclude $\tilde{M}$ is non-orientable. Recall $G$ is non-orientable. Therefore $\tilde{M} \in N_nI$ or $\tilde{M} \in N_nIII$. Furthermore, $\tilde{M} \in N_nI$ if and only if $\omega|\pi_1(F_0)$ trivializes the bundle of $e'$.

(c) Assume $n$ is an even number and every cycle of $\omega(v_j)$ has even order for all $j = 1, \ldots, g$. Then we have the following cases:

- If $\text{Im}(\omega|\pi_1(F_0))$ is $\frac{n}{2}$ 2–imprimitive. Then
  1. Suppose $\omega|\pi_1(F_0))$ trivializes the bundle of $e'$. Then $\tilde{e}$ is trivial (Theorem 2.1.1). Thus, if $\omega|\pi_1(F_0)$ trivializes the bundle of $w_1(M_0)$ then $\tilde{M} \in OO$. Otherwise, $\tilde{M} \in N_nI$.
  2. Suppose $\omega|\pi_1(F_0))$ does not trivialize the bundle of $e'$. Then $\tilde{e}$ is non-trivial (Theorem 2.1.1). Therefore, if $\omega|\pi_1(F_0)$ trivializes the bundle of $w_1(F_0)$, then $w_1(G_0)$ and $w_1(G)$ are trivial (Theorem 2.1.1). Thus $G$ is orientable and we conclude $\tilde{M} \in No$; Otherwise, if $\omega$ does not trivialize
the bundle \( w_1(F_0) \), then \( \tilde{M} \in NnIII \) or \( \tilde{M} \in On \). Again we can decide if \( \tilde{M} \in On \) by means of Theorem 2.1.1 applied to \( w_1(M_0) \).

- If \( \text{Im}(\omega|\pi_1(F_0)) \) is not \( \frac{n}{2} \)-imprimitive, we proceed as before in (2). \( \square \)

To finish our study about representations of Seifert manifolds that send a regular fiber into the identity we prove the following Theorem which let us to compute the Seifert symbol for \( \tilde{M} \).

**Theorem 2.3.8** Let \( M = (Xx, g; \frac{\beta_1}{\alpha_1}, \ldots, \frac{\beta_r}{\alpha_r}) \) be a Seifert manifold with orbit projection \( p : M \to F \), where \( Xx \in \{Oo, On, No, NnI, NnII, NnIII\} \). Suppose that \( F \) is the orbit surface of \( M \) and let \( g \) be the genus of \( F \). Consider \( \{v_j\} \) a basis for \( \pi_1(F) \) such that every curve \( v_j \) is orientation reversing in \( F \), if \( F \) is non-orientable. Let \( h \) be a regular fiber of \( M \).

Write \( M_0 = M - \bigcup_{i=1}^{r} V_i \), where each \( V_i \) is a fibered neighborhood of the fiber corresponding to \( \beta_i/\alpha_i \), for \( i = 1, \ldots, r \). Note that \( \partial M_0 \) is the union of \( r \) tori, \( T_1 \sqcup \cdots \sqcup T_r \). Let \( q_i = p(T_i) \), for \( i = 1, \ldots, r \). Let \( n \in \mathbb{N} \) and \( \omega : \pi_1(M_0) \to S_n \) be a transitive representation defined by

\[
\begin{align*}
\omega(h) &= (1), \\
\omega(q_i) &= \sigma_{i,1} \cdots \sigma_{i,\ell_i}, \text{ for } i = 1, \ldots, r \text{ and } \\
\omega(v_j) &= \rho_{j,1} \cdots \rho_{j,s_j},
\end{align*}
\]

where \( \sigma_{i,1} \cdots \sigma_{i,\ell_i} \) and \( \rho_{j,1} \cdots \rho_{j,s_j} \) are the disjoint cycle decompositions of \( \omega(q_i) \) and \( \omega(v_j) \), respectively. Let \( \varphi : \tilde{M} \to M \) be the covering associated to \( \omega \). Let \( \tilde{p} : \tilde{M} \to G \) be the orbit projection of \( \tilde{M} \) and suppose that \( G \) has genus \( \tilde{g} \).

**a)** Suppose \( F \) is non-orientable, then \( \tilde{M} \) is the manifold

\[
(Yy, \tilde{g}; \frac{B_{1,1}}{A_{1,1}}, \ldots, \frac{B_{r,1}}{A_{r,1}}, \cdots, \frac{B_{r,\ell_r}}{A_{r,\ell_r}}),
\]

where \( Yy \in \{Oo, On, No, NnI, NnII, NnIII\} \) is determined by Theorems 2.3.3, 2.3.5, 2.3.6 and 2.3.7. If \( G \) is orientable, then

\[
\tilde{g} = 1 - \frac{n(2 - g) + \sum_{i=1}^{r} \ell_i - nr}{2};
\]

otherwise,

\[
\tilde{g} = n(g - 2) + 2 + nr - \sum_{i=1}^{r} \ell_i.
\]
b) If $F$ is orientable, then $\tilde{M}$ is the manifold

$$(Yy, \tilde{g}; B_{1,1}/A_{1,1}, \ldots, B_{1,\ell_1}/A_{1,\ell_1}, \ldots, B_{r,1}/A_{r,1}, \ldots, B_{r,\ell_r}/A_{r,\ell_r})$$

where $Yy \in \{Oo, No\}$ is determined by Theorems 2.3.2 and 2.3.4; and

$$\tilde{g} = 1 + n(g - 1) + \frac{nr - \sum_{i=1}^{r} \ell_i}{2}.$$

The numbers $B_{i,k}$ and $A_{i,k}$ in the Seifert symbol for $\tilde{M}$ in both (a) and (b) are given by:

$$B_{i,k} = \frac{\text{order}(\sigma_{i,k}) \cdot \beta_i}{\gcd\{\alpha_i, \text{order}(\sigma_{i,k})\}},$$

$$A_{i,k} = \frac{\alpha_i}{\gcd\{\alpha_i, \text{order}(\sigma_{i,k})\}},$$

where $\gcd\{\alpha_i, \text{order}(\sigma_{i,k})\}$ denotes the greatest common divisor of $\alpha_i$ and $\text{order}(\sigma_{i,k})$.

Proof.

The genus of $G$, $\tilde{g}$, is determined by Lemma 2.3.2 and the class $Yy$ is determined by Theorems 2.3.2, 2.3.3, 2.3.4, 2.3.5, 2.3.6 and 2.3.7.

Now we compute the numbers $B_{i,k}$ and $A_{i,k}$.

Recall that $G_0 = \varphi^{-1}(F_0)$ and also recall that we have a covering $\varphi : G_0 \to F_0$. The representation associated to $\varphi : G_0 \to F_0$ is $\omega : \pi_1(F_0) \to S_n$.

The manifold $M$ is obtained from $M_0$ by gluing a solid tori $U_i$ to $T_i\partial M_0$ with homeomorphisms $f_i : \partial U_i \to T_i$ such that $f_i(m_i) = q_i^m h^\beta$, where $m_i$ is a meridian of $\partial U_i$.

If $i \in \{1, \ldots, r\}$ and we consider the torus $T_i = q_i \times h$, then $\varphi^{-1}(T_i)$ has $\ell_i$ components for $\varphi : G_0 \to F_0$ is a covering and $\omega(q_i)$ is a product of $\ell_i$ cycles, in particular, $\varphi^{-1}(q_i)$ has $\ell_i$ components.

Let $T_{i,k}$ be a component of $\varphi^{-1}(T_i)$, for $k \in \{1, \ldots, \ell_i\}$. Note that $T_{i,k}$ is a torus and that $\varphi$ induces a covering $\varphi_{i,k} : T_{i,k} \to T_i$ with $\text{order}(\sigma_{i,k})$ sheets such that, if $\tilde{h}$ is a component
of \(\varphi^{-1}(h)\) and \(\tilde{q}_{i,k}\) is the pre-image of \(q_i\) in the torus \(T_{i,k}\), then \(\{\tilde{h}, \tilde{q}_{i,k}\}\) is a basis for \(\pi_1(T_{i,k})\) for \(\varphi|: G \to F\) is a covering. Note that \(\tilde{q}_{i,k}\) is the union of order \(\text{ord}(\sigma_{i,k})\) liftings of \(q_i\). Then \(\varphi_{i,k}(\tilde{h}) = h\) and \(\varphi_{i,k}(\tilde{q}_{i,k}) = q_i^{\text{ord}(\sigma_{i,k})}\). Since \(\{\tilde{h}, \tilde{q}_{i,k}\}\) is a basis for \(\pi_1(T_{i,k})\), if \(\tilde{m}_{i,k} \subset \varphi_{i,k}^{-1}(m_i)\) then there are \(A_{i,k}\) and \(B_{i,k}\) integer numbers such that
\[
\varphi_{i,k}(\tilde{m}_{i,k}) = \varphi_{i,k}(\tilde{q}_{i,k}^{A_{i,k}} h^{B_{i,k}}) = q_i^{\text{ord}(\sigma_{i,k})A_{i,k}} h^{B_{i,k}}.
\] (2.2)

On the other hand, associated to \(\varphi_{i,k}\) we have a representation \(\omega_{i,k}: T_i \to S_{\text{ord}(\sigma_{i,k})}\) such that \(\omega(h) = (1)\), the identity permutation in \(S_{\text{ord}(\sigma_{i,k})}\), and \(\omega(q_i) = \varepsilon_{\text{ord}(\sigma_{i,k})}\), the standard \(\text{ord}(\sigma_{i,k})\)-cycle in \(S_{\text{ord}(\sigma_{i,k})}\). Note that \(\omega_{i,k}\) satisfies that \(\omega_{i,k}(m_i) = \omega_{i,k}(q_\alpha h^\beta) = (\sigma_{i,k})^{\alpha_i}\).

This implies
\[
\varphi_{i,k}(\tilde{m}_{i,k}) = m_i^{\text{ord}(\sigma_{i,k})^{\alpha_i}} = (q_i^{\alpha_i \text{ord}(\sigma_{i,k})^{\alpha_i}})(h^{\beta_i \text{ord}(\sigma_{i,k})^{\alpha_i}}).
\] (2.3)

But in fact \(\text{ord}(\sigma_{i,k})^{\alpha_i} = \frac{\text{ord}(\sigma_{i,k})}{\gcd(\alpha_i, \text{ord}(\sigma_{i,k}))}\), hence by recalling Equations 2.2 and 2.3, we obtain
\[
B_{i,k} = \frac{\text{ord}(\sigma_{i,k}) \cdot \beta_i}{\gcd\{\alpha_i, \text{ord}(\sigma_{i,k})\}},
\]
and
\[
A_{i,k} = \frac{\alpha_i}{\gcd\{\alpha_i, \text{ord}(\sigma_{i,k})\}}
\]
for \(k = 1, \ldots, l_i\) and either \(i = 1, \ldots, g\), if \(F\) is non-orientable or \(i = 1, \ldots, 2g\), if \(F\) is orientable.

### 2.3.2 The case \(\omega(h) = \varepsilon_n\), the standard \(n\)-cycle

Suppose \(M\) is a Seifert manifold and \(h\) is a regular fiber of \(M\), in this section we focus in representations \(\omega: \pi_1(M_0) \to S_n\) such that \(\omega(h) = \varepsilon_n\), where \(\varepsilon_n\) is the standard \(n\)-cycle of \(S_n\).

**Definition 2.3.2** Let \(P\) be an \(n\)-sided regular polygon with vertices labeled with the numbers from 1 to \(n\). A reflection \(\rho\) in \(S_n\) is a permutation determined by a reflection of \(P\) restricted to the vertices of \(P\).
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Note that by definition a reflection $\rho$ has order 2.

We say that $\sigma \in S_n$ anticommutes with $\varepsilon_n$ if $\sigma \varepsilon_n \sigma^{-1} = \varepsilon_n^{-1}$.

**Lemma 2.3.7** Let $\sigma \in S_n$. Then $\sigma$ anticommutes with $\varepsilon_n$ if and only if $\sigma$ is a reflection.

**Proof.**

Let $P$ be a $n$–sided regular polygon and $\sigma \in S_n$ be a reflection. Note that $\varepsilon_n$ is induced by a rotation of $P$ through an angle $2\pi/n$; by inspections it is easy to see that $\sigma$ anticommutates with $\varepsilon_n$.

In a $n$–sided regular polygon $P$ we have $n$ reflections, then if $A = \{h \in S_n : h\varepsilon_n h^{-1} = \varepsilon_n^{-1}\}$ we have that $|A| \geq n$.

Now we prove $|A| = n$.

Suppose $\rho \in A$, then $\rho \varepsilon_n \rho^{-1} = \varepsilon_n^{-1}$. Let $\cdot : S_n \times S_n \to S_n$ be the group action defined by $g \cdot h = ghg^{-1}$. With this action the stabilizer of $\varepsilon_n$ is the subgroup $Stabilizer(\varepsilon_n) = \{g \in S_n : g \varepsilon_n g^{-1} = \varepsilon_n\}$. Consider $S_n/\text{Stabilizer}(\varepsilon_n) = \{g(\text{Stabilizer}(\varepsilon_n)) : g \in S_n\}$ and note that $r \in \rho(\text{Stabilizer}(\varepsilon_n))$ if and only if $r \varepsilon_n r^{-1} = \rho \varepsilon_n \rho^{-1}$. Thus $\sigma(\text{Stabilizer}(\varepsilon_n)) = \{r \in S_n | r \varepsilon_n r^{-1} = \varepsilon_n^{-1}\} = A$.

On the other hand, the orbit of $\varepsilon_n$ under this action is the set $O_{\varepsilon_n} = \{h \in S_n | h = g\varepsilon_n g^{-1} \text{ for some } g \in S_n\}$. Note that $O_{\varepsilon_n}$ is the set of $n$–cycles for the conjugates of an $n$-cycle have also order $n$. 

Figure 2.1: Reflections
We have a bijection $S_n/\text{Stabilizer}(\varepsilon_n) \rightarrow O_{\varepsilon_n}$. Then $n! = |S_n| = (|\text{Stabilizer}(\varepsilon_n)|)(|O_{\varepsilon_n}|)$. Since $|O_{\varepsilon_n}| = (n - 1)!$, we obtain $|\text{Stabilizer}(\varepsilon_n)| = n$.

Therefore $|A| = n$ because $|A| = |\rho(\text{Stabilizer}(\varepsilon_n))| = |\text{Stabilizer}(\varepsilon_n)| = n$. □

Lemma 2.3.8 Let $\sigma \in S_n$. Then $\sigma$ commutes with $\varepsilon_n$ if and only if there is $k \in \mathbb{Z}$ such that $\sigma = \varepsilon_n^k$.

Proof.

Consider again the group action $\cdot : S_n \times S_n \rightarrow S_n$ given by $g \cdot h = ghg^{-1}$. Recall from the proof of the previous lemma that $|\text{Stabilizer}(\varepsilon_n)| = n$. Since $\{(1), \varepsilon_n, \ldots, \varepsilon_n^{n-1}\} \subset \text{Stabilizer}(\varepsilon_n)$ we obtain $\text{Stabilizer}(\varepsilon_n) = \{(1), \varepsilon_n, \ldots, \varepsilon_n^{n-1}\}$. Therefore, $\sigma = \varepsilon_n^k$, for some $k \in \mathbb{Z}$. □

Lemma 2.3.9 (Torus Lemma)[N-RL] Let $T$ be a torus and let $h, q \subset T$ be a basis for $\pi_1(T)$. Let $n \in \mathbb{Z}$ and assume that $\omega : \pi_1(T) \rightarrow S_n$ is the representation such that

$$
\begin{align*}
\omega(h) &= \varepsilon_n, \\
\omega(q) &= \varepsilon_n^k,
\end{align*}
$$

where $\varepsilon_n = (1, 2, \ldots, n)$ is the standard $n$–cycle. Suppose that $\varphi : \tilde{T} \rightarrow T$ is the covering space defined by $\omega$. Then there exist a basis $\tilde{h}, \tilde{q} \subset \tilde{T}$ for $\pi_1(\tilde{T})$ such that $\varphi(\tilde{h}) = h^n$ and $\varphi(\tilde{q}) = qh^{-k}$.

Proof.

Cut $T$ along $h$ and $q$ to get the identification square $S$ shown in Figure 2.2.

The boundary of $S$ is the union of $h^+, h^-, q^+$ and $q^-$. If $S(1), \ldots, S(n)$ are $n$ copies of $S$ and the boundary of $S(i)$ is the union of $h(i)^+, h(i)^-, q(i)^+, q(i)^-$, we can construct $\tilde{T}$ by glueing $q(i)^+ \subset S(i)$ with $q(\varepsilon_n(i))^- \subset S(\varepsilon_n(i))$ and $h(i)^+$ with $h(\varepsilon_n(i))^-$.
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Figure 2.2: Square S

Figure 2.3: \( \tilde{T} \)

Suppose \( x \in h(1)^+ \) and let \( y \in h(k+1)^+ \) be the image of \( x \) under the identification. Let \( \tilde{h} = \varphi^{-1}(h) \) and \( \tilde{q} \) a shortest curve in \( S(1) \cup \cdots \cup S(n) \) connecting \( x \) and \( y \), as shown in Figure 2.3. Observe that \( \tilde{h} \cap \tilde{q} = \{x\} \), then it is clear that \( \tilde{h}, \tilde{q} \subset \tilde{T} \) is a basis for \( \pi_1(T) \). By construction \( \varphi(\tilde{h}) = h^n \) and \( \varphi(\tilde{q}) = qh^{-k} \).

\[ \square \]

Lemma 2.3.10 (Klein Bottle Lemma) Let \( K \) be a Klein bottle with \( \pi_1(K) = \langle h, v : vhv^{-1} = h^{-1} \rangle \). Consider a representation \( \omega : \pi_1(K) \rightarrow S_n \) such that \( \omega(h) = \varepsilon_n \), where \( \varepsilon_n = (1, 2, \ldots, n) \). Assume \( \varphi : \tilde{K} \rightarrow K \) is the covering associated to \( \omega \). Then \( \omega(v) \) is a reflection \( \rho \), the covering space \( \tilde{K} \) is also a Klein bottle and, if \( \rho(1) = t \), then there exists a basis \( \{\tilde{h}, \tilde{v}\} \) for \( \tilde{K} \) such that \( \varphi(\tilde{h}) = h^n \) and \( \varphi(\tilde{v}) = vh^{-(t-1)} \).

Proof.

Note that \( \omega(v)\varepsilon_n\omega(v)^{-1} = \varepsilon^{-1} \), for \( \omega(h) = \varepsilon_n \) and \( vhv^{-1} = h^{-1} \). By Lemma 2.3.7, \( \omega(v) \) is
a reflection $\rho$. The surface $\tilde{K}$ is a closed surface. Also $\chi(\tilde{K}) = n\chi(K) = 0$ for $\chi(K) = 0$, where $\chi(\tilde{K})$ and $\chi(K)$ are the Euler characteristic of $\tilde{K}$ and $K$, respectively. Thus $\tilde{K}$ could be either a Klein bottle or a torus.

To construct $\tilde{K}$, cut $K$ along $h$ and $v$ to get the identification square $S$ shown in Figure 2.4.

![Figure 2.4: Square S](image)

The boundary of $S$ is the union of $h^+, h^-, v^+$ and $v^-$. If $S(1), \ldots, S(n)$ are $n$ copies of $S$ and the boundary of $S(i)$ is the union of $h(i)^+, h(i)^-, v(i)^+, v(i)^-$, then $\tilde{K}$ is constructed by glueing $v(i)^+ \subset S(i)$ along $v(\varepsilon_n(i))^- \subset S(\varepsilon_n(i))$ and $h(i)^+$ with $h(\rho(i))^-$.

![Figure 2.5: $\tilde{T}$](image)

Suppose $x \in h(1)^+$ and let $y \in h(t)^-$ be the image of $x$ under the identification. Let $\tilde{h} = \varphi^{-1}(h)$ and $\tilde{v}$ be a shortest curve in $S(1) \cup \cdots \cup S(n)$ connecting $x$ and $y$, as shown in the Figure 2.5 Then $\varphi\#(\tilde{h}) = h^n$, $\varphi\#(\tilde{v}) = v h^{-(t-1)}$ by construction.
Notice that

\[ \varphi_\#(\tilde{v} \tilde{h} \tilde{v}^{-1} \tilde{h}) = \varphi_\#(\tilde{v}) \varphi_\#(\tilde{h}) \varphi_\#(\tilde{v}^{-1}) \varphi_\#(\tilde{h}) \]
\[ = (vh^{-1})^{n} h^{n} \]
\[ = vh^{n}v^{-1}h^{n} \]
\[ = \underbrace{vhv^{-1}vhv^{-1} \cdots vhv^{-1}}_{n\text{-times}} h^{n} \]
\[ = h^{-n}h^{n} \text{ (because of the relation } v_{j}hv-j^{-1}=h^{-1}) \]
\[ = 1. \]

Thus \( \tilde{v} \tilde{h} \tilde{v}^{-1} = \tilde{h}^{-1} \) for \( \varphi_\# \) is injective.

Observe that \( \tilde{h} \) intersects transversally \( \tilde{v} \) only in one single point, thus \( \tilde{K} \) must be a Klein bottle. Otherwise, \( \{ \tilde{h}, \tilde{v} \} \) would be a non-commuting pair in \( \pi_{1}(K) \), the fundamental group of the torus \( \tilde{K} \). Finally, \( \{ \tilde{h}, \tilde{v} \} \) is a basis for \( \pi_{1} (\tilde{K}) \) because the complement of these curves is a 2-disk, by construction. \( \square \)

**Remark 2.3.2** Suppose \( M \) is a Seifert manifold with orbit projection \( p : M \to F \). Assume \( F \) is of genus \( g \). Let \( \{ h_{i} \}_{i=1}^{r} \) be a set of fibers of \( M \) which contains all the exceptional fibers and a finite number of regular fibers. Recall each fiber has a neighborhood \( \tilde{V}_{i} \) fiber preserving homeomorphic to a fibered solid torus \( T(\beta_{i}/\alpha_{i}) \).

Write \( M_{0} = \overline{M - \cup V_{i}} \). Note that we have a quotient \( p| : M_{0} \to F_{0} \), where \( F_{0} \) is a surface with boundary. Recall \( F_{0} = F \cap M_{0} \). The boundary of \( F_{0} \) has \( r \) components, one for each component of \( \partial M_{0} \). Let \( q_{1}, \ldots, q_{r} \) be the components of \( \partial F_{0} \) and \( h \) be a regular fiber in \( M_{0} \).

Suppose \( \{ v_{j} \} \) is a basis for \( \pi_{1}(F) \) such that \( v_{j} \) is orientation reversing in \( F \), if \( F \) is non-orientable.
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• Assume $M \in O_o$, a presentation for $\pi_1(M_0)$ is

$$
\pi_1(M_0) \cong \langle v_1, \ldots, v_g, q_1, \ldots, q_r, h; [h, v_j] = 1, [h, q_i] = 1, q_1 q_2 \cdots q_r = \prod_{j=1}^g [v_{2j-1}, v_{2j}] \rangle.
$$

Let $\omega : \pi_1(M_0) \to S_n$ be a representation such that $\omega(h) = \varepsilon_n$, where $\varepsilon_n = (1, 2, \ldots, n)$. Then $\omega(v_j)$ and $\omega(q_i)$ commute with $\varepsilon_n$, for $[h, v_j] = [h, q_i] = 1$, $j = 1, \ldots, 2g$ and $i = 1, \ldots, r$. By Lemma (2.3.8), there are integer numbers $k_i$ and $s_j$ such that

$$
\omega(q_i) = \varepsilon_n^{k_i}, \text{ for } i = 1, \ldots, r \text{ and }
\omega(v_j) = \varepsilon_n^{s_j}, \text{ for } j = 1, \ldots, 2g.
$$

In $\pi_1(M_0)$ we have the relation $q_1 \cdots q_r = \prod [v_{2j-1}, v_{2j}]$. Then

$$
\omega(q_1 \cdots q_r (\prod [v_{2j-1}, v_{2j}])^{-1}) = \varepsilon^{\sum k_i} = (1).
$$

Since $\varepsilon_n$ has order $n$, there is an integer number $p$ such that $\sum k_i = np$. Define $k_i' = k_i - np$ and $k_j' = k_j$, if $j \neq 1$. Then we get a representation $\omega' : \pi_1(M_0) \to S_n$ such that

$$
\omega'(h) = \varepsilon_n,
\omega'(q_i) = \varepsilon_n^{k_i'}, \text{ for } i = 1, \ldots, r \text{ and }
\omega'(v_j) = \varepsilon_n^{s_j'}, \text{ for } j = 1, \ldots, 2g.
$$

Clearly $\sum k_i' = 0$ and $\varepsilon_n^{k_1} = \varepsilon_n^{k_j'}$ because $\varepsilon_n$ has order $n$. Therefore $\omega' = \omega$ and we can always assume $\sum k_i = 0$.

• If $M \in O_n$, then a presentation for $\pi_1(M_0)$ is

$$
\pi_1(M_0) \cong \langle v_1, \ldots, v_g, q_1, \ldots, q_r, h; v_j h v_j^{-1} = h^{-1}, [h, q_i] = 1, q_1 q_2 \cdots q_r = \prod_{j=1}^g v_{2j}^3 \rangle.
$$

Let $\omega : \pi_1(M_0) \to S_n$ be a representation such that $\omega(h) = \varepsilon_n$, where $\varepsilon_n = (1, 2, \ldots, n)$. Note that $\omega(v_j)$ anticommutes with $\varepsilon_n$, that is, $\omega(v_j) \varepsilon_n \omega(v_j)^{-1} = e^{-1}$, and $\omega(q_i)$ commute
with \( \varepsilon_n \), since we have that relations \( v_j h v_j^{-1} = h^{-1} \) and \( [h, q_i] = 1 \), \( j = 1, \ldots, 2g \) and \( i = 1, \ldots, r \). By Lemmas 2.3.8 and 2.3.7 there are integer numbers \( k_i \) and reflections \( \rho_j \) such that \( \omega : \pi_1(M_0) \to S_n \) is defined by
\[
\begin{align*}
\omega(h) &= \varepsilon_n \\
\omega(q_i) &= \varepsilon_n^{k_i}, \text{ for } i = 1, \ldots, r \text{ and} \\
\omega(v_j) &= \rho_j, \text{ for } j = 1, \ldots, g.
\end{align*}
\]

Since we have the relation \( q_1 \cdots q_r = \prod v_j^2 \) in \( \pi_1(M_0) \) and reflections have order 2, then
\[
\omega(q_1 \cdots q_r (\prod v_j^2)^{-1}) = \varepsilon^{\sum k_i} = (1).
\]

Therefore there is an integer number \( p \) such that \( \sum k_i = np \). Let \( k'_i = k_i - np \) and \( k'_j = k_j \), if \( j \neq 1 \). We define a representation \( \omega' : \pi_1(M_0) \to S_n \) by
\[
\begin{align*}
\omega'(h) &= \varepsilon_n \\
\omega'(q_i) &= \varepsilon_n^{k'_i}, \text{ for } i = 1, \ldots, r \text{ and} \\
\omega'(v_j) &= \rho_j, \text{ for } j = 1, \ldots, g.
\end{align*}
\]

Note that \( \omega' = \omega \) and \( \sum k'_i = 0 \). Therefore we can always assume \( \sum k_i = 0 \).

- If \( M \in N_o \), then a presentation for \( \pi_1(M_0) \) is
\[
\pi_1(M_0) \cong \langle v_1, \ldots, v_{2g}, q_1, \ldots, q_r, h; q_1 q_2 \cdots q_r = \prod_{j=1}^{g} [v_{2j-1}, v_{2j}] \rangle,
\]
\( [h, q_i] = 1, v_1 h v_1^{-1} = h^{-1}, [v_j, h] = 1 \) for \( j \geq 2 \).

Assume \( \omega : \pi_1(M_0) \to S_n \) is a representation such that \( \omega(h) = \varepsilon_n \), where \( \varepsilon_n = (1, 2, \ldots, n) \).

Then \( \omega(v_1) \) anticommutes with \( \varepsilon_n \) for \( v_1 h v_1^{-1} \); \( \omega(v_j) \) and \( \omega(q_i) \) commute with \( \varepsilon_n \), for \( [h, v_j] = [h, q_i] = 1, j = 2, \ldots, 2g \) and \( i = 1, \ldots, r \), By Lemma 2.3.7, there is a reflection \( \rho_1 \) and by Lemma 2.3.8 there are integer numbers \( k_1, \ldots, k_r, s_2, s_3, \ldots, s_{2g-1} \) and \( s_{2g} \) such that \( \omega : \pi_1(M_0) \to S_n \) is defined by
\[
\begin{align*}
\omega(h) &= \varepsilon_n \\
\omega(q_i) &= \varepsilon_n^{k_i}, \text{ for } i = 1, \ldots, r \text{ and} \\
\omega(v_1) &= \rho_1 \\
\omega(v_j) &= \varepsilon_n^{s_j}, \text{ for } j = 2, \ldots, 2g.
\end{align*}
\]

In $\pi_1(M_0)$ we have the relation $q_1 \cdots q_r = \prod [v_{2j-1}, v_{2j}]$. Then

$$\omega(q_1 \cdots q_r (\prod [v_{2j-1}, v_{2j}])^{-1}) = \varepsilon^{\sum k_i + 2s_2} = (1).$$

Thus there is an integer number $p$ such that $\sum k_i + 2s_2 = np$. Define $k'_1 = k_1 - np$ and $k'_j = k_j$, if $j \neq 1$. We get a representation $\omega': \pi_1(M_0) \to S_n$ such that

$$\omega'(h) = \varepsilon_n,$$
$$\omega'(q_i) = \varepsilon^{k'_i}, \text{ for } i = 1, \ldots, r \text{ and}$$
$$\omega'(v_1) = \rho_1,$$
$$\omega'(v_j) = \varepsilon^{s_j}, \text{ for } j = 2, \ldots, 2g.$$

It is easy to see $\sum k'_i + 2s_2 = 0$ and $\varepsilon_n^{k'_1} = \varepsilon_n^{k'_i}$ for $\varepsilon_n$ has order $n$. Therefore $\omega' = \omega$ and we can always assume $\sum k_i + 2s_2 = 0$.

* If $M \in NnI$, then a presentation for $\pi_1(M_0)$ is

$$\pi_1(M_0) \cong \langle v_1, \ldots, v_g, q_1, \ldots, q_r, h; [v_j, h] = 1, [h, q_i] = 1, q_1q_2\cdots q_r = \prod_{j=1}^{g} v_{2j}^{s_j} \rangle.$$

Suppose $\omega : \pi_1(M_0) \to S_n$ is a representation such that $\omega(h) = \varepsilon_n$, where $\varepsilon_n = (1, 2, \ldots, n)$. Then $\omega(v_j)$ and $\omega(q_i)$ commute with $\varepsilon_n$, for $[h, v_j] = [h, q_i] = 1$. By Lemma 2.3.8, $j = 1, \ldots, 2g$ and $i = 1, \ldots, r$, there are integer numbers $k_i$ and $s_j$ such that

$$\omega(q_i) = \varepsilon^{k_i}, \text{ for } i = 1, \ldots, r \text{ and}$$
$$\omega(v_j) = \varepsilon^{s_j}, \text{ for } j = 1, \ldots, g.$$

Recall in $\pi_1(M_0)$ we have the relation $q_1 \cdots q_r = \prod v_{2j}^{s_j}$. Then

$$\omega(q_1 \cdots q_r (\prod v_{2j}^{s_j})^{-1}) = \varepsilon^{\sum k_i - 2\sum s_j} = (1).$$

Since $\varepsilon_n$ has order $n$, there is an integer number $p$ such that $\sum k_i - 2\sum s_j = np$. Define $k'_1 = k_1 - np$ and $k'_j = k_j$, if $j \neq 1$. Then we get a representation $\omega' : \pi_1(M_0) \to S_n$ such
that

\[ \omega'(h) = \varepsilon_n \]
\[ \omega'(q_i) = \varepsilon_n^{k_i}, \quad \text{for } i = 1, \ldots, r \]
\[ \omega'(v_j) = \varepsilon_n^{s_j}, \quad \text{for } j = 1, \ldots, g. \]

Clearly \( \sum k_i' - 2 \sum s_j = 0 \) and \( \varepsilon_n^{k_i} = \varepsilon_n^{k_i'} \) because \( \varepsilon_n \) has order \( n \). Therefore \( \omega' = \omega \) and we can always assume \( \sum k_i - 2 \sum s_j = 0 \).

• If \( M \in NnII \), then a presentation for \( \pi_1(M_0) \) is

\[
\pi_1(M_0) \cong \langle v_1, \ldots, v_g, q_1, \ldots, q_r, h; [h, q_1] = 1, q_1 q_2 \cdots q_r = \prod_{j=1}^g v_j^2, [v_1, h] = 1, v_j h v_j^{-1} = h^{-1}, \text{ for each } j \geq 2 \rangle.
\]

Assume \( \omega : \pi_1(M_0) \rightarrow S_n \) is a representation such that \( \omega(h) = \varepsilon_n \), where \( \varepsilon_n = (1, 2, \ldots, n) \). Then \( \omega(v_1) \) and \( \omega(q_i) \) commute with \( \varepsilon_n \) for \( [v_1, h] = [h, q_i] = 1 \); if \( j \geq 2 \), then \( \omega(v_j) \) anticommutes with \( \varepsilon_n \) because \( [h, v_j] = [h, q_i] = 1 \), for \( j \geq 2 \). By Lemma 2.3.7 and 2.3.8, there are reflections \( \rho_j \), \( j \geq 2 \), and there are integer numbers \( k_i \) and \( s_1 \) such that \( \omega : \pi_1(M_0) \rightarrow S_n \) is defined by

\[ \omega(h) = \varepsilon_n \]
\[ \omega(q_i) = \varepsilon_n^{k_i}, \quad \text{for } i = 1, \ldots, r \]
\[ \omega(v_1) = \varepsilon_n^{s_1}, \quad \text{and} \]
\[ \omega(v_j) = \rho_j, \quad \text{for } j = 2, \ldots, g. \]

Note that

\[ \omega(q_1 \cdots q_r ( \prod v_j^2 )^{-1} ) = \varepsilon^{\sum k_i - 2s_1} = 1 \]

because of relation \( q_1 \cdots q_r = \prod v_j^2 \) and because reflections have order 2.

Thus there is an integer number \( p \) such that \( \sum k_i - 2s_1 = np \). Define \( k_1' = k_1 - np \) and
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$k'_j = k_j$, if $j \neq 1$. We get a representation $\omega' : \pi_1(M_0) \to S_n$ such that

\[
\begin{align*}
\omega'(h) &= \varepsilon_n \\
\omega'(q_i) &= \varepsilon_{k'_i}, \text{ for } i = 1, \ldots, r; \\
\omega'(v_1) &= \varepsilon_{s_1}^n, \text{ and} \\
\omega'(v_j) &= \rho_j, \text{ for } j = 2, \ldots, g.
\end{align*}
\]

It is easy to see $\sum k'_i - 2s_1 = 0$ and $\varepsilon_{k'_i} = \varepsilon_{s_1}$ since $\varepsilon_n$ has order $n$. Therefore $\omega' = \omega$ and we can always assume $\sum k_i - 2s_1 = 0$.

• If $M \in N_{nIII}$, then a presentation for $\pi_1(M_0)$ is

\[
\pi_1(M_0) \cong \langle v_1, \ldots, v_g, q_1, \ldots, q_r, h; [h, q_i] = 1, q_1q_2 \cdots q_r = \prod_{j=1}^{g} v_j^2, \rangle \\
[v_1, h] = 1, [v_2, h] = 1, v_j h v_j^{-1} = h^{-1}, \text{ for each } j \geq 3 \rangle.
\]

Suppose $\omega : \pi_1(M_0) \to S_n$ is a representation such that $\omega(h) = \varepsilon_n$, where $\varepsilon_n = (1, 2, \ldots, n)$. Then $\omega(v_1)$, $\omega(v_2)$ and $\omega(q_i)$ commute with $\varepsilon_n$ for $[v_1, h] = [v_2, h] = [h, q_i] = 1$; if $j \geq 3$, then $\omega(v_j)$ anticommutes with $\varepsilon_n$ for if $j \geq 3$ then $[h, v_j] = [h, q_i] = 1$. By Lemma 2.3.7 and 2.3.8, there are reflections $\rho_j$, $j \geq 3$, and there are integer numbers $k_i$, $s_1$ and $s_2$ such that $\omega : \pi_1(M_0) \to S_n$ is defined by

\[
\begin{align*}
\omega(h) &= \varepsilon_n \\
\omega(q_i) &= \varepsilon_{k_i}, \text{ for } i = 1, \ldots, r \\
\omega(v_1) &= \varepsilon_{s_1}^n, \\
\omega(v_2) &= \varepsilon_{s_2}^n, \text{ and} \\
\omega(v_j) &= \rho_j, \text{ for } j = 3, \ldots, g.
\end{align*}
\]

Note that

\[
\omega(q_1 \cdots q_r (\prod_{j} v_j^2)^{-1}) = \varepsilon^{\sum k_i - 2s_1 - 2s_2} = 1
\]

since $q_1 \cdots q_r = \prod v_j^2$ and because reflections have order 2.
Thus there is an integer number $p$ such that $\sum k_i - 2s_1 - 2s_2 = np$. Let $k'_1 = k_1 - np$ and $k'_j = k_j$, if $j \neq 1$. We obtain a representation $\omega' : \pi_1(M_0) \to S_n$ such that

$$
\begin{align*}
\omega'(h) &= \varepsilon_n \\
\omega'(q_i) &= \varepsilon_{k'_i}^n, \text{ for } i = 1, \ldots, r \\
\omega'(v_1) &= \varepsilon_{s_1}^n, \\
\omega'(v_2) &= \varepsilon_{s_2}^n, \text{ and} \\
\omega'(v_j) &= \rho_j, \text{ for } j = 3, \ldots, g;
\end{align*}
$$

It is easy to see $\sum k'_i - 2s_1 - 2s_2 = 0$ and $\varepsilon_{k'_i}^n = \varepsilon_{k_i}^n$ for $\varepsilon_n$ has order $n$. Therefore $\omega' = \omega$ and we can always assume $\sum k_i - 2s_1 - 2s_2 = 0$.

Lemma 2.3.11 Let $M$ be a Seifert manifold. Assume $M_0$, $F$ and $F_0$ are as in last remark. Suppose $h$ is a regular fiber of $M$ and $\omega : \pi_1(M_0) \to S_n$ is a representation such that $\omega(h) = \varepsilon_n$. Let $\varphi : \tilde{M} \to M$ be the covering of $M$ branched along fibers of $M$ determined by $\omega$. Assume $\tilde{p} : \tilde{M} \to G$ is the orbit projection of $\tilde{M}$. Then $F \cong G$.

Proof.

Let $\tilde{M}_0 = \varphi^{-1}(M_0)$, $\tilde{F}_0 = \varphi^{-1}(F_0)$ and $G_0 = \tilde{p}(\tilde{M}_0)$. Then $\varphi| : \tilde{F}_0 \to F_0$ is a covering space of $n$ sheets. Since $\omega(h) = \varepsilon_n$, each fiber of $\tilde{M}_0$ is the preimage of a fiber $h'$ in $M_0$ under $\varphi$. Thus the projection $\tilde{p}| : \tilde{F}_0 \to G_0$ is also an $n$-fold covering for each fiber of $\tilde{M}_0$ intersects $\tilde{F}_0$ in $n$ points. Suppose that $\tilde{x}, \tilde{y} \in \tilde{F}_0$ and $\tilde{p}(\tilde{x}) = \tilde{p}(\tilde{y})$. Then there is one fiber $\tilde{h}$ in $\tilde{M}_0$ such that $\tilde{x}, \tilde{y} \in \tilde{h} \cap \tilde{F}_0$. Also there is a fiber $h'$ of $M_0$ such that $\varphi(\tilde{h}) = (h')^n$ for $\omega(h) = \varepsilon_n$. We conclude $\varphi|((\tilde{x}) = \varphi|((\tilde{y})$ for $\varphi|((\tilde{x}), \varphi|((\tilde{y}) \in h' \cap F_0$ and each fiber intersects $F_0$ in one single point. Thus there exists the following commutative diagram:

$$
\begin{array}{ccc}
\tilde{F}_0 & \xrightarrow{\varphi} & G_0 \\
\downarrow{\tilde{p}} & & \downarrow{\tilde{p}} \\
F_0 & \xrightarrow{\varphi} & G_0 \\
\end{array}
$$
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The map $\varphi_0 : G_0 \to F_0$ is defined as usual: Let $x \in G_0$ and consider $\tilde{x} \in (\tilde{p})^{-1}(x)$ then $\varphi_0(x) = \varphi((\tilde{x}))$. Of course, $\varphi_0(x)$ does not depend on $\tilde{x}$ because $(\varphi)((\tilde{p})^{-1}(x))$ is one point. Note that $\varphi_0$ is a covering of 1 sheet for $\tilde{p} : \tilde{M} \to G$ and $\varphi : \tilde{F} \to F$ are $n$-fold coverings and for the diagram above is a commutative diagram. Thus $\varphi_0$ is a homeomorphism. Therefore there is a homeomorphism $\varphi : G \to F$. □

Note that in this context $\tilde{M}$ is no longer a pullback.

Lemma 2.3.12 Let $M$ be a Seifert manifold and $\varphi : \tilde{M} \to M$ be a covering of $M$ branched along fibers. Assume $\tilde{p} : \tilde{M} \to G$ and $p : M \to F$ are the orbit projections of $\tilde{M}$ and $M$, respectively. Let $h$ be a regular fiber of $M$. Let $\omega : \pi_1(M_0) \to S_n$ be the representation determined by $\varphi$. Suppose $\omega(h) = \varepsilon_n$. Let $G_0$ and $F_0$ be as in the proof of the previous lemma. Let $\varphi_0 : G_0 \to F_0$ be the homeomorphism obtained in the previous lemma. Recall $\pi_1(F) \to \mathbb{Z}_2$ is the valuation homomorphism. Let $\tilde{v} \subset G_0$ and $v \subset F_0$ be simple closed curves such that $\varphi_0(\tilde{v}) = v$. Then:

(a) The map $\varphi : \tilde{p}^{-1}(\tilde{v}) \to p^{-1}(v)$ is an $n$-fold covering space.

(b) If $e(v) = +1$, then $\tilde{e}(\tilde{v}) = +1$.

(c) If $e(v) = -1$, Then $\tilde{e}(\tilde{v}) = -1$.

Proof.

(a) Note that the following diagram commutes.

$$
\begin{array}{ccc}
\tilde{M}_0 & \xrightarrow{\varphi} & M_0 \\
\downarrow{\tilde{p}} & & \downarrow{p} \\
G_0 & \xrightarrow{\varphi} & F_0
\end{array}
$$

Thus $\varphi : \tilde{p}^{-1}(\tilde{v}) \to p^{-1}(v)$ is a covering space and

$\omega' : \pi_1(p^{-1}(v)) \to S_r = S(\{a_1, \ldots, a_r\})$, the representation associated to this covering,
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sends $h$ into $\varepsilon_n$. Note that $\tilde{p}^{-1}(\tilde{v})$ and $p^{-1}(v)$ are $S^1$-bundles over the simple closed curves $\tilde{v}$ and $v$, respectively. Then $\tilde{p}^{-1}(\tilde{v})$ and $p^{-1}(v)$ are either tori or Klein bottles depending on the triviality of the $S^1$-bundles.

(b) Since $e(v) = +1$, then $p^{-1}(v)$ is a torus and $\tilde{p}^{-1}(\tilde{v})$ is a torus. Thus $\tilde{e}(\tilde{v}) = +1$ for $\tilde{p}^{-1}(\tilde{v})$ is an $S^1$-bundle over $\tilde{v}$.

(c) If $e(v) = -1$, then $p^{-1}(v)$ is a Klein bottle. According to Lemma 2.3.10, we conclude $\tilde{e}(\tilde{v}) = -1$.

\begin{align*}
\text{Theorem 2.3.9} & \quad \text{Assume } M = (Oo, g; \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r) \text{ is a Seifert manifold. Let } v_j \text{ and } q_i \text{ be as in Remark 2.3.2 and } \omega : \pi_1(M_0) \to S_n \text{ be a representation defined by } \\
\omega(h) &= \varepsilon_n \\
\omega(q_i) &= \varepsilon_{ki}^n, \text{ for } i = 1, \ldots, r \text{ and } \\
\omega(v_j) &= \varepsilon_{sj}^n, \text{ for } j = 1, \ldots, 2g; \\
\text{where } \sum k_i &= 0.
\end{align*}

Let $\varphi : \tilde{M} \to M$ be the covering defined by $\omega$. Then $\tilde{M} \in Oo$.

Proof.

Let $p : M \to F$ be the orbit projection of $M$ and let $\tilde{p} : \tilde{M} \to G$ be the orbit projection of $\tilde{M}$. By Lemma 2.3.11, there exists a homeomorphism $\varphi : G \to F$. Then $G$ is orientable. Let $\tilde{M}_0 = \varphi^{-1}(M_0)$. Since $\varphi| : \tilde{M}_0 \to M_0$ is a covering and $M_0$ is orientable, then $\tilde{M}_0$, and consequently, $\tilde{M}$ are orientable by Lemma 2.3.5 and Corollary 2.1.2. Therefore $\tilde{M} \in Oo$.

\begin{align*}
\text{Theorem 2.3.10} & \quad \text{Assume } M = (On, g; \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r) \text{ is a Seifert manifold. Let } v_j \text{ and } q_i \text{ be as in Remark 2.3.2 and } \omega : \pi_1(M_0) \to S_n \text{ be a representation defined by } \\
\omega(h) &= \varepsilon_n \\
\omega(q_i) &= \varepsilon_{ki}^n, \text{ for } i = 1, \ldots, r \text{ and } \\
\omega(v_j) &= \rho_j, \text{ for } j = 1, \ldots, g;
\end{align*}
where $\sum k_i = 0$ and $\rho_j$ is a reflection, for $j = 1, \ldots, g$.

Let $\varphi : \tilde{M} \to M$ be the covering defined by $\omega$. Then $\tilde{M} \in \text{On}$.

Proof.

Let $p : M \to F$ be the orbit projection of $M$ and let $\tilde{p} : \tilde{M} \to G$ be the orbit projection of $\tilde{M}$.

By Lemma 2.3.11, there exists a homeomorphism $\varphi : G \to F$. Then $G$ is non-orientable. Let $\tilde{M}_0 = \varphi^{-1}(M_0)$. Since $\varphi| : \tilde{M}_0 \to M_0$ is a covering and $M_0$ is orientable, then $\tilde{M}_0$ is orientable; $\tilde{M}$ as also orientable by Lemma 2.3.5 and Corollary 2.1.2. Therefore $\tilde{M} \in \text{On}$. $\square$

Theorem 2.3.11

Assume $M = (\text{No}, g; \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r)$ is a Seifert manifold. Let $v_j$ and $q_i$ be as in Remark 2.3.2 and $\omega : \pi_1(M_0) \to S_n$ be a representation defined by

$$
\begin{align*}
\omega(h) &= \varepsilon_n \\
\omega(q_i) &= \varepsilon_k, \text{ for } i = 1, \ldots, r \text{ and } \\
\omega(v_1) &= \rho_1 \\
\omega(v_j) &= \varepsilon^s_j, \text{ for } j = 2, \ldots, 2g;
\end{align*}
$$

where $\sum k_i + 2s_2 = 0$ and $\rho_1$ is a reflection. Suppose $\rho_1(1) = t_1 \in \{1, \ldots, n\}$.

Let $\varphi : \tilde{M} \to M$ be the covering defined by $\omega$. Then $\tilde{M} \in \text{No}$.

Proof.

Let $p : M \to F$ be the orbit projection of $M$ and let $\tilde{p} : \tilde{M} \to G$ be the orbit projection of $\tilde{M}$. Recall $e : \pi_1(F) \to \mathbb{Z}_2$, the valuation homomorphism of $M$, is defined by $e(v_i) = -1$ and $e(v_2) = +1$, for $i = 2, \ldots, 2g$. By Lemma 2.3.11, there is a homeomorphism $\varphi : G \to F$. Thus $G$ is orientable. Let $\{v'_j\}_{j=1}^{2g}$ be a basis for $\pi_1(G)$ such that $\varphi(v'_j) = v_j$. By Lemma (2.3.12), the map $\varphi| : \tilde{p}^{-1}(v'_j) \to p^{-1}(v_j)$ is a covering and $\tilde{e}(v'_j) = e(v_j)$, for $j = 1, \ldots, 2g$, where $\tilde{e} : \pi_1(G) \to \mathbb{Z}_2$ is the valuation homomorphism of $\tilde{M}$. Therefore $\tilde{M} \in \text{No}$. $\square$
Theorem 2.3.12 Assume $M = (NnI, g; \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r)$ is a Seifert manifold. Let $v_j$ and $q_i$ be as in Remark 2.3.2 and $\omega: \pi_1(M_0) \to S_n$ be a representation defined by
\[
\omega(h) = \varepsilon_n \\
\omega(q_i) = \varepsilon_{k_i}^n, \text{ for } i = 1, \ldots, r \text{ and} \\
\omega(v_j) = \varepsilon_{s_j}^n, \text{ for } j = 1, \ldots, g;
\]
where $\sum k_i - 2 \sum s_j = 0$.

Let $\varphi: \tilde{M} \to M$ be the covering defined by $\omega$. Then $\tilde{M} \in NnI$.

Proof.
Let $p: M \to F$ be the orbit projection of $M$ and let $\tilde{p}: \tilde{M} \to G$ be the orbit projection of $\tilde{M}$.
Recall $\{v_j\}$ is a basis of orientation reversing curves for $\pi_1(F)$ and $e: \pi_1(F) \to \mathbb{Z}_2$, the valuation homomorphism of $M$, is trivial. By Lemma 2.3.11, there is an homeomorphism $\varphi: G \to F$. Thus $G$ is non-orientable. Since $\varphi$ is a homeomorphism, there exists a basis $\{v_j'\}_{j=1}^g$ of orientation reversing curves for $\pi_1(G)$ such that $\varphi(v_j') = v_j$. By Lemma 2.3.12, the map $\varphi|: \tilde{p}^{-1}(v_j') \to p^{-1}(v_j)$ is a covering and $\tilde{e}: \pi_1(G) \to \mathbb{Z}_2$ is trivial, where $\tilde{e}: \pi_1(G) \to \mathbb{Z}_2$ is the valuation homomorphism of $\tilde{M}$. Therefore $\tilde{M} \in NnI$. $\square$

Theorem 2.3.13 Assume $M = (NnII, g; \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r)$ is a Seifert manifold. Let $v_j$ and $q_i$ be as in Remark 2.3.2 and $\omega: \pi_1(M_0) \to S_n$ be a representation defined by
\[
\omega(h) = \varepsilon_n \\
\omega(q_i) = \varepsilon_{k_i}^n, \text{ for } i = 1, \ldots, r \\
\omega(v_1) = \varepsilon_{s_1}^n, \text{ and} \\
\omega(v_j) = \rho_j, \text{ for } j = 2, \ldots, g;
\]
where $\sum k_i - 2s_1 = 0$ and $\rho_j$ is a reflection, for all $j = 2, \ldots, g$.

Let $\varphi: \tilde{M} \to M$ be the covering defined by $\omega$. Then $\tilde{M} \in NnII$. 
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Proof.

Let \( p : M \to F \) be the orbit projection of \( M \) and let \( \tilde{p} : \tilde{M} \to G \) be the orbit projection of \( \tilde{M} \).

Recall \( \{ v_j \} \) is a basis of orientation reversing curves for \( \pi_1(F) \) and \( e : \pi_1(F) \to \mathbb{Z}_2 \), the valuation homomorphism of \( M \), is defined by \( e(v_1) = +1 \) and \( e(v_j) = -1 \), for \( j = 2, \ldots, g \). By Lemma 2.3.11, there is an homeomorphism \( \varphi : G \to F \). Then \( G \) is non-orientable. Also there exists a basis \( \{ v'_j \}_{j=1}^g \) of orientation reversing curves for \( \pi_1(G) \) such that \( \varphi(v'_j) = v_j \), because \( \varphi \) is a homeomorphism. By Lemma 2.3.12, the map \( \varphi| : \tilde{p}^{-1}(v'_j) \to p^{-1}(v_j) \) is a covering and \( \tilde{e}(v'_j) = e(v_j) \), for \( j = 1, \ldots, g \), where \( \tilde{e} : \pi_1(G) \to \mathbb{Z}_2 \) is the valuation homomorphism of \( \tilde{M} \). Therefore \( \tilde{M} \in N_{nII} \). \( \square \)

Theorem 2.3.14 Assume \( M = (N_{nIII}, g; \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r) \) is a Seifert manifold. Let \( v_j \) and \( q_i \) be as in Remark 2.3.2 and \( \omega : \pi_1(M_0) \to S_n \) be a representation defined by

\[
\begin{align*}
\omega(h) &= \varepsilon_n \\
\omega(q_i) &= \varepsilon_n^{k_i}, \text{ for } i = 1, \ldots, r \\
\omega(v_1) &= \varepsilon_n^{s_1}, \\
\omega(v_2) &= \varepsilon_n^{s_2}, \text{ and} \\
\omega(v_j) &= \rho_j, \text{ for } j = 3, \ldots, g;
\end{align*}
\]

where \( \sum k_i - 2s_1 - 2s_2 = 0 \) and \( \rho_j \) is a reflection, for \( j = 3, \ldots, g \).

Let \( \varphi : \tilde{M} \to M \) be the covering defined by \( \omega \). Then \( \tilde{M} \in N_{nIII} \).

Proof.

Let \( p : M \to F \) be the orbit projection of \( M \) and let \( \tilde{p} : \tilde{M} \to G \) be the orbit projection of \( \tilde{M} \).

Recall \( \{ v_j \} \) is a basis of orientation reversing curves for \( \pi_1(F) \) and \( e : \pi_1(F) \to \mathbb{Z}_2 \), the valuation homomorphism of \( M \), is defined by \( e(v_1) = +1 \) and \( e(v_j) = -1 \), for \( j = 2, \ldots, g \). By Lemma 2.3.11, there is an homeomorphism \( \varphi : G \to F \). Then \( G \) is non-orientable. Also there exists a basis \( \{ v'_j \}_{j=1}^g \) of orientation reversing curves for \( \pi_1(G) \) such that \( \varphi(v'_j) = v_j \), for
\( \varphi \) is a homeomorphism. By Lemma 2.3.12, the map \( \varphi| : \tilde{p}^{-1}(v'_j) \to p^{-1}(v_j) \) is a covering and \( \tilde{e}(v'_j) = e(v_j) \), for \( j = 1, \ldots, g \), where \( \tilde{e} : \pi_1(G) \to \mathbb{Z}_2 \) is the valuation homomorphism of \( \tilde{M} \). Therefore \( \tilde{M} \in NnIII. \)

\( \square \)

**Corollary 2.3.1** Let \( M = (Xx, g; \beta_1/\alpha_1, \ldots, \alpha_r/\beta_r) \) and \( M_0 \) as in Remark 2.3.2. Assume \( h \) is a regular fiber of \( M \). Let \( \omega : \pi_1(M_0) \to S_n \) be a representation such that \( \omega(h) = \varepsilon_n \) and let \( \varphi : \tilde{M} \to M \) be covering space determined by \( \omega \).

Then \( \tilde{M} \) is in the same class of \( M \).

Now let us compute some special Orbit Surfaces for the coverings.

**Lemma 2.3.13** Suppose \( M = (Oo, g; \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r) \) is a Seifert manifold. Assume \( h \) is a regular fiber of \( M \). Let \( \omega : \pi_1(M_0) \to S_n \) such that \( \omega(h) = \varepsilon_n \), where \( \varepsilon_n = (1, 2, \ldots, n) \). By Remark 2.3.2, \( \omega : \pi_1(M_0) \to S_n \) is defined by

\[
\begin{align*}
\omega(h) &= \varepsilon_n \\
\omega(q_i) &= \varepsilon_n^{k_i}, \text{ for } i = 1, \ldots, r \quad \text{and} \\
\omega(v_j) &= \varepsilon_n^{s_j}, \text{ for } j = 1, \ldots, 2g;
\end{align*}
\]

where \( v_j \) and \( q_i \) are considered as in Remark 2.3.2 and \( \sum k_i = 0 \).

Let \( \varphi : \tilde{M} \to M \) be the covering defined by \( \omega \).

Then there are an orbit surface \( G'_0 \) of \( \tilde{M} \) and a basis \( \tilde{v}_1, \ldots, \tilde{v}_g \) for \( \pi_1(G'_0) \) and curves \( \tilde{q}_i \) in the boundary of \( G'_0 \) such that \( \varphi_\#(\tilde{q}_i) = q_i h^{-k_i}, \varphi_\#(\tilde{v}_j) = v_j h^{-s_j} \), for all \( j \).

In particular, we have an orbit surface \( G' \) of \( \tilde{M} \) such that \( \tilde{v}_1, \ldots, \tilde{v}_g \) is a basis for \( \pi_1(G') \).

**Proof.**

Let \( p : M \to F \) be the orbit projection of \( M \) and let \( \tilde{p} : \tilde{M} \to G \) be the orbit projection of \( \tilde{M} \).
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Recall \( F_0 = p(M_0) \). By Lemma 2.3.11, there exists a homeomorphism \( \varphi_0 : G_0 \to F_0 \), where \( F_0 = p(M_0) \) and \( G_0 = \tilde{p}(\varphi^{-1}(M_0)) \). Then there exists a basis \( \{v'_j, q'_i\} \), where \( j = 1, \ldots, 2g \) and \( i = 1, \ldots, r \), for \( \pi_1(G_0) \) such that \( \varphi_0(v'_j) = v_j \) and \( \varphi_0(q'_i) = q_i \), for all \( j = 1, \ldots, 2g \) and for \( i = 1, \ldots, r \).

Recall \( e : \pi_1(F) \to \mathbb{Z}_2 \), the valuation homomorphism of \( M \), is trivial. By Lemma 2.3.12 \( \tilde{e}(v'_j) = \tilde{e}(q'_i) = +1 \), where \( \tilde{e} : \pi_1(G) \to \mathbb{Z}_2 \) is the valuation homomorphism of \( \tilde{M} \).

By Lemma 2.3.12, \( \varphi| : \tilde{p}^{-1}(q'_i) \to p^{-1}(q_i) \) is a covering space; using Lemma 2.3.9 we obtain a basis \( \{h, \tilde{q}_i\} \) for \( \pi_1(\tilde{p}^{-1}(q'_i)) \) such that \( \varphi_#(h) = h^n \) and \( \varphi_#(\tilde{q}_i) = q_i h^{-k_i} \).

Analogously, there is a basis \( \{\tilde{v}_j, \tilde{h}\} \) for \( \pi_1(\tilde{p}^{-1}(v'_j)) \) such that \( \varphi_#(\tilde{h}) = h^n \) and \( \varphi_#(\tilde{v}_j) = v_j h^{-s_j} \), for all \( j \). Note that, by construction, \( \tilde{v}_j \) and \( \tilde{q}_i \) intersect every fiber of \( \tilde{p}^{-1}(v'_j) \) and \( \tilde{p}^{-1}(q'_i) \), respectively, in exactly one point.

Since \( h \) commutes with \( v_j \), for \( j = 1, \ldots, 2g \), we obtain

\[
\varphi_#(\tilde{q}_1 \cdots \tilde{q}_r (\prod [\tilde{v}_2 j-1, \tilde{v}_2 j])^{-1}) \simeq q_1 h^{-k_1} \cdots q_r h^{-k_r} (\prod [v_{2l-1}, v_{2l}])^{-1} \\
\simeq h^{-\sum k_i} q_1 \cdots q_r (\prod [v_{2l-1}, v_{2l}])^{-1} \quad (\text{recall } \sum k_i = 0) \\
\simeq q_1 \cdots q_r (\prod [v_{2l-1}, v_{2l}])^{-1} \\
\simeq 1,
\]

where all homotopies are \( rel \partial I \). Thus \( \tilde{q}_1 \cdots \tilde{q}_r (\prod [\tilde{v}_2 j-1, \tilde{v}_2 j])^{-1} \simeq 1 \) for \( \varphi_# \) is injective.

Then the curves \( \tilde{q}_1, \ldots, \tilde{q}_r \) span a surface \( G'_0 \) in \( M_0 \). After some isotopies of \( G'_0 \) in \( \tilde{M} \) fixing \( \partial G'_0 \), we obtain \( G'_0 \) is an orbit surface. After filling the holes of \( \tilde{M}_0 \), \( G'_0 \) gives rise to \( G' \) as required. \( \square \)

**Lemma 2.3.14** Suppose \( M = (O_n, g; \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r) \) is a Seifert manifold. Assume \( h \) is a regular fiber of \( M \). Let \( M_0 \) be as in Remark 2.3.2 and \( \omega : \pi_1(M_0) \to S_n \) such that
\( \omega(h) = \varepsilon_n \), where \( \varepsilon_n = (1, 2, \ldots, n) \). By Remark 2.3.2, \( \omega : \pi_1(M_0) \to S_n \) is defined by

\[
\begin{align*}
\omega(h) &= \varepsilon_n \\
\omega(q_i) &= \varepsilon_n^{k_i}, \text{ for } i = 1, \ldots, r \text{ and} \\
\omega(v_j) &= \rho_j, \text{ for } j = 1, \ldots, g;
\end{align*}
\]

where \( \sum k_i = 0 \) and \( \rho_j \) is a reflection, for \( j = 1, \ldots, g \). Suppose \( \rho_j(1) = t_j \in \{1, \ldots, n\} \), for \( j = 1, \ldots, g \).

Let \( \varphi : \tilde{M} \to M \) be the covering defined by \( \omega \).

Then there are an orbit surface \( G'_0 \) of \( \tilde{M} \) and a basis \( \tilde{v}_1, \ldots, \tilde{v}_g \) for \( \pi_1(G'_0) \) and curves \( \tilde{q}_i \) in the boundary of \( G'_0 \) such that \( \varphi_{\#}(\tilde{q}_i) = q_i h^{-k_i} \), \( \varphi_{\#}(\tilde{v}_j) = v_j h^{-(t_j - 1)} \), for all \( j \).

In particular, we have an orbit surface \( G' \) of \( \tilde{M} \) such that \( \tilde{v}_1, \ldots, \tilde{v}_g \) is a basis for \( \pi_1(G') \).

Proof.

Let \( p : M \to F \) be the orbit projection of \( M \) and let \( \tilde{p} : \tilde{M} \to G \) be the orbit projection of \( \tilde{M} \).

Recall \( F_0 = p(M_0) \) and \( \{v_j\} \) is a basis of orientation reversing curves for \( \pi_1(F) \). By Lemma 2.3.11, there exists a homeomorphism \( \varphi_0 : G_0 \to F_0 \), where \( F_0 = p(M_0) \) and \( G_0 = \tilde{p}(\varphi^{-1}(M_0)) \).

Then there exists a basis \( \{v'_j, q'_i\} \), where \( j = 1, \ldots, g \) and \( i = 1, \ldots, r \), for \( \pi_1(G_0) \) such that \( \varphi_0(v'_j) = v_j \) and \( \varphi_0(q'_i) = q_i \), for all \( j = 1, \ldots, g \) and for \( i = 1, \ldots, r \).

Recall \( e : \pi_1(F) \to \mathbb{Z}_2 \), the valuation homomorphism of \( M \), is defined by \( e(v_j) = -1 \), for \( j = 1, \ldots, g \), and \( e(q_i) = +1 \), for \( i = 1, \ldots, r \). Let \( \tilde{e} : \pi_1(G) \to \mathbb{Z}_2 \) be the valuation homomorphism of \( \tilde{M} \); by Lemma 2.3.12 we have that \( \varphi| : \tilde{p}^{-1}(q'_i) \to p^{-1}(q_i) \) is a covering, \( \tilde{e}(v'_j) = -1 \) and \( \tilde{e}(q'_i) = +1 \).

From Lemma 2.3.9 it follows that we have a basis \( \{\tilde{h}, \tilde{q}_i\} \) for \( \pi_1(\tilde{p}^{-1}(q'_i)) \) such that \( \varphi_{\#}(\tilde{h}) = h^n \) and \( \varphi_{\#}(\tilde{q}_i) = q_i h^{-k_i} \).
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Recall \(\rho_j(1) = t_j\). By Lemma 2.3.10 there is a basis \(\{\tilde{v}_j, \tilde{h}\}\) for \(\pi_1(\tilde{p}^{-1}(v_j'))\) such that \(\varphi_#(\tilde{h}) = h^n\) and \(\varphi_#(\tilde{v}_j) = v_j h^{-(t_j - 1)}\), for \(j = 1, \ldots, g\).

Note that, by construction, \(\tilde{v}_j\) and \(\tilde{q}_i\) intersect each fiber of \(\tilde{p}^{-1}(v'_j)\) and \(\tilde{p}^{-1}(q'_i)\), respectively, in exactly one point.

Since \(h\) anticommutes with \(v_j\), we obtain \(v_j h^{-(t_j - 1)} = h^{(t_j - 1)}v_j\) and \(v_j h(t_j - 1) = h^{-(t_j - 1)}v_j\), for \(j = 1, \ldots, 2g\). Then \(v_j h^{-(t_j - 1)}v_j h^{-(t_j - 1)} = h^{(t_j - 1) - (t_j - 1)}v_j^2 = v_j^2\).

Note that
\[
\varphi_# \left( \tilde{q}_1 \cdots \tilde{q}_r (\prod \tilde{v}_j^2)^{-1} \right) \simeq q_1 h^{-k_1} \cdots q_r h^{-k_r} (\prod (v_j h^{-(t_j - 1)})^2)^{-1} \\
\simeq h^{-\sum k_i} q_1 \cdots q_r (\prod v_j h^{-(t_j - 1)} v_j h^{-(t_j - 1)})^{-1}, \text{ (recall } \sum k_i = 0.\text{)} \\
\simeq q_1 \cdots q_r (\prod v_j^2)^{-1}, \\
\simeq 1.
\]

Thus \(\tilde{q}_1 \cdots \tilde{q}_r (\prod \tilde{v}_j^2)^{-1} \simeq 1\) because for \(\varphi_#\) is injective.

Then the curves \(\tilde{q}_1, \ldots, \tilde{q}_r\) span a surface \(G'_0\) in \(M_0\). After some isotopies of \(G'_0\) in \(\tilde{M}\) fixing \(\partial G'_0\), we obtain \(G'_0\) is an orbit surface. After filling the holes of \(\tilde{M}_0\), \(G'_0\) gives rise to \(G'\) as required. \(\square\)

Lemma 2.3.15 Suppose \(M = (N, g; \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r)\) is a Seifert manifold. Assume \(h\) is a regular fiber of \(M\). Let \(M_0\) be as in Remark 2.3.2 and \(\omega : \pi_1(M_0) \to S_n\) such that \(\omega(h) = \varepsilon_n\), where \(\varepsilon_n = (1, 2, \ldots, n)\). Let \(\omega : \pi_1(M_0) \to S_n\) be a representation is defined by
\[
\omega(h) = \varepsilon_n \\
\omega(q_i) = \varepsilon_n^{k_i}, \text{ for } i = 1, \ldots, r \text{ and } \\
\omega(v_1) = \rho_1 \\
\omega(v_j) = \varepsilon_n^{s_j}, \text{ for } j = 2, \ldots, 2g;
\]
where \(\sum k_i + 2s_2 = 0\) and \(\rho_1\) is a reflection. Suppose \(\rho_1(1) = t_1 \in \{1, \ldots, n\}\).
Let $\varphi : \tilde{M} \to M$ be the covering defined by $\omega$.

Then there are an orbit surface $G'_0$ of $\tilde{M}_0$ and a basis $\tilde{v}_1, \ldots, \tilde{v}_g$ for $\pi_1(G'_0)$ and curves $\tilde{q}_i$ in the boundary of $G'_0$ such that $\varphi_\#(\tilde{q}_i) = q_i h^{-k_i}$, $\varphi_\#(\tilde{v}_1) = v_1 h^{-(t_1-1)}$ and $\varphi_\#(\tilde{v}_j) = v_j h^{-s_j}$, for $j = 2, \ldots, 2g$.

In particular, we have an orbit surface $G'$ of $\tilde{M}$ such that $\tilde{v}_1, \ldots, \tilde{v}_g$ is a basis for $\pi_1(G')$.

Proof.

Let $p : M \to F$ be the orbit projection of $M$ and let $\tilde{p} : \tilde{M} \to G$ be the orbit projection of $\tilde{M}$.

Recall $F_0 = p(M_0)$. By Lemma 2.3.11, there exists a homeomorphism $\varphi_0 : G_0 \to F_0$, where $F_0 = p(M_0)$ and $G_0 = \tilde{p}(\varphi^{-1}(M_0))$. Then there exists a basis $\{v'_j, q'_i\}$, where $j = 1, \ldots, g$ and $i = 1, \ldots, r$, for $\pi_1(G_0)$ such that $\varphi_0(v'_j) = v_j$ and $\varphi_0(q'_i) = q_i$, for $j = 1, \ldots, g$ and for $i = 1, \ldots, r$.

Recall $e(v_1) = -1$, $e(v_j) = +1$, for $j = 2, \ldots, 2g$, and $e(q_i) = +1$, for $i = 1, \ldots, r$, where $e : \pi_1(F) \to \mathbb{Z}_2$ is the valuation homomorphism of $M$. Let $\tilde{e} : \pi_1(G) \to \mathbb{Z}_2$ be the valuation homomorphism of $\tilde{M}$; by Lemma 2.3.12 we have that $\varphi| : \tilde{p}^{-1}(q'_i) \to p^{-1}(q_i)$ is a covering space, $\tilde{e}(v'_j) = -1$, $\tilde{e}(v'_j) = +1$, for $j = 2, \ldots, 2g$ and $\tilde{e}(q'_i) = +1$, for $i = 1, \ldots, r$.

From Lemma 2.3.9 it follows we have basis $\{\tilde{h}, \tilde{v}_j\}$ and $\{\tilde{h}, \tilde{q}_i\}$ for $\pi_1(\tilde{p}^{-1}(v'_j))$ and $\pi_1(\tilde{p}^{-1}(q'_i))$, respectively, such that $\varphi_\#(\tilde{h}) = h^n$, $\varphi_\#(\tilde{v}_j) = v_j h^{-s_j}$ and $\varphi_\#(\tilde{q}_i) = q_i h^{-k_i}$, for $j = 2, \ldots, 2g$ and for $i = 1, \ldots, r$.

Recall $\rho_1(1) = t_1$. By Lemma 2.3.10 there is a basis $\{\tilde{v}_1, \tilde{h}\}$ for $\pi_1(\tilde{p}^{-1}(v'_1))$ such that $\varphi_\#(\tilde{h}) = h^n$ and $\varphi_\#(\tilde{v}_1) = v_1 h^{-(t_1-1)}$. By construction, $\tilde{v}_j$ and $\tilde{q}_i$ intersect each fiber of $\tilde{p}^{-1}(v'_j)$ and $\tilde{p}^{-1}(q'_i)$, respectively, in exactly one point.
Since \( h \) anticommutes with \( v_1 \) we obtain 
\[ v_1^{-1}h^{s_2} = h^{-s_2}v_1^{-1}. \]
Then
\[ v_1h^{-(t_1-1)}v_2h^{-s_2}h^{(t_1-1)}v_1^{-1}h^{s_2}v_2^{-1} = v_1v_2v_1^{-1}v_2^{-1}h^{2s_2} \]
because \( h \) commutes with \( v_2 \).

Thus
\[
\varphi_\# \left( \tilde{q}_1 \cdots \tilde{q}_r \left( \prod_{j=1}^g [\tilde{v}_{2j-1}, \tilde{v}_{2j}] \right)^{-1} \right) \simeq q_1h^{-k_1} \cdots q_rh^{-k_r} \left( \prod_{j=1}^g [\varphi_\#(\tilde{v}_{2j-1}), \varphi_\#(\tilde{v}_{2j})] \right)^{-1} \\
\simeq h^{-\sum k_1}q_1 \cdots q_rh^{-2s_2} \left( \prod_{j=1}^g [v_{2j-1}, v_{2j}] \right)^{-1} \\
\simeq h^{-\sum k_i}q_1 \cdots q_rh^{-2s_2} \left( \prod_{j=1}^g [v_{2j-1}, v_{2j}] \right)^{-1} \\
\simeq 1 \ (\text{for } \sum k_i + 2s_2 = 0). 
\]

Thus \( \tilde{q}_1 \cdots \tilde{q}_r \left( \prod_{j=1}^g [\tilde{v}_{2j-1}, \tilde{v}_{2j}] \right)^{-1} \simeq 1 \) for \( \varphi_\# \) is injective. Then the curves \( \tilde{q}_1, \ldots, \tilde{q}_r \) span a surface \( G'_0 \) in \( M_0 \). After some isotopies of \( G'_0 \) in \( \tilde{M} \) fixing \( \partial G'_0 \), we obtain \( G'_0 \) is an orbit surface. After filling the holes of \( \tilde{M}_0 \), \( G'_0 \) gives rise to \( G' \) as required. \( \square \)

**Lemma 2.3.16** Suppose \( M = (\text{NnI}, g; \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r) \) is a Seifert manifold. Assume \( h \) is a regular fiber of \( M \). Let \( \omega : \pi_1(M_0) \to S_n \) be a representation such that \( \omega(h) = \varepsilon_n \), where \( \varepsilon_n = (1, 2, \ldots, n) \). By Remark 2.3.2, \( \omega : \pi_1(M_0) \to S_n \) is defined by
\[
\omega(h) = \varepsilon_n \\
\omega(q_i) = \varepsilon_n^{k_i}, \text{ for } i = 1,\ldots,r \text{ and} \\
\omega(v_j) = \varepsilon_n^{s_j}, \text{ for } j = 1,\ldots,g.
\]
where \( \sum k_i - 2 \sum s_j = 0 \).

Let \( \varphi : \tilde{M} \to M \) be the covering defined by \( \omega \).

Then there are an orbit surface \( G'_0 \) of \( \tilde{M}_0 \) and a basis \( \tilde{v}_1,\ldots,\tilde{v}_g \) for \( \pi_1(G'_0) \) and curves \( \tilde{q}_i \) in the boundary of \( G'_0 \) such that \( \varphi_\#(\tilde{q}_i) = q_ih^{-k_i}, \varphi_\#(\tilde{v}_j) = v_jh^{-s_j} \), for all \( j = 1,\ldots,g \).

In particular, we have an orbit surface \( G' \) of \( \tilde{M} \) such that \( \tilde{v}_1,\ldots,\tilde{v}_g \) is a basis for \( \pi_1(G') \).
Proof.

Let $p : M \to F$ be the orbit projection of $M$ and let $\tilde{p} : \tilde{M} \to G$ be the orbit projection of $\tilde{M}$.

Recall $F_0 = p(M_0)$ and $\{v_j\}$ is a basis of orientation reversing curves for $\pi_1(F)$. By Lemma 2.3.11, there exists a homeomorphism $\varphi_0 : G_0 \to F_0$, where $F_0 = p(M_0)$ and $G_0 = \tilde{p}(\varphi^{-1}(M_0))$. Then there exists a basis $\{v'_j, q'_i\}$, where $j = 1, \ldots, g$ and $i = 1, \ldots, r$, for $\pi_1(G_0)$ such that $\varphi_0(v'_j) = v_j$ and $\varphi_0(q'_i) = q_i$, for all $j = 1, \ldots, g$ and for $i = 1, \ldots, r$.

Recall the valuation homomorphism of $M$, $e : \pi_1(F) \to \mathbb{Z}_2$, is trivial. Let $\tilde{e} : \pi_1(G) \to \mathbb{Z}_2$ be the valuation homomorphism of $\tilde{M}$; by Lemma 2.3.12 we have that $\varphi| : \tilde{p}^{-1}(q'_i) \to p^{-1}(q_i)$ is a covering, $\tilde{e}(v'_j) = \tilde{e}(q'_i) = +1$, for $j = 1, \ldots, g$ and $i = 1, \ldots, r$.

From Lemma 2.3.9 it follows we have a basis $\{\tilde{h}, \tilde{q}_i\}$ for $\pi_1(\tilde{p}^{-1}(q'_i))$ such that $\varphi_{\#}(\tilde{h}) = h^n$ and $\varphi_{\#}(\tilde{q}_i) = q_i h^{-k_i}$.

Analogously, there is a basis $\{\tilde{v}_j, \tilde{h}\}$ for $\pi_1(\tilde{p}^{-1}(v'_j))$ such that $\varphi_{\#}(\tilde{h}) = h^n$ and $\varphi_{\#}(\tilde{v}_j) = v_j h^{-s_j}$, for $j = 1, \ldots, g$. Note that, by construction, $\tilde{v}_j$ and $\tilde{q}_i$ intersect each fiber of $\tilde{p}^{-1}(v'_j)$ and $\tilde{p}^{-1}(q'_i)$, respectively, in exactly one point.

Since $h$ commutes with $v_j$ and $q_i$, then:

$$\varphi_{\#}(\tilde{q}_1 \cdots \tilde{q}_r (\prod \tilde{v}_j^2)^{-1}) \simeq q_1 h^{-k_1} \cdots q_r h^{-k_r} (\prod (v_j h^{-s_j})^2)^{-1}$$

$$\simeq h^{-\sum k_i + 2 \sum s_j} q_1 \cdots q_r (\prod v_j^2)^{-1}, \text{ (recall } \sum k_i - 2 \sum s_j = 0.)$$

$$\simeq q_1 \cdots q_r (\prod v_j^2)^{-1},$$

$$\simeq 1.$$

Thus $\tilde{q}_1 \cdots \tilde{q}_r (\prod \tilde{v}_j^2)^{-1} \simeq 1$ for $\varphi_{\#}$ is injective.

Then the curves $\tilde{q}_1, \ldots, \tilde{q}_r$ span a surface $G'_0$ in $M_0$. After some isotopies of $G'_0$ in $\tilde{M}$ fixing $\partial G'_0$, we obtain $G'_0$ is an orbit surface. After filling the holes of $\tilde{M}_0$, $G'_0$ gives rise to $G'$ as required. \qed
Lemma 2.3.17 Suppose $M = (NnII, g; \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r)$ is a Seifert manifold. Assume $h$ is a regular fiber of $M$. Let $\omega : \pi_1(M_0) \to S_n$ be a representation such that $\omega(h) = \varepsilon_n$, where $\varepsilon_n = (1, 2, \ldots, n)$. By Remark 2.3.2, $\omega : \pi_1(M_0) \to S_n$ is defined by

$$
\begin{align*}
\omega(h) &= \varepsilon_n \\
\omega(q_i) &= \varepsilon_{n}^{k_i}, \text{ for } i = 1, \ldots, r, \\
\omega(v_1) &= \varepsilon_{n}^{s_1}, \\
\omega(v_j) &= \rho_j, \text{ for } j = 2, \ldots, g;
\end{align*}
$$

where $\sum k_i - 2s_1 = 0$ and $\rho_j$ is a reflection, for $j = 2, \ldots, g$. Assume $\rho_j(1) = t_j$, for $j = 2, \ldots, g$.

Let $\phi : \tilde{M} \to M$ be the covering defined by $\omega$.

Then there are an orbit surface $G'_0$ of $\tilde{M}_0$ and a basis $\tilde{v}_1, \ldots, \tilde{v}_g$ for $\pi_1(G'_0)$ such that $\varphi_\#(\tilde{q}_i) = q_i h^{-k_i}$, $\varphi_\#(\tilde{v}_1) = v_1 h^{-(s_1)}$ and $\varphi_\#(\tilde{v}_j) = v_j h^{-(t_j - 1)}$, for all $j = 2, \ldots, g$.

In particular, we have an orbit surface $G'$ of $\tilde{M}$ such that $\tilde{v}_1, \ldots, \tilde{v}_g$ is a basis for $\pi_1(G')$.

Proof.

Let $p : M \to F$ be the orbit projection of $M$ and let $\tilde{p} : \tilde{M} \to G$ be the orbit projection of $\tilde{M}$.

Recall $F_0 = p(M_0)$ and $\{v_j\}$ is a basis of orientation reversing curves for $\pi_1(F)$. By Lemma 2.3.11, there exists a homeomorphism $\varphi_0 : G_0 \to F_0$, where $F_0 = p(M_0)$ and $G_0 = \tilde{p}(\varphi^{-1}(M_0))$. Then there exists a basis $\{v'_j, q'_i\}$, where $j = 1, \ldots, g$ and $i = 1, \ldots, r$, for $\pi_1(G_0)$ such that $\varphi_0(v'_j) = v_j$ and $\varphi_0(q'_i) = q_i$, for all $j = 1, \ldots, g$ and for $i = 1, \ldots, r$.

Recall also the valuation homomorphism of $M$, $e : \pi_1(F) \to \mathbb{Z}_2$, is defined by $e(v_1) = +1$ and $e(v_j) = -1$, for $j = 2, \ldots, g$. Let $\tilde{e} : \pi_1(G) \to \mathbb{Z}_2$ be the valuation homomorphism of $\tilde{M}$; by Lemma 2.3.12 we have that $\varphi| : \tilde{p}^{-1}(q'_i) \to p^{-1}(q_i)$ is a covering, $\tilde{e}(v'_1) = \tilde{e}(q'_i) = +1$, for
i = 1, \ldots, r, and \partial_0(v'_j) = -1, if j = 2, \ldots, g.

By Lemma 2.3.9, we have basis \{h, v_i\} and \{\tilde{h}, \tilde{q}_i\} for \pi_1(\tilde{p}^{-1}(v'_i)) and \pi_1(\tilde{p}^{-1}(q'_i)) respectively, such that \varphi_#(\tilde{h}) = h^n, \varphi_#(v_i) = v_1 h^{-s_1} and \varphi_#(\tilde{q}_i) = q_i h^{-k_i}. Note that there is also a basis \{\tilde{v}_j, \tilde{h}\} for \pi_1(\tilde{p}^{-1}(v'_j)) such that \varphi_#(\tilde{h}) = h^n and \varphi_#(\tilde{v}_j) = v_j h^{-(t_j-1)}, for j = 2, \ldots, g, for Lemma 2.3.10. By construction, \tilde{v}_j and \tilde{q}_i intersect each fiber of \tilde{p}^{-1}(v'_j) and \tilde{p}^{-1}(q'_i), respectively, in exactly one point.

Since h anticommutes with v_1, then h^{-(t_j-1)}v_j = v_j h^{(t_j-1)} and h^{-2s_1}v_j = v_j h^{2s_1}. Consequently h^{-(t_j-1)}v_j h^{-(t_j-1)} = v_j, h^{-2s_1}v_j^2 = v_j^2 h^{-2s_1} and

\[
\varphi_#(\tilde{q}_1 \cdots \tilde{q}_r (\prod_{j=1}^r \tilde{v}_j^2)^{-1}) \simeq q_1 h^{-k_1} \cdots q_r h^{-k_r} ((v_1 h^{-s_1})^2 \prod_{j=2}^r v_j h^{-(t_j-1)} v_j h^{-(t_j-1)})^{-1} \\
\simeq h^{-\sum k_i + 2s_1} q_1 \cdots q_r (\prod_{j=1}^r \tilde{v}_j^2)^{-1}, \text{ (recall } \sum k_i - 2s_1 = 0. \text{)} \\
\simeq q_1 \cdots q_r (\prod \tilde{v}_j^2)^{-1}, \\
\simeq 1.
\]

Thus \tilde{q}_1 \cdots \tilde{q}_r (\prod \tilde{v}_j^2)^{-1} \simeq 1 for \varphi_# is injective.

Then the curves \tilde{q}_1, \ldots, \tilde{q}_r span a surface \tilde{G}'_0 in \tilde{M}_0. After some isotopies of \tilde{G}'_0 in \tilde{M} fixing \partial \tilde{G}'_0, we obtain \tilde{G}'_0 is an orbit surface. After filling the holes of \tilde{M}_0, \tilde{G}'_0 gives rise to \tilde{G}' as required. \hfill \Box

Lemma 2.3.18 Suppose \( M = (NnIII, g; \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r) \) is a Seifert manifold with orbit projection \( p: M \to F \). Assume \( h \) is a regular fiber of \( M \). Let \( \omega: \pi_1(M_0) \to S_n \) be a representation such that \( \omega(h) = \varepsilon_n \), where \( \varepsilon_n = (1, 2, \ldots, n) \). By Remark 2.3.2, \( \omega: \pi_1(M_0) \to S_n \) is defined by

\[
\omega(h) = \varepsilon_n \\
\omega(q_i) = \varepsilon^k_n, \text{ for } i = 1, \ldots, r, \\
\omega(v_1) = \varepsilon^{s_1}_n, \\
\omega(v_2) = \varepsilon^{s_2}_n, \text{ and} \\
\omega(v_j) = \rho_j, \text{ for } j = 3, \ldots, g;
\]
where $\sum k_i - 2s_1 - 2s_2 = 0$ and $\rho_j$ is a reflection, for $j = 3, \ldots, g$. Assume $\rho_j(1) = t_j$, for $j = 2, \ldots, g$.

Let $\varphi : \tilde{M} \to M$ be the covering defined by $\omega$.

Then there exists an orbit surface $G'_0$ of $\tilde{M}_0$ and a basis $\tilde{v}_1, \ldots, \tilde{v}_g$ for $\pi_1(G'_0)$ and curves $\tilde{q}_i$ in the boundary of $G'_0$ such that $\varphi_#(\tilde{q}_i) = q_i h^{-k_i}, \varphi_#(\tilde{v}_1) = v_1 h^{-(s_1)}, \varphi_#(\tilde{v}_2) = v_2 h^{-(s_2)}, \varphi_#(\tilde{v}_j) = v_j h^{-(t_j - 1)}$, for all $j = 3, \ldots, g$.

In particular, we have an orbit surface $G'$ of $\tilde{M}$ such that $\tilde{v}_1, \ldots, \tilde{v}_g$ is a basis for $\pi_1(G')$.

Proof.

Let $p : M \to F$ be the orbit projection of $M$ and let $\tilde{p} : \tilde{M} \to G$ be the orbit projection of $\tilde{M}$.

Recall $F_0 = p(M_0)$ and $\{v_j\}$ is a basis of orientation reversing curves for $\pi_1(F)$. By Lemma 2.3.11, there exists a homeomorphism $\overline{\varphi}_0 : G_0 \to F_0$, where $F_0 = p(M_0)$ and $G_0 = \tilde{p}(\varphi^{-1}(M_0))$. Then there exists a basis $\{v'_j, q'_i\}$, where $j = 1, \ldots, g$ and $i = 1, \ldots, r$, for $\pi_1(G_0)$ such that $\overline{\varphi}_0(v'_j) = v_j$ and $\overline{\varphi}_0(q'_i) = q_i$, for all $j = 1, \ldots, g$ and for $i = 1, \ldots, r$.

The valuation homomorphism of $M$, $e : \pi_1(F) \to \mathbb{Z}_2$, is defined by $e(v_1) = e(V_2) = +1$ and $e(v_j) = -1$, for $j = 3, \ldots, g$. Let $\tilde{e} : \pi_1(G) \to \mathbb{Z}_2$ be the valuation homomorphism of $\tilde{M}$; by Lemma 2.3.12 we have $\varphi : \tilde{p}^{-1}(q'_i) \to p^{-1}(q_i)$ is a covering, $\tilde{e}(v'_j) = \tilde{e}(v'_2) = \tilde{e}(q'_i) = +1$, for $i = 1, \ldots, r$, and $\tilde{e}(v'_j) = -1$, if $j = 3, \ldots, g$.

By Lemma 2.3.9, we have basis $\{\tilde{h}, \tilde{v}_1\}, \{\tilde{h}, \tilde{v}_2\}$ and $\{\tilde{h}, \tilde{q}_i\}$ for $\pi_1(\tilde{p}^{-1}(v'_i)), \pi_1(\tilde{p}^{-1}(v'_2))$ and $\pi_1(\tilde{p}^{-1}(q'_i))$, respectively, such that $\varphi_#(\tilde{h}) = h^n, \varphi_#(\tilde{v}_1) = v_1 h^{-s_1}, \varphi_#(\tilde{v}_2) = v_2 h^{-s_2}$ and $\varphi_#(\tilde{q}_i) = q_i h^{-k_i}$. Note that by Lemma 2.3.10 there is also a basis $\{\tilde{v}_j, \tilde{h}\}$ for $\pi_1(\tilde{p}^{-1}(v'_j))$ such that $\varphi_#(\tilde{h}) = h^n$ and $\varphi_#(\tilde{v}_j) = v_j h^{-(t_j - 1)}$, for $j = 3, \ldots, g$. By construction, $\tilde{v}_j$ and $\tilde{q}_i$ intersect each fiber of $\tilde{p}^{-1}(v'_j)$ and $\tilde{p}^{-1}(q'_i)$, respectively, in exactly one point.
Note that
\[
\varphi\left(\bar{q}_1 \cdots \bar{q}_r (\prod_{j=1}^g v_j^2)^{-1}\right) \simeq q_1 h^{-k_1} \cdots q_r h^{-k_r} ((v_1 h^{-s_1})^2 \prod_{j=2}^g v_j h^{-(t_j-1)} v_j h^{-(t_j-1)})^{-1} \\
\simeq h^{-\sum k_i + 2s_1} q_1 \cdots q_r (\prod_{j=1}^g v_j^2)^{-1}, \text{ (recall } \sum k_i - 2s_1 = 0 \text{.)} \\
\simeq q_1 \cdots q_r (\prod v_j^2)^{-1},
\]
because \( h \) commutes with \( v_1, v_2 \) and \( q_i \); and \( h \) anticommutes with \( v_j \), for \( j = 3, \ldots, g \).

Thus \( \bar{q}_1 \cdots \bar{q}_r (\prod v_j^2)^{-1} \simeq 1 \) because \( \varphi\# \) is injective.

Then the curves \( \bar{q}_1, \ldots, \bar{q}_r \) span a surface \( G'_0 \) in \( M_0 \). After some isotopies of \( G'_0 \) in \( \tilde{M} \) fixing \( \partial G'_0 \), we obtain \( G'_0 \) is an orbit surface. After filling the holes of \( \tilde{M} \), \( G'_0 \) gives rise to \( G' \) as required. \( \square \)

**Theorem 2.3.15** Let \( M = (X, g; \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r) \) be a Seifert manifold, where \( X \in \{Oo, On, No, NnI, NnII, NnIII\} \). Let \( h \) be a regular fiber of \( M \). Write \( M_0 = \tilde{M} - \bigcup_{i=1}^r V_i \), where each \( V_i \) is a fibered neighborhood of an exceptional fiber or a fibered neighborhood of a regular fiber, for \( i = 1, \ldots, r \), and \( V_i \) is homeomorphic (under a fiber preserving homeomorphism) to the torus \( T(\beta_i/\alpha_i) \). Assume \( n \in \mathbb{N} \). Let \( \omega : \pi_1(M_0) \to S_n \) be a representation such that \( \omega(h) = \varepsilon_n \), where \( \varepsilon_n = (1, 2, \ldots, n) \). Then
\[
\omega(q_i) = \varepsilon_n^{k_i}, \text{ for } i = 1, \ldots, r \text{ and} \\
\omega(v_j) = \tau_j,
\]
where \( \{h, v_j, q_i\} \) is a standard system of generators of \( \pi_1(M_0) \), and \( \tau_j \) is a power of \( \varepsilon_n \) if \( v_j \) commutes with \( h \), or a reflection if \( v_j \) anticommutes with \( h \).

Let \( \varphi : \tilde{M} \to M \) be the covering of \( M \) branched along fibers determined by \( \omega \). Then \( \tilde{M} \) is in the same class of \( M \) and the Seifert symbol of \( \tilde{M} \) is:
\[
(X, g; B_1/A_1, \ldots, B_r/A_r),
\]
with
\[ B_i = \frac{\beta_i + k_i \alpha_i}{\gcd\{n, \beta_i + k_i \alpha_i\}}, \]
\[ A_i = \frac{n \alpha_i}{\gcd\{n, \beta_i + k_i \alpha_i\}}, \]
where \( \gcd\{n, \beta_i + k_i \alpha_i\} \) denotes the greatest common divisor of \( n \) and \( \beta_i + k_i \alpha_i \).

**Proof.**

By Remark 2.3.2, \( \omega \) is defined as stated. Also \( \tilde{M} \) is in the same class of \( M \) because of Corollary 2.3.1.

Suppose that \( F \), of genus \( g \), is the orbit surface of \( M \). Recall \( F_0 = p(M_0), \tilde{M}_0 = \varphi^{-1}(M_0) \) and \( G_0 = \tilde{p}(\tilde{M}_0) \), where \( \tilde{p} : \tilde{M} \to G \) is the orbit projection of \( \tilde{M} \).

Let \( G \) be the orbit surface of \( \tilde{M} \).

By Lemma 2.3.11, there exists a homeomorphism \( \varphi_0 : G_0 \to F_0 \). Thus \( \partial G_0 \) has \( r \) components because \( \partial F_0 \) has \( r \) components. Therefore \( \partial \tilde{M}_0 \) has \( r \) components.

Note that we can obtain \( M \) from \( M_0 \) by gluing solid tori \( U_i \) to \( T_i \) with homeomorphisms \( f_i : \partial U_i \to T_i \) such that \( f_i(m_i) = q_i^{\alpha_i} h^{\beta_i} \), where \( m_i \) is a meridian of \( \partial V_i \).

Let \( G' \) be the orbit surface of \( \tilde{M} \) obtained in Lemmas 2.3.13, 2.3.14, 2.3.15, 2.3.16, 2.3.17 and 2.3.18. Recall that Lemmas 2.3.13, 2.3.14, 2.3.15, 2.3.16, 2.3.17 and 2.3.18 give us a basis \( \{\tilde{e}_j\} \) for \( \pi_1(G) \) and curves \( \tilde{q}_i \) in \( G \), such that, \( \varphi_{\#}(\tilde{q}_i) = q_i h^{-k_i} \).

Now we compute \( B_i \) and \( A_i \).

Because of \( m_i \sim q_i^{\alpha_i} h^{\beta_i} \), we have that \( \omega(m_i) = \omega(q_i^{\alpha_i} h^{\beta_i}) = e^{\beta_i + k_i \alpha_i} \). Let \( d_i = \gcd\{n, \beta_i + k_i \alpha_i\} \). Note that the order of \( \omega(m_i) \) is \( n/d_i \) and that \( \varphi^{-1}(m_i) \) has \( d_i \) components. Let \( \tilde{m}_i \) be a
component of $\varphi^{-1}(m_i)$, then

$$\varphi(\tilde{m}_i) = m_i^{n/d_i} = q_i^{\alpha_i/d_i} h^{n\beta_i/d_i}. \quad (2.4)$$

On the other hand, $\tilde{m}_i = q_i^{A_i} h^{B_i}$ for some $A_i$ and $B_i$ positive integer numbers such that $gcd\{A_i, B_i\} = 1$, then

$$\varphi(\tilde{m}_i) = (q_i h^{-k_i})^{A_i} h^{n B_i} = q_i^{A_i} h^{-A_i k_i + n B_i}. \quad (2.5)$$

Equating (2.4) and (2.5) we get that

$$B_i = \frac{\beta_i + k_i \alpha_i}{gcd\{n, \beta_i + k_i \alpha_i\}}, \text{ and}$$

$$A_i = \frac{n \alpha_i}{gcd\{n, \beta_i + k_i \alpha_i\}}.$$

□
Chapter 3

Heegaard genera of coverings of Seifert manifolds branched along fibers

3.1 Heegaard genera of Seifert manifolds

Theorem 3.1.1 [B-Z]

Let $M = (Oo, g; \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r)$ be a Seifert manifold; assume $\alpha_i > 1$, and $1 \leq i \leq r$.

i) If $M = (Oo, 0; 1/2, 1/2, \ldots, 1/2, \beta_r/(2\lambda+1))$, with $\lambda > 0$, $r$ even and $r \geq 4$, then $\text{rank}(\pi_1(M)) = r - 2 \leq h(M) \leq r - 1$.

ii) Suppose that $M$ does not belong to the case (i) and $r \geq 3$, then $\text{rank}(\pi_1(M)) = h(M) = 2g + r - 1$.

ii’) If $g > 0$ and $r = 2$, then $\text{rank}(\pi_1(M)) = h(M) = 2g + 1$.

iii) If $r = 1$, then $\text{rank}(\pi_1(M)) = h(M) = 2g$ if $\beta_1 = \pm 1$.

Otherwise, $\text{rank}(\pi_1(M)) = h(M) = 2g + 1$.

iii’) If $r = 0$, then $\text{rank}(\pi_1(M)) = h(M) = 2g$ if $\beta_1 = \pm 1$.

Otherwise $\text{rank}(\pi_1(M)) = h(M) = 2g + 1$.  

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CHAPTER 3. HEEGAARD GENERA OF COVERINGS OF SEIFERT MANIFOLDS

Theorem 3.1.2 \[B-Z\]

Let \( M = (On, g; \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r) \) be a Seifert Manifold; suppose \( \alpha_i > 1 \) and \( 1 \leq i \leq r \).

i) If \( r \geq 2 \), then \( h(M) = g + r - 1 \).

ii) Suppose \( r = 1 \).

(a) If \( \beta_1 = \pm 1 \), then \( h(M) = g \).

(b) If \( \beta_1 \neq \pm 1 \) is even, then \( h(M) = g + 1 \).

iii) If \( r = 0 \), then \( h(M) = g \) if \( \beta_1 = \pm 1 \); otherwise, \( h(M) = g + 1 \).

Remark 3.1.1 In Theorem 3.1.2, if \( \beta_1 \neq \pm 1 \) is odd, Boileau and Zieschang claimed but did not prove that \( h(M) = g + 1 \). According to \[Nu1\] this claim is correct.

Theorem 3.1.3 \[Nu\]

Let \( M \) be a non-orientable Seifert manifold.

(i) If \( M = (No, g; \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r) \), where \( \alpha_i > 1 \), then

(a) If \( r \geq 2 \), then \( h(M) = 2g + r - 1 \).

(b) Suppose \( r = 1 \). If \( \beta_1 \) is even, then \( h(M) = 2g + 1 \). If \( \beta_1 = 1 \), then \( h(M) = 2g \).

(c) Suppose \( r = 0 \). If \( \beta_1 \) is even then \( h(M) = 2g + 1 \). If \( \beta_1 \) is odd, then \( h(M) = 2g \).

Also, if \( r = 1 \) and \( \beta_1 \neq 1 \) is odd, then \( 2g \leq h(M) \leq 2g + 1 \).

(ii) If \( M = (Xx, g; \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r) \), where \( Xx \in \{NnI, NnII, NnIII\} \), and \( \alpha_i > 1 \); then:

(a) If \( r \geq 2 \), then \( h(M) = g + r - 1 \).

(b) Suppose \( r = 1 \). If \( \beta_1 \) is even, then \( h(M) = g + 1 \). If \( \beta_1 = 1 \), then \( h(M) = g \).

(c) Suppose \( r = 0 \). If \( \beta_1 \) is even, then \( h(M) = g + 1 \). If \( \beta_1 \) is odd, then \( h(M) = g \).

Also, if \( r = 1 \) and \( \beta_1 \neq 1 \) is odd, then \( g \leq h(M) \leq g + 1 \).
3.2 Heegaard genera of coverings

Let $M$ be a Seifert manifold with orbit projection $p : M \to F$. Assume $\varphi : \tilde{M} \to M$ is a covering of $M$ branched along fibers. In this section we compare the Heegaard genus of $\tilde{M}$, $h(\tilde{M})$, with the Heegaard genus of $M$, $h(M)$. We always will assume that $M$ is not in the following list:

(a) $M = (\text{On}, 1; \beta_1/\alpha_1)$, $\alpha_1 \geq 1$

(b) $M = (\text{Oo}, 0; \beta_1/\alpha_1, \beta_2/\alpha_2)$, $\alpha_i \geq 1$

(c) $M = (\text{Oo}, 0; \beta_1/2, \beta_2/2, \beta_3/m)$

(d) $M = (\text{Oo}, 0; \beta_1/2, \beta_2/3, \beta_3/3)$

(e) $M = (\text{Oo}, 0; \beta_1/2, \beta_2/3, \beta_3/4)$

(f) $M = (\text{Oo}, 0; \beta_1/2, \beta_2/3, \beta_3/5)$

We take out the cases (a) – (f) because these manifolds have finite fundamental group and in this cases $S^3$ is the universal covering of $M$. Thus $h(M) > h(S^3) = 0$ if $\pi_1(M) \neq 1$.

(g) $M = (\text{Oo}, 0; 1/2, 1/2, \ldots, 1/2, \beta_r/(2\lambda + 1))$, with $\lambda > 0$, $r$ even and $r \geq 4$.

(h) $M = (\text{Zz}, g; \beta/\alpha)$, with $\text{Zz} \in \{\text{No}, \text{NnI}, \text{NnII}, \text{NnIII}\}$, $\beta \neq 1$, $\beta$ odd and $\alpha \geq 2$. (Non-orientable Seifert manifolds with exactly one exceptional fiber and $\beta \neq 1$ odd.)

We rule out (g) y (h) because we can not compute $h(M)$ precisely. In case (g), we only know $r - 2 \leq h(M) \leq r - 1$ and in case (h), $h(M)$ satisfies $2g \leq h(M) \leq 2g + 1$.

Let $M$ be a Seifert manifold and $\{h_i\}_{i=1}^r$ be a set of fibers of $M$ which contains all the exceptional fibers and a finite number of regular fibers. Recall each fiber has a neighborhood $V_i$ fiber preserving homeomorphic to a solid fibered torus $T(\beta_i/\alpha_i)$ be the fibered solid torus homeomorphic to $V_i$, for $i = 1, \ldots, r$. Note that $\alpha_i$ and $\beta_i$ are coprime numbers and $\alpha_i \geq 1$. 
Define \( M_0 = M - \bigcup V_i \).

Suppose \( \varphi : \tilde{M} \to M \) is a covering of \( M \) branched along fibers and \( \tilde{M} \) is connected. By Theorem 2.3.1, we know that there are \( \psi : \tilde{M} \to M' \) and \( \zeta : M' \to M \) branched coverings such that the following diagram is commutative

\[
\begin{array}{ccc}
\tilde{M} & \xrightarrow{\psi} & M' \\
\downarrow{\varphi} & & \downarrow{\zeta} \\
M & & \\
\end{array}
\]

Also if \( \omega_\psi \) and \( \omega_\zeta \) are the representations associated to \( \psi \) and \( \zeta \), respectively, we have that \( \omega_\psi(h') = \varepsilon_t \) and \( \omega_\zeta(h) = (1) \), where \( (1) \) is the identity permutation in \( S_n \) and \( \varepsilon_t = (1, 2, \ldots, t) \); \( h \) and \( h' \) are regular fibers of \( M \) and \( M' \), respectively. Thus we will only consider representations \( \omega(\pi_1(M_0)) \to S_n \) such that \( \omega(h) = (1) \) and \( \omega(h) = \varepsilon_n \), where \( h \) is a regular fiber of \( M \).

Along this section we use the following notation:

- \( M \) is a Seifert manifold with orbit projection \( p : M \to F \), and \( h \) is a regular fiber of \( M \).
- The surface \( F \) has genus \( g \). Let \( \{h_i\}_{i=1}^r \) be a set of fibers of \( M \) which contains all the exceptional fibers and some regular fibers. Recall each fiber has a neighborhood \( V_i \) fiber preserving homeomorphic to a fibered solid torus \( T(\beta_i/\alpha_i) \), for \( i = 1, \ldots, r \).
- \( \{v_j\} \) is a basis for \( \pi_1(F) \) and we assume \( v_j \) is orientation reversing if \( F \) is non-orientable, for each \( j \).
- \( M_0 = M - \bigcup_{i=1}^r V_i \).
  Note that \( \partial M_0 \) has \( r \) components; \( T_1, \ldots, T_r \)
- \( q_i = p(T_i) \).
- \( \omega : \pi_1(M_0) \to S_n \) is a transitive representation.
3.2. HEEGAARD GENERA OF COVERINGS

- The identity permutation in $S_n$ is denoted by (1) and the standard $n$–cycle $(1, \ldots, n)$ is denoted by $\varepsilon_n$.

- $\varphi : \tilde{M} \to M$ is the covering branched along fibers of $M$ associated to the representation $\omega : \pi_1(M_0) \to S_n$ and $\tilde{p} : \tilde{M} \to G$ is the orbit projection of $\tilde{M}$.

- The surface $G$ has genus $\tilde{g}$.

- The natural number $n$ is always greater than 2. Otherwise, if $n = 1$ then $\varphi$ would be a homeomorphism.

- The Heegaard genus of $M$ is denoted by $h(M)$.

### 3.2.1 Heegaard genera when $\omega(h) = (1)$

Let $M = (Xx, g; \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r)$ be a Seifert manifold, where $Xx \in \{Oo, On, No, NnI, NnII, NnIII\}$. Suppose that $\omega : \pi_1(M_0) \to S_n$ is a transitive representation defined by $\omega(h) = (1)$, $\omega(q_i) = \sigma_{i,1} \cdots \sigma_{i,\ell_i}$, for $i = 1, \ldots, r$ and $\omega(v_j) = \rho_{j,1} \cdots \rho_{j,s_j}$; where $\sigma_{i,1} \cdots \sigma_{i,\ell_i}$ and $\rho_{j,1} \cdots \rho_{j,s_j}$ are the disjoint cycle decompositions of $\omega(q_i)$ and $\omega(v_j)$, respectively.

By Theorem 2.3.8,

- **a)** If $F$ is non-orientable, $\tilde{M}$ is the manifold

  $$(Yy, \tilde{g}; \frac{B_{1,1}}{A_{1,1}}, \ldots, \frac{B_{1,\ell_1}}{A_{1,\ell_1}}, \ldots, \frac{B_{r,1}}{A_{r,1}}, \ldots, \frac{B_{r,\ell_r}}{A_{r,\ell_r}}),$$

  where $Yy \in \{Oo, On, No, NnI, NnII, NnIII\}$ and it is determined by Theorems 2.3.3, 2.3.5, 2.3.6 and 2.3.7. If $G$ is orientable, then

  $$\tilde{g} = 1 - \frac{n(2 - g)}{2} + \sum_{i=1}^r \ell_i - nr.$$
If $G$ is non-orientable, then
\[ \tilde{g} = n(g - 2) + 2 + nr - \sum_{i=1}^{r} \ell_i. \]

b) If $F$ is orientable, then $\tilde{M}$ is the manifold
\[ (Y y, \tilde{g}; \frac{B_{1,1}}{A_{1,1}}, \ldots, \frac{B_{1,\ell_1}}{A_{1,\ell_1}}, \ldots, \frac{B_{r,1}}{A_{r,1}}, \ldots, \frac{B_{r,\ell_r}}{A_{r,\ell_r}}), \]
where $Y y \in \{Oo, No\}$ and it is determined by Theorems 2.3.2 and 2.3.4; and
\[ \tilde{g} = 1 + n(g - 1) + \frac{nr - \sum_{i=1}^{r} \ell_i}{2}. \]

The numbers $B_{i,k}$ and $A_{i,k}$ in the Seifert symbol for $\tilde{M}$ in (a) and (b) are:
\[ B_{i,k} = \frac{\text{order}(\sigma_{i,k}) \cdot \beta_i}{\text{gcd}\{\alpha_i, \text{order}(\sigma_{i,k})\}}, \text{ and} \]
\[ A_{i,k} = \frac{\alpha_i}{\text{gcd}\{\alpha_i, \text{order}(\sigma_{i,k})\}}, \]
where $\text{gcd}\{\alpha_i, \text{order}(\sigma_{i,k})\}$ denotes the greatest common divisor of $\alpha_i$ and $\text{order}(\sigma_{i,k})$.

We highlight the following equations for future reference.

Note that $n \geq \ell_i \geq 1$, for all $i = 1, \ldots, r,$ \hspace{1cm} (3.1)
because $\ell_i$ is the number of disjoint cycles of $\omega(q_i)$ and
\[ A_{i,k} = 1, \text{ if and only if, } \alpha_i | \text{order}(\sigma_{i,k}) \hspace{1cm} (3.2) \]
since the definition of $A_{i,k}$.

Let $a$ be a positive number. Assume $n > 1$. Then
\[ n(a - 2) + 2 \geq a \text{ if and only if } a \geq 2 \hspace{1cm} (3.3) \]
and
\[ 2 + 2n(a - 1) \geq 2a \text{ if and only if } a \geq 1. \hspace{1cm} (3.4) \]
Lemma 3.2.1 Let $M = (Xx, g; \beta_1/1)$, where $Xx \in \{Oo, On, No, NnI, NnII, NnIII\}$. Consider a transitive representation $\omega : \pi_1(M_0) \to S_n$ defined by

\[
\begin{align*}
\omega(h) &= (1), \\
\omega(q_1) &= \sigma_1 \cdots \sigma_{\ell_1}, \text{ and} \\
\omega(v_j) &= \rho_{j,1} \cdots \rho_{j,s_j},
\end{align*}
\]

where $\sigma_{i,1} \cdots \sigma_{i,\ell_i}$ and $\rho_{j,1} \cdots \rho_{j,s_j}$ are the disjoint cycle decompositions of $\omega(q_i)$ and $\omega(v_j)$, respectively.

By Theorem 2.3.8, we have that $\tilde{M} = (Yy, \tilde{g}; B_1/A_1, \cdots, B_{\ell_1}/A_{\ell_1})$, with $B_k = \text{order}(\sigma_k) \cdot \beta_1$ and $A_k = 1$, for $k = 1, \ldots, \ell_1$. Let $p : M \to F$ be the orbit projection of $M$. Let $g$ be the genus of $F$. Then:

(a) If $F$ is non-orientable, then $h(\tilde{M}) = n(g - 2) + n - \ell_1 + 3$.

(b) If $F$ is orientable, then $h(\tilde{M}) = 2n(g - 1) + n - \ell_1 + 3$

Proof.

By Theorem 2.2.1, we can assume $\tilde{M} = (Yy, \tilde{g}; n\beta_1/1)$. Note that $n\beta_1 \neq 1$ for $n \geq 2$ and $\beta_1$ is an integer number. Also $n\beta_1$ is even if $\beta_1$ is even, this implies that we can compute $h(\tilde{M})$, if $\tilde{M}$ is non-orientable.

(a) Suppose $F$ is non-orientable.

(i) If $G$ is non-orientable, then $\tilde{g} = n(g - 2) + 2 + n - \ell_1$, by Lemma 2.3.8. Since $n\beta_1 \neq 1$, then

\[h(\tilde{M}) = \tilde{g} + 1 = n(g - 2) + n - \ell_1 + 3.\]

(ii) If $G$ is orientable, by Lemma 2.3.8, $2\tilde{g} = n(g - 2) + 2 + n - \ell_1$. Thus

\[h(\tilde{M}) = 2\tilde{g} + 1 = n(g - 2) + n - \ell_1 + 3,\]

for $n\beta_1 \neq 1$. 
Therefore
\[ h(\tilde{M}) = 2\tilde{g} + 1 = n(g - 2) + n - \ell_1 + 3. \]

(b) Suppose \( F \) is orientable. Then \( G \) is orientable and by Lemma 2.3.8 we know \( 2\tilde{g} = 2n(g - 1) + n - \ell_1 + 2. \) Since \( n\beta_1 \neq 1 \) we obtain
\[ h(\tilde{M}) = 2\tilde{g} + 1 = 2\tilde{g} = 2n(g - 1) + n - \ell_1 + 3. \]

\[ \square \]

**Corollary 3.2.1** Let \( M = (Xx, g; \beta_1/1) \), where \( Xx \in \{O_o, On, No, NnI, NnII, NnIII\} \).

Consider a transitive representation \( \omega: \pi_1(M_0) \rightarrow S_n \) defined by
\[
\begin{align*}
\omega(h) &= (1), \\
\omega(q_1) &= \sigma_1 \cdots \sigma_{\ell_1}, \text{ and} \\
\omega(v_j) &= \rho_{j,1} \cdots \rho_{j,s_j},
\end{align*}
\]
where \( \sigma_{i,1} \cdots \sigma_{i,\ell_i} \) and \( \rho_{j,1} \cdots \rho_{j,s_j} \) are the disjoint cycle decompositions of \( \omega(q_i) \) and \( \omega(v_j) \), respectively.

Let \( \varphi: \tilde{M} \rightarrow M \) be the covering of \( M \) branched along fibers associated to \( \omega \). Then \( h(\tilde{M}) \geq h(M) \)

**Proof.**

Consider the following cases:

**First case.** \( F \) is non-orientable. By Lemma 3.2.1, \( h(\tilde{M}) = 2\tilde{g} + 1 = n(g - 2) + n - \ell_1 + 3. \)

Recalling Equations 3.3 and 3.1 we conclude \( h(\tilde{M}) \geq h(M) \).

**Second case.** \( F \) is orientable. Then \( h(\tilde{M}) = 2\tilde{g} + 1 = 2\tilde{g} = 2n(g - 1) + n - \ell_1 + 3 \) for Lemma 3.2.1. By Equation 3.4 we obtain \( h(\tilde{M}) \geq h(M) \).
Lemma 3.2.2 Let $M = (X, g; \beta_1/\alpha_1)$ with $\alpha_1 \geq 2$. Consider a transitive representation $\omega: \pi_1(M_0) \rightarrow S_n$ defined by
\[
\begin{align*}
\omega(h) &= (1), \\
\omega(q_1) &= \sigma_1 \cdots \sigma_{\ell_1}, \quad \text{and} \\
\omega(v_j) &= \rho_{j,1} \cdots \rho_{j,s_j},
\end{align*}
\]
where $\sigma_{1,1} \cdots \sigma_{1,\ell_1}$ and $\rho_{j,1} \cdots \rho_{j,s_j}$ are the disjoint cycle decompositions of $\omega(q_1)$ and $\omega(v_j)$, respectively.

Let $\varphi: \tilde{M} \rightarrow M$ be covering associated to $\omega$. By Theorem 2.3.8, we have
\[
\tilde{M} = (Y, \tilde{g}; B_1/A_1, \cdots, B_{\ell_1}/A_{\ell_1}),
\]
where
\[
B_k = \frac{\text{order}(\sigma_k) \cdot \beta_1}{\text{gcd}\{\alpha_1, \text{order}(\sigma_k)\}},
\]
and
\[
A_k = \frac{\alpha_1}{\text{gcd}\{\alpha_1, \text{order}(\sigma_k)\}}.
\]
Recall $\text{gcd}\{\alpha_1, \text{order}(\sigma_k)\}$ denotes the greatest common divisor of $\alpha_1$ and $\text{order}(\sigma_k)$.

Let $k_1 = \#\{\sigma_k : \alpha_1 \nmid \text{order}(\sigma_k)\}$. Then:

(a) Assume $F$ is non-orientable.

1. Suppose $k_1 = 0$. If $\beta_1 = 1$, $n = \alpha_1$ and $\omega(q_1) = (1, 2, \ldots, \alpha_1)$, then
   \[h(\tilde{M}) = n(g - 2) + n - \ell_1 + 3.\]
   Otherwise, $h(\tilde{M}) = n(g - 2) + n - \ell_1 + 3$.

2. Suppose $k_1 = 1$. Then $h(\tilde{M}) = n(g - 2) + n - \ell_1 + 3$

3. Suppose $k_1 \geq 2$, then $h(\tilde{M}) = n(g - 2) + n - \ell_1 + k_1 + 1$.

(b) Assume $F$ is orientable.

1. Suppose $k_1 = 0$. If $\beta_1 = 1$, $n = \alpha_1$ and $\omega(q_1) = (1, 2, \ldots, \alpha_1)$, then $h(\tilde{M}) = 2n(g - 1) + n - \sum \ell_1 + 2$.
   Otherwise, $h(\tilde{M}) = 2n(g - 1) + n - \ell_1 + 3$.

2. Suppose $k_1 = 1$, then $h(\tilde{M}) = 2n(g - 1) + n - \ell_1 + 3$.

3. Suppose $k_1 \geq 2$, then $h(\tilde{M}) = 2n(g - 1) + n - \ell_1 + k_1 + 1$. 
CHAPTER 3. HEEGAARD GENERA OF COVERINGS OF SEIFERT MANIFOLDS

Proof.

Note that $A_i = 1$ if and only if $\alpha_1 | \text{order}(\sigma_i)$. Thus $k_1$ is the number of exceptional fibers of $\tilde{M}$. Let $G$ be the orbit surface of $\tilde{M}$ and let $\tilde{g}$ of $G$.

(a) Suppose $F$ is non-orientable.

1. Assume $k_1 = 0$. Then $\alpha_1 | \text{order}(\sigma_k)$, for all $k = 1, \ldots, \ell_1$. Thus there are integer numbers $p_k > 0$ such that $\text{order}(\sigma_k) = p_k \alpha_1$. Hence, by Theorem 2.2.1 we can assume that $\tilde{M} = (Y y, \tilde{\gamma}; B/1)$, where $B = \beta_1 \sum p_k$. Also, if $\beta_1$ is even then $B$ is even; then it is possible to compute the Heegaard genus of $\tilde{M}$ when $\beta_1$ is even. Note that $B = 1$ if and only if $\beta_1 = 1, n = \alpha_1$ and $\omega(q_1) = (1, 2, \ldots, \alpha_1)$.

(i) If $G$ is non-orientable, then $\tilde{g} = n(g - 2) + 2 + n - \ell_1$ due to Theorem 2.3.8.

Therefore, from Theorems 3.1.1, 3.1.2 and 3.1.3 we obtain that $h(\tilde{M}) = \tilde{g} = n(g - 2) + n - \ell_1 + 2$, if $\beta_1 = 1, n = \alpha_1$ and $\omega(q_1) = (1, 2, \ldots, \alpha_1)$; Otherwise, $h(\tilde{M}) = \tilde{g} + 1 = n(g - 2) + n - \ell_1 + 3$.

(ii) If $G$ is orientable, then $2\tilde{g} = n(g - 2) + 2 + n - \ell_1$ due to Theorem 2.3.8.

Therefore, from Theorem 3.1.1, 3.1.2 and 3.1.3 we obtain that $h(\tilde{M}) = \tilde{g} = n(g - 2) + n - \ell_1 + 2$, if $n = \alpha_1$ and $\omega(q_1) = (1, 2, \ldots, \alpha_1)$; Otherwise, $h(\tilde{M}) = \tilde{g} + 1 = n(g - 2) + n - \ell_1 + 3$.

2. Assume $k_1 = 1$. By renumbering the indices, if necessary, we can assume that $A_1 \geq 2$ and $A_m = 1$, for each $m = 2, \ldots, \ell_1$. Then there are integer numbers $p_m > 0$ such that $\text{order}(\sigma_m) = p_m \alpha_1$, for all $m \in \{2, \ldots, \ell_1\}$. Thus, by Theorem 2.2.1 we have that $\tilde{M} = (Y y, \tilde{\gamma}; B/A_1)$, where

$$B = B_1 + \beta_1 A_1 \sum p_m = \frac{\beta_1 (\text{order}(\sigma_1) + \alpha_1 \sum p_m)}{\gcd(\alpha_1, \text{order}(\sigma_1))}$$

Note that $B$ is an even number if $\beta_1$ is even. Then we always can compute the Heegaard genus of $\tilde{M}$.

Suppose that $B = 1$. Then $\gcd(\alpha_1, \text{order}(\sigma_1)) = \beta_1 (\text{order}(\sigma_1) + \alpha_1 \sum p_m)$. From this fact we obtain $\beta_1 | \alpha_1$ and $(\text{order}(\sigma_1) + \alpha_1 \sum p_m) | \text{order}(\sigma_1)$, consequently, $\beta_1 = 1$
and \( \alpha_1 \sum p_m = 0 \). Since \( \alpha_1 > 0 \) we conclude \( \sum p_m = 0 \). Thus \( p_m = 0 \). This contradicts our assumption of \( p_m > 0 \).

Therefore \( B \neq 1 \).

(i) If \( G \) is non-orientable, then \( \tilde{g} = n(g - 2) + n - \ell_1 + 1 \). Hence by Theorems 3.1.1, 3.1.2 and 3.1.3 we obtain \( h(\tilde{M}) = 2\tilde{g} + 1 = n(g - 2) + n - \ell_1 + 3 \).

(ii) If \( G \) is orientable, then \( 2\tilde{g} = n(g - 2) + n - \ell_1 + 1 \). By Theorems 3.1.1, 3.1.2 and 3.1.3 we conclude \( h(\tilde{M}) = \tilde{g} + 1 = n(g - 2) + n - \ell_1 + 3 \).

3. Assume \( k_1 \geq 2 \). Recall \( k_1 \) is the number of exceptional fibers of \( \tilde{M} \).

(i) If \( G \) is non-orientable, from Theorem 2.3.8 we obtain that \( \tilde{g} = n(g - 2) + n - \ell_1 + 2 \). By Theorems 3.1.1, 3.1.2 and 3.1.3 we conclude \( h(\tilde{M}) = \tilde{g} + k_1 - 1 = n(g - 2) + n - \ell_1 + k_1 + 1 \).

(ii) If \( G \) is orientable, by Theorem 2.3.8 we know that \( 2\tilde{g} = n(g - 2) + n - \ell_1 + 2 \). Since \( k_1 \) is the number of exceptional fibers of \( \tilde{M} \) we have \( h(\tilde{M}) = 2\tilde{g} + k_1 - 1 = n(g - 2) + n - \ell_1 + k_1 + 1 \).

(b) Suppose \( F \) is orientable, then \( G \) is orientable and \( 2\tilde{g} = 2n(g - 1 + n - \ell_1) + 2 \) due to Theorem 2.3.8.

1. If \( k_1 = 0 \), then \( \alpha_1 |o(\sigma_k) \), for all \( k = 1, \ldots, \ell_1 \). Thus there are integer numbers \( p_k > 0 \) such that \( \text{order}(\sigma_k) = p_k\alpha_1 \). Hence, by Theorem 2.2.1 we can assume that \( \tilde{M} = (Y, \tilde{g}; B/1) \), where \( B = \beta_1 \sum p_k \). Also, if \( \beta_1 \) is even then \( B \) is even; then it is possible to compute the Heegaard genus of \( \tilde{M} \) when \( \beta_1 \) is even. Note that \( B = 1 \) if and only if \( \beta_1 = 1, n = \alpha_1 \) and \( \omega(q_1) = (1, 2, \ldots, \alpha_1) \). Therefore \( h(\tilde{M}) = 2\tilde{g} = 2n(g - 1) + n - \ell_1 + 2 \), if \( n = \alpha_1 \) and \( \omega(q_1) = (1, 2, \ldots, \alpha_1) \). Otherwise, \( h(\tilde{M}) = 2\tilde{g} + 1 = 2n(g - 1) + n - \ell_1 + 3 \).

2. If \( k_1 = 1 \), by renumbering the indices, if necessary, we can suppose that \( A_1 \geq 2 \) and \( A_m = 1 \), for each \( m = 2, \ldots, \ell_1 \). Then there exist integer numbers \( p_m > 0 \) such that \( \text{order}(\sigma_m) = p_m\alpha_1 \), for all \( m \in \{2, \ldots, \ell_1\} \). By Theorem 2.2.1, we can assume
\[ \tilde{M} = (Y, \tilde{g}; B/A_1), \]

where

\[
B = B_1 + \beta_1 A_1 \sum p_m = \frac{\beta_1 (\text{order}(\sigma_1) + \alpha_1 \sum p_m)}{\text{gcd}(\alpha_1, \text{order}(\sigma_1))}
\]

Note that \( B \) is an even number if \( \beta_1 \) is even. Then we always can compute the Heegaard genus of \( \tilde{M} \).

Suppose that \( B = 1 \). Then \( \text{gcd}(\alpha_1, \text{order}(\sigma_1)) = \beta_1 (\text{order}(\sigma_1) + \alpha_1 \sum p_m) \). From this fact we obtain \( \beta_1|\alpha_1 \) and \( (\text{order}(\sigma_1) + \alpha_1 \sum p_m)|\text{order}(\sigma_1) \), consequently, \( \beta_1 = 1 \) and \( \alpha_1 \sum p_m = 0 \). Since \( \alpha_1 > 0 \) we conclude \( \sum p_m = 0 \). Thus \( p_m = 0 \) and we obtain a contradiction to our assumption \( p_m > 0 \).

Therefore \( B \neq 1 \) and \( h(\tilde{M}) = 2\tilde{g} + 1 = 2n(g - 1) + n - \ell_1 + 3 \).

3. If \( k_1 \geq 2 \), then \( h(\tilde{M}) = 2\tilde{g} + k_1 - 1 \) since \( k_1 \) is the number of exceptional fibers. Therefore \( h(\tilde{M}) = 2n(g - 1) + n - \ell_1 + k_1 + 1 \). \( \square \)

**Corollary 3.2.2** Let \( M = (X, g; \beta_1/\alpha_1) \) where \( X \in \{\text{Oo, On, No, NnI, NnII, NnIII}\} \) and \( \alpha_1 \geq 2 \). Consider a transitive representation \( \omega : \pi_1(M_0) \to S_n \) defined by

\[
\begin{align*}
\omega(h) &= (1), \\
\omega(q_1) &= \sigma_1 \cdots \sigma_{\ell_1}, \quad \text{and} \\
\omega(v_j) &= \rho_{j,1} \cdots \rho_{j,s_j},
\end{align*}
\]

where \( \sigma_1 \cdots \sigma_{\ell_1} \) and \( \rho_{j,1} \cdots \rho_{j,s_j} \) are the disjoint cycle decompositions of \( \omega(q_1) \) and \( \omega(v_j) \), respectively.

Let \( \varphi : \tilde{M} \to M \) be covering associated to \( \omega \). Then \( h(\tilde{M}) \geq h(M) \).

**Proof.**
Recall $F$ and $G$ are the orbit surfaces of $M$ and $\tilde{M}$, respectively. Let $k_1$ be as in previous lemma.

(a) Suppose $F$ is non-orientable. Then $g \geq 2$ because $g = 1$ implies $M$ has finite fundamental group.

1. Assume $k_1 = 0$.

If $\beta_1 = 1$, $n = \alpha_1$ and $\omega(q_1) = (1, \ldots, \alpha_1)$, then $h(\tilde{M}) = n(g-2) + n - \ell_1 + 2$, by Lemma 3.2.2. Notice that $h(M) = g$ because $\beta = 1$. From Equation 3.3 we get that $n(g-2) + 2 \geq g$. Equation 3.1 yields to $n \geq \ell_1$. Therefore $h(\tilde{M}) \geq h(M)$.

If $\beta_1 \neq 1$ or $n \neq \alpha_1$ or $\omega(q_1) \neq (1, \ldots, \alpha_1)$, then $h(\tilde{M}) = n(g-2) + n - \ell_1 + 3$. Recalling Equations 3.3 and 3.1 we obtain that $n(g-2) + 2 \geq g$ and $n - \ell_1 \geq 0$. Therefore $h(\tilde{M}) \geq g + 1 \geq h(M)$.

2. Assume $k_1 = 1$. From Lemma 3.2.2 we know that $h(\tilde{M}) = n(g-2) + n - \ell_1 + 3$.

Using again Equations 3.3 and 3.1 we conclude $h(\tilde{M}) \geq g + 1 \geq h(M)$.

3. Assume $k_1 \geq 2$. Then $h(\tilde{M}) = n(g-2) + n - \ell_1 + k_1 + 1$ because of Lemma 3.2.2. Since $k_1 \geq 2$, Equation 3.3 implies that $n(g-2) + k_1 \geq g$. By Equation 3.1, we conclude that $h(\tilde{M}) \geq h(M)$ as we stated.

(b) Suppose $F$ is orientable. Note that $F$ is not $S^2$, otherwise $M$ would be a Seifert manifold with finite fundamental group and we do not want $M$ with finite fundamental group. Thus $g \geq 1$.

1. Suppose $k_1 = 0$.

If $\beta = 1$, $n = \alpha_1$ and $\omega(q_1) = (1, \ldots, \alpha_1)$, then $h(\tilde{M}) = 2n(g-1) + n - \ell_1 + 2$ for Lemma 3.2.2. Also $h(M) = 2g$ because $\beta = 1$. Since $g \geq 1$, using Equation 3.4 we obtain that $2n(g-1) + 2 \geq 2g$. From Equation 3.1 we conclude $h(\tilde{M}) \geq h(M)$. 
CHAPTER 3. HEEGAARD GENERA OF COVERINGS OF SEIFERT MANIFOLDS

If $\beta \neq 1$ or $n \neq \alpha_1$ or $\omega(q_1) \neq (1, \ldots, \alpha_1)$, then $h(\tilde{M}) = 2n(g-1) + n - \ell_1 + 3$. By Equations 3.4 and 3.1, we conclude $h(\tilde{M}) \geq 2g + 1 \geq h(M)$.

2. Suppose $k_1 = 1$. In this case, $h(\tilde{M}) = 2n(g-1) + n - \ell_1 + 3$. Hence Equations 3.4 and 3.1 let us conclude that $h(\tilde{M}) \geq 2g + 1 \geq h(M)$.

3. Suppose $k_1 \geq 2$. From Lemma 3.2.2 we obtain that $h(\tilde{M}) = 2n(g-1) + n - \ell_1 + k_1$. Equation 3.4 yields to $2n(g-1) + k_1 \geq 2g$. From Equation 3.1 we obtain $h(\tilde{M}) \geq h(M)$.

Lemma 3.2.3 Let $M = (Xx, g; \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r)$, where $Xx \in \{Oo, On, No, NnI, NnII, NnIII\}$, $\alpha_i \geq 2$, for each $i \in \{1, \ldots, r\}$, and $r \geq 2$ (a Seifert manifold with at least two exceptional fibers). Consider the transitive representation $\omega: \pi_1(M_0) \to S_n$ defined by

- $\omega(h) = (1)$,
- $\omega(q_i) = \sigma_{i,1} \cdots \sigma_{i,\ell_i}$, for $i = 1, \ldots, r$ and
- $\omega(v_j) = \rho_{j,1} \cdots \rho_{j,s_j}$,

where $\sigma_{i,1} \cdots \sigma_{i,\ell_i}$ and $\rho_{j,1} \cdots \rho_{j,s_j}$ are the disjoint cycle decompositions of $\omega(q_i)$ and $\omega(v_j)$, respectively.

Let $\varphi: \tilde{M} \to M$ be the covering associated to $\omega$. By Theorem 2.3.8,

$$\tilde{M} = (Yy, \tilde{g}; A_{1,1}, B_{1,1}, A_{1,\ell_1}, B_{1,\ell_1}, \ldots, A_{r,1}, B_{r,1}, A_{r,\ell_r}, B_{r,\ell_r}),$$

where

$$B_{i,k} = \frac{\text{order}(\sigma_{i,k}) \cdot \beta_i}{\gcd\{\alpha_i, \text{order}(\sigma_{i,k})\}},$$

and

$$A_{i,k} = \frac{\alpha_i}{\gcd\{\alpha_i, \text{order}(\sigma_{i,k})\}}.$$

Let $k_i = \#\{\sigma_{i,s} \in \omega(q_i) : \alpha_i \nmid \text{order}(\sigma_{i,s})\}$. By renumbering the indices, if necessary, we can assume that $\omega(q_i) = \sigma_{i,1} \cdots \sigma_{i,k_i} \cdots \sigma_{i,\ell_i}$ in such way that $\alpha_i \nmid \text{order}(\sigma_{i,k})$, for $k = 1, \ldots, k_i$.

(a) Assume $F$ is non-orientable.
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1. Suppose \( \sum_{i=1}^{r} k_i = 0 \). Note that \( \alpha_i | \text{order}(\sigma_i, s) \), for \( i = 1, \ldots, r \) and for \( s = 1, \ldots, \ell_i \). Assume that \( p_i, s \) are integer numbers such that \( \text{order}(\sigma_i, s) = p_i, s \alpha_i \). Write \( B = \sum_{s=1}^{r} \sum_{i=1}^{\ell_i} p_i, s \beta_i \).

Then \( h(\tilde{M}) = n(g - 2) + nr - \sum \ell_i + 2 \), if \( B = \pm 1 \); Otherwise, \( h(\tilde{M}) = n(g - 2) + nr - \sum \ell_i + 3 \).

2. Suppose \( \sum_{i=1}^{r} k_i = 1 \). By renumbering indices, if necessary, in this case we can assume that \( \alpha_1 \nmid \text{order}(\sigma_{1, s}) \), for \( s = 2, \ldots, \ell_1 \), and \( \alpha_i | \text{order}(\sigma_{i, s}) \), for \( i = 2, \ldots, r \) and for \( s = 1, \ldots, \ell_i \). Assume \( p_{i, s}' \), for \( s = 2, \ldots, \ell_1 \) and \( p_{i, s} \), for \( i = 2, \ldots, r \) and for \( s = 1, \ldots, \ell_i \), are integers numbers such that \( \text{order}(\sigma_{1, s}) = p_{1, s}' \alpha_1 \), for \( s = 2, \ldots, \ell_1 \), and \( \text{order}(\sigma_{i, s}) = p_{i, s} \alpha_i \), for \( i = 2, \ldots, r \) and for \( s = 1, \ldots, \ell_i \). Define

\[
B = B_{1,1} + A_{1,1}(\beta_1 \sum_{s=2}^{\ell_1} p_{1, s}' + \sum_{i=2}^{r} \sum_{s=1}^{\ell_i} p_{i, s} \beta_i).
\]

Then \( h(\tilde{M}) = n(g - 2) + nr - \sum \ell_i + 2 \), if \( B = \pm 1 \); Otherwise, \( h(\tilde{M}) = n(g - 2) + nr - \sum \ell_i + 3 \).

3. Suppose \( \sum_{i=1}^{r} k_i \geq 2 \). Then \( h(\tilde{M}) = n(g - 2) + nr - \sum \ell_i + \sum k_i + 1 \).

(b) Assume \( F \) is orientable.

1. Suppose \( \sum_{i=1}^{r} k_i = 0 \). Note that \( \alpha_i | \text{order}(\sigma_i, s) \), for \( i = 1, \ldots, r \) and for \( s = 1, \ldots, \ell_i \). Let \( p_i, s \) be integer numbers such that \( \text{order}(\sigma_i, s) = p_i, s \alpha_i \). Define \( B = \sum_{i=1}^{r} \sum_{s=1}^{\ell_i} p_i, s \beta_i \). Then \( h(\tilde{M}) = 2n(g - 1) + nr - \sum \ell_i + 2 \), if \( B = \pm 1 \); Otherwise, \( h(\tilde{M}) = 2n(g - 1) + nr - \sum \ell_i + 3 \).

2. Suppose \( \sum_{i=1}^{r} k_i = 1 \). We can assume that \( \alpha_1 \nmid \text{order}(\sigma_{1, s}) \), for \( s = 2, \ldots, \ell_1 \), and \( \alpha_i | \text{order}(\sigma_{i, s}) \), for \( i = 2, \ldots, r \) and for \( s = 1, \ldots, \ell_i \). Assume that \( p_{1, s}' \), for \( s = 2, \ldots, \ell_1 \) and \( p_{i, s} \), for \( i = 2, \ldots, r \) and for \( s = 1, \ldots, \ell_i \), are integers numbers such that \( \text{order}(\sigma_{1, s}) = p_{1, s}' \alpha_1 \), for \( s = 2, \ldots, \ell_1 \), and \( \text{order}(\sigma_{i, s}) = p_{i, s} \alpha_i \), for \( i = 2, \ldots, r \) and for \( s = 1, \ldots, \ell_i \).
for \( i = 2, \ldots, r \) and for \( s = 1, \ldots, \ell_i \). Write

\[ B = B_{1,1} + A_{1,1} \left( \beta_1 \sum_{s=2}^{r} p'_{1,s} + \sum_{i=2}^{r} \sum_{s=1}^{\ell_i} p_{i,s} \beta_i \right). \]

Then \( h(\tilde{M}) = 2n(g-1) + nr - \sum \ell_i + 2 \), if \( B = \pm 1 \). Otherwise, \( h(\tilde{M}) = 2n(g-1) + nr - \sum \ell_i + 3 \).

3. Suppose \( \sum_{i=1}^{r} k_i \geq 2 \). Then \( h(\tilde{M}) = 2n(g-1) + nr - \sum \ell_i + \sum k_i + 1 \).

Proof.

Note that \( \sum k_i \) is the number of exceptional fibers of \( \tilde{M} \) because \( A_{i,k} = \frac{\alpha_i}{\gcd(\alpha_i, \text{order}(\sigma_{i,k}))} = 1 \) if and only if \( \alpha_i | \text{order}(\sigma_{i,k}) \). We proceed case by case.

(a) Suppose \( F \) is non-orientable.

1. Assume \( \sum k_i = 0 \). Recall \( p_{i,s} \) are integer numbers such that \( \text{order}(\sigma_{i,s}) = p_{i,s} \alpha_i \).

From definition of \( B_{i,k} \), \( A_{i,k} \) and from Theorem 2.2.1 we can assume that \( \tilde{M} = (Y,y,\tilde{g};B/A) \), where \( B = \sum_{i=1}^{r} \sum_{s=1}^{\ell_i} p_{i,s} \beta_i \).

(i) If \( G \) is non-orientable, then \( \tilde{g} = n(g-2) + nr - \sum \ell_i + 2 \). Therefore \( h(\tilde{M}) = \tilde{g} = n(g-2) + nr - \sum \ell_i + 2 \), if \( B = \pm 1 \). Otherwise, \( h(\tilde{M}) = \tilde{g} + 1 = n(g-2) + nr - \sum \ell_i + 3 \).

(ii) If \( G \) is orientable, then \( 2\tilde{g} = n(g-2) + nr - \sum \ell_i + 2 \). Then \( h(\tilde{M}) = 2\tilde{g} = n(g-2) + nr - \sum \ell_i + 2 \), if \( B = \pm 1 \). Otherwise, \( h(\tilde{M}) = 2\tilde{g} + 1 = n(g-2) + nr - \sum \ell_i + 3 \).

2. Assume \( \sum k_i = 1 \). Recall \( B = B_{1,1} + A_{1,1} \left( \beta_1 \sum_{s=2}^{r} p'_{1,s} + \sum_{i=2}^{r} \sum_{s=1}^{\ell_i} p_{i,s} \beta_i \right) \), where \( p'_{1,s} \), for \( s = 2, \ldots, \ell_1 \) and \( p_{i,s} \), for \( i = 2, \ldots, r \) and for \( s = 1, \ldots, \ell_i \), are integers numbers such that \( \text{order}(\sigma_{1,s}) = p'_{1,s} \alpha_1 \), for \( s = 2, \ldots, \ell_1 \), and \( \text{order}(\sigma_{i,s}) = p_{i,s} \alpha_i \), for \( i = 2, \ldots, r \) and for \( s = 1, \ldots, \ell_i \). Then

\( \tilde{M} = (Y,y,\tilde{g};B_{1,1}/A_{1,1},B_{1,2}/1, \ldots, B_{1,\ell_1}/1, \ldots, B_{r,1}/1, \ldots, B_{r,\ell_r}/1). \)

By Theorem 2.2.1 and Definition of \( B_{i,k} \), we can consider \( \tilde{M} = (Y,y,\tilde{g};B/A_{1,1}). \)

(i) If \( G \) is non-orientable, then \( \tilde{g} = n(g-2) + nr - \sum \ell_i + 2 \). Thus \( h(\tilde{M}) = \tilde{g} = n(g-2) + nr - \sum \ell_i + 2 \), if \( B = \pm 1 \). Otherwise, \( h(\tilde{M}) = \tilde{g} + 1 = n(g-2) + nr - \sum \ell_i + 3 \).
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(ii) If $G$ is orientable, then $2\tilde{g} = n(g - 2) + nr - \sum \ell_i + 2$ and we can conclude that

$$h(\tilde{M}) = n(g - 2) + nr - \sum \ell_i + 2,$$

if $B = \pm 1$. Otherwise, $h(\tilde{M}) = n(g - 2) + nr - \sum \ell_i + 3$.

3. Assume $\sum k_i \geq 2$. Note that if $G$ is non-orientable then $\tilde{g} = n(g - 2) + nr - \sum \ell_i + 2$, and if $G$ is orientable then $2\tilde{g} = n(g - 2) + nr - \sum \ell_i + 2$. Since $\sum k_i$ is the number of exceptional fibers then $h(\tilde{M}) = \tilde{g} + \sum k_i - 1$, if $F$ is non-orientable and $h(\tilde{M}) = 2\tilde{g} + \sum k_i - 1$, if $F$ is orientable. Then it is clear that $h(\tilde{M}) = n(g - 2) + nr - \sum \ell_i + \sum k_i + 1$.

(b) Suppose $F$ is orientable. Then $2\tilde{g} = 2n(g - 1) + nr - \sum \ell_i + 2$, by Theorem 2.3.8.

1. Assume $\sum k_i = 0$. Recall $p_{i,s}$ are integer numbers such that $\text{order}(\sigma_{1,s}) = p_{i,s}\alpha_i$.

From definition of $B_{i,k}$, $A_{i,k}$ and from Theorem 2.2.1 we obtain that $\tilde{M} = (Yy, \tilde{g}; B/1)$, where $B = \sum_{i=1}^{r} \sum_{s=1}^{\ell_i} p_{i,s} \beta_i$. Thus $h(\tilde{M}) = 2\tilde{g} = 2n(g-1)+nr-\ell_i+2$, if $B = \pm 1$. Otherwise, $h(\tilde{M}) = 2\tilde{g} + 1 = 2n(g-1)+nr-\ell_i+3$.

2. Assume $\sum k_i = 1$. Recall $B = B_{1,1} + A_{1,1} \left( \beta_1 \sum_{s=2}^{\ell_1} p_{1,s} + \sum_{i=2}^{r} \sum_{s=1}^{\ell_i} p_{i,s} \beta_i \right)$, where $p_{1,s}$, for $s = 2, \ldots, \ell_1$ and $p_{i,s}$, for $i = 2, \ldots, r$ and for $s = 1, \ldots, \ell_i$, are integers numbers such that $\text{order}(\sigma_{1,s}) = p_{1,s}\alpha_1$, for $s = 2, \ldots, \ell_1$, and $\text{order}(\sigma_{i,s}) = p_{i,s}\alpha_i$, for $i = 2, \ldots, r$ and for $s = 1, \ldots, \ell_i$. Then

$$\tilde{M} = (Yy, \tilde{g}; B_{1,1}/A_{1,1}, B_{1,2}/1, \ldots, B_{1,\ell_1}/1, \ldots, B_{r,1}/1, \ldots, B_{r,\ell_r}/1).$$

By Theorem 2.2.1 and Definition of $B_{i,k}$, we can consider $\tilde{M} = (Yy, \tilde{g}; B/A_{1,1})$. Thus $h(\tilde{M}) = 2\tilde{g} = 2n(g-1)+nr-\sum \ell_i + 2$, if $B = \pm 1$. Otherwise, $h(\tilde{M}) = 2\tilde{g} + 1 = 2n(g-1)+nr-\sum \ell_i + 3$.

3. Assume $\sum k_i \geq 2$. Then $h(\tilde{M}) = 2n(g - 1) + nr - \sum \ell_i + \sum k_i + 1$ for $\sum k_i$ is the number of exceptional fibers of $\tilde{M}$. □

**Corollary 3.2.3** Let $M = (Xx, g; \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r)$ where $Xx \in \{Oo, On, Nn, NnI, NnII, NnIII\}$, and $g \neq 0$, and $\alpha_i \geq 2$, for each $i \in \{1, \ldots, r\}$, and
Consider the transitive representation \( \omega : \pi_1(M_0) \rightarrow S_n \) defined by
\[
\begin{align*}
\omega(h) &= (1), \\
\omega(q_i) &= \sigma_{i,1} \cdots \sigma_{i,\ell_i}, \text{ for } i = 1, \ldots, r \text{ and} \\
\omega(v_j) &= \rho_{j,1} \cdots \rho_{j,s_j},
\end{align*}
\]
where \( \sigma_{i,1} \cdots \sigma_{i,\ell_i} \) and \( \rho_{j,1} \cdots \rho_{j,s_j} \) are the disjoint cycle decompositions of \( \omega(q_i) \) and \( \omega(v_j) \), respectively.

Let \( \varphi : \tilde{M} \rightarrow M \) be the covering associated to \( \omega \). Then \( h(\tilde{M}) \geq h(M) \).

**Proof.**

Let \( r \) be the number of exceptional fibers of \( M \). Since \( M \) has at least two exceptional fibers, then \( h(M) = 2g + r - 1 \) or \( h(M) = g + r - 1 \), if \( F \) is orientable or not, respectively. Let \( k_i \) be as in previous lemma. Recall \( \sum k_i \) is the number of exceptional fibers of \( \tilde{M} \). Again we proceed case by case.

(a) If \( F \) is non-orientable. Recall \( \tilde{g} = n(g - 2) + 2 + nr - \sum_{i=1}^{r} \ell_i \), if \( G \) is non-orientable; otherwise, if \( G \) is orientable we have \( 2\tilde{g} = n(g - 2) + 2 + nr - \sum_{i=1}^{r} \ell_i \).

1. If \( \sum k_i = 0 \), then \( h(\tilde{M}) \geq n(g-2)+nr-\sum_{i=1}^{r} \ell_i+2 \). Recall \( \alpha_i \geq 2 \) and \( \alpha_i \mid \text{order} (\sigma_{i,k}) \), for all \( i, k \), then each cycle of \( \omega(q_i) \) has order at least 2. Thus \( \ell_i \leq \frac{n}{2} \). Also \( \ell_i \leq n-1 \) since \( n - 1 \geq \frac{n}{2} \), if \( n \geq 2 \). Then \( \sum_{i=1}^{r} (n-1)(r-2) \).

Hence
\[
\sum_{i=1}^{r} \ell_i \leq (n-1)(r-2) + \frac{n}{2} + \frac{n}{2} = (n-1)(r-2) + n
\]
because \( \ell_{r-1} \leq \frac{n}{2} \) and \( \ell_r \leq \frac{n}{2} \).

Note that \((n-1)(r-2) + n = (n-1)(r-1) + 1\).

From the facts
\[
\left[ n(g-2) + 2 + nr - \sum_{i=1}^{r} \ell_i \right] - h(M) = (n-1)(g-2) + (n-1)r - \sum \ell_i + 1,
\]
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\[ (n - 1)(r - 2) + n = (n - 1)(r - 1) + 1 \]
and \( h(\tilde{M}) \geq [n(g - 2) + 2 + nr - \sum_{i=1}^{r} \ell_i] \),
it follows that:

- If \( g = 1 \), then
  \[ [n(g - 2) + 2 + nr - \sum_{i=1}^{r} \ell_i] - h(M) = (n - 1)(r - 1) - \sum \ell_i + 1 \geq 0. \]
  Thus \( h(\tilde{M}) \geq h(M) \).

- If \( g \geq 2 \), then
  \[ [n(g - 2) + 2 + nr - \sum_{i=1}^{r} \ell_i] - h(M) \geq (n - 1)(g - 2) + (n - 1)(r - 1) - \sum \ell_i + 1 \geq 0. \]
  Thus \( h(\tilde{M}) \geq h(M) \).

Therefore \( h(\tilde{M}) \geq h(M) \).

2. If \( \sum k_i = 1 \), then

\[ [n(g - 2) + nr - \sum_{i=1}^{r} \ell_i + 2] - h(M) = (n - 1)(g - 2) + (n - 1)r - \sum \ell_i + 1. \]

Recall \( h(\tilde{M}) \geq n(g - 2) + nr - \sum \ell_i + 2 \) and \( \ell_1 \) is the number of cycles of \( \omega(q_1) \).

From previous lemma, we can suppose \( \alpha_{1,1} \nmid \text{order}(\sigma_{1,1}) \), \( \alpha_{1,1} \mid \text{order}(\sigma_{1,s}) \), for \( s = 2, \ldots, \ell_1 \), and \( \alpha_i \mid \text{order}(\sigma_{i,k}) \), for \( i = 2, \ldots, r \) and for \( k = 1, \ldots, \ell_i \). Then \( \text{order}(\sigma_{1,s}) \geq 2 \), if \( s \neq 1 \); and \( \text{order}(\sigma_{i,k}) \geq 2 \), for \( i = 2, \ldots, r \) and for all \( k \).

(i) Assume \( n = 2 \). Then \( \tilde{M} \) has exactly one exceptional fiber if and only if
\[ M = (X, x; g; \beta_1/\alpha_1, \beta_2/2, \ldots, \beta_r/2), \]
where \( \alpha_1 > 2 \) and \( \omega(q_i) = (1, 2) \), for \( i = 1, \ldots, r \). Thus \( \tilde{M} = (Y, y; B_{1,1}/A_{1,1}, \beta_2/1, \ldots, \beta_r/1) \). It is easy to see in this case that \( \sum_{i=1}^{r} \ell_i = r \) Then \( [n(g - 2) + nr - \sum_{i=1}^{r} \ell_i + 2] - h(M) = g - 1. \)
Recalling \( g \neq 0 \) we conclude \( h(\tilde{M}) \geq h(M) \).

(ii) Assume \( n \geq 3 \). In this case we have that \( \ell_i \leq \frac{n}{2} \leq n - 1 \), for all \( i = 2, \ldots, r \), since \( \text{order}(\sigma_{i,k}) \geq 2 \), for \( i \geq 2 \). Thus \( \sum_{i=3}^{r} \ell_i \leq (n - 1)(r - 3) \).
Now note that
\[ \ell_1 \leq \frac{n - \text{order}(\sigma_{1,1})}{2} + 1 \]
for \( \omega(q_1) \) contains the cycle \( \sigma_{1,1} \) and the cycles \( \sigma_{1,s} \), for \( s = 2, \ldots, r \), but the cycles \( \sigma_{1,s} \), for \( s = 2, \ldots, r \), have order at least 2 then we have at most \( \frac{n - \text{order}(\sigma_{1,1})}{2} + 1 \) cycles in \( \omega(q_1) \). Also, we have that the inequality \( \frac{n - \text{order}(\sigma_{1,1})}{2} + 1 \leq \frac{n-1}{2} + 1 \) follows since \( \text{order}(\sigma_{1,1}) \geq 1 \). Thus \( \ell_1 \leq \frac{n-1}{2} + 1 \).

Then
\[ \sum_{i=1}^{r} \ell_i \leq \frac{n-1}{2} + 1 + \frac{n}{2} + (n-1)(r-3) = (n-1)(r-3) + n + \frac{1}{2} \]
because \( \ell_2 \leq n/2 \) and \( \ell_1 \leq \frac{n-1}{2} + 1 \). Since \( (n-1)(r-3) + n + 1/2 \leq (n-1)(r-1)+1 \) we obtain
\[ (n-1)(r-1) + 1 - \sum_{i=1}^{r} \ell_i \geq 0. \]

Last inequality together the fact \( h(\tilde{M}) \geq [n(g-2) + nr - \sum_{i=1}^{r} \ell_i + 2] \) allow us to get the following:

- If \( g = 1 \), then
  \[ \left[ n(g-2) + nr - \sum_{i=1}^{r} \ell_i + 2 \right] - h(M) = (n-1)(r-1) - \sum_{i=1}^{r} \ell_i + 1 \geq 0. \]
  Thus \( h(\tilde{M}) \geq h(M) \).

- If \( g \geq 2 \), then
  \[ \left[ n(g-2) + nr - \sum_{i=1}^{r} \ell_i + 2 \right] - h(M) = (n-1)(g-2) + (n-1)r - \sum_{i=1}^{r} \ell_i + 1 \geq 0. \]
  Thus \( h(\tilde{M}) \geq h(M) \).

Therefore \( h(\tilde{M}) \geq h(M) \).

3. If \( \sum k_i \geq 2 \), notice that
\[ h(\tilde{M}) - h(M) = (n-1)(g-2) + (n-1)r - \left( \sum_{i=1}^{r} \ell_i - \sum_{i=1}^{r} k_i \right) \]
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The inequality

\[ \ell_i \leq \frac{n - \sum_{i=1}^{k_i} \text{order}(\sigma_{i,s})}{2} + k_i \]

follows since \( \ell_i \) is the number of cycles of \( \omega(q_i) \) and \( \text{order}(\sigma_{i,j}) \geq 2 \) for \( j = k+1, \ldots, r \); also the inequality

\[ \frac{n - \sum_{i=1}^{k_i} \text{order}(\sigma_{i,s})}{2} + k_i \leq \frac{n - 1}{2} + k_i \]

follows since \( \sum_{i=1}^{k_i} \text{order}(\sigma_{i,s}) \geq 1 \).

Then \( \sum_{i=1}^{r} \ell_i - \sum_{i=1}^{r} k_i \leq \frac{(n - 1)r}{2} \). On the other hand, \( r/2 \leq r - 1 \) for \( r \geq 2 \). Thus

\[ \frac{(n - 1)(r - 1)}{2} - \left( \sum_{i=1}^{r} \ell_i - \sum_{i=1}^{r} k_i \right) \geq 0 \]

and we obtain

\[ (n - 1)(r - 1) - \left( \sum_{i=1}^{r} \ell_i - \sum_{i=1}^{r} k_i \right) \geq 0. \]

Finally, we have that:

- If \( g = 1 \), then

\[ h(\tilde{M}) - h(M) = (n - 1)(r - 1) - \left( \sum_{i=1}^{r} \ell_i - \sum_{i=1}^{r} k_i \right) \geq 0. \]

- If \( g \geq 2 \), then

\[ h(\tilde{M}) - h(M) \geq (n - 1)(g - 2) + (n - 1)(r - 1) - \left( \sum_{i=1}^{r} \ell_i - \sum_{i=1}^{r} k_i \right) \geq 0. \]

Therefore \( h(\tilde{M}) \geq h(M) \).

(b) Assume \( F \) is orientable. In this case, \( G \) is orientable and \( 2\tilde{g} = 2n(g - 1) + nr - \sum_{i=1}^{r} \ell_i + 2 \).

1. If \( \sum k_i = 0 \), then

\[ h(\tilde{M}) \geq 2\tilde{g} = 2n(g - 1) + nr - \sum_{i=1}^{r} \ell_i + 2. \]
Recall $\alpha_i \geq 2$ and $\alpha_i|\text{order}(\sigma_{i,k})$, for all $i, k$, then each cycle of $\omega(q_i)$ has order at least 2. Thus $\ell_i \leq n/2$. Also $\ell_i \leq n - 1$ since $n - 1 \geq n/2$, if $n \geq 2$. Then $\sum_{i=1}^{r-2} \ell_i \leq (n - 1)(r - 2)$.

Hence
\[
\sum_{i=1}^{r} \ell_i \leq (n - 1)(r - 2) + \frac{n}{2} + \frac{n}{2}
\]
because $\ell_{r-1} \leq n/2$ and $\ell_r \leq n/2$.

It is clear that $(n - 1)(r - 2) + n = (n - 1)(r - 1) + 1$.

Since $2\tilde{g} - h(M) = 2(n - 1)(g - 1) + (n - 1)r - \sum_{i=1}^{r} \ell_i + 1$, we have that
\[
2\tilde{g} - h(M) \geq 2(n - 1)(g - 1) + (n - 1)(r - 1) - \sum_{i=1}^{r} \ell_i + 1 \geq 0.
\]

Therefore $h(\tilde{M}) \geq h(M)$.

2. If $\sum k_i = 1$, recall $h(\tilde{M}) \geq 2\tilde{g}$. Then
\[
2\tilde{g} - h(M) = 2(n - 1)(g - 1) + (n - 1)r - \sum_{i=1}^{r} \ell_i + 1.
\]

By previous lemma, we can suppose $\alpha_{1,1} \nmid \text{order}(\sigma_{1,1})$, $\alpha_{1,1}|\text{order}(\sigma_{1,s})$, for $s = 2, \ldots, \ell_1$, and $\alpha_i|\text{order}(\sigma_{i,k})$, for $i = 2, \ldots, r$ and for $k = 1, \ldots, \ell_i$. Then $\text{order}(\sigma_{1,s}) \geq 2$, if $s \neq 1$; and $\text{order}(\sigma_{i,k}) \leq 2$, for $i = 2, \ldots, r$ and for all $k$.

(i) Assume $n = 2$. Then $\tilde{M}$ has exactly one exceptional fiber if and only if
\[
M = (Xx, g; \beta_1/\alpha_1, \beta_2/2, \ldots, \beta_r/2), \text{ where } \alpha_1 > 2 \text{ y } \omega(q_i) = (1, 2), \text{ for } i = 1, \ldots, r. \text{ Thus } \tilde{M} = (Yy, \tilde{g}; B_{1,1}/A_{1,1}, \beta_2/1, \ldots, \beta_r/1). \text{ It is easy to see in this case that } \sum \ell_i = r. \text{ Then } 2\tilde{g} - h(M) = 2(g - 1) + 1 \text{ and we conclude } h(\tilde{M}) \geq h(M) \text{ since } g \neq 0.
\]

(ii) Assume $n \geq 3$. In this case we have that $\ell_i \leq n/2 \leq n - 1$, for all $i = 2, \ldots, r$, since $\text{order}(\sigma_{i,k}) \geq 2$, for $i \geq 2$. Thus $\sum_{i=3}^{r} \ell_i \leq (n - 1)(r - 3)$. Now note that
\[
\ell_1 \leq \frac{n - \text{order}(\sigma_{1,1})}{2} + 1 \leq \frac{n - 1}{2} + 1.
\]
The first inequality $\ell_1 \leq \frac{n - \text{order}(\sigma_{1,1})}{2} + 1$ follows for $\ell_1$ is the number of cycles in $\omega(q_1)$; in $\omega(q_1)$ we have the cycle $\sigma_{1,1}$ and the cycles $\sigma_{j,k}$, for $j = 2, \ldots, r$, but the cycles $\sigma_{j,k}$ have order at least 2, for $j = 2, \ldots, r$, then we have at most $\frac{n - \text{order}(\sigma_{1,1})}{2} + 1$ cycles in $\omega(q_1)$. The second inequality $\frac{n - \text{order}(\sigma_{1,1})}{2} + 1 \leq \frac{n - 1}{2} + 1$ follows because $\text{order}(\sigma_{1,1}) \geq 1$.

Then

$$\sum_{i=1}^{r} \ell_i \leq (n - 1)(r - 3) + \frac{n}{2} + \frac{n - 1}{2} + 1 = (n - 1)(r - 3) + n + \frac{1}{2}$$

for $\ell_2 \leq n/2$ and $\ell_1 \leq \frac{n - 1}{2} + 1$. Since $(n - 1)(r - 3) + n + 1/2 \leq (n - 1)(r - 1) + 1$ we obtain

$$(n - 1)(r - 1) + 1 - \sum_{i=1}^{r} \ell_i \geq 0.$$ 

Therefore $h(\tilde{M}) \geq 2\tilde{g} \geq h(M)$.

3. If $\sum k_i \geq 2$, then

$$h(\tilde{M}) - h(M) = 2(n - 1)(g - 1) + (n - 1)r - \left( \sum_{i=1}^{r} \ell_i - \sum_{i=1}^{r} k_i \right).$$

Note that

$$\ell_i \leq \frac{n - \sum_{i=1}^{k_i} \text{order}(\sigma_{i,s})}{2} + k_i$$

because $\ell_i$ is the number of cycles of $\omega(q_i)$ and $\text{order}(\sigma_{i,j}) \geq 2$ for $j = k + 1, \ldots, r$; note also that

$$\frac{n - \sum_{i=1}^{k_i} \text{order}(\sigma_{i,s})}{2} + k_i \leq \frac{n - 1}{2} + k_i$$

since $\sum_{i=1}^{k_i} \text{order}(\sigma_{i,s}) \geq 1$. 

Therefore \[ \frac{(n - 1)(r - 1)}{2} - \left( \sum_{i=1}^{r} \ell_i - \sum_{i=1}^{r} k_i \right) \geq 0. \]

Because of \( r \geq 2 \), then \( \frac{r}{2} \leq r - 1 \). Thus

\[ (n - 1)(r - 1) - \left( \sum_{i=1}^{r} \ell_i - \sum_{i=1}^{r} k_i \right) \geq 0. \]

Therefore \( h(\tilde{M}) \geq h(M) \). \( \square \)

**Corollary 3.2.4** Assume \( r \) is an even non-negative number such that \( r \geq 4 \). Consider the Seifert manifold

\[ M = (Oo, 0; (-2r + 3)/4, 1/2, 1/2, \ldots, 1/2) \]

and note that \( \pi_1(M) \) is infinite. Let \( \omega : \pi_1(M_0) \to S_2 \) be the representation defined by

\[ \begin{align*}
\omega(h) &= (1) \\
\omega(q_1) &= \varepsilon_2 \\
& \vdots \\
\omega(q_r) &= \varepsilon_2.
\end{align*} \]

Let \( \varphi : \tilde{M} \to M \) be the (unbranched) covering associated to \( \omega \).

Then \( h(\tilde{M}) < h(M) \).

**Proof.**

First we have to highlight that the representation \( \omega : \pi_1(M_0) \to S_2 \) extends to a representation \( \omega : \pi_1(M) \to S_2 \) for \( \omega(q_i h^{2k}) = (1) \). Also, it is easy to see that \( h(M) = r - 1 \), by Theorem 3.1.1. Now note that \( h(\tilde{M}) = 2((r/2) - 1) = r - 2 \) since

\[ \tilde{M} = (Oo, (r/2) - 1; (-2r + 3)/2, 1/1, \ldots, 1/1) \]

by Theorem 2.3.8

\[ = (Oo, (r/2) - 1; 1/2). \]

Hence \( h(\tilde{M}) < h(M) \). \( \square \)
Remark 3.2.1 Of course, there are also manifolds $M = (O_0, O_0; \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r)$ with at least two exceptional fibers and infinite fundamental group, admitting representations $\omega : \pi_1(M_0) \to S_n$ such that $\omega(h) = (1)$ and the covering $\tilde{M}$ determined by $\omega$ satisfies that $h(\tilde{M}) \geq h(M)$, for example:

Assume $r$ is an even non-negative number such that $r \geq 4$. Consider the Seifert manifold

$$M = (O_0, 0; \frac{1}{4}, \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})^{r \text{times}}$$

and note that $h(M) = r - 1$ and $\pi_1(M)$ is infinite. Let $\omega : \pi_1(M_0) \to S_2$ be the representation defined by

$$\omega(h) = (1),$$
$$\omega(q_1) = \varepsilon_2,$$
$$\vdots$$
$$\omega(q_r) = \varepsilon_2$$

Then

$$\tilde{M} = (O_0, (r/2) - 1; \frac{1}{4}, \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})^{(r-1) \text{times}}$$

by Theorem 2.3.8

$$= (O_0, (r/2) - 1; (1 + 2(r - 1))/2)$$

and we have that $h(\tilde{M}) = 2((r/2) - 1) + 1 = r - 1$ since $1 + 2(r - 1) \neq 1$.

Therefore $h(\tilde{M}) = h(M)$. □

We can summarize some of the previous Corollaries in the following Theorem.

Theorem 3.2.1 Let $M = (X, g; \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r)$ where $X \in \{O_0, On, No, NnI, NnII, NnIII\}$ and $g \neq 0$. Let $n \in \mathbb{N}$ and $\omega : \pi_1(M_0) \to S_n$ be a transitive representation defined by

$$\omega(h) = (1),$$
$$\omega(q_i) = \sigma_{i,1} \cdots \sigma_{i,\ell_i}, \forall i = 1, \ldots, r$$
$$\omega(v_j) = \rho_{j,1} \cdots \rho_{j,s_j},$$
where $\sigma_{i,1} \cdots \sigma_{i,\ell_i}$ and $\rho_{j,1} \cdots \rho_{j,s_j}$ are the disjoint cycle decompositions of $\omega(q_i)$ and $\omega(v_j)$, respectively, and $\{h, v_j, q_i\}$ is a standard system of generators of $\pi_1(M_0)$.

Then $h(\tilde{M}) \geq h(M)$.

Proof.

The result follows from Corollaries 3.2.1, 3.2.2 and 3.2.3.

\[\Box\]

### 3.2.2 Heegaard genus when $\omega(h) = \varepsilon_n$

Recall $\varepsilon_n = (1, 2, \ldots, n) \in S_n$. Given a Seifert manifold $M = (Xx, g; \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r)$, where $Xx \in \{Oo, On, No, NnI, NnII, NnIII\}$, with orbit projection $p : M \to F$, where $F$ has genus $g$, and given a representation $\omega : \pi_1(M_0) \to S_n$ defined by

\[
\begin{align*}
\omega(h) &= \varepsilon_n, \\
\omega(q_i) &= \varepsilon_{n_k}^{k_i}, \forall i = 1, \ldots, r \quad \text{and} \\
\omega(v_j) &= \tau_j,
\end{align*}
\]

where $\tau_j$ is a power of the $n$-cycle $\varepsilon_n$, if $e(v_j) = +1$ or $\tau_j$ is a reflection $\rho_j$, if $e(v_j) = -1$. Then, if $\varphi : \tilde{M} \to M$ is the covering determined by $\omega$, by Theorem 2.3.15 we have that $\tilde{M} = (Xx, g; B_1/A_1, \ldots, B_r/A_r)$, where

\[
B_i = \frac{\beta_i + k_i\alpha_i}{\gcd\{n, \beta_i + k_i\alpha_i\}}
\]

and

\[
A_i = \frac{n\alpha_i}{\gcd\{n, \beta_i + k_i\alpha_i\}}.
\]

Recall $\gcd\{n, \beta_i + k_i\alpha_i\}$ denotes the greatest common divisor of $n$ and $\beta_i + k_i\alpha_i$.

Note that $\alpha_i \geq 2$ implies that $A_i \geq 2$. 
Lemma 3.2.4 Let $M = (X, g; \beta_1/\alpha_1)$ be a Seifert manifold, where $X \in \{Oo, On, No, NnI, NnII, NnIII\}$ where $\alpha_1 \geq 1$. Suppose that $n \in \mathbb{N}$ and $\omega : \pi_1(M_0) \to S_n$ is the representation defined by

$$\omega(h) = \varepsilon_n, \quad \omega(q_1) = \varepsilon_n^{k_1}, \quad \omega(v_j) = \tau_j,$$

where $\{h, q_i, v_j\}$ is a standard system of generators of $\pi_1(M_0)$, and $\tau_j$ is a power of $\varepsilon_n$, if $v_j$ commutes with $h$; otherwise, if $v_j$ anticommutes with $h$, $\tau_j$ is a reflection $\rho_j$.

Suppose $\varphi : \tilde{M} \to M$ is the covering determined by $\omega$.

- Assume $(\beta_1 + k_1\alpha_1) \nmid n$. Then $h(\tilde{M}) = 2g + 1$ or $h(\tilde{M}) = g + 1$, if $F$ is orientable or $F$ is non-orientable, respectively. Also $h(\tilde{M}) \geq h(M)$.

- Assume $(\beta_1 + k_1\alpha_1)|n$. Then $h(\tilde{M}) = 2g$, if $F$ is orientable; Otherwise, if $F$ is non-orientable, then $h(\tilde{M}) = g$. Furthermore, $h(\tilde{M}) = h(M)$ or $h(\tilde{M}) < h(M)$, if $\beta_1 = \pm 1$ or $\beta_1 \neq \pm 1$, respectively.

Proof.

Observe that $\tilde{M} = (X, g; B_1/A_1)$, with $B_1 = \frac{\beta_1 + k_1\alpha_1}{\gcd\{n, \beta_1 + k_1\alpha_1\}}$ and $A_1 = \frac{n\alpha_1}{\gcd\{n, \beta_1 + k_1\alpha_1\}}$. It is clear that $B_1 = \pm 1$ if and only if $(\beta_1 + k_1\alpha_1)|n$. Of course, through this proof, if $M$ is non-orientable we ask $\beta_1 + k_1\alpha_1$ be even, in order, to compute $h(\tilde{M})$.

- If $(\beta_1 + k_1\alpha_1) \nmid n$, then $B_1 \neq \pm 1$ and

$$h(\tilde{M}) = \begin{cases} 2g + 1, & \text{if } F \text{ is orientable, or} \\ g + 1, & \text{otherwise.} \end{cases}$$

On the other hand, it is clear that $h(M) \leq 2g + 1$ or $h(M) \leq g + 1$, if $F$ is orientable or $F$ is non-orientable, respectively. Hence $h(\tilde{M}) \geq h(M)$.

- Suppose $(\beta_1 + k_1\alpha_1)|n$. Then $\tilde{M} = (X, g; \pm 1/A_1)$ and we conclude that $h(\tilde{M}) = 2g$ or $h(\tilde{M}) = g$, if $F$ is orientable or $F$ is non-orientable, respectively.

On the other hand, note that:
(a) If $\beta_1 = \pm 1$, then $h(M) = 2g$ or $h(M) = g$, if $F$ is orientable or $F$ is non-orientable, respectively. Thus $h(\tilde{M}) = h(M)$.

(b) If $\beta_1 \neq \pm 1$, then $h(M) = 2g + 1$ or $h(M) = g + 1$, if $F$ is orientable or $F$ is non-orientable, respectively. Thus $h(\tilde{M}) < h(M)$.

□

Corollary 3.2.5 Let $\beta_1$ be an even number and consider the Seifert manifold $M = (Xx, g; \beta_1/\alpha_1)$, where $Xx \in \{Oo, On, No, NnI, NnII, NnIII\}$ and $\alpha_1 \geq 1$. Let $\omega : \pi_1(M) \to S_{|\beta_1|}$ be the representation defined by

$$
\omega(h) = \varepsilon_{|\beta_1|},
\omega(q_1) = (1), \text{ and }
\omega(v_j) = \tau_j,
$$

where $\tau_j$ is a power of $\varepsilon_{|\beta_1|}$ or a reflection $\rho_j$ depending on if $v_j$ commutes or anticommutes with $h$, respectively. If $\varphi : \tilde{M} \to M$ is the covering branched along fibers of $M$ determined by $\omega$, then $\varphi : \tilde{M} \to M$ is an (unbranched) covering of $M$ and $h(\tilde{M}) < h(M)$.

Proof. Since $\omega(q_1^a h_{\beta_1}) = \varepsilon_{|\beta_1|^a} = (1)$ then $\omega : \pi_1(M_0) \to S_{|\beta_1|}$ extends to a representation $\omega : \pi_1(M) \to S_{|\beta_1|}$. Therefore $\varphi : \tilde{M} \to M$ is an unbranched covering of $M$. By Lemma 3.2.4 we conclude that $h(\tilde{M}) < h(M)$.

□

Lemma 3.2.5 Let $M = (Xx, g; \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r)$ be a Seifert manifold, where $Xx \in \{Oo, On, No, NnI, NnII, NnIII\}$ such that $\alpha_i \geq 2$ and $r \geq 2$. Consider a representation $\omega : \pi_1(M_0) \to S_n$ defined by

$$
\omega(h) = \varepsilon_n,
\omega(q_i) = \varepsilon_{k_i}, \forall i = 1, \ldots, r \text{ and }
\omega(v_j) = \tau_j,
$$

such that $\tau_j$ is a power of $\varepsilon_n$, if $v_j$ commutes with $h$; otherwise, $\tau_j$ is a reflection $\rho_j$, if $v_j$ anticommutes with $h$. 

Let $\varphi : \tilde{M} \to M$ be the covering associated to $\omega$.

Then $h(\tilde{M}) = h(M)$.

**Proof.**

Let $F$ and $G$ be the orbit surfaces of $M$ and $\tilde{M}$, respectively. If $g$ is the genus of $F$, then $G$ also has genus $g$ since $F$ and $G$ are homeomorphic because of Theorem 2.3.15. Note that $\alpha_i \geq 2$ implies that $A_i \geq 2$, thus the number of exceptional fibers of $\tilde{M}$ is equal to $r$. Therefore $h(\tilde{M}) = h(M)$. □

Now we are able to prove the following theorem.

**Theorem 3.2.2** Consider $M = (Xx, g; \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r)$ a Seifert manifold, where $Xx \in \{Oo, On, No, NnI, NnII, NnIII\}$ and assume $\omega : \pi_1(M_0) \to S_n$ is a representation defined by

\[
\begin{align*}
\omega(h) &= \varepsilon_n, \\
\omega(q_i) &= \varepsilon_n^{k_i}, \forall i = 1, \ldots, r \text{ and} \\
\omega(v_j) &= \tau_j,
\end{align*}
\]

such that $\tau_j$ is a power of $\varepsilon_n$ if $v_j$ commutes with $h$; otherwise, $\tau_j$ is a reflection $\rho_j$, if $v_j$ anticommutes with $h$.

Suppose $\varphi : \tilde{M} \to M$ is the covering determined by $\omega$.

If $M = (Xx, g; \beta_1/\alpha_1)$, where $\alpha_1 \geq 1$, $(\beta_1 + k_1 \alpha_1)|n$ and $\beta_1 \neq \pm 1$, then $h(\tilde{M}) < h(M)$.

Otherwise, $h(\tilde{M}) \geq h(M)$.

**Proof.**

The result follows from Lemma 3.2.4 and Lemma 3.2.5. □
Bibliography


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