Centro de Investigación en Matemáticas A.C.

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Supergrupos de Lie sobre GL(2) asociados a la representación adjunta

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Aclaración. Nos disculpamos con el lector cuya lengua nativa es el español, al presentar este documento en inglés. Sin embargo, tenemos buenas razones para hacerlo de esta forma. Hace dos meses apliqué para una posición posdoctoral en el extranjero. En particular, el Prof. D. Vogan del M.I.T. me pidió una versión preliminar de esta Tesis y fue hasta ese momento en que tuve la oportunidad de recolectar los resultados y unificar la presentación en un solo documento. Pero tenía que ser en inglés, por razones obvias. Inmediatamente después de presentar esa versión preliminar empezamos a trabajar en algunos problemas relacionados con los que obtuvimos en esta tesis con el Prof. Finlay Thompson. Así que usamos esa versión preliminar en inglés como referencia para colaborar con él. Por último, dado que el Prof. V. Pestov de la Universidad Victoria de Nueva Zelanda ha sido invitado a participar en el Comité de la Defensa de Tesis, y puesto que hay algún precedente en CIMAT de presentar la Tesis en inglés, decidimos mantenerla en esta forma.
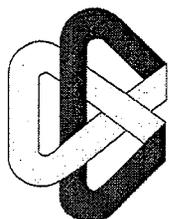
Centro de Investigación en Matemáticas A.C.

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Lie supergroups supported over GL(2) associated to the adjoint representation

Thesis

By
Ramón Peniche Mena

December 14, 2001   Guanajuato, Gto. Mexico

To Jesus, because He lives.

Neither is there salvation in any other: for there is none other name under heaven given among men, whereby we must be saved.

Acts 4:12

The fear of the LORD *is* the beginning of wisdom: and the knowledge of the holy *is* understanding.

If any of you lack wisdom, let him ask of God, that giveth to all *men* liberally, and upbraideth not; and it shall be given him.

And further, by these, my son, be admonished: of making many books *there is* no end; and much study *is* a weariness of the flesh.

Let us hear the conclusion of the whole matter: Fear God, and keep his commandments: for this *is* the whole *duty* of man.

Proverbs 9:10

James 1:5

Ecclesiastes 12:12,13

LIE SUPERGROUPS SUPPORTED OVER GL_2
ASSOCIATED TO THE ADJOINT REPRESENTATION

R. PENICHE

December 14, 2001

CONTENTS

Disclaimer. We apologize to the reader whose native language is Spanish, for presenting this document in English. We had a good reason, however, for doing it so. Some two months ago I applied for a postdoctoral position abroad. In particular, Prof. D. Vogan from M.I.T., requested a preliminary version of this thesis and it was at that point in time that I took the opportunity to collect the results and unify the presentation in a single document. But it had to be done in English for the obvious reasons. Right immediately after I submitted that preliminary version, we started to work on some problems related to those approached in this thesis, with Prof. Finlay Thompson. So we used that English preliminary version for reference and general background for the collaboration with him. Finally, since Prof. V. Pestov from the Victoria University of New Zealand has been invited to participate in the Thesis Defense Committee, and since there has been some precedent at CIMAT of presenting the Ph.D. Thesis in English, we decided to keep it in this form.

INTRODUCTION

The main purpose of this work is to understand the structure of the (real and complex) Lie supergroups having GL_2 as their underlying Lie group and having $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, with $\mathfrak{g}_0 = \mathfrak{gl}_2 = \mathfrak{g}_1$, as their Lie superalgebra, further restricted by the condition that the action of the even $\mathfrak{gl}_2$ on the odd $\mathfrak{gl}_2$ is given by the adjoint representation. We also give a description of their compact real forms and look at their maximal tori.

In other words, we solve Lie's problem of finding the Lie group associated to a Lie algebra within the category of supermanifolds for a specific family of Lie superalgebras. The exposition is organized in such a way that we have kept in the main part of this presentation (§2 to §6) the results that, up to our knowledge, are new in the existing literature of the subject. The exceptions are §1 and the Appendix. In fact, §1 contains results from the Ph.D. Thesis of my 'PhD-seminar-mate' Gil Salgado (see [15]) that are included here for the sake of completeness and self-containedness. The Appendix, on the other hand, has been included for the benefit of the reader, as it provides a quick reference to the basics on Lie supergroup theory and to the theory of Ordinary Differential Equations in supermanifolds —a main tool used in this thesis.

A good deal of motivation for studying Lie superalgebras of the form $\mathfrak{h} \oplus \mathfrak{h}$, for a given Lie algebra $\mathfrak{h}$, has been given in [6]. The Lie superalgebras studied there correspond to those for which the Lie bracket of any pair of odd elements is identically zero, as they represent the Lie superalgebra generated by Lie derivatives $\mathcal{L}_X$, and contractions i_Y on the graded algebra of differential forms on a smooth manifold. On the other hand, from [15] we know that for the Lie algebra $\mathfrak{h} = \mathfrak{gl}_2$ defined over the complex numbers, one actually obtains eight different isomorphism classes of Lie superalgebras satisfying the constraints mentioned above (ten different classes of them over the real field). Only one of these classes, of course, yield all brackets of odd elements equal to zero.

Other motivation for looking at the superalgebras $\mathfrak{h} \oplus \mathfrak{h}$ is that they provide us with the simplest supervector space model on which we can actually study nontrivial equivariant 'change of parity maps' going from the even copy of $\mathfrak{h}$ into the odd one, with minimum hypotheses. A bit more of motivation comes from [1], where in order to understand some physically relevant Lie superalgebras, the authors classify

Lie superalgebras $\mathfrak{g}_0 \oplus \mathfrak{g}_1$ under the hypotheses of having a reductive $\mathfrak{g}_0 = \mathfrak{s} \oplus \mathfrak{m}$ but constraining its action on $\mathfrak{g}_1$ (eg, $\mathfrak{m}$ acts trivially), as well as the image of the odd brackets (eg, they must be contained in $\mathfrak{m}$). A simple nontrivial reductive $\mathfrak{g}_0 = \mathfrak{s} \oplus \mathfrak{m}$ we may think of is $\mathfrak{gl}_2$, so that $\mathfrak{s} = \mathfrak{sl}_2$ and $\mathfrak{m}$ is the one-dimensional subspace generated by the multiples of the identity 2×2-matrix. For us then, $\mathfrak{g}_1 = \mathfrak{gl}_2 = \mathfrak{g}_0$ and the adjoint representation gives the $\mathfrak{gl}_2$-action in the odd Lie-module.

The general algebraic problem of determining all *Lie superalgebras* of the form $\mathfrak{gl}_n \oplus \mathfrak{gl}_n$ defined by means of the adjoint representation of the even $\mathfrak{gl}_n$ into the odd $\mathfrak{gl}_n$ has been approached and completely solved in [15]. But we are mainly interested in the *Lie supergroup structures* arising from the family of Lie superalgebras supported over $\mathfrak{gl}_2$ under the conditions imposed. We want to stress the fact that all of them can be put on equal footing and that each one of them is susceptible of further geometric considerations.

To give a more precise idea of what is involved in the statement of 'understanding the Lie supergroups having GL_2 as their underlying Lie group, ...' of the opening paragraph, let us first consider the Lie algebra $\mathfrak{gl}_2$ generated, as usual, by the 2×2-matrices $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, which we will denote here by x_0, x_1, x_2 and x_3, respectively. In considering the Lie superalgebra $\mathfrak{gl}_2 \oplus \mathfrak{gl}_2$ we think of the odd generators as $\pi(I)$, $\pi(H)$, $\pi(E)$, $\pi(F)$, respectively, where $\pi(X)$ stands for X with its $\mathbb{Z}_2$-parity reversed. The fact that the action of the even $\mathfrak{gl}_2$ in the odd $\mathfrak{gl}_2$ is given by the adjoint representation is written in terms of π as $\pi([x_i, x_j]) = [x_i, \pi(x_j)] = -[\pi(x_j), x_i]$, where $[\cdot, \cdot]$ stands for the Lie algebra bracket. To complete the Lie superalgebra description we must give a symmetric bilinear map $\Gamma : \mathfrak{gl}_2 \times \mathfrak{gl}_2 \to \mathfrak{gl}_2$ representing the bracket of any pair of odd elements. Writing $\Gamma(x_i, x_j) = [\pi(x_i), \pi(x_j)]$, it is a straightforward matter to check that the Jacobi identities for the Lie superalgebra imply that,

$$\Gamma(x_0, x_0) = \lambda x_0$$

$$\Gamma(x_0, x_1) = \mu x_1 \qquad \Gamma(x_1, x_1) = 2\nu x_0$$

$$\Gamma(x_0, x_2) = \mu x_2 \qquad \Gamma(x_1, x_2) = 0 \qquad \Gamma(x_2, x_2) = 0$$

$$\Gamma(x_0, x_3) = \mu x_3 \qquad \Gamma(x_1, x_3) = 0 \qquad \Gamma(x_2, x_3) = \nu x_0 \qquad \Gamma(x_3, x_3) = 0,$$

for arbitrary parameters λ, μ and ν in the ground field. A different symmetric bilinear map $\Gamma' : \mathfrak{gl}_2 \times \mathfrak{gl}_2 \to \mathfrak{gl}_2$ would yield a different set of parameters; say λ', μ' and ν', respectively. Let us denote by $\mathfrak{gl}_2(\mathbb{F}; \lambda, \mu, \nu)$ the $\mathbb{F}$-Lie superalgebra ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$) $\mathfrak{gl}_2 \oplus \mathfrak{gl}_2$ defined by the parameter values (λ, μ, ν). It was proved in [15] that the Lie superalgebras $\mathfrak{gl}_2(\mathbb{F}; \lambda, \mu, \nu)$ and $\mathfrak{gl}_2(\mathbb{F}; \lambda', \mu', \nu')$ are isomorphic if and only if there is a Lie algebra automorphism $T : \mathfrak{gl}_2 \to \mathfrak{gl}_2$ and an $\mathbb{F}$-linear isomorphism $S : \mathfrak{gl}_2 \to \mathfrak{gl}_2$ satisfying

$$[T(x), S(y)] = S([x, y]) \qquad \text{and} \qquad \Gamma'(S(x), S(y)) = T(\Gamma(x, y)),$$

for any x and y in the Lie algebra $\mathfrak{gl}_2$. This is the case (see [15]) if and only if there are nonzero constants α_0, α_1 and α_2 in the ground field $\mathbb{F}$, such that,

$$\lambda' = \lambda \frac{\alpha_1^2}{\alpha_0}, \qquad \mu' = \mu \frac{\alpha_1 \alpha_2}{\alpha_0}, \qquad \nu' = \nu \, \alpha_0 \alpha_2^2.$$

It follows that either, the three parameters λ, μ and ν are equal to zero, or exactly two of them are zero, or exactly one is zero, or none of them is zero. That is how the eight isomorphism classes over $\mathbb{C}$ arise. In the real case, one further sees that the product $\lambda'\nu'$ is equal to $\lambda\nu$ times a positive constant. Therefore, the sign of this product must remain constant, thus giving ten real isomorphism classes. Concrete representatives for the different classes can be given. In the complex case we shall agree on choosing the parameter representatives in such a way that, when different from zero, $\lambda = 2$, $\mu = 2$ and $\nu = 1$.

We are then interested in finding out explicitly the actual Lie supergroup whose underlying Lie group is GL_2 and whose Lie superalgebra is $\mathfrak{gl}_2(\mathbb{F}; \lambda, \mu, \nu)$. Let us denote this Lie supergroup by $\mathrm{GL}_2(\mathbb{F}; \lambda, \mu, \nu)$. We proceed by following Lie's techniques: That is, we first find a faithful representation of the Lie superalgebra $\mathfrak{gl}_2(\mathbb{F}; \lambda, \mu, \nu)$ inside the Lie superalgebra of supervector fields on a given supermanifold. We then look at the integral flows of the supervector fields which are images of the Lie superalgebra generators x_i and $\pi(x_i)$. The main resource for finding out the integral flows of the various supervector fields is the existence and uniqueness theorem proved in [13] —and thanks to the ODE theory in supermanifolds developed there, we may follow Lie's original method (which is non-obvious *a priori* in the $\mathbb{Z}_2$-graded category) for finding the supergroup structure out of the Lie superalgebra. We then consider the composition of the various integral flows regarding the integration parameters as independent variables; they will eventually be interpreted as local coordinates on the Lie supergroup. The multiplication law will emerge after composing two different sets of integral flows.

The problem of giving a faithful representation of the Lie superalgebra $\mathfrak{gl}_2(\mathbb{F}; \lambda, \mu, \nu)$ inside the Lie superalgebra of supervector fields on a given supermanifold and the problem of finding the integral flows of the image generators have some interesting implications. The integral flows actually describe a (local) action of $\mathrm{GL}_2(\mathbb{F}; \lambda, \mu, \nu)$ on the supermanifold. Since supermanifolds come equipped with a natural forgetful (covariant) functor that recovers for each object the underlying smooth (or holomorphic) manifold over which the 'super' structure sheaf is defined, the natural supermanifolds to consider for a faithful representation of $\mathfrak{gl}_2(\mathbb{F}; \lambda, \mu, \nu)$ are those having $\mathbb{F}^2$ as their underlying manifold. We would then expect the Lie supergroup action of $\mathrm{GL}_2(\mathbb{F}; \lambda, \mu, \nu)$ on it to yield the ordinary linear action of GL_2 on $\mathbb{F}^2$ upon application of such a functor. What we have found, however, is that all but two isomorphism classes of the Lie superalgebras we obtained can be represented as supervector fields on the $(2,2)$-dimensional supermanifold $\mathbb{F}^{2|2}$ in such a way that we can recover the $\mathbb{F}$-linear GL_2-action on $\mathbb{F}^2$. The classes corresponding to $[\lambda \neq 0, \mu = 0, \nu = 0]$ and to $[\lambda = 0, \mu = 0, \nu \neq 0]$ have to be represented as supervector fields in $\mathbb{F}^{3|3}$ and the interpretation of the $\mathrm{GL}_2(\mathbb{F}; \lambda, \mu, \nu)$ action is more subtle.

We recall that a Lie supergroup structure over GL_2 is thought of as a sheaf of $\mathbb{Z}_2$-graded algebras. Actually, for Lie supergroups this structure sheaf is easy to describe (see [8]) and in particular, for the Lie supergroups we are interested in, the structure sheaf —which we denote by $\mathcal{GL}_2(\mathbb{F}; \lambda, \mu, \nu)$— is isomorphic to the sheaf of sections of the exterior algebra bundle associated to the trivial vector bundle $\mathrm{GL}_2 \times \mathfrak{gl}_2 \to \mathrm{GL}_2$. In particular, $\mathrm{GL}_2(\mathbb{F}; \lambda, \mu, \nu) = (\mathrm{GL}_2, \mathcal{GL}_2(\mathbb{F}; \lambda, \mu, \nu))$ is a $(4,4)$-dimensional supermanifold and a set of local coordinates $\{g_{ij}; \gamma_{ij}\}$ $(i, j \in \{1, 2\})$ can

be chosen in such a way that under the natural sheaf epimorphism (ie, the source of the forgetful functor mentioned above) $\mathcal{GL}_2(\mathbb{F}; \lambda, \mu, \nu) \to \mathcal{C}^\infty_{\mathrm{GL}_2}$, $g_{ij} \mapsto \widetilde{g}_{ij}$, $\{\widetilde{g}_{ij}\}$ is a set of local coordinates on GL_2. Moreover, the set $\{\gamma_{ij}\}$ yields at each point in GL_2 a frame in the corresponding fiber of the trivial bundle $\mathrm{GL}_2 \times \mathfrak{gl}_2 \to \mathrm{GL}_2$. To say that $\mathrm{GL}_2(\mathbb{F}; \lambda, \mu, \nu)$ is a Lie supergroup also means that there is a multiplication morphism (composition law) $\mathrm{m} : \mathrm{GL}_2(\mathbb{F}; \lambda, \mu, \nu) \times \mathrm{GL}_2(\mathbb{F}; \lambda, \mu, \nu) \to \mathrm{GL}_2(\mathbb{F}; \lambda, \mu, \nu)$ and we would like it to be described by thinking of the local coordinates g_{ij} and γ_{ij} as 'elements in the supergroup', identified with 2×2 matrices $\mathbf{g}$ and $\boldsymbol{\gamma}$ respectively, that can be 'multiplied'. We show that the Lie supergroup multiplication map can be cast into a composition law that looks as follows:

$$(\mathbf{g}', \boldsymbol{\gamma}') \cdot (\mathbf{g}, \boldsymbol{\gamma}) = (\mathbf{g}'\, \mathbf{g}(1 + \mathcal{F}_0), \boldsymbol{\gamma} + \mathrm{Ad}(\mathbf{g}^{-1})(\boldsymbol{\gamma}' + \mathcal{F}_1)),$$

where the $\mathcal{F}_i$'s are matrices —depending on the parameters λ, μ and ν, as well as on the matrices $\mathbf{g}'$, $\boldsymbol{\gamma}'$, $\mathbf{g}$ and $\boldsymbol{\gamma}$— taking their values in the nilpotent ideal of the exterior algebra generated by the entries of $\boldsymbol{\gamma}$ and $\boldsymbol{\gamma}'$. Note that we can immediately read off that the composition law starts, at the level of the even coordinates, as the usual matrix multiplication for GL_2. At the level of the odd coordinates it starts as the $\mathfrak{gl}_2$-component of the semidirect product of $\mathrm{GL}_2 \times \mathfrak{gl}_2$ associated to the adjoint representation. The composition law will be the semidirect product precisely when $\mathcal{F}_0$ and $\mathcal{F}_1$ are identically zero, which corresponds to the case when all the brackets of the odd elements are zero; ie, $\lambda = \mu = \nu = 0$.

We call the reader's attention to the fact that the actual determination of the multiplication law above looks formally like the determination of the *co-multiplication map* on the $\mathbb{Z}_2$-commutative $\mathbb{Z}_2$-graded algebra generated by the local coordinates in the supergroup (see §3 below). This co-multiplication map will depend on the parameters λ, μ and ν. Since the inversion morphism defined with the Lie supergroup gives rise to an antipode map for the co-multiplicative structure, the final product might also be approached within the theory of *quantum groups* (see the papers by Woronowicz, [20], [21] and [22]). The main reason for choosing the differential-geometric approach rather than the Hopf-algebra approach is that we also wanted to understand the supermanifold structure of the Lie supergroups $\mathrm{GL}_2(\mathbb{F}; \lambda, \mu, \nu)$ within the spirit of Lie's fundamental theory, making extensive use of ODE's and having in the background Frobenius Theorem (see [12]).

Let us mention that we have succeeded in finding general composition laws (ie, depending on arbitrary values of the parameters (λ, μ, ν)) for only four of the eight isomorphism classes over the complex numbers. We have been able to give the other four composition laws only for specific representatives inside the isomorphism class. The difficult cases for giving a composition law for arbitrary parameter values are those corresponding to the Lie superalgebras having $\nu \neq 0$.

Once we reach at the point of identifying the multiplication law $(\mathbf{g}', \boldsymbol{\gamma}') \cdot (\mathbf{g}, \boldsymbol{\gamma})$, we should be able to verify that the Lie superalgebra of left-invariant supervector fields on $\mathrm{GL}_2(\mathbb{F}; \lambda, \mu, \nu)$ is actually isomorphic to the abstract Lie superalgebra $\mathfrak{gl}_2(\mathbb{F}; \lambda, \mu, \nu)$ we started with. In order to do this we have to provide an appropriate commutative diagram of supermanifold morphisms capable of stating the left-invariance property. This is a common resource in supermanifolds theory: Since supermanifold morphisms are not determined by their values on the points of the underlying manifolds involved, one must be careful each time one needs to leave

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an argument fixed in a two-argument morphism (an example of this technique is given by the 'evaluation map' introduced in [13] to deal with the uniqueness of the integral flows of supervector fields). We have given such a commutative diagram and showed that it really corresponds to the left-invariance property at the level of points. We have also used it to verify the assertion that the Lie algebra of left-invariant supervector fields on $\mathrm{GL}_2(\mathbb{F}; \lambda, \mu, \nu)$ is isomorphic to $\mathfrak{gl}_2(\mathbb{F}; \lambda, \mu, \nu)$.

At this point, the essence of the classical Lie's Theorems finds a concrete verification for the Lie supergroups $\mathrm{GL}_2(\mathbb{F}; \lambda, \mu, \nu)$. Other aspects of Lie's theory and some applications can be observed for our particular family of examples by looking at their compact real forms. In particular, the problem that called our attention first was to understand in what sense we could realize the analogue (or analogues, if there was more than one possibility) of the Hopf fibration, once we have concrete descriptions of the 'super' versions of the groups U_2 and SU_2.

In order to approach this problem we first consider the real Lie superalgebras $\mathfrak{u}_2(\lambda, \mu, \nu)$ —whose underlying supervector space is $\mathfrak{u}_2 \oplus \mathfrak{u}_2$— that arise after changing the basis in $\mathfrak{gl}_2(\mathbb{C}; \lambda, \mu, \nu)$, so as to have the even copy of $\mathfrak{u}_2$ generated over the real field by $w_0 = iI$, $w_3 = iH$, $w_2 = E - F$ and $w_1 = i(E + F)$, as usual. We let π be as before, so that the symmetric bilinear equivariant map $\Gamma : \mathfrak{u}_2 \times \mathfrak{u}_2 \to \mathfrak{u}_2$ that gives the Lie bracket of any pair of odd elements via $\Gamma(z, w) = [\pi(z), \pi(w)]$ is

$$\Gamma(w_0, w_0) = i\lambda w_0$$

$$\Gamma(w_0, w_3) = i\mu w_3 \quad \Gamma(w_3, w_3) = 2i\nu w_0$$

$$\Gamma(w_0, w_2) = i\mu w_2 \quad \Gamma(w_3, w_2) = 0 \qquad \Gamma(w_2, w_2) = 2i\nu w_0$$

$$\Gamma(w_0, w_1) = i\mu w_1 \quad \Gamma(w_3, w_1) = 0 \quad \cdot \quad \Gamma(w_2, w_1) = 0 \qquad \Gamma(w_1, w_1) = 2i\nu w_0.$$

Therefore, λ, μ and ν have to be restricted from taking arbitrary complex values in $\mathfrak{gl}_2(\mathbb{C}; \lambda, \mu, \nu)$ to purely imaginary values on $\mathfrak{u}_2(\lambda, \mu, \nu)$. The maximal toral subalgebra of $\mathfrak{u}_2(\lambda, \mu, \nu)$ is generated by w_0, w_3, $\pi(w_0)$ and $\pi(w_3)$. We find the integral flows of the appropriate supervector fields which are images of these generators and also find the composition law for the maximal torus $\mathbb{T}^2(\lambda, \mu, \nu) \subset \mathrm{U}_2(\lambda, \mu, \nu)$. In this case we have succeeded in writing it down for arbitrary parameter values of λ, μ and ν, regardless of the isomorphism class $[\lambda, \mu, \nu]$.

The maximal tori arising from the different isomorphism classes of the unitary supergroups brings to the foreground the general problem of classifying all the real Lie supergroup structures that can be defined over the torus $\mathbb{T}^2 = S^1 \times S^1$, whose 'odd sector' comes from the adjoint representation of $\mathfrak{t}_2 = \mathrm{Lie}(\mathbb{T}^2)$. This problem fits into the general spirit of the first part of this work and can be solved by using the same methods; ie, by classifying first the Lie superalgebra structures on $\mathfrak{t}_2 \oplus \mathfrak{t}_2$ associated to the adjoint representation. Since $\mathfrak{t}_2$ is Abelian, the Jacobi identities for the various combinations of homogeneous elements are all trivial and, therefore, there are no conditions imposed on the symmetric bilinear map $\Gamma : \mathfrak{t}_2 \times \mathfrak{t}_2 \to \mathfrak{t}_2$.

Once a basis of $\mathfrak{t}_2$ is given (and the basis of the odd direct summand is the same but with the understanding that its parity has been reversed), the problem of classifying those symmetric bilinear maps $\Gamma : \mathfrak{t}_2 \times \mathfrak{t}_2 \to \mathfrak{t}_2$ that yield isomorphic Lie superalgebras on $\mathfrak{t}_2 \oplus \mathfrak{t}_2$ comes down to the problem of classifying pairs (θ^1, θ^2) of real symmetric bilinear forms under the action of $\mathrm{GL}_2(\mathbb{R}) \times \mathrm{GL}_2(\mathbb{R})$ given by

$$(T, S) \cdot (\theta^1, \theta^2) = \left(T_{11} S^{-1} \cdot \theta^1 + T_{12} S^{-1} \cdot \theta^2, \, T_{21} S^{-1} \cdot \theta^1 + T_{22} S^{-1} \cdot \theta^2 \right),$$

where T and S belong to $\mathrm{GL}_2(\mathbb{R})$, $S^{-1} \cdot \theta^i = S^{-1}\theta^i(S^{-1})^t$ and the indicated matrix entries are referred to the chosen basis. It is proved that there are seven different orbits for this action (only four if one would pose the same problem over the complex field) and it is shown that there is a surjection from the equivalence classes of tori we have found for the various superunitary groups, onto the equivalence classes of Lie superalgebras obtained this way.

Actually, one could be slightly more general and remove the condition that the Lie superalgebras be associated to the adjoint representation only. In fact, one can classify the Lie superalgebra structures that can be supported on $t_2 \oplus \mathbb{F}^2$; that is, with the only condition that the odd module be a 2-dimensional representation space for the torus. This problem has been solved in general in a recent joint work of this author with F. Thompson and O.A. Sánchez-Valenzuela.

Some of our future goals are: To look at some fibrations arising from the different Lie subsupergroups $\mathrm{GL}_2(\mathbb{F}; \lambda, \mu, \nu)$ and giving rise to nice supermanifold quotients. In fact, we have begun this study by looking at the various Lie subsuperalgebras (and their corresponding Lie subsupergroups) having the maximal parabolic subgroup $P \subset \mathrm{GL}_2$ as their underlying Lie group. We have also started to look at the fibrations over the various superspheres obtained as the quotient and we have also taken a look at the problem of defining and realizing in a concrete fashion some left-invariant geometric structures —like $\mathbb{Z}_2$-graded Riemannian metrics and $\mathbb{Z}_2$-graded connections— for the principal fibrations obtained. In particular, the Maurer-Cartan form and Cartan's structure equations can be easily described for all the supergroups $\mathrm{GL}_2(\mathbb{F}; \lambda, \mu, \nu)$, regardless of their isomorphism class. We are also looking at the Hopf fibration in this $\mathbb{Z}_2$-graded category, which seems to be more illuminating and more accessible now that we have explicit descriptions for the various Lie supergroup structures involved. These results will soon be fully developed and published elsewhere.

1. Classification of Lie superalgebras based on $\mathfrak{gl}_2$ whose odd module is $\mathfrak{gl}_2$ itself under the adjoint action

Let $\mathbb{F}$ be either $\mathbb{R}$ or $\mathbb{C}$. Let $\mathfrak{gl}_2$ be the Lie algebra of 2×2-matrices with entries in $\mathbb{F}$. We want to classify the Lie superalgebra structures on the supervector space

$$
(1) \qquad \mathfrak{gl}_{2|2} = \mathfrak{g}_0 \oplus \mathfrak{g}_1, \qquad \text{where} \qquad \mathfrak{g}_0 = \mathfrak{gl}_2 \quad \text{and} \quad \mathfrak{g}_1 = \mathfrak{gl}_2,
$$

for which the action of the even Lie algebra $\mathfrak{g}_0 = \mathfrak{gl}_2$ on the odd Lie module $\mathfrak{g}_1 = \mathfrak{gl}_2$ is the adjoint action. To classify them all amounts to classify the symmetric bilinear maps $\Gamma : \mathfrak{gl}_2 \times \mathfrak{gl}_2 \to \mathfrak{gl}_2$, satisfying the equivariant property,

$$
(2) \qquad \Gamma([x,y],z) + \Gamma(y,[x,z]) = [x,\Gamma(y,z)]
$$

and the *odd* Jacobi identity,

$$
(3) \qquad [x,\Gamma(y,z)] + [z,\Gamma(x,y)] + [y,\Gamma(z,x)] = 0.
$$

Warning. In the following sections we have a slight change of notation with respect to that used in the introduction above. Here we have written y_i instead of $\pi(x_i)$.

Let $\{x_i\}$ be a basis for $\mathfrak{g}_0 = \mathfrak{gl}_2$ and write $y_i = \pi(x_i)$ so that $\{y_i\}$ becomes a basis of $\mathfrak{g}_1 = \mathfrak{gl}_2$. We shall refer ourselves to $\{x_i\}$ as the set of *even generators* and to $\{y_i\}$ as the set of *odd generators*.

Convention. It will be assumed that the bases of the even and the odd direct summands, of all the Lie superalgebras that appear throughout this work, are related this way.

Let $\Gamma : \mathfrak{gl}_2 \times \mathfrak{gl}_2 \to \mathfrak{gl}_2$ be a symmetric bilinear map satisfying (2) and (3) above. Following the standard notation in Lie superalgebras we shall write $[y_i, y_j]$ instead of $\Gamma(x_i, x_j)$ and we write the Lie superalgebra bracket

$$
[\cdot, \cdot] : \mathfrak{gl}_2 \oplus \mathfrak{gl}_2 \times \mathfrak{gl}_2 \oplus \mathfrak{gl}_2 \to \mathfrak{gl}_2 \oplus \mathfrak{gl}_2
$$

in terms of the graded basis $\{x_i, y_i\}$ as

$$
(4) \qquad [x_i, x_j] = \sum_{k=0}^{3} C_{ikj} x_k, \qquad [x_i, y_j] = \sum_{k=0}^{3} C_{ikj} y_k, \qquad [y_i, y_j] = \sum_{k=0}^{3} \Gamma_{ikj} x_k,
$$

where the C_{ikj}'s are the structure constants for the Lie algebra $\mathfrak{gl}_2$ and the Γ_{ikj}'s are obtained from Γ after setting $[y_i, y_j] = \Gamma(x_i, x_j)$, and writing them as linear combinations of the even generators.

It is a straightforward matter to verify that (2) implies (3). Now, using (2) and the first two equations in (4), one may explicitly find the Γ_{ikj}'s and show that they depend on three scalar parameters (λ, μ, ν). More concretely, we use the identification,

$$
(5) \qquad x_0 \leftrightarrow I, \qquad x_1 \leftrightarrow H, \qquad x_2 \leftrightarrow E \qquad \text{and} \qquad x_3 \leftrightarrow F,
$$

where I is the identity 2×2 matrix and $\{H, E, F\}$ is the standard basis for $\mathfrak{sl}_2$. Similarly, we set,

(6) $\qquad y_0 \leftrightarrow I, \qquad y_1 \leftrightarrow H, \qquad y_2 \leftrightarrow E \qquad \text{and} \qquad y_3 \leftrightarrow F.$

Then,

(7)
$$
\begin{aligned}
&[y_0, y_0] = \lambda x_0 \\
&[y_0, y_1] = \mu x_1 \qquad [y_1, y_1] = 2\nu x_0 \\
&[y_0, y_2] = \mu x_2 \qquad [y_1, y_2] = 0 \qquad\qquad [y_2, y_2] = 0 \\
&[y_0, y_3] = \mu x_3 \qquad [y_1, y_3] = 0 \qquad\qquad [y_2, y_3] = \nu x_0 \qquad [y_3, y_3] = 0.
\end{aligned}
$$

Thus, writing Γ_i for the matrix whose (j, k) entry is Γ_{ijk}, we have

(8)
$$
\Gamma_0 = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \mu & 0 & 0 \\ 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & \mu \end{pmatrix}, \qquad
\Gamma_1 = \begin{pmatrix} 0 & 2\nu & 0 & 0 \\ \mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
$$
$$
\Gamma_2 = \begin{pmatrix} 0 & 0 & 0 & \nu \\ 0 & 0 & 0 & 0 \\ \mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad
\Gamma_3 = \begin{pmatrix} 0 & 0 & \nu & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \mu & 0 & 0 & 0 \end{pmatrix}.
$$

Now, the Lie superalgebra corresponding to the parameter values (λ, μ, ν) is *isomorphic* to the Lie superalgebra corresponding to (λ', μ', ν') if and only if there is a pair of *invertible* $\mathbb{F}$-linear maps $S : \mathfrak{gl}_2 \to \mathfrak{gl}_2$ and $T : \mathfrak{gl}_2 \to \mathfrak{gl}_2$ such that, for $j = 0, 1, 2$ and 3,

(9) $\qquad TC_jT^{-1} = \displaystyle\sum_{i=0}^{3} T_{ij}C_i, \qquad SC_jS^{-1} = \sum_{i=0}^{3} T_{ij}C_i, \qquad T\Gamma_jS^{-1} = \sum_{i=0}^{3} S_{ij}\Gamma_i',$

where Γ_i' is the matrix corresponding to the parameter values (λ', μ', ν'); on the other hand, C_0 is the zero 4×4-matrix and

(10) $\quad C_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}, \qquad C_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad C_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix}.$

Using the first equation in (9), one immediately finds that $T_{i0} = 0$ for $i = 1, 2, 3$. Therefore the structure of the linear map T can be decomposed in blocks as

$$
T = \begin{pmatrix} \alpha_0 & \mathbf{t} \\ 0 & t \end{pmatrix} \qquad t \in \mathrm{Aut}(\mathfrak{sl}_2), \qquad \mathbf{t} = (t_{01}, t_{02}, t_{03}) \in (\mathfrak{sl}_2)^*, \qquad \alpha_0 \in \mathbb{F} - \{0\}.
$$

This triangular block form of the matrix T allows us to explicitly find its inverse. Furthermore, let us write $\mathrm{ad}_{\mathfrak{sl}}(x_i) : \mathfrak{sl}_2 \to \mathfrak{sl}_2$ for the restriction to $\mathfrak{sl}_2$ of the action of x_i, $i = 1, 2, 3$. Then, the LHS of the first equation in (9) says that,

$$
TC_jT^{-1} = \begin{pmatrix} 0 & \mathbf{t}\,\mathrm{ad}_{\mathfrak{sl}}(x_j)t^{-1} \\ 0 & t\,\mathrm{ad}_{\mathfrak{sl}}(x_j)t^{-1} \end{pmatrix} \qquad j = 1, 2, 3.
$$

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On the other hand, the RHS of the same equation yields,

$$\sum_{i=1}^{3} T_{ij} C_i = \begin{pmatrix} 0 & 0 \\ 0 & \sum_{i=1}^{3} T_{ij} \operatorname{ad}_{\mathfrak{sl}}(x_i) \end{pmatrix} \qquad j = 1, 2, 3.$$

It then follows, on the one hand, that

(11) $$t \operatorname{ad}_{\mathfrak{sl}}(x_j) t^{-1} = \sum_{i=1}^{3} T_{ij} \operatorname{ad}_{\mathfrak{sl}}(x_i), \quad j = 1, 2, 3.$$

This says that $t \in \operatorname{Aut}(\mathfrak{sl}_2)$. A straightforward computation shows that t satisfies (11) if and only if

$$\det(t)\, t^{-1} = \kappa^{-1} t^T \kappa, \qquad \text{where} \quad \kappa = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

In particular, taking determinants on both sides, we get $\det(t) = 1$, and therefore,

(12) $\quad t \in \operatorname{Aut}(\mathfrak{sl}_2) \iff t \in SO_\kappa(\mathfrak{sl}_2) \iff t^{-1} = \kappa^{-1} t^T \kappa, \quad \det(t) = 1,$

where κ is the Cartan-Killing metric in $\mathfrak{sl}_2$, $\kappa(z, w) = \operatorname{Tr}\big(\operatorname{ad}(z) \circ \operatorname{ad}(w)\big)$, and the matrix appearing above is taken with respect to the basis $\{H, E, F\}$.

On the other hand, we also have $\mathbf{t}\operatorname{ad}_{\mathfrak{sl}}(x_j) t^{-1} = 0$, for $j = 1, 2, 3$, which easily implies that $\mathbf{t} = 0$. In summary,

(13) $$T = \begin{pmatrix} \alpha_0 & 0 \\ 0 & t \end{pmatrix}, \qquad t \in \operatorname{Aut}(\mathfrak{sl}_2), \quad \alpha_0 \in \mathbb{F} - \{0\}.$$

Now, using the first two equations in (9), note that they both imply that $T C_j T^{-1} = S C_j S^{-1}$; that is, $S^{-1} T C_j = C_j S^{-1} T$, for $j = 1, 2, 3$. Let M be any 4×4 matrix that commutes with C_1, C_2 and C_3. It is a straightforward matter to see that,

$$M C_j = C_j M, \quad j = 1, 2, 3 \qquad \iff \qquad M = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \mathbb{1}_{3 \times 3} \end{pmatrix},$$

where α_1 and α_2 are arbitrary scalars and $\mathbb{1}_{3 \times 3}$ stands for the 3×3 unit matrix. Then, for our problem,

$$S^{-1} T = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \mathbb{1}_{3 \times 3} \end{pmatrix}, \qquad \alpha_1 \alpha_2 \neq 0,$$

and it follows that

$$S = \begin{pmatrix} \alpha_0 \alpha_1^{-1} & 0 \\ 0 & \alpha_2^{-1} t \end{pmatrix}.$$

We now finally look at the third set of equations in (9). The equation for $j = 0$ yields the relations

$$\lambda \alpha_1^2 = \lambda' \alpha_0 \qquad \text{and} \qquad \mu \alpha_1 \alpha_2 = \mu' \alpha_0,$$

whereas the equations for $j = 1, 2, 3$ implies that

$$\alpha_0 \nu = \frac{\nu'}{\alpha_2^2} \qquad \text{and} \qquad \mu \alpha_1 \alpha_2 = \mu' \alpha_0,$$

where use of (12) has to be made. We obtain the following:

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1.1 Proposition. *(1) A Lie superalgebra based on* $\mathfrak{gl}_2$ *whose odd module is* $\mathfrak{gl}_2$ *itself under the adjoint action is defined by giving three independent parameters* $(\lambda, \mu, \nu) \in \mathbb{F}^3$, *in such a way that the given generators,* $\{x_i\}$ *and* $\{y_i\}$, *where* $y_i = \pi(x_i)$, *satisfy the commutation relations (4) with the* Γ_{ikj}'s *and* C_{ikj}'s *as in (8) and (10), respectively.*

2) Let $\mathfrak{g}(\lambda, \mu, \nu)$ *be the Lie superalgebra obtained from the parameter values* (λ, μ, ν). *Then,* $\mathfrak{g}(\lambda, \mu, \nu)$ *is isomorphic to* $\mathfrak{g}(\lambda', \mu', \nu')$ *if and only if there are nonzero constants* α_0, α_1 *and* α_2, *such that,*

$$\lambda' = \lambda \, \frac{\alpha_1^{\,2}}{\alpha_0}, \qquad \mu' = \mu \, \frac{\alpha_1 \alpha_2}{\alpha_0}, \qquad \nu' = \nu \, \alpha_0 \alpha_2^{\,2}.$$

(3) Let $\mathbb{R}$ *be the ground field. There are, up to isomorphism, ten different Lie superalgebras based on* $\mathfrak{gl}_2$ *whose odd module is equal to* $\mathfrak{gl}_2$ *under the adjoint action.*

(4) Let $\mathbb{C}$ *be the ground field. There are, up to isomorphism, eight different Lie superalgebras based on* $\mathfrak{gl}_2$ *whose odd module is equal to* $\mathfrak{gl}_2$ *under the adjoint action.*

Proof. What we have done so far is the work of showing why the first two statements are true. For the third statement note that if $\mathbb{R}$ is the ground field, then the product $\lambda'\nu'$ is equal to the product $\lambda\nu$ times the positive constant $\alpha_1^{\,2}\alpha_2^{\,2}$. Therefore, the sign of the product $\lambda\nu$ must remain invariant, and concrete representatives can be found with parameter values of (λ, μ, ν) satisfying the conditions given in the following list:

$\lambda\nu > 0$, $\mu \neq 0$	$\lambda\nu > 0$, $\mu = 0$
$\lambda\nu < 0$, $\mu \neq 0$	$\lambda\nu < 0$, $\mu = 0$
$\nu\mu \neq 0$, $\lambda = 0$	$\lambda\mu \neq 0$, $\nu = 0$
$\nu \neq 0$, $\mu = \lambda = 0$	$\lambda \neq 0$, $\mu = \nu = 0$
$\mu \neq 0$, $\lambda = \nu = 0$	$\lambda = \mu = \nu = 0$.

When the ground field is $\mathbb{C}$ there is no sign of the product $\lambda\nu$ to care for. Therefore, concrete representatives can be found satisfying the conditions given in the following list:

$\lambda\nu \neq 0$, $\mu \neq 0$	$\lambda\nu \neq 0$, $\mu = 0$
$\nu\mu \neq 0$, $\lambda = 0$	$\lambda\mu \neq 0$, $\nu = 0$
$\nu \neq 0$, $\mu = \lambda = 0$	$\lambda \neq 0$, $\mu = \nu = 0$
$\mu \neq 0$, $\lambda = \nu = 0$	$\lambda = \mu = \nu = 0$.

2. Representations of $\mathfrak{gl}_2(\mathbb{F}; \lambda, \mu, \nu)$ by means of supervector fields on $\mathbb{F}^{2|2}$ and $\mathbb{F}^{3|3}$

We now want to find representatives of the different classes of Lie superalgebras just found, but realized as supervector fields on an appropriate supermanifold. In order to do that, let us first note that if $\{z^b, \zeta^\nu\}$ are local coordinates on a given open subset of a supermanifold (with z^b even and ζ^ν odd) and, if we write the even and odd supervector field generators in their simplest (linear) form

$$x_i \mapsto X_i = \sum_{a,b} A(x_i)_{ab} z^a \frac{\partial}{\partial z^b} + \sum_{\mu,\nu} D(x_i)_{\mu\nu} \zeta^\mu \frac{\partial}{\partial \zeta^\nu}$$

and

$$y_i \mapsto Y_i = \sum_{\mu,b} C(y_i)_{\mu b} \zeta^\mu \frac{\partial}{\partial z^b} + \sum_{a,\nu} B(y_i)_{a\nu} z^a \frac{\partial}{\partial \zeta^\nu},$$

we obtain

$$[X_i, X_j] = \sum_{a,b} \big(A(x_i) A(x_j) - A(x_j) A(x_i) \big)_{ab} z^a \frac{\partial}{\partial z^b}$$
$$+ \sum_{\mu,\nu} \big(D(x_i) D(x_j) - D(x_j) D(x_i) \big)_{\mu\nu} \zeta^\mu \frac{\partial}{\partial \zeta^\nu},$$

$$[X_i, Y_j] = \sum_{b,\mu} \big(D(x_i) C(y_j) - C(y_j) A(x_i) \big)_{\mu b} \zeta^\mu \frac{\partial}{\partial z^b}$$
$$+ \sum_{a,\nu} \big(A(x_i) B(y_j) - B(y_j) D(x_i) \big)_{a\nu} z^a \frac{\partial}{\partial \zeta^\nu},$$

$$[Y_i, Y_j] = \sum_{a,b} \big(B(y_i) C(y_j) + B(y_j) C(y_i) \big)_{ab} z^a \frac{\partial}{\partial z^b}$$
$$+ \sum_{\mu,\nu} \big(C(y_i) B(y_j) + C(y_j) B(y_i) \big)_{\mu\nu} \zeta^\mu \frac{\partial}{\partial \zeta^\nu}.$$

Therefore, this suggests to look for matrix representations of the form

$$x_i \mapsto X_i = \begin{pmatrix} A(x_i) & 0 \\ 0 & D(x_i) \end{pmatrix} \quad \text{and} \quad y_i \mapsto Y_i = \begin{pmatrix} 0 & B(y_i) \\ C(y_i) & 0 \end{pmatrix},$$

where the $\mathbb{Z}_2$-graded Lie bracket is given by the usual commutator when the elements are both even, or one even and one odd, and it is given by the anticommutator if both elements are odd. One observes that $A : \mathfrak{gl}_2 \to \text{End } V_0$ and $D : \mathfrak{gl}_2 \to \text{End } V_1$ are ordinary representations of the Lie algebra $\mathfrak{gl}_2$ and, therefore, they restrict themselves to representations of $\mathfrak{sl}_2$. We therefore find explicit representations which we state in the following theorems:

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2.1 Theorem. *(1) Let the ground field be* $\mathbb{R}$. *Lie superalgebras in the equivalence classes of*

$$\lambda\nu > 0 , \ \mu \neq 0 \qquad\qquad \lambda\nu < 0 , \ \mu \neq 0$$
$$\lambda\nu < 0 , \ \mu = 0$$
$$\nu\mu \neq 0 , \ \lambda = 0 \qquad\qquad \lambda\mu \neq 0 , \ \nu = 0$$
$$\mu \neq 0 , \ \lambda = \nu = 0 \qquad\qquad \lambda = \mu = \nu = 0$$

admit a matrix representation in the supervector space $\mathbb{R}^2 \oplus \mathbb{R}^2$ *of the following form:*

$$X \in \mathfrak{gl}_2 \quad |X| = 0 \qquad X \in \mathfrak{sl}_2 \quad |X| = 1 \qquad X = I \in \mathfrak{gl}_2 \quad |X| = 1$$

$$\begin{pmatrix} X & 0 \\ 0 & X \end{pmatrix} \qquad\qquad \begin{pmatrix} 0 & dX \\ eX & 0 \end{pmatrix} \qquad\qquad \begin{pmatrix} 0 & gI \\ kI & 0 \end{pmatrix},$$

where $\lambda = 2gk$, $\mu = eg + dk$ *and* $\nu = ed$. *The real Lie superalgebras lying in the equivalence class of* $\lambda\nu > 0$ *and* $\mu = 0$ *admit a similar matrix representation, but in the supervector space* $\mathbb{C}^2 \oplus \mathbb{C}^2$.

(2) Let the ground field be $\mathbb{C}$. *Lie superalgebras in the equivalence classes of*

$$\lambda\nu \neq 0 , \ \mu \neq 0 \qquad\qquad \lambda\nu \neq 0 , \ \mu = 0$$
$$\nu\mu \neq 0 , \ \lambda = 0 \qquad\qquad \lambda\mu \neq 0 , \ \nu = 0$$
$$\mu \neq 0 , \ \lambda = \nu = 0 \qquad\qquad \lambda = \mu = \nu = 0$$

admit a matrix representation of the type above in the supervector space $\mathbb{C}^2 \oplus \mathbb{C}^2$.

In either case, their explicit realizations in terms of supervector fields in the supermanifold $\mathbb{R}^{2|2}$ *or* $\mathbb{C}^{2|2}$ *with local coordinates* $\{ z^1, z^2 ; \zeta^1, \zeta^2 \}$ *are given by*

$$X_0 = z^1 \frac{\partial}{\partial z^1} + z^2 \frac{\partial}{\partial z^2} + \zeta^1 \frac{\partial}{\partial \zeta^1} + \zeta^2 \frac{\partial}{\partial \zeta^2}$$

$$X_1 = z^1 \frac{\partial}{\partial z^1} - z^2 \frac{\partial}{\partial z^2} + \zeta^1 \frac{\partial}{\partial \zeta^1} - \zeta^2 \frac{\partial}{\partial \zeta^2}$$

$$X_2 = z^1 \frac{\partial}{\partial z^2} + \zeta^1 \frac{\partial}{\partial \zeta^2}$$

$$X_3 = z^2 \frac{\partial}{\partial z^1} + \zeta^2 \frac{\partial}{\partial \zeta^1}$$

$$Y_0 = k \left(\zeta^1 \frac{\partial}{\partial z^1} + \zeta^2 \frac{\partial}{\partial z^2} \right) + g \left(z^1 \frac{\partial}{\partial \zeta^1} + z^2 \frac{\partial}{\partial \zeta^2} \right)$$

$$Y_1 = e \left(\zeta^1 \frac{\partial}{\partial z^1} - \zeta^2 \frac{\partial}{\partial z^2} \right) + d \left(z^1 \frac{\partial}{\partial \zeta^1} - z^2 \frac{\partial}{\partial \zeta^2} \right)$$

$$Y_2 = e\, \zeta^1 \frac{\partial}{\partial z^2} + d\, z^1 \frac{\partial}{\partial \zeta^2}$$

$$Y_3 = e\, \zeta^2 \frac{\partial}{\partial z^1} + d\, z^2 \frac{\partial}{\partial \zeta^1}.$$

2.2 Theorem. *Let* $\mathbb{F}$ *be the ground field, which can be either* $\mathbb{R}$ *or* $\mathbb{C}$. *Lie superalgebras in the equivalence class of* $\nu \neq 0$ *with* $\mu = \lambda = 0$ *admit a matrix representation in the supervector space* $\mathbb{F}^3 \oplus \mathbb{F}^3$ *as follows:*

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$$X \mapsto \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & & X \\ 0 & & \\ 0 & 0 & 0 \\ 0 & & 0 \\ 0 & & \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & & 0 \\ 0 & & \\ 0 & 0 & 0 \\ 0 & & X \\ 0 & & \end{pmatrix} \right) \qquad X \in \mathfrak{gl}_2 \quad |X| = 0,$$

$$X \mapsto \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & & 0 \\ 0 & & \\ 0 & 0 & 0 \\ 0 & & \nu X \\ 0 & & \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & & X \\ 0 & & \\ 0 & 0 & 0 \\ 0 & & 0 \\ 0 & & \end{pmatrix} \right) \qquad X \in \mathfrak{sl}_2 \quad |X| = 1,$$

$$I \mapsto \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & & 0 \\ 0 & & \\ 0 & 0 & 0 \\ 0 & & 0 \\ 0 & & \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & & 0 \\ 0 & & \\ 0 & 0 & 0 \\ 0 & & 0 \\ 0 & & \end{pmatrix} \right) \qquad I \in \mathfrak{gl}_2 \quad |I| = 1.$$

Furthermore, their explicit realizations in terms of supervector fields in the super-manifold $\mathbb{F}^{3|3}$ with local coordinates $\{ z^0, z^1, z^2 ; \zeta^0, \zeta^1, \zeta^2 \}$ are given by the same expressions for X_k $(k = 0, 1, 2, 3)$ and Y_ℓ $(\ell = 1, 2, 3)$ in the theorem above (corresponding to the parameter values $d = 1$ and $e = \nu$), together with,

$$Y_0 = z^0 \frac{\partial}{\partial \zeta^0}.$$

2.3 Theorem. *Let $\mathbb{F}$ be the ground field, which can be either $\mathbb{R}$ or $\mathbb{C}$. Lie superalgebras in the equivalence class of $\lambda \neq 0$ with $\mu = \nu = 0$ admit a matrix representation in the supervector space $\mathbb{F}^3 \oplus \mathbb{F}^3$ as follows:*

$$X \mapsto \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & & X \\ 0 & & \\ 0 & 0 & 0 \\ 0 & & 0 \\ 0 & & \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & & 0 \\ 0 & & \\ 0 & 0 & 0 \\ 0 & & X \\ 0 & & \end{pmatrix} \right) \qquad X \in \mathfrak{sl}_2 \quad |X| = 0,$$

$$I \mapsto \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & & 0 \\ 0 & & \\ 0 & 0 & 0 \\ 0 & & 0 \\ 0 & & \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & & 0 \\ 0 & & \\ 1 & 0 & 0 \\ 0 & & 0 \\ 0 & & \end{pmatrix} \right) \qquad I \in \mathfrak{gl}_2 \quad |I| = 0,$$

$$X \mapsto \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & & 0 \\ 0 & & \\ 0 & 0 & 0 \\ 0 & & 0 \\ 0 & & \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & & X \\ 0 & & \\ 0 & 0 & 0 \\ 0 & & 0 \\ 0 & & \end{pmatrix} \right) \qquad X \in \mathfrak{sl}_2 \quad |X| = 1,$$

$$I \quad \mapsto \quad \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & & 0 \\ 0 & & 0 \end{pmatrix} \quad \begin{pmatrix} \frac{\lambda}{2} & 0 & 0 \\ 0 & & 0 \\ 0 & 0 & 0 \end{pmatrix} \right) \qquad I \in \mathfrak{gl}_2 \quad |I| = 1.$$
$$\left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & & 0 \\ 0 & & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & & 0 \\ 0 & & 0 \end{pmatrix} \right)$$

Furthermore, their explicit realizations in terms of supervector fields in the super-manifold $\mathbb{F}^{3|3}$ with local coordinates $\{ z^0, z^1, z^2 ; \zeta^0, \zeta^1, \zeta^2 \}$ are given by the same expressions for X_k $(k = 1,2,3)$ and Y_ℓ $(\ell = 1,2,3)$ in the theorem above (corresponding to the parameter values $d = 1$ and $e = 0$), together with,

$$X_0 = z^0 \frac{\partial}{\partial z^0} + \zeta^0 \frac{\partial}{\partial \zeta^0}$$

and

$$Y_0 = \zeta^0 \frac{\partial}{\partial z^0} + \frac{\lambda}{2} z^0 \frac{\partial}{\partial \zeta^0}.$$

The proofs. We shall now proceed to prove these theorems: We first consider $V_0 = V_1 = \mathbb{F}^3$ and a 3-dimensional representation

$$A = \rho_{(a,b,c)} : \mathfrak{gl}_2 \to \operatorname{End} \mathbb{F}^3$$

depending on the parameters $(a, b, c) \in \mathbb{F}^3$, where

$$\rho_{(a,b,c)}(x_0) = \begin{pmatrix} a & c(b-a) & 0 \\ 0 & b & 0 \\ 0 & 0 & b \end{pmatrix}, \qquad \rho_{(a,b,c)}(x_1) = \begin{pmatrix} 0 & c & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$\rho_{(a,b,c)}(x_2) = \begin{pmatrix} 0 & 0 & c \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \rho_{(a,b,c)}(x_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Similarly, we consider

$$D = \rho_{(a',b',c')} : \mathfrak{gl}_2 \to \operatorname{End} \mathbb{F}^3.$$

Now, the conditions that the odd module is equal to the adjoint representation are the following:

$$2B(y_2) = \rho_{(a,b,c)}(x_1)B(y_2) - B(y_2)\rho_{(a',b',c')}(x_1)$$
$$-2B(y_3) = \rho_{(a,b,c)}(x_1)B(y_3) - B(y_3)\rho_{(a',b',c')}(x_1)$$
$$B(y_1) = \rho_{(a,b,c)}(x_2)B(y_3) - B(y_3)\rho_{(a',b',c')}(x_2)$$
$$= B(y_2)\rho_{(a',b',c')}(x_3) - \rho_{(a,b,c)}(x_3)B(y_2)$$

together with

$$2C(y_2) = \rho_{(a',b',c')}(x_1)C(y_2) - C(y_2)\rho_{(a,b,c)}(x_1)$$
$$-2C(y_3) = \rho_{(a',b',c')}(x_1)C(y_3) - C(y_3)\rho_{(a,b,c)}(x_1)$$
$$C(y_1) = \rho_{(a',b',c')}(x_2)C(y_3) - C(y_3)\rho_{(a,b,c)}(x_2)$$
$$= C(y_2)\rho_{(a,b,c)}(x_3) - \rho_{(a',b',c')}(x_3)C(y_2)$$

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and it is easy to check that

$$B(y_i) = d\rho_{(a,b,c)}(y_i) \qquad \text{and} \qquad C(y_i) = e\rho_{(a',b',c')}(y_i), \qquad i = 1, 2, 3,$$

whereas

$$B(y_0) = \begin{pmatrix} f & cg - c'f & 0 \\ 0 & g & 0 \\ 0 & 0 & g \end{pmatrix}, \qquad C(y_0) = \begin{pmatrix} h & c'k - ch & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{pmatrix}.$$

Therefore, the equations that have to be satisfied for these matrices to define a representation of $\mathfrak{gl}_2(\mathbb{F}; \lambda, \mu, \nu)$ are the following:

$$2B(y_0)C(y_0) = \lambda\rho_{(a,b,c)}(x_0) \qquad \text{and} \qquad 2C(y_0)B(y_0) = \lambda\rho_{(a',b',c')}(x_0),$$

from which it follows that

$$2fh = \lambda a = \lambda a' \qquad \text{and} \qquad 2gk = \lambda b = \lambda b'.$$

Then,

$$\mu\rho_{(a,b,c)}(x_i) = B(y_0)C(y_i) + B(y_i)C(y_0)$$
$$\mu\rho_{(a',b',c')}(x_i) = C(y_0)B(y_i) + C(y_i)B(y_0).$$

From which it follows that

$$\mu = eg + dk.$$

Then

$$B(y_1)C(y_1) = \nu\rho_{(a,b,c)}(x_0) \qquad \text{and} \qquad C(y_1)B(y_1) = \nu\rho_{(a',b',c')}(x_0),$$

implies

$$\nu a = \nu a' = 0 \qquad \text{and} \qquad \nu b = \nu b' = ed$$

which are also the same relations implied by

$$\nu\rho_{(a,b,c)}(x_0) = B(y_2)C(y_3) + B(y_3)C(y_2)$$
$$\nu\rho_{(a',b',c')}(x_0) = C(y_2)B(y_3) + C(y_3)B(y_2).$$

Finally, the relations

$$0 = B(y_1)C(y_2) + B(y_2)C(y_1) \qquad 0 = C(y_1)B(y_2) + C(y_2)B(y_1)$$

$$0 = B(y_1)C(y_3) + B(y_3)C(y_1) \qquad 0 = C(y_1)B(y_3) + C(y_3)B(y_1)$$

and

$$B(y_2)C(y_2) = 0 \qquad C(y_2)B(y_2) = 0 \qquad B(y_3)C(y_3) = 0 \qquad C(y_3)B(y_3) = 0$$

are automatically satisfied.

We can now proceed to see how the concrete representatives given in Proposition 1.1 can be realized via this family of representations. The equations that have to be solved are posed in the following table:

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$$\lambda\nu \neq 0 \,,\ \mu \neq 0 \qquad \mu = eg + dk \qquad a = a' = fh = 0 \qquad b = b' = \frac{2gk}{\lambda} = \frac{ed}{\nu}$$

$$\lambda\nu \neq 0 \,,\ \mu = 0 \qquad 0 = eg + dk \qquad a = a' = fh = 0 \qquad b = b' = \frac{2gk}{\lambda} = \frac{ed}{\nu}$$

$$\nu\mu \neq 0 \,,\ \lambda = 0 \qquad \mu = eg + dk \qquad a = a' = fh = 0 \qquad b = b' = \frac{ed}{\nu} \qquad gk = 0$$

$$\lambda\mu \neq 0 \,,\ \nu = 0 \qquad \mu = eg + dk \qquad a = a' = \frac{2fh}{\lambda} \qquad b = b' = \frac{2gk}{\lambda} \qquad ed = 0$$

$$\nu \neq 0 \,,\ \mu = \lambda = 0 \qquad 0 = eg + dk \qquad a = a' = fh = 0 \qquad b = b' = \frac{2gk}{\lambda} = \frac{ed}{\nu}$$

$$\lambda \neq 0 \,,\ \mu = \nu = 0 \qquad 0 = eg + dk \qquad a = a' = \frac{2fh}{\lambda} \qquad b = b' = \frac{2gk}{\lambda} \qquad ed = 0$$

$$\mu \neq 0 \,,\ \lambda = \nu = 0 \qquad \mu = eg + dk \qquad fh = 0 \qquad gk = 0 \qquad ed = 0$$

$$\lambda = \mu = \nu = 0 \qquad 0 = eg + dk \qquad fh = 0 \qquad gk = 0 \qquad ed = 0.$$

A word must be said about the class of $\lambda\nu > 0$ and $\mu = 0$ over the reals. It is a straightforward matter to see that the equations to be solved require imaginary numbers. That means that the Lie superalgebras coming from that class need to be represented on a complex supermanifold, which nevertheless may be regarded as a real supermanifold with twice as many even and odd dimensions.

3. The associated Lie supergroups $GL_2(\mathbb{F}; \lambda, \mu, \nu)$

It is a straightforward matter to find the integral flows of each of the represented supervector fields. The techniques introduced in [13] are particularly simple to apply in this case. We shall start with the supervector fields that can be realized in the $(2,2)$-dimensional supermanifolds $\mathbb{F}^{2|2}$. Thus, let $\Gamma_{x_i} : \mathbb{R}^{1|1} \times \mathbb{F}^{2|2} \to \mathbb{F}^{2|2}$ be the integral flow of the *even* supervector field X_i that represents x_i. According to the general theory in [13] $\Gamma_{x_i}{}^* = \mathrm{Exp}(t_i X_i)$, where t_i is the *even parameter* resulting from the integration process. By computing the effect of $\Gamma_{x_i}{}^* = \mathrm{Exp}(t_i X_i)$ on the coordinates $z^1, z^2, \zeta^1, \zeta^2$, it is easily seen the action of $\Gamma_{x_i}{}^*$ has the same effect as the 4×4 matrix $\mathrm{Exp}(t_i X_i)$ does (X_i being the 4×4 matrix associated to x_i via the representation) on the unit column vectors

$$
z^1 \leftrightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad
z^2 \leftrightarrow \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \qquad
\zeta^1 \leftrightarrow \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \qquad
\zeta^2 \leftrightarrow \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.
$$

Since $X_0 z^i = z^i$, $X_0 \zeta^i = \zeta^i$, and since $X_1 z^i = (-1)^{i-1} z^i$, $X_0 \zeta^i = (-1)^{i-1} \zeta^i$, for $i = 1, 2$, it is easy to see that

$$
\Gamma_{x_0}{}^* = \mathrm{Exp}(t_0 X_0) : \begin{cases} z^1 \mapsto e^{t_0} z^1 \\ z^2 \mapsto e^{t_0} z^2 \\ \zeta^1 \mapsto e^{t_0} \zeta^1 \\ \zeta^2 \mapsto e^{t_0} \zeta^2 \end{cases} \qquad
\Gamma_{x_1}{}^* = \mathrm{Exp}(t_1 X_1) : \begin{cases} z^1 \mapsto e^{t_1} z^1 \\ z^2 \mapsto e^{-t_1} z^2 \\ \zeta^1 \mapsto e^{t_1} \zeta^1 \\ \zeta^2 \mapsto e^{-t_1} \zeta^2. \end{cases}
$$

In a similar way, since $X_j \circ X_j z^i = 0$ and $X_j \circ X_j \zeta^i = 0$, for $i = 1, 2$ and $j = 2, 3$, we have

$$
\Gamma_{x_2}{}^* = \mathrm{Exp}(t_2 X_2) : \begin{cases} z^1 \mapsto z^1 \\ z^2 \mapsto z^2 + t_2 z^1 \\ \zeta^1 \mapsto \zeta^1 \\ \zeta^2 \mapsto \zeta^2 + t_2 \zeta^1 \end{cases} \qquad
\Gamma_{x_3}{}^* = \mathrm{Exp}(t_3 X_3) : \begin{cases} z^1 \mapsto z^1 + t_3 z^2 \\ z^2 \mapsto z^2 \\ \zeta^1 \mapsto \zeta^1 + t_3 \zeta^2 \\ \zeta^2 \mapsto \zeta^2 \end{cases}
$$

and we set up the correspondences

$$
\Gamma_{x_0}{}^* \leftrightarrow \begin{pmatrix} e^{t_0} & 0 & 0 & 0 \\ 0 & e^{t_0} & 0 & 0 \\ 0 & 0 & e^{t_0} & 0 \\ 0 & 0 & 0 & e^{t_0} \end{pmatrix} \leftrightarrow \begin{pmatrix} e^{t_0} & 0 \\ 0 & e^{t_0} \end{pmatrix} = \mathfrak{m}(t_0; x_0)
$$

$$
\Gamma_{x_1}{}^* \leftrightarrow \begin{pmatrix} e^{t_1} & 0 & 0 & 0 \\ 0 & e^{-t_1} & 0 & 0 \\ 0 & 0 & e^{t_1} & 0 \\ 0 & 0 & 0 & e^{-t_1} \end{pmatrix} \leftrightarrow \begin{pmatrix} e^{t_1} & 0 \\ 0 & e^{-t_1} \end{pmatrix} = \mathfrak{m}(t_1; x_1)
$$

$$
\Gamma_{x_2}{}^* \leftrightarrow \begin{pmatrix} 1 & t_2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & t_2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & t_2 \\ 0 & 1 \end{pmatrix} = \mathfrak{m}(t_2; x_2)
$$

$$\Gamma_{x_3}{}^* \leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ t_3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & t_3 & 1 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 0 \\ t_3 & 1 \end{pmatrix} = \mathrm{m}(t_3; x_3).$$

We can now compose any two morphisms in some prescribed order in order to see what the effect of the composition is and to identify the final result with *the rule* to compose the group *even coordinates* t_0, t_1, t_2 and t_3. That is,

$$\mathrm{m}(t_i; x_i) \cdot \mathrm{m}(t'_j; x_j) \qquad \left\{ \begin{array}{l} \text{must correspond to} \\ \text{the composition of} \end{array} \right\} \qquad \left\{ \begin{array}{l} \Gamma_{x_i}{}^* = \mathrm{Exp}(t_i X_i) \\ \text{and} \\ \Gamma_{x_j}{}^* = \mathrm{Exp}(t'_j X_j) \end{array} \right.$$

in some appropriate order. By computing directly with the integral flows $\Gamma_{x_i}{}^* = \mathrm{Exp}(t_i X_i)$, where the X_i are taken as the even supervector fields in Theorem 1 in §2 above, we see that, *the appropriate order is*

$$\mathrm{m}(t_i; x_i) \cdot \mathrm{m}(t'_j; x_j) \quad \longleftrightarrow \quad \mathrm{Exp}(t_i X_i) \circ \mathrm{Exp}(t'_j X_j)$$

because it is in this, and only this way, that the composition law for the parameters t_i, expressed in matrix form as above, actually corresponds to the usual rule for matrix multiplication.

More generally, we may perform a change of parameters and transform $\mathbf{t} = (t_0, t_1, t_2, t_3)$ into a new set of parameters $\mathbf{g} = (\alpha, \beta, \gamma, \delta)$ in such way that if

$$\mathrm{Exp}(t_0 X_0) \circ \mathrm{Exp}(t_1 X_1) \circ \mathrm{Exp}(t_2 X_2) \circ \mathrm{Exp}(t_3 X_3) = \Gamma_{\mathbf{g}}^*$$

then,

$$\Gamma_{\mathbf{g}}^* : \left\{ \begin{array}{l} z^1 \mapsto \alpha z^1 + \gamma z^2 \\ z^2 \mapsto \beta z^1 + \delta z^2 \\ \zeta^1 \mapsto \alpha \zeta^1 + \gamma \zeta^2 \\ \zeta^2 \mapsto \beta \zeta^1 + \delta \zeta^2. \end{array} \right.$$

That is,

$$\alpha = (1 + t_2 t_3)\, \mathrm{e}^{t_0 + t_1}, \ \ \beta = t_2\, \mathrm{e}^{t_0 + t_1}, \ \ \gamma = t_3\, \mathrm{e}^{t_0 - t_1} \ \text{ and } \ \delta = \mathrm{e}^{t_0 - t_1}.$$

Therefore, from

$$\mathrm{Exp}(t'_0 X_0) \circ \mathrm{Exp}(t'_1 X_1) \circ \mathrm{Exp}(t'_2 X_2) \circ \mathrm{Exp}(t'_3 X_3) = \Gamma_{\mathbf{g}'}^*$$

and $\Gamma_{\mathbf{g}''}^* = \Gamma_{\mathbf{g}'}^* \circ \Gamma_{\mathbf{g}}^*$ one concludes that

$$\mathbf{g}' \leftrightarrow \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix}, \quad \mathbf{g} \leftrightarrow \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \implies \mathbf{g}'' \leftrightarrow \begin{pmatrix} \alpha'\alpha + \beta'\gamma & \alpha'\beta + \beta'\delta \\ \gamma'\alpha + \delta'\gamma & \gamma'\beta + \delta'\delta \end{pmatrix}.$$

Remark. What we have accomplished by proceeding this way is to actually recover the (local) Lie group multiplication law between any two generators in the identity component. Note that this procedure only yields a *multiplication table* for the group generators. However, this table has been obtained from the actual composition of

integral flows, by recording the overall effect on the local coordinates. Therefore, the multiplication law obtained this way is associative. Finally, by going into the group ring associated to this multiplication law, and writing down the general 2×2 matrix in the usual form (in terms of new coordinate parameters), one recovers (locally) the usual law for matrix multiplication as the associated group operation.

Even though this remark is very well understood in the classical Lie theory, we now want to see how it should be applied to the integral flows of the odd vector fields representing the odd Lie algebra generators y_0, y_1, y_2 and y_3. As mentioned before, the techniques introduced in [13] can be readily applied and in this case, the integral flow $\Gamma_{y_i} : \mathbb{F}^{1|1} \times \mathbb{F}^{2|2} \to \mathbb{F}^{2|2}$ depends on an odd parameter τ_i, as $\Gamma_{y_i}{}^* = \mathrm{Exp}(\tau_i Y_i) = id + \tau_i Y_i$. We may then immediately compute its effect on the coordinates z^1, z^2, ζ^1, ζ^2, and obtain

$$
\Gamma_{y_0}{}^* = \mathrm{Exp}(\tau_0 Y_0) : \begin{cases} z^1 \mapsto z^1 + k\tau_0\zeta^1 \\ z^2 \mapsto z^2 + k\tau_0\zeta^2 \\ \zeta^1 \mapsto \zeta^1 + g\tau_0 z^1 \\ \zeta^2 \mapsto \zeta^2 + g\tau_0 z^2 \end{cases} \quad \Gamma_{y_1}{}^* = \mathrm{Exp}(\tau_1 Y_1) : \begin{cases} z^1 \mapsto z^1 + e\tau_1\zeta^1 \\ z^2 \mapsto z^2 - e\tau_1\zeta^2 \\ \zeta^1 \mapsto \zeta^1 + d\tau_1 z^1 \\ \zeta^2 \mapsto \zeta^2 - d\tau_1 z^2 \end{cases}
$$

$$
\Gamma_{y_2}{}^* = \mathrm{Exp}(\tau_2 Y_2) : \begin{cases} z^1 \mapsto z^1 \\ z^2 \mapsto z^2 + e\tau_2\zeta^1 \\ \zeta^1 \mapsto \zeta^1 \\ \zeta^2 \mapsto \zeta^2 + d\tau_2 z^1 \end{cases} \quad \Gamma_{y_3}{}^* = \mathrm{Exp}(\tau_3 Y_3) : \begin{cases} z^1 \mapsto z^1 + e\tau_3\zeta^2 \\ z^2 \mapsto z^2 \\ \zeta^1 \mapsto \zeta^1 + d\tau_3 z^2 \\ \zeta^2 \mapsto \zeta^2. \end{cases}
$$

In order to find the multiplication law for the supergroup in terms of its own local coordinates (actually, the integration parameters t_i and τ_i), we choose a definite sequence for the integral flows: We shall write

$$
\Psi(\mathbf{g}; \tau_0, \tau_1, \tau_2, \tau_3) := \Gamma_{\mathbf{g}}{}^* \circ \Gamma_{y_0}{}^* \circ \Gamma_{y_1}{}^* \circ \Gamma_{y_2}{}^* \circ \Gamma_{y_3}{}^*
$$

and, from

$$
\Psi(\mathbf{g}''; \tau_0'', \tau_1'', \tau_2'', \tau_3'') = \Psi(\mathbf{g}'; \tau_0', \tau_1', \tau_2', \tau_3') \circ \Psi(\mathbf{g}; \tau_0, \tau_1, \tau_2, \tau_3),
$$

we shall deduce the Lie supergroup multiplication law. We compute first the composition law for integral flows depending on the odd generators. We thus get,

$$
\mathrm{Exp}(\tau_2 Y_2) \circ \mathrm{Exp}(\tau_3 Y_3) : \begin{cases} z^1 \mapsto \mathrm{Exp}(\tau_2 Y_2)(z^1 + e\tau_3\zeta^2) \\ z^2 \mapsto \mathrm{Exp}(\tau_2 Y_2)(z^2) \\ \zeta^1 \mapsto \mathrm{Exp}(\tau_2 Y_2)(\zeta^1 + d\tau_3 z^2) \\ \zeta^2 \mapsto \mathrm{Exp}(\tau_2 Y_2)(\zeta^2). \end{cases}
$$

For the sake of illustration, let us first compute this carefully, using the fact that that $\mathrm{Exp}(\tau_2 Y_2) = id + \tau_2 Y_2$ and the fact that Y_2 is an odd derivation:

$$
\begin{aligned}
\mathrm{Exp}(\tau_2 Y_2)\left(z^1 + e\tau_3\zeta^2\right) &= z^1 + e\tau_3\zeta^2 + \tau_2 Y_2\left(z^1 + e\tau_3\zeta^2\right) \\
&= z^1 + e\tau_3\zeta^2 + \tau_2 Y_2\left(z^1\right) + \tau_2 Y_2\left(e\tau_3\zeta^2\right) \\
&= z^1 + e\tau_3\zeta^2 - e\,\tau_2 Y_2(\zeta^2\tau_3) \\
&= z^1 + e\tau_3\zeta^2 - e\,\tau_2\left(Y_2(\zeta^2)\,\tau_3 - \zeta^2 Y_2(\tau_3)\right) \\
&= z^1 + e\tau_3\zeta^2 - e\,\tau_2\,dz^1\,\tau_3 \\
&= z^1 + e\tau_3\zeta^2 - ed\tau_2\tau_3\,z^1.
\end{aligned}
$$

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Note that we have used the fact that $Y_2(\tau_3) = 0$. The final result shows that $\text{Exp}(\tau_2 Y_2)(e\tau_3 \zeta^2) = e\tau_3 \text{Exp}(\tau_2 Y_2)(\zeta^2)$ as it should be, since for each fixed value of the odd section τ_2, $\text{Exp}(\tau_2 Y_2)$ must be an algebra isomorphism and, therefore, the constants — even the odd constants like τ_3 — must be preserved by it. At the light of this, it is very easy to prove that

$$\text{Exp}(\tau_2 Y_2) \circ \text{Exp}(\tau_3 Y_3): \begin{cases} z^1 \mapsto (1 - ed\tau_2\tau_3)z^1 + e\tau_3\zeta^2 \\ z^2 \mapsto z^2 + e\tau_2\zeta^1 \\ \zeta^1 \mapsto (1 - ed\tau_2\tau_3)\zeta^1 + d\tau_3 z^2 \\ \zeta^2 \mapsto \zeta^2 + d\tau_2 z^1. \end{cases}$$

By proceeding in this way, a straightforward computation shows that $\text{Exp}(\tau_0 Y_0) \circ \text{Exp}(\tau_1 Y_1) \circ \text{Exp}(\tau_2 Y_2) \circ \text{Exp}(\tau_3 Y_3)$ is a morphism which transforms the coordinates $z^1, z^2, \zeta^1, \zeta^2$ as follows:

$$\begin{aligned}
z^1 \mapsto &(1 - eg\tau_0\tau_1)(1 - ed\tau_2\tau_3)z^1 - e(g\tau_0 - d\tau_1)\tau_3 z^2 \\
&+ (k\tau_0 + e\tau_1)(1 - ed\tau_2\tau_3)\zeta^1 + e(1 + kd\tau_0\tau_1)\tau_3\zeta^2 \\
z^2 \mapsto &- e(g\tau_0 + d\tau_1)\tau_2 z^1 + (1 + eg\tau_0\tau_1)z^2 \\
&+ e(1 - kd\tau_0\tau_1)\tau_2\zeta^1 + (k\tau_0 - e\tau_1)\zeta^2 \\
\zeta^1 \mapsto &(1 - dk\tau_0\tau_1)(1 - ed\tau_2\tau_3)\zeta^1 - d(k\tau_0 - e\tau_1)\tau_3\zeta^2 \\
&+ (g\tau_0 + d\tau_1)(1 - ed\tau_2\tau_3)z^1 + d(1 + eg\tau_0\tau_1)\tau_3 z^2 \\
\zeta^2 \mapsto &- d(k\tau_0 + e\tau_1)\tau_2\zeta^1 + (1 + dk\tau_0\tau_1)\zeta^2 \\
&+ d(1 - eg\tau_0\tau_1)\tau_2 z^1 + (g\tau_0 - d\tau_1)z^2.
\end{aligned}$$

3.1. CASE $[\lambda, \mu, \nu = 0]$

Let us consider the case when $e = 0$. The effect of $\Psi(g; \tau_0, \tau_1, \tau_2, \tau_3)$ on the local coordinates $z^1, z^2, \zeta^1, \zeta^2$ is

$$\begin{aligned}
z^1 \mapsto &\alpha z^1 + \gamma z^2 + k\alpha\tau_0\zeta^1 + k\gamma\tau_0\zeta^2 \\
z^2 \mapsto &\beta z^1 + \delta z^2 + k\beta\tau_0\zeta^1 + k\delta\tau_0\zeta^2 \\
\zeta^1 \mapsto &[(g\tau_0 + d\tau_1)\alpha + d\beta\tau_3]\, z^1 + [(g\tau_0 + d\tau_1)\gamma + d\delta\tau_3]\, z^2 \\
&+ [(1 - dk\tau_0\tau_1)\alpha - dk\beta\tau_0\tau_3]\, \zeta^1 + [(1 - dk\tau_0\tau_1)\gamma - dk\delta\tau_0\tau_3]\, \zeta^2 \\
\zeta^2 \mapsto &[d\alpha\tau_2 + (g\tau_0 - d\tau_1)\beta]\, z^1 + [d\gamma\tau_2 + (g\tau_0 - d\tau_1)\delta]\, z^2 \\
&+ [-dk\alpha\tau_0\tau_2 + (1 + dk\tau_0\tau_1)\beta]\, \zeta^1 + [-dk\gamma\tau_0\tau_2 + (1 + dk\tau_0\tau_1)\delta]\, \zeta^2.
\end{aligned}$$

Writing Ψ as a shorthand notation for $\Psi(g; \tau_0, \tau_1, \tau_2, \tau_3)$, the results above can be rewritten as

$$\begin{aligned}
\Psi \begin{pmatrix} z^1 \\ z^2 \end{pmatrix} &= g \begin{pmatrix} z^1 \\ z^2 \end{pmatrix} + k\tau_0\, g \begin{pmatrix} \zeta^1 \\ \zeta^2 \end{pmatrix} \\
\Psi \begin{pmatrix} \zeta^1 \\ \zeta^2 \end{pmatrix} &= g(g\tau_0 \mathbb{1} + d\tau) \begin{pmatrix} z^1 \\ z^2 \end{pmatrix} + g(\mathbb{1} - dk\tau_0\tau) \begin{pmatrix} \zeta^1 \\ \zeta^2 \end{pmatrix},
\end{aligned}$$

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where $\tau = \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_3 & -\tau_1 \end{pmatrix}$ and $\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. In a similar way, if $\Psi' := \Psi(g'; \tau_0', \tau_1', \tau_2', \tau_3')$,

$$\Psi' \begin{pmatrix} z^1 \\ z^2 \end{pmatrix} = g' \begin{pmatrix} z^1 \\ z^2 \end{pmatrix} + k\tau_0' \, g' \begin{pmatrix} \zeta^1 \\ \zeta^2 \end{pmatrix}$$

$$\Psi' \begin{pmatrix} \zeta^1 \\ \zeta^2 \end{pmatrix} = g'(g\tau_0' \mathbb{1} + d\tau') \begin{pmatrix} z^1 \\ z^2 \end{pmatrix} + g'(\mathbb{1} - dk\tau_0'\tau') \begin{pmatrix} \zeta^1 \\ \zeta^2 \end{pmatrix},$$

where $g' = \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix}$ and $\tau' = \begin{pmatrix} \tau_1' & \tau_2' \\ \tau_3' & -\tau_1' \end{pmatrix}$. We now want to compute the composition $\Psi' \circ \Psi$ and it is easy to see that

(1)
$$\Psi' \circ \Psi \begin{pmatrix} z^1 \\ z^2 \end{pmatrix} = g' \left\{ (1 - gk\tau_0'\tau_0)\mathbb{1} + dk\tau_0\tau' \right\} g \begin{pmatrix} z^1 \\ z^2 \end{pmatrix}$$

$$+ k\, g' \left\{ (\tau_0' + \tau_0)\mathbb{1} + dk\tau_0'\tau_0\tau' \right\} g \begin{pmatrix} \zeta^1 \\ \zeta^2 \end{pmatrix}$$

(2)
$$\Psi' \circ \Psi \begin{pmatrix} \zeta^1 \\ \zeta^2 \end{pmatrix} = g' \left\{ g(\tau_0' + \tau_0)\, g + d\tau'\, g\, (\mathbb{1} - dk\tau_0\tau) \right.$$

$$+ g\tau d(1 - gk\tau_0'\tau_0) \} \begin{pmatrix} z^1 \\ z^2 \end{pmatrix}$$

$$+ g' \left\{ (1 - gk\tau_0'\tau_0)\, g - dk(\tau_0' + \tau_0)\, g\tau \right.$$

$$\left. - dk\tau_0'\tau'\, g\, (\mathbb{1} - dk\tau_0\tau) \right\} \begin{pmatrix} \zeta^1 \\ \zeta^2 \end{pmatrix}.$$

On the other hand, we want to compare $\Psi'' := \Psi(g''; \tau_0'', \tau_1'', \tau_2'', \tau_3'')$ with $\Psi' \circ \Psi$, where

(3)
$$\Psi'' \begin{pmatrix} z^1 \\ z^2 \end{pmatrix} = g'' \begin{pmatrix} z^1 \\ z^2 \end{pmatrix} + k\tau_0'' \, g'' \begin{pmatrix} \zeta^1 \\ \zeta^2 \end{pmatrix}$$

(4)
$$\Psi'' \begin{pmatrix} \zeta^1 \\ \zeta^2 \end{pmatrix} = g''(g\tau_0'' \mathbb{1} + d\tau'') \begin{pmatrix} z^1 \\ z^2 \end{pmatrix} + g''(\mathbb{1} - dk\tau_0''\tau'') \begin{pmatrix} \zeta^1 \\ \zeta^2 \end{pmatrix}.$$

Equations (1), (2), (3) and (4) are equivalent to

(5)
$$g'' = (1 - gk\tau_0'\tau_0)\, g'\, g + dk\tau_0\, g'\, \tau'\, g$$

(6)
$$k\tau_0''\, g'' = k(\tau_0' + \tau_0)\, g'\, g + dk^2\tau_0'\tau_0\, g'\, \tau'\, g,$$

which come from $\Psi'' \begin{pmatrix} z^1 \\ z^2 \end{pmatrix} = \Psi' \circ \Psi \begin{pmatrix} z^1 \\ z^2 \end{pmatrix}$. In a similar way,

(7)
$$g''(g\tau_0'' \mathbb{1} + d\tau'') = g' \left\{ g(\tau_0' + \tau_0)\, g + d\tau'\, g(\mathbb{1} - dk\tau_0\tau) \right.$$
$$\left. + d(1 - gk\tau_0'\tau_0)\, g\tau \right\}$$

(8)
$$g''(\mathbb{1} - dk\tau_0''\tau'') = g' \left\{ (1 - gk\tau_0'\tau_0)\, g - dk(\tau_0' + \tau_0)\, g\tau \right.$$
$$\left. - dk\tau_0'\tau'\, g(\mathbb{1} - dk\tau_0\tau) \right\},$$

which come from $\Psi'' \begin{pmatrix} \zeta^1 \\ \zeta^2 \end{pmatrix} = \Psi' \circ \Psi \begin{pmatrix} \zeta^1 \\ \zeta^2 \end{pmatrix}.$

We know from (5) that

$$\mathbf{g}'' = \mathbf{g}'\,\mathbf{g} + k\tau_0\,\mathbf{g}'\,(g\tau_0'\mathbb{1} + d\tau')\,\mathbf{g},$$

and then, from (6), that

$$\tau_0'' = \tau_0' + \tau_0.$$

Moreover, it is clear that

$$(\mathbf{g}'')^{-1} = \mathbf{g}^{-1}(\mathbf{g}')^{-1} - k\tau_0\,\mathbf{g}^{-1}(g\tau_0'\mathbb{1} + d\tau')(\mathbf{g}')^{-1}$$

so, we can rewrite (7) as

$$\begin{aligned}
g\tau_0''\mathbb{1} + d\tau'' =& \{\mathbf{g}^{-1} - k\tau_0\,\mathbf{g}^{-1}(g\tau_0'\mathbb{1} + d\tau')\}\cdot \\
& \{g\tau_0''\,\mathbf{g} + d\tau'\,\mathbf{g}(\mathbb{1} - dk\tau_0\tau) + d(1 - gk\tau_0'\tau_0)\,\mathbf{g}\,\tau\},
\end{aligned}$$

which implies

$$\tau'' = \tau + \mathbf{g}^{-1}\,\tau'\,\mathbf{g} - dk\tau_0(\mathbf{g}^{-1}\,\tau'\,\mathbf{g})^2.$$

Finally, after a straightforward computation, one verifies that (8) is satisfied.

Notation and abstract form for the multiplication morphism. We shall think of the Lie supergroup coordinates as if it were actual *elements* of the supergroup and represent them by $(\mathbf{g}, \tau_0, \tau)$. So, if $(\mathbf{g}', \tau_0', \tau')$ is another *element*, we realize that the multiplication law $(\mathbf{g}', \tau_0', \tau') \cdot (\mathbf{g}, \tau_0, \tau)$ is given by

$$\left(\mathbf{g}'\,\mathbf{g} - \frac{\lambda}{2}\tau_0'\tau_0\,\mathbf{g}'\,\mathbf{g} + \mu\tau_0\,\mathbf{g}'\,\tau'\,\mathbf{g}, \ \tau_0' + \tau_0, \ \tau + \mathbf{g}^{-1}\,\tau'\,\mathbf{g} - \mu\tau_0(\mathbf{g}^{-1}\,\tau'\,\mathbf{g})^2 \right),$$

where $\lambda = 2gk$ and $\mu = dk$. Even more, we shall define $\gamma = \begin{pmatrix} \tau_0 + \tau_1 & \tau_2 \\ \tau_3 & \tau_0 - \tau_1 \end{pmatrix}$ in order to write the coordinate elements as pairs $(\mathbf{g}, \gamma)$. Then, the multiplication law is separated into even and odd components and, if $(\mathbf{g}', \gamma')$ is another element, then $(\mathbf{g}', \gamma') \cdot (\mathbf{g}, \gamma)$ is given by

$$\begin{aligned}
&\left(\mathbf{g}'\,\mathbf{g} + (\mu - \frac{\lambda}{2}) \left(\frac{\gamma_{11}' + \gamma_{22}'}{2} \right) \left(\frac{\gamma_{11} + \gamma_{22}}{2} \right) \mathbf{g}'\,\mathbf{g} + \mu \left(\frac{\gamma_{11} + \gamma_{22}}{2} \right) \mathbf{g}'\,\gamma'\,\mathbf{g}, \right. \\
&\left. \gamma + \mathbf{g}^{-1}\,\gamma'\,\mathbf{g} - \mu \left(\frac{\gamma_{11} + \gamma_{22}}{2} \right) (\mathbf{g}^{-1}\,\gamma'\,\mathbf{g})^2 \right).
\end{aligned}$$

It is a straightforward matter to check that the associativity law holds true for this multiplication (although this was something we already knew by first principles) and one may also compute the left-invariant vector fields for it. But before doing that, we note that $(\mathbb{1}, \mathbf{0})$ is the supergroup's identity element, where $\mathbf{0}$ is the 2×2 zero matrix. A straightforward computation shows that the inverse element for $(\mathbf{g}, \gamma)$, which we shall write as $(\mathbf{g}, \gamma)^{-1}$, is given by

$$(\mathbf{g}, \gamma)^{-1} = \left(\mathbf{g}^{-1} + \mu \left(\frac{\gamma_{11} + \gamma_{22}}{2} \right) \gamma\,\mathbf{g}^{-1}, \ -\mathbf{g}\gamma\mathbf{g}^{-1} + \mu \left(\frac{\gamma_{11} + \gamma_{22}}{2} \right) (\mathbf{g}\gamma\mathbf{g}^{-1})^2 \right).$$

Even more, we can write the multiplication morphism as follows: Let x_{ij} and ξ_{ij} the projection maps defined by $x_{ij}(\mathbf{g}, \gamma) = \mathbf{g}_{ij}$ and $\xi_{ij}(\mathbf{g}, \gamma) = \gamma_{ij}$. Then it is easy to prove that

$$\text{m}^* x_{ij} = \left\{ 1 + (\mu - \frac{\lambda}{2}) \left(\frac{p_1^* \xi_{11} + p_1^* \xi_{22}}{2} \right) \left(\frac{p_2^* \xi_{11} + p_2^* \xi_{22}}{2} \right) \right\} \sum_{k=1}^{2} p_1^* x_{ik} p_2^* x_{kj}$$

$$+ \mu \left(\frac{p_2^* \xi_{11} + p_2^* \xi_{22}}{2} \right) \sum_{k=1}^{2} p_1^* x_{ik} p_1^* \xi_{k\ell} p_2^* x_{\ell j},$$

$$\text{m}^* \xi_{ij} = p_2^* \xi_{ij} + \sum_{k,\ell=1}^{2} p_2^* y_{ik} p_1^* \xi_{k\ell} p_2^* x_{\ell j}$$

$$- \mu \left(\frac{p_2^* \xi_{11} + p_2^* \xi_{22}}{2} \right) \left(\sum_{k,\ell=1}^{2} p_2^* y_{ik} p_1^* \xi_{k\ell} p_2^* x_{\ell j} \right)^2,$$

where $(y_{ij}) = (x_{11} x_{22} - x_{12} x_{21})^{-1} \begin{pmatrix} x_{22} & -x_{12} \\ -x_{21} & x_{11} \end{pmatrix}$ and $i, j \in \{1, 2\}$.

3.2. CASE $[\lambda \neq 0, \mu = 0, \nu = 0]$

From Theorem 3 in §2 we know that the class for which $\lambda \neq 0$ and $\mu = \nu = 0$ admit a realization in terms of supervector fields in the supermanifold $\mathbb{F}^{3|3}$ with local coordinates $\{z^0, z^1, z^2, \zeta^0, \zeta^1, \zeta^2\}$ given by

$$X_0 = z^0 \frac{\partial}{\partial z^0} + \zeta^0 \frac{\partial}{\partial \zeta^0}$$

$$X_1 = z^1 \frac{\partial}{\partial z^1} - z^2 \frac{\partial}{\partial z^2} + \zeta^1 \frac{\partial}{\partial \zeta^1} - \zeta^2 \frac{\partial}{\partial \zeta^2}$$

$$X_2 = z^1 \frac{\partial}{\partial z^2} + \zeta^1 \frac{\partial}{\partial \zeta^2}$$

$$X_3 = z^2 \frac{\partial}{\partial z^1} + \zeta^2 \frac{\partial}{\partial \zeta^1}$$

$$Y_0 = \zeta^0 \frac{\partial}{\partial z^0} + \frac{\lambda}{2} z^0 \frac{\partial}{\partial \zeta^0}$$

$$Y_1 = z^1 \frac{\partial}{\partial \zeta^1} - z^2 \frac{\partial}{\partial \zeta^2}$$

$$Y_2 = z^1 \frac{\partial}{\partial \zeta^2}$$

$$Y_3 = z^2 \frac{\partial}{\partial \zeta^1}.$$

Let $\Gamma_{x_i} : \mathbb{F}^{1|1} \times \mathbb{F}^{3|3} \to \mathbb{F}^{3|3}$ be the integral flow of the *even* vector field X_i that represents x_i. Just as in the last section, and according to the general theory in [13], $\Gamma_{x_i}{}^* = \text{Exp}(t_i X_i)$, where t_i is the *even parameter* resulting from the integration process. By computing the effect of $\Gamma_{x_i}{}^* = \text{Exp}(t_i X_i)$ on the coordinates z^0, z^1, z^2, ζ^0, ζ^1, ζ^2, it is easily seen that the action of $\Gamma_{x_i}{}^*$ has the same effect as the 6×6 matrix $\text{Exp}(t_i X_i)$ has on the unit column vectors (Note: X_i is the 6×6 matrix

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associated to x_i via the representation),

$$z^0 \leftrightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad z^1 \leftrightarrow \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad z^2 \leftrightarrow \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$\zeta^0 \leftrightarrow \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \zeta^1 \leftrightarrow \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \zeta^2 \leftrightarrow \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

It is easy to see that

$$\Gamma_{x_0}{}^* = \mathrm{Exp}(t_0 X_0) : \begin{cases} z^0 \mapsto e^{t_0} z^0 \\ z^1 \mapsto z^1 \\ z^2 \mapsto z^2 \\ \zeta^0 \mapsto e^{t_0} \zeta^0 \\ \zeta^1 \mapsto \zeta^1 \\ \zeta^2 \mapsto \zeta^2 \end{cases} \qquad \Gamma_{x_1}{}^* = \mathrm{Exp}(t_1 X_1) : \begin{cases} z^0 \mapsto z^0 \\ z^1 \mapsto e^{t_1} z^1 \\ z^2 \mapsto e^{-t_1} z^2 \\ \zeta^0 \mapsto \zeta^0 \\ \zeta^1 \mapsto e^{t_1} \zeta^1 \\ \zeta^2 \mapsto e^{-t_1} \zeta^2 \end{cases}$$

$$\Gamma_{x_2}{}^* = \mathrm{Exp}(t_2 X_2) : \begin{cases} z^0 \mapsto z^0 \\ z^1 \mapsto z^1 \\ z^2 \mapsto z^2 + t_2 z^1 \\ \zeta^0 \mapsto \zeta^0 \\ \zeta^1 \mapsto \zeta^1 \\ \zeta^2 \mapsto \zeta^2 + t_2 \zeta^1 \end{cases} \qquad \Gamma_{x_3}{}^* = \mathrm{Exp}(t_3 X_3) : \begin{cases} z^0 \mapsto z^0 \\ z^1 \mapsto z^1 + t_3 z^2 \\ z^2 \mapsto z^2 \\ \zeta^0 \mapsto \zeta^0 \\ \zeta^1 \mapsto \zeta^1 + t_3 \zeta^2 \\ \zeta^2 \mapsto \zeta^2 \end{cases}$$

and we set up the correspondences

$$\Gamma_{x_0}{}^* \leftrightarrow \begin{pmatrix} e^{t_0} & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{t_0} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \leftrightarrow \begin{pmatrix} e^{t_0} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathfrak{m}(t_0; x_0)$$

$$\Gamma_{x_1}{}^* \leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{t_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{-t_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{t_1} & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{-t_1} \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{t_1} & 0 \\ 0 & 0 & e^{-t_1} \end{pmatrix} = \mathfrak{m}(t_1; x_1)$$

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$$\Gamma_{x_2}{}^* \leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & t_2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & t_2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t_2 \\ 0 & 0 & 1 \end{pmatrix} = \mathrm{m}(t_2; x_2)$$

$$\Gamma_{x_3}{}^* \leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & t_3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & t_3 & 1 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & t_3 & 1 \end{pmatrix} = \mathrm{m}(t_3; x_3).$$

As in the last subsection, we now compose any two morphisms in some prescribed order in order to see what the effect of the composition is and then try to identify the final result with *the rule* to compose the group's *even coordinates* t_0, t_1, t_2 and t_3. We perform a change of parameters and transform $\mathrm{t} = (t_0, t_1, t_2, t_3)$ into a new set of even parameters $\mathrm{g} = (\epsilon, \alpha, \beta, \gamma, \delta)$ in such way that, if

$$\mathrm{Exp}(t_0 X_0) \circ \mathrm{Exp}(t_1 X_1) \circ \mathrm{Exp}(t_2 X_2) \circ \mathrm{Exp}(t_3 X_3) = \Gamma_{\mathrm{g}}^*$$

then,

$$\Gamma_{\mathrm{g}}^* : \begin{cases} z^0 \mapsto \epsilon z^0 \\ z^1 \mapsto \alpha z^1 + \gamma z^2 \\ z^2 \mapsto \beta z^1 + \delta z^2 \\ \zeta^0 \mapsto \epsilon \zeta^0 \\ \zeta^1 \mapsto \alpha \zeta^1 + \gamma \zeta^2 \\ \zeta^2 \mapsto \beta z^1 + \delta \zeta^2. \end{cases}$$

That is,

$$\epsilon = \mathrm{e}^{t_0}, \quad \alpha = (1 + t_2 t_3)\, \mathrm{e}^{t_1}, \quad \beta = t_2\, \mathrm{e}^{t_1}, \quad \gamma = t_3\, \mathrm{e}^{-t_1}, \quad \delta = \mathrm{e}^{-t_1},$$

where $\alpha\delta - \beta\gamma = 1$. Therefore, from $\Gamma_{\mathrm{g}''}^* = \Gamma_{\mathrm{g}'}^* \circ \Gamma_{\mathrm{g}}^*$, one concludes that

$$\mathrm{g}' \leftrightarrow \begin{pmatrix} \epsilon' & 0 & 0 \\ 0 & \alpha' & \beta' \\ 0 & \gamma' & \delta' \end{pmatrix}, \mathrm{g} \leftrightarrow \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & \gamma & \delta \end{pmatrix} \Longrightarrow \mathrm{g}'' \leftrightarrow \begin{pmatrix} \epsilon'\epsilon & 0 & 0 \\ 0 & \alpha'\alpha + \beta'\gamma & \alpha'\beta + \beta'\delta \\ 0 & \gamma'\alpha + \delta'\gamma & \gamma'\beta + \delta'\delta \end{pmatrix}.$$

But now, we want to see how it should be applied to the integral flows of the odd vector fields representing the odd Lie algebra generators y_0, y_1, y_2 and y_3. If $\Gamma_{y_i} : \mathbb{F}^{1|1} \times \mathbb{F}^{3|3} \to \mathbb{F}^{3|3}$ is the integral flow of any of the odd supervector fields, $\Gamma_{y_i}{}^* = \mathrm{Exp}(\tau_i Y_i) = id + \tau_i Y_i$, for an odd parameter τ_i. Then, we may immediately compute their effect on the coordinates z^0, z^1, z^2, ζ^0, ζ^1, ζ^2 and obtain

$$\Gamma_{y_0}{}^* = \mathrm{Exp}(\tau_0 Y_0) : \begin{cases} z^0 \mapsto z^0 + \tau_0 \zeta^0 \\ z^1 \mapsto z^1 \\ z^2 \mapsto z^2 \\ \zeta^0 \mapsto \zeta^0 + \frac{\lambda}{2}\tau_0 z^0 \\ \zeta^1 \mapsto \zeta^1 \\ \zeta^2 \mapsto \zeta^2 \end{cases} \qquad \Gamma_{y_1}{}^* = \mathrm{Exp}(\tau_1 Y_1) : \begin{cases} z^0 \mapsto z^0 \\ z^1 \mapsto z^1 \\ z^2 \mapsto z^2 \\ \zeta^0 \mapsto \zeta^0 \\ \zeta^1 \mapsto \zeta^1 + \tau_1 z^1 \\ \zeta^2 \mapsto \zeta^2 - \tau_1 z^2 \end{cases}$$

$$\Gamma_{y_2}{}^* = \mathrm{Exp}(\tau_2 Y_2) : \begin{cases} z^0 \mapsto z^0 \\ z^1 \mapsto z^1 \\ z^2 \mapsto z^2 \\ \zeta^0 \mapsto \zeta^0 \\ \zeta^1 \mapsto \zeta^1 \\ \zeta^2 \mapsto \zeta^2 + \tau_2 z^1 \end{cases} \qquad \Gamma_{y_3}{}^* = \mathrm{Exp}(\tau_3 Y_3) : \begin{cases} z^0 \mapsto z^0 \\ z^1 \mapsto z^1 \\ z^2 \mapsto z^2 \\ \zeta^0 \mapsto \zeta^0 \\ \zeta^1 \mapsto \zeta^1 + \tau_3 z^2 \\ \zeta^2 \mapsto \zeta^2. \end{cases}$$

In order to find the multiplication law of the underlying Lie supergroups acting on $\mathbb{F}^{3|3}$ via the eight integral flows we have found, we choose a definite sequence as before and write

$$\Psi(\mathbf{g}; \tau_0, \tau_1, \tau_2, \tau_3) := \Gamma_{\mathbf{g}}{}^* \circ \Gamma_{y_0}{}^* \circ \Gamma_{y_1}{}^* \circ \Gamma_{y_2}{}^* \circ \Gamma_{y_3}{}^*$$

and, from

$$\Psi(\mathbf{g}''; \tau_0'', \tau_1'', \tau_2'', \tau_3'') = \Psi(\mathbf{g}'; \tau_0', \tau_1', \tau_2', \tau_3') \circ \Psi(\mathbf{g}; \tau_0, \tau_1, \tau_2, \tau_3),$$

we deduce the Lie supergroup multiplication law.

We compute the effect of $\Psi = \Psi(\mathbf{g}; \tau_0, \tau_1, \tau_2, \tau_3)$ on the local coordinates z^0, z^1, z^2, ζ^0, ζ^1, ζ^2:

$$z^0 \mapsto \epsilon z^0 + \epsilon \tau_0 \zeta^0$$
$$z^1 \mapsto \alpha z^1 + \gamma z^2$$
$$z^2 \mapsto \beta z^1 + \delta z^2$$
$$\zeta^0 \mapsto \frac{\lambda}{2} \epsilon \tau_0 z^0 + \epsilon \zeta^0$$
$$\zeta^1 \mapsto (\alpha \tau_1 + \beta \tau_3) z^1 + (\gamma \tau_1 + \delta \tau_3) z^2 + \alpha \zeta^1 + \gamma \zeta^2$$
$$\zeta^2 \mapsto (\alpha \tau_2 - \beta \tau_1) z^1 + (\gamma \tau_2 - \delta \tau_1) z^2 + \beta \zeta^1 + \delta \zeta^2.$$

By defining

$$\mathbf{a} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad \tau = \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_3 & -\tau_1 \end{pmatrix}, \quad \mathbf{z} = \begin{pmatrix} z^0 \\ z^1 \\ z^2 \end{pmatrix}, \quad \zeta = \begin{pmatrix} \zeta^0 \\ \zeta^1 \\ \zeta^2 \end{pmatrix},$$

we obtain

$$\Psi(\mathbf{z}) = \begin{pmatrix} \epsilon & 0 \\ 0 & \mathbf{a} \end{pmatrix} \mathbf{z} + \begin{pmatrix} \epsilon \tau_0 & 0 \\ 0 & 0 \end{pmatrix} \zeta$$

$$\Psi(\zeta) = \begin{pmatrix} \frac{\lambda}{2} \epsilon \tau_0 & 0 \\ 0 & \mathbf{a}\tau \end{pmatrix} \mathbf{z} + \begin{pmatrix} \epsilon & 0 \\ 0 & \mathbf{a} \end{pmatrix} \zeta,$$

with similar expressions for $\Psi' = \Psi(\mathbf{g}'; \tau_0', \tau_1', \tau_2', \tau_3')$ and $\Psi'' = \Psi(\mathbf{g}''; \tau_0'', \tau_1'', \tau_2'', \tau_3'')$. A very simple computation shows that

$$\Psi' \circ \Psi(\mathbf{z}) = \begin{pmatrix} \epsilon'\epsilon(1 - \frac{\lambda}{2}\tau_0'\tau_0) & 0 \\ 0 & \mathbf{a}'\mathbf{a} \end{pmatrix} \mathbf{z} + \begin{pmatrix} \epsilon'\epsilon(\tau_0' + \tau_0) & 0 \\ 0 & 0 \end{pmatrix} \zeta$$

$$\Psi' \circ \Psi(\zeta) = \begin{pmatrix} \epsilon'\epsilon\frac{\lambda}{2}(\tau_0' + \tau_0) & 0 \\ 0 & \mathbf{a}'\mathbf{a}\tau + \mathbf{a}'\tau'\mathbf{a} \end{pmatrix} \mathbf{z} + \begin{pmatrix} \epsilon'\epsilon(1 - \frac{\lambda}{2}\tau_0'\tau_0) & 0 \\ 0 & \mathbf{a}'\mathbf{a} \end{pmatrix} \zeta.$$

From $\Psi' \circ \Psi = \Psi''$ it follows that $\epsilon'' = \epsilon'\epsilon(1 - \frac{\lambda}{2}\tau_0'\tau_0)$, $\mathbf{a}'' = \mathbf{a}'\mathbf{a}$, $\tau_0'' = \tau_0' + \tau_0$ and $\tau'' = \tau + \mathbf{a}^{-1}\tau'\mathbf{a}$.

Notation and abstract form for the multiplication morphism. Now set
$\mathbf{g} = \begin{pmatrix} \epsilon & 0 \\ 0 & \mathbf{a} \end{pmatrix}$ and $\gamma = \begin{pmatrix} \tau_0 & 0 \\ 0 & \tau \end{pmatrix}$ and realize that the Lie supergroup coordinate elements can be written as pairs $(\mathbf{g}, \gamma)$ in such way that, if $(\mathbf{g}', \gamma')$ is another such coordinate element, their product is given by

$$(\mathbf{g}', \gamma') \cdot (\mathbf{g}, \gamma) = \left(\begin{pmatrix} \epsilon'\epsilon(1 - \frac{\lambda}{2}\tau_0'\tau_0) & 0 \\ 0 & \mathbf{a}'\mathbf{a} \end{pmatrix}, \begin{pmatrix} \tau_0' + \tau_0 & 0 \\ 0 & \tau + \mathbf{a}^{-1}\tau'\mathbf{a} \end{pmatrix} \right).$$

The identity element is $\left(\begin{pmatrix} 1 & 0 \\ 0 & \mathbb{1} \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{0} \end{pmatrix} \right)$ and the inverse element of $(\mathbf{g}, \gamma)$ as above, written as $(\mathbf{g}, \gamma)^{-1}$, is given by $\left(\begin{pmatrix} \epsilon^{-1} & 0 \\ 0 & \mathbf{a}^{-1} \end{pmatrix}, \begin{pmatrix} -\tau_0 & 0 \\ 0 & -\mathbf{a}\tau\mathbf{a}^{-1} \end{pmatrix} \right).$

Actually, we can also write down the multiplication morphism: First we write the projections

$$x_{00}(\mathbf{g}, \gamma) = \epsilon, \quad x_{11}(\mathbf{g}, \gamma) = \alpha, \quad x_{12}(\mathbf{g}, \gamma) = \beta, \quad x_{21}(\mathbf{g}, \gamma) = \gamma,$$
$$\xi_{00}(\mathbf{g}, \gamma) = \tau_0, \quad \xi_{11}(\mathbf{g}, \gamma) = \tau_1, \quad \xi_{12}(\mathbf{g}, \gamma) = \tau_2, \quad \xi_{21}(\mathbf{g}, \gamma) = \tau_3,$$

where $\xi_{22}(\mathbf{g}, \gamma) = -\tau_1$ and

$$x_{22}(\mathbf{g}, \gamma) = \delta = \frac{1 + \beta\gamma}{\alpha} = \frac{1 + x_{12}(\mathbf{g}, \gamma)x_{21}(\mathbf{g}, \gamma)}{x_{11}(\mathbf{g}, \gamma)}.$$

Then, the multiplication morphism is given by

$$\mathbf{m}^* x_{00} = p_1^* x_{00} p_2^* x_{00} - \frac{\lambda}{2} p_1^* x_{00} p_2^* x_{00} p_1^* \xi_{00} p_2^* \xi_{00},$$

$$\mathbf{m}^* x_{ij} = \sum_{k=1}^{2} p_1^* x_{ik} p_2^* x_{kj},$$

$$\mathbf{m}^* \xi_{00} = p_1^* \xi_{00} + p_2^* \xi_{00},$$

$$\mathbf{m}^* \xi_{ij} = p_2^* \xi_{ij} + \sum_{k,\ell=1}^{2} p_2^* y_{ik} p_1^* \xi_{k\ell} p_2^* x_{\ell j},$$

where $(y_{ij}) = \begin{pmatrix} x_{22} & -x_{12} \\ -x_{21} & x_{11} \end{pmatrix}$ and $i, j \in \{1, 2\}$.

3.3. CASE [$\lambda = 0, \mu = 0, \nu \neq 0$]

We proceed exactly as before. By computing the effect of $\Psi = \Psi(\mathbf{g}; \tau_0, \tau_1, \tau_2, \tau_3)$ on the local coordinates z^0, z^1, z^2, ζ^0, ζ^1, ζ^2, we found

$z^0 \mapsto z^0$

$z^1 \mapsto [\alpha(1 - \nu\tau_2\tau_3) + \beta\nu\tau_1\tau_3] z^1 + [\gamma(1 - \nu\tau_2\tau_3) + \delta\nu\tau_1\tau_3] z^2$
$\qquad + [\alpha\nu\tau_1(1 - \nu\tau_2\tau_3) + \beta\nu\tau_3] \zeta^1 + [\gamma\nu\tau_1(1 - \nu\tau_2\tau_3) + \delta\nu\tau_3] \zeta^2$

$z^2 \mapsto (\beta - \nu\alpha\tau_1\tau_2)z^1 + (\delta - \nu\gamma\tau_1\tau_2)z^2 + (\nu\alpha\tau_2 - \nu\beta\tau_1)\zeta^1 + (\nu\gamma\tau_2 - \nu\delta\tau_1)\zeta^2$

$\zeta^0 \mapsto \tau_0 z^0 + \zeta^0$

$\zeta^1 \mapsto [\alpha\tau_1(1 - \nu\tau_2\tau_3) + \beta\tau_3] z^1 + [\gamma\tau_1(1 - \nu\tau_2\tau_3) + \delta\tau_3] z^2$
$\qquad + [\alpha(1 - \nu\tau_2\tau_3) + \beta\nu\tau_1\tau_3] \zeta^1 + [\gamma(1 - \nu\tau_2\tau_3) + \delta\nu\tau_1\tau_3] \zeta^2$

$\zeta^2 \mapsto (\alpha\tau_2 - \beta\tau_1)z^1 + (\gamma\tau_2 - \delta\tau_1)z^2 + (\beta - \alpha\nu\tau_1\tau_2)\zeta^1 + (\delta - \gamma\nu\tau_1\tau_2)\zeta^2.$

Defining

$$\mathbf{p} = \begin{pmatrix} 1 - \nu\tau_2\tau_3 & -\nu\tau_1\tau_2 \\ \nu\tau_1\tau_3 & 1 \end{pmatrix} \qquad \text{and} \qquad \mathbf{q} = \begin{pmatrix} \tau_1 - \nu\tau_1\tau_2\tau_3 & \tau_2 \\ \tau_3 & -\tau_1 \end{pmatrix}$$

we have

$$\Psi(\mathbf{z}) = \begin{pmatrix} 1 & 0 \\ 0 & \mathbf{g}\,\mathbf{p} \end{pmatrix} \mathbf{z} + \begin{pmatrix} 0 & 0 \\ 0 & \nu\,\mathbf{g}\,\mathbf{q} \end{pmatrix} \zeta$$

$$\Psi(\zeta) = \begin{pmatrix} \tau_0 & 0 \\ 0 & \mathbf{g}\,\mathbf{q} \end{pmatrix} \mathbf{z} + \begin{pmatrix} 1 & 0 \\ 0 & \mathbf{g}\,\mathbf{p} \end{pmatrix} \zeta.$$

Similarly, writing $\Psi' = \Psi(\mathbf{g}'; \tau_0', \tau_1', \tau_2', \tau_3')$ we have

$$\Psi'(\mathbf{z}) = \begin{pmatrix} 1 & 0 \\ 0 & \mathbf{g}'\,\mathbf{p}' \end{pmatrix} \mathbf{z} + \begin{pmatrix} 0 & 0 \\ 0 & \nu\,\mathbf{g}'\,\mathbf{q}' \end{pmatrix} \zeta$$

$$\Psi'(\zeta) = \begin{pmatrix} \tau_0' & 0 \\ 0 & \mathbf{g}'\,\mathbf{q}' \end{pmatrix} \mathbf{z} + \begin{pmatrix} 1 & 0 \\ 0 & \mathbf{g}'\,\mathbf{p}' \end{pmatrix} \zeta$$

and we find the composition $\Psi' \circ \Psi$

$$\Psi' \circ \Psi(\mathbf{z}) = \begin{pmatrix} 1 & 0 \\ 0 & \mathbf{g}'\,\mathbf{p}'\,\mathbf{g}\,\mathbf{p} - \nu\,\mathbf{g}'\,\mathbf{q}'\,\mathbf{g}\,\mathbf{q} \end{pmatrix} \mathbf{z} + \begin{pmatrix} 0 & 0 \\ 0 & \nu\,\mathbf{g}'\,\mathbf{q}'\,\mathbf{g}\,\mathbf{p} + \nu\,\mathbf{g}'\,\mathbf{p}'\,\mathbf{g}\,\mathbf{q} \end{pmatrix} \zeta$$

$$\Psi' \circ \Psi(\zeta) = \begin{pmatrix} \tau_0' + \tau_0 & 0 \\ 0 & \mathbf{g}'\,\mathbf{p}'\,\mathbf{g}\,\mathbf{q} + \mathbf{g}'\,\mathbf{q}'\,\mathbf{g}\,\mathbf{p} \end{pmatrix} \mathbf{z} + \begin{pmatrix} 1 & 0 \\ 0 & \mathbf{g}'\,\mathbf{p}'\,\mathbf{g}\,\mathbf{p} - \nu\,\mathbf{g}'\,\mathbf{q}'\,\mathbf{g}\,\mathbf{q} \end{pmatrix} \zeta,$$

where $\mathbf{z} = \begin{pmatrix} z^0 \\ z^1 \\ z^2 \end{pmatrix}$ and $\zeta = \begin{pmatrix} \zeta^0 \\ \zeta^1 \\ \zeta^2 \end{pmatrix}$. So we have $\tau_0'' = \tau_0' + \tau_0$,

$$\mathbf{g}''\,\mathbf{p}'' = \mathbf{g}'\,\mathbf{p}'\,\mathbf{g}\,\mathbf{p} - \nu\,\mathbf{g}'\,\mathbf{q}'\,\mathbf{g}\,\mathbf{q} \qquad \text{and} \qquad \mathbf{g}''\,\mathbf{q}'' = \mathbf{g}'\,\mathbf{p}'\,\mathbf{g}\,\mathbf{q} + \mathbf{g}'\,\mathbf{q}'\,\mathbf{g}\,\mathbf{p}.$$

These equations are equivalent to

(1) $$\mathbf{g}'' = (\mathbf{g}'\,\mathbf{p}'\,\mathbf{g}\,\mathbf{p} - \nu\,\mathbf{g}'\,\mathbf{q}'\,\mathbf{g}\,\mathbf{q})(\mathbf{p}'')^{-1}$$

(2) $$(\mathbf{p}'')^{-1}\mathbf{q}'' = (\mathbf{g}'\,\mathbf{p}'\,\mathbf{g}\,\mathbf{p} - \nu\,\mathbf{g}'\,\mathbf{q}'\,\mathbf{g}\,\mathbf{q})^{-1}(\mathbf{g}'\,\mathbf{p}'\,\mathbf{g}\,\mathbf{q} + \mathbf{g}'\,\mathbf{q}'\,\mathbf{g}\,\mathbf{p}),$$

where it is easy to prove that $(\mathbf{p}'')^{-1}\mathbf{q}'' = \tau'' + \nu\tau_1''\tau_2''\tau_3''\,\mathbb{1}$. In order to find RHS of (2), we note that $\mathbf{p} = \mathbb{1} - \nu\mathbf{p}_{(2)}$, where $\mathbf{p}_{(2)} = \begin{pmatrix} \tau_2\tau_3 & \tau_1\tau_2 \\ -\tau_1\tau_3 & 0 \end{pmatrix}$ (similar expression for $\mathbf{p}'$) and we can write $\mathbf{g}'\,\mathbf{p}'\,\mathbf{g}\,\mathbf{p} - \nu\,\mathbf{g}'\,\mathbf{q}'\,\mathbf{g}\,\mathbf{q} = \mathbf{g}'\,\mathbf{g}(\mathbb{1} - \nu\mathbf{x})$, where $\mathbf{x} = \mathbf{p}_{(2)} + \mathbf{g}^{-1}\,\mathbf{p}_{(2)}'\,\mathbf{g} + (\mathbf{g}^{-1}\,\mathbf{q}'\,\mathbf{g})\mathbf{q} - \nu(\mathbf{g}^{-1}\,\mathbf{p}_{(2)}'\,\mathbf{g})\mathbf{p}_{(2)}$. Then, from $(\mathbb{1} - \nu\mathbf{x})^{-1} = \mathbb{1} + \nu\mathbf{x} + \nu^2\mathbf{x}^2 + \nu^3\mathbf{x}^3$ we have

$$(\mathbf{g}'\,\mathbf{p}'\,\mathbf{g}\,\mathbf{p} - \nu\,\mathbf{g}'\,\mathbf{q}'\,\mathbf{g}\,\mathbf{q})^{-1} = (\mathbb{1} + \nu\mathbf{x} + \nu^2\mathbf{x}^2 + \nu^3\mathbf{x}^3)(\mathbf{g}'\,\mathbf{g})^{-1}$$

thus, RHS of (2) is

$$\mathbf{r} = (\mathbb{1} + \nu\mathbf{x} + \nu^2\mathbf{x}^2 + \nu^3\mathbf{x}^3)\left(\mathbf{g}^{-1}\,\mathbf{p}'\,\mathbf{g}\,\mathbf{q} + \mathbf{g}^{-1}\,\mathbf{q}'\,\mathbf{g}\,\mathbf{p}\right)$$

One finds explicitly $\tau_1'' = \frac{1}{2}(\mathbf{r}_{11} - \mathbf{r}_{22})$, $\tau_2'' = \mathbf{r}_{12}$ and $\tau_3'' = \mathbf{r}_{21}$. With these results, we find $\mathbf{g}''$ from (1), because $(\mathbf{p}'')^{-1} = \mathbb{1} + \nu\mathbf{p}_{(2)}''$.

3.4. Remarks when $\nu \neq 0$

We show a procedure for the general case: We know from §3 what the morphism $\mathrm{Exp}\,(\tau_0 Y_0) \circ \mathrm{Exp}\,(\tau_1 Y_1) \circ \mathrm{Exp}\,(\tau_2 Y_2) \circ \mathrm{Exp}\,(\tau_3 Y_3)$ is; set $\Psi = \Gamma_{\mathbf{g}}^* \circ \mathrm{Exp}\,(\tau_0 Y_0) \circ \mathrm{Exp}\,(\tau_1 Y_1) \circ \mathrm{Exp}\,(\tau_2 Y_2) \circ \mathrm{Exp}\,(\tau_3 Y_3)$ and realize that

$$\Psi \mathbf{z} = A\mathbf{z} + C\zeta \qquad \text{and} \qquad \Psi \zeta = B\mathbf{z} + D\zeta,$$

where A and D are invertible matrices with even entries, whereas B and C are matrices with odd entries. Using similar expressions for Ψ' and Ψ'' we can check that the condition $\Psi'' = \Psi' \circ \Psi$ implies

$$
\begin{aligned}
A'' &= A'A + B'C & B'' &= A'B + B'D \\
C'' &= C'A + D'C & D'' &= C'B + D'D.
\end{aligned}
$$

Then from the equations above we must find $\mathbf{g}''$, τ_0'' and τ'' in terms of other values. This task, however, may not be an easy one. We have seen on the example $\lambda = 0$, $\mu = 0$ and $\nu \neq 0$ that, as soon as the ν parameter is nonzero, we find serious computational problems. We shall then leave the remaining cases with $\nu \neq 0$ for a later attempt (one with a slightly different approach). In sections §3.1 and §3.2 we found a multiplication law for the Lie supergroups with arbitrary values for λ, μ and $\nu = 0$. We now choose some values for these parameters in order to find a multiplication law for at least one representative in the remaining isomorphism classes. Thus, for example, if $\lambda\mu\nu \neq 0$, we choose $\lambda = \mu = 2$ and $\nu = 1$ and with these selections we have the following:

3.1 Proposition. *Let $\mathbb{C}$ be the ground field, $\gamma_{11}, \gamma_{12}, \gamma_{21}$ and γ_{22} be odd elements and let $\mathrm{GL}_2(\mathbb{C}; 2, 2, 1)$ be the group of 2×2 matrices with entries in $\wedge[\gamma_{11}, \gamma_{12}, \gamma_{21}, \gamma_{22}]$ —the exterior algebra generated by $\gamma_{11}, \gamma_{12}, \gamma_{21}, \gamma_{22}$. Let $\mathbf{g} + \gamma$ be an element in $\mathrm{GL}_2(\mathbb{C}; 2, 2, 1)$, and let x_{ij} and ξ_{ij} be the projection maps defined by $x_{ij}(\mathbf{g} + \gamma) = \mathbf{g}_{ij}$ and $\xi_{ij}(\mathbf{g} + \gamma) = \gamma_{ij}$. Define a multiplication law in $\mathrm{GL}_2(\mathbb{C}; 2, 2, 1)$ by*

$$\mathrm{m}^*(x_{ij}) = \sum_{k=1}^{2} p_1^*(x_{ik})p_2^*(x_{kj}) + (-1)^{i+k}p_1^*(\xi_{ik})p_2^*(\xi_{kj}),$$

$$\mathrm{m}^*(\xi_{ij}) = \sum_{k=1}^{2} (-1)^{i+k}p_1^*(x_{ik})p_2^*(\xi_{kj}) + p_1^*(\xi_{ik})p_2^*(x_{kj}).$$

The left invariant supervector fields associated to this multiplication are

$$X_{pq} = \sum_{k=1}^{2} x_{kp}\frac{\partial}{\partial x_{kq}} + \xi_{kp}\frac{\partial}{\partial \xi_{kq}}$$

$$Y_{pq} = \sum_{k=1}^{2} (-1)^{k}x_{kp}\frac{\partial}{\partial \xi_{kq}} + (-1)^{k+1}\xi_{kp}\frac{\partial}{\partial x_{kq}}$$

satisfying

$$
\begin{aligned}
[X_{pq}, X_{rs}] &= \delta_{rq}X_{ps} - \delta_{ps}X_{rq}, \\
[X_{pq}, Y_{rs}] &= \delta_{rq}Y_{ps} - \delta_{ps}Y_{rq}, \\
[Y_{pq}, Y_{rs}] &= -\delta_{rq}X_{ps} - \delta_{ps}X_{rq}
\end{aligned}
$$

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and, by setting

$$x_0 = -X_{11} - X_{22}, \ x_1 = -X_{11} + X_{22}, \ x_2 = -X_{12}, \ x_3 = -X_{21},$$
$$y_0 = \ \ Y_{11} + Y_{22}, \ y_1 = \ \ Y_{11} - Y_{22}, \ y_2 = \ \ Y_{12}, \ y_3 = \ \ Y_{21},$$

we recover the Lie superalgebra associated to the parameters $\lambda = \mu = 2$ and $\nu = 1$.

Sketch of proof. It is a straightforward matter to check that the given multiplication morphism is associative. The identity morphism *id* is given by

$$id^* x_{ij} = x_{ij} \qquad \text{and} \qquad id^* \xi_{ij} = \xi_{ij},$$

whereas the inversion morphism α is given by

$$\alpha^*(x_{ij} + \xi_{ij}) = \mathbf{y}_{ij} - \left((\mathbf{y}\xi)\mathbf{y}\right)_{ij} + \left((\mathbf{y}\xi)^2\mathbf{y}\right)_{ij} - \left((\mathbf{y}\xi)^3\mathbf{y}\right)_{ij} + \left((\mathbf{y}\xi)^4\mathbf{y}\right)_{ij},$$

where

$$\mathbf{y} = (\mathbf{y}_{ij}) = (x_{11}x_{22} - x_{12}x_{21})^{-1} \begin{pmatrix} x_{22} & -x_{12} \\ -x_{21} & x_{11} \end{pmatrix} \qquad \text{and} \qquad \xi = (\xi_{ij}).$$

We shall see in §4.3 below how to compute the left-invariant supervector fields for this multiplication morphism and verify that they are indeed those given in the statement. The important thing to note is that the correspondence $x_0 \mapsto -X_{11} - X_{22}$, etc., given in the statement, sets up a bijection with the Lie superalgebra $\mathfrak{gl}_2(\mathbb{C}; 2, 2, 1)$.

Remark. The multiplication law given in this proposition was taken from [17]. It has been shown there that the special form of this matrix product, actually corresponds to the composition law for two endomorphisms on the graded vector space of dimension $(2, 2)$ (see also other references by the same author in [17]). Note that the supergroup defined by this multiplication law has sometimes appeared in the literature under the name $Q(2)$.

4. Left-invariant supervector fields in Lie supergroups

We want to determine the left-invariant supervector fields for each multiplication law we have found. In order to do that, we first have to know what conditions must such supervector fields satisfy in a coordinate-free manner and encapsulate that information inside some appropriate commutative diagram. Before we explain that, we remind ourselves that every supergroup comes equipped with a special morphism that plays the role of the identity element: $\varepsilon : (G, \mathcal{A}_G) \to (G, \mathcal{A}_G)$ such that $\mathrm{m} \circ (id, \varepsilon) = id = \mathrm{m} \circ (\varepsilon, id)$ (see [3]). Let us use a similar morphism $\varepsilon : G \to G$ in the C^∞-category to understand first what the commutative diagram for left-invariance should be in ordinary Lie group theory.

Let G be an ordinary Lie group. A vector field $X : C^\infty(G) \to C^\infty(G)$ in G is completely determined locally by the values $X x_j$ on a given coordinate system x_j. In order to find these functions, we define the vector field $\widehat{X} : C^\infty(G \times G) \to C^\infty(G \times G)$ in such way that $\widehat{X} p_1^* f = 0$ and $\widehat{X} p_2^* f = p_2^* X f$, for every $f \in C^\infty(G)$, and define $\varepsilon^{(2)^*} : C^\infty(G \times G) \to C^\infty(G)$ by $\varepsilon^{(2)^*} p_1^* = id^*$ and $\varepsilon^{(2)^*} p_2^* = \varepsilon^*$. So, X is a left invariant vector field if the diagram

$$
\begin{array}{ccccc}
C^\infty(G \times G) & \xrightarrow{(p_1, \mathrm{m})^*} & C^\infty(G \times G) & \xrightarrow{\widehat{X}} & C^\infty(G \times G) \\
\widehat{X} \downarrow & & & & \downarrow \varepsilon^{(2)^*} \\
C^\infty(G \times G) & \xrightarrow[(p_1, \mathrm{m})^*]{} & C^\infty(G \times G) & \xrightarrow[\varepsilon^{(2)^*}]{} & C^\infty(G)
\end{array}
$$

commutes, ie, if $\varepsilon^{(2)^*} \circ \widehat{X} \circ (p_1, \mathrm{m})^* = \varepsilon^{(2)^*} \circ (p_1, \mathrm{m})^* \circ \widehat{X}$. We claim that a completely analogous diagram can be used in the theory of Lie supergroups to determine the left-invariant supervector fields.

4.1. Case $GL_2(\mathbb{F}; \lambda, \mu, 0)$

Let $X = \sum_{m,n} A_{mn} \frac{\partial}{\partial x_{mn}} + B_{mn} \frac{\partial}{\partial \xi_{mn}}$ be a supervector field. In order to find the superfunctions A_{mn} and B_{mn}, we will use the above assertion. According to the multiplication law for this case (see §3.1) it is easy to find that $\varepsilon^* x_{ij} = \delta_{ij}$ and $\varepsilon^* \xi_{ij} = 0$, and then, $\varepsilon^* f = \widetilde{f}(\mathbb{1})$.

Defining $\widehat{X} = \sum_{m,n} p_2^* A_{mn} \frac{\partial}{\partial p_2^* x_{mn}} + p_2^* B_{mn} \frac{\partial}{\partial p_2^* \xi_{mn}}$, a straightforward computation shows

$$
A_{ij}(\mathbf{x}, \boldsymbol{\xi}) = \sum_{k=1}^{2} x_{ik} \widetilde{A}_{kj}(\mathbb{1}) + \mu \left(\frac{\widetilde{B}_{11}(\mathbb{1}) + \widetilde{B}_{22}(\mathbb{1})}{2} \right) x_{ik} \xi_{kj}
$$

$$
- (\mu - \frac{\lambda}{2}) \left(\frac{\widetilde{B}_{11}(\mathbb{1}) + \widetilde{B}_{22}(\mathbb{1})}{2} \right) \left(\frac{\xi_{11} + \xi_{22}}{2} \right) x_{ij},
$$

$$
B_{ij}(\mathbf{x}, \boldsymbol{\xi}) = \widetilde{B}_{ij}(\mathbb{1}) + \sum_{k=1}^{2} \xi_{ik} \widetilde{A}_{kj}(\mathbb{1}) - \widetilde{A}_{ik}(\mathbb{1}) \xi_{kj}
$$

$$
- \mu \left(\frac{\widetilde{B}_{11}(\mathbb{1}) + \widetilde{B}_{22}(\mathbb{1})}{2} \right) \xi_{ik} \xi_{kj}.
$$

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So, we can write $X = \sum_{pq} \widetilde{A}_{pq}(\mathbb{1})X_{pq} + \widetilde{B}_{pq}(\mathbb{1})Y_{pq}$, where

$$X_{pq} = \sum_{k=1}^{2} x_{kp}\frac{\partial}{\partial x_{kq}} + \xi_{kp}\frac{\partial}{\partial \xi_{kq}} - \xi_{qk}\frac{\partial}{\partial \xi_{pk}},$$

$$Y_{pq} = \frac{\partial}{\partial \xi_{pq}} + \frac{\delta_{pq}}{2}\left\{ \sum_{i,j=1}^{2} \left((\frac{\lambda}{2} - \mu)\left(\frac{\xi_{11} + \xi_{22}}{2}\right)x_{ij} + \mu\sum_{k=1}^{2} x_{ik}\xi_{kj} \right) \frac{\partial}{\partial x_{ij}} \right.$$

$$\left. -\mu\sum_{i,j,k=1}^{2} \xi_{ik}\xi_{kj}\frac{\partial}{\partial \xi_{ij}} \right\}$$

and they satisfy

$$[X_{pq}, X_{rs}] = \delta_{rq}X_{ps} - \delta_{ps}X_{rq},$$
$$[X_{pq}, Y_{rs}] = \delta_{rq}Y_{ps} - \delta_{ps}Y_{rq}.$$

By our previous considerations, if $Z = \sum_{p,q} a_{pq}X_{pq}$ and $W = \sum_{r,s} b_{rs}X_{rs}$ are supervector fields, then $[Z, W] = \sum_{i,j}([a,b])_{ij}X_{ij}$, where $a = (a_{pq})$ and $b = (b_{rs})$. Then, by setting

$$x_0 = X_{11} + X_{22} \qquad x_1 = X_{11} - X_{22} \qquad x_2 = X_{12} \qquad x_3 = X_{21}$$
$$y_0 = Y_{11} + Y_{22} \qquad y_1 = Y_{11} - Y_{22} \qquad y_2 = Y_{12} \qquad y_3 = Y_{21}$$

we recover the Lie superalgebra with parameters $[\lambda, \mu, \nu = 0]$.

4.2. CASE $GL_2(\mathbb{F}; \lambda \neq 0, 0, 0)$

Let $X = \sum_{m,n} A_{mn}\frac{\partial}{\partial x_{mn}} + B_{mn}\frac{\partial}{\partial \xi_{mn}}$ be a supervector field. According to the multiplication morphism found in §3.2, a straightforward computation shows that X will be a left-invariant supervector field if and only if

$$X = \widetilde{A}_{00}(\mathbb{1})X_{00} + \widetilde{A}_{ij}(\mathbb{1})X_{ij} + \widetilde{B}_{00}(\mathbb{1})Y_{00} + \widetilde{B}_{ij}(\mathbb{1})Y_{ij},$$

where

$$X_{00} = x_{00}\frac{\partial}{\partial x_{00}},$$

$$X_{ij} = \sum_{k=1}^{2} x_{ki}\frac{\partial}{\partial x_{kj}} + \xi_{ki}\frac{\partial}{\partial \xi_{kj}} - \xi_{jk}\frac{\partial}{\partial \xi_{ik}},$$

$$Y_{00} = \frac{\partial}{\partial \xi_{00}} - \frac{\lambda}{2}x_{00}\xi_{00}\frac{\partial}{\partial x_{00}},$$

$$Y_{ij} = \frac{\partial}{\partial \xi_{ij}},$$

Actually, this result came from $\varepsilon^* x_{00} = 1$, $\varepsilon^* x_{ij} = \delta_{ij}$, $\varepsilon^* \xi_{00} = 0$, $\varepsilon^* \xi_{ij} = 0$ and then $\varepsilon^* f = \widetilde{f}(\mathbb{1})$. Furthermore, it is easy to prove that

$$[X_{00}, X_{rs}] = 0, \qquad [X_{pq}, X_{rs}] = \delta_{rq}X_{ps} - \delta_{ps}X_{rq},$$
$$[X_{00}, Y_{00}] = 0, \qquad [X_{pq}, Y_{00}] = 0,$$
$$[X_{00}, Y_{rs}] = 0, \qquad [X_{pq}, Y_{rs}] = \delta_{rq}Y_{ps} - \delta_{ps}Y_{rq},$$
$$[Y_{00}, Y_{00}] = -\lambda X_{00}, \qquad [Y_{00}, Y_{rs}] = 0,$$
$$[Y_{pq}, Y_{rs}] = 0,$$

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thus realizing the $[\lambda \neq 0, \mu = 0, \nu = 0]$ Lie superalgebra equivalence class.

4.3. CASE $GL_2(\mathbb{C}; 2, 2, 1)$

Let $X = \sum_{m,n} A_{mn} \dfrac{\partial}{\partial x_{mn}} + B_{mn} \dfrac{\partial}{\partial \xi_{mn}}$ be a supervector field. According to the multiplication morphism given in Proposition 1 §3.4, we can check that $\varepsilon^* x_{ij} = \delta_{ij}$ and $\varepsilon^* \xi_{ij} = 0$, from where we know that $\varepsilon^*(f) = \tilde{f}(\mathbb{1})$, for any given superfunction f. A straightforward computation shows that X will be a left-invariant supervector field if and only if

$$\varepsilon^{(2)^*} \circ \widehat{X} \circ (p_1, \mathrm{m})^* p_2^* x_{ij} = \varepsilon^{(2)^*} \circ (p_1, \mathrm{m})^* \circ \widehat{X} p_2^* x_{ij},$$

$$\varepsilon^{(2)^*} \circ \widehat{X} \circ (p_1, \mathrm{m})^* p_2^* \xi_{ij} = \varepsilon^{(2)^*} \circ (p_1, \mathrm{m})^* \circ \widehat{X} p_2^* \xi_{ij}.$$

Now, the first of these conditions leads to

$$\begin{aligned}
\varepsilon^{(2)^*} \circ \widehat{X} \circ (p_1, \mathrm{m})^* p_2^* x_{ij} &= \varepsilon^{(2)^*} \circ \widehat{X} \left\{ p_1^* x_{ik} p_2^* x_{kj} + (-1)^{i+k} p_1^* \xi_{ik} p_2^* \xi_{kj} \right\} \\
&= \varepsilon^{(2)^*} \left\{ p_2^* A_{kj} p_1^* x_{ik} + (-1)^{i+k+1} p_2^* B_{kj} p_1^* \xi_{ik} \right\} \\
&= x_{ij} \widetilde{A}_{kj}(\mathbb{1}) + (-1)^{i+k+1} \xi_{ik} \widetilde{B}_{kj}(\mathbb{1}).
\end{aligned}$$

Similarly, the second condition leads to

$$\varepsilon^{(2)^*} \circ \widehat{X} \circ (p_1, \mathrm{m})^* p_2^* \xi_{ij} = (-1)^{i+k} x_{ij} \widetilde{B}_{kj}(\mathbb{1}) + \xi_{ik} \widetilde{A}_{kj}(\mathbb{1}).$$

On the other hand,

$$\varepsilon^{(2)^*} \circ (p_1, \mathrm{m})^* \circ \widehat{X} p_2^* x_{ij} = A_{ij},$$

$$\varepsilon^{(2)^*} \circ (p_1, \mathrm{m})^* \circ \widehat{X} p_2^* \xi_{ij} = B_{ij}.$$

So, we conclude that X is a left-invariant supervector field if and only if

$$X = \sum_{ij} \widetilde{A}_{ij}(\mathbb{1}) X_{ij} + \widetilde{B}_{ij}(\mathbb{1}) Y_{ij},$$

where

$$X_{ij} = \sum_{k=1}^{2} x_{ki} \frac{\partial}{\partial x_{kj}} + \xi_{ki} \frac{\partial}{\partial \xi_{kj}},$$

$$Y_{ij} = (-1)^i \left(\sum_{k=1}^{2} (-1)^{k+1} \xi_{ki} \frac{\partial}{\partial x_{kj}} + (-1)^k x_{ki} \frac{\partial}{\partial \xi_{kj}} \right).$$

where (-1) appearing in Y_{ij} can be included into $\widetilde{B}_{ij}(\mathbb{1})$. As we mentioned in Proposition 1 §3.4, these supervector fields define the Lie superalgebra on the equivalence class of $\lambda = 2, \mu = 2$ and $\nu = 1$.

5. COMPACT REAL FORMS

Let us consider the real Lie superalgebras $u_2(\lambda, \mu, \nu)$ with underlying Lie algebra u_2, that arise after changing the basis in $gl_2(\mathbb{C}; \lambda, \mu, \nu)$ by $w_0 = iI$, $w_3 = iH$, $w_2 = E - F$ and $w_1 = i(E + F)$, as usual. By letting π as before, a change of parity map, we have that the symmetric bilinear equivariant map $\Gamma : u_2 \times u_2 \to u_2$ that gives the Lie bracket for any pair of odd elements, where $\Gamma(z, w) = [\pi(z), \pi(w)]$, is

$$\Gamma(w_0, w_0) = i\lambda w_0$$

$$\Gamma(w_0, w_3) = i\mu w_3 \quad \Gamma(w_3, w_3) = 2i\nu w_0$$

$$\Gamma(w_0, w_2) = i\mu w_2 \quad \Gamma(w_3, w_2) = 0 \qquad \Gamma(w_2, w_2) = 2i\nu w_0$$

$$\Gamma(w_0, w_1) = i\mu w_1 \quad \Gamma(w_3, w_1) = 0 \qquad \Gamma(w_2, w_1) = 0 \qquad \Gamma(w_1, w_1) = 2i\nu w_0.$$

Then, in order to have the compact real form for GL_2, λ, μ and ν have to be restricted so as to be purely imaginary.

As in §2, we have faithful representations for all these Lie superalgebras in supervector fields of the supermanifolds $\mathbb{F}^{2|2}$ and $\mathbb{F}^{3|3}$:

5.1 Proposition. *Lie superalgebras in the equivalence classes of*

$$\lambda\nu > 0 \ , \ \mu \neq 0 \qquad \lambda\nu < 0 \ , \ \mu \neq 0$$
$$\lambda\nu < 0 \ , \ \mu = 0$$
$$\nu\mu \neq 0 \ , \ \lambda = 0 \qquad \lambda\mu \neq 0 \ , \ \nu = 0$$
$$\mu \neq 0 \ , \ \lambda = \nu = 0 \qquad \lambda = \mu = \nu = 0$$

admit an explicit realization in terms of supervector fields in the supermanifold $\mathbb{R}^{2|2}$ with local coordinates $\{z^1, z^2 ; \zeta^1, \zeta^2\}$ given by

$$W_0 = i\left(z^1 \frac{\partial}{\partial z^1} + z^2 \frac{\partial}{\partial z^2} + \zeta^1 \frac{\partial}{\partial \zeta^1} + \zeta^2 \frac{\partial}{\partial \zeta^2}\right)$$

$$W_3 = i\left(z^1 \frac{\partial}{\partial z^1} - z^2 \frac{\partial}{\partial z^2} + \zeta^1 \frac{\partial}{\partial \zeta^1} - \zeta^2 \frac{\partial}{\partial \zeta^2}\right)$$

$$W_2 = \quad z^1 \frac{\partial}{\partial z^2} - z^2 \frac{\partial}{\partial z^1} + \zeta^1 \frac{\partial}{\partial \zeta^2} - \zeta^2 \frac{\partial}{\partial \zeta^1}$$

$$W_1 = i\left(z^2 \frac{\partial}{\partial z^1} + z^1 \frac{\partial}{\partial z^2} + \zeta^2 \frac{\partial}{\partial \zeta^1} + \zeta^1 \frac{\partial}{\partial \zeta^2}\right)$$

$$Z_0 = ik\left(\zeta^1 \frac{\partial}{\partial z^1} + \zeta^2 \frac{\partial}{\partial z^2}\right) + ig\left(z^1 \frac{\partial}{\partial \zeta^1} + z^2 \frac{\partial}{\partial \zeta^2}\right)$$

$$Z_3 = ie\left(\zeta^1 \frac{\partial}{\partial z^1} - \zeta^2 \frac{\partial}{\partial z^2}\right) + id\left(z^1 \frac{\partial}{\partial \zeta^1} - z^2 \frac{\partial}{\partial \zeta^2}\right)$$

$$Z_2 = \quad e\left(\zeta^1 \frac{\partial}{\partial z^2} - \zeta^2 \frac{\partial}{\partial z^1}\right) + \quad d\left(z^1 \frac{\partial}{\partial \zeta^2} - z^2 \frac{\partial}{\partial \zeta^1}\right)$$

$$Z_1 = ie\left(\zeta^2 \frac{\partial}{\partial z^1} + \zeta^1 \frac{\partial}{\partial z^2}\right) + id\left(z^2 \frac{\partial}{\partial \zeta^1} + z^1 \frac{\partial}{\partial \zeta^2}\right),$$

where $\lambda = 2gk$, $\mu = eg + dk$ and $\nu = ed$. The Lie superalgebras lying in the equivalence class of $\lambda\nu > 0$ and $\mu = 0$ admit a realization as above but in $\mathbb{C}^{2|2}$.

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5.2 Proposition. *Lie superalgebras in the equivalence class of $\nu \neq 0$ with $\mu = \lambda = 0$ admit explicit realizations in terms of supervector fields in the supermanifold $\mathbb{R}^{3|3}$ with local coordinates $\{z^0, z^1, z^2 ; \zeta^0, \zeta^1, \zeta^2\}$ given by the same expressions for W_k ($k = 0, 1, 2, 3$) and Z_ℓ ($\ell = 1, 2, 3$) in the proposition above (corresponding to the parameter values $d = 1$ and $e = \nu$), together with,*

$$Z_0 = iz^0 \frac{\partial}{\partial \zeta^0}.$$

5.3 Proposition. *Lie superalgebras in the equivalence class of $\lambda \neq 0$ with $\mu = \nu = 0$ admit explicit realizations in terms of supervector fields in the supermanifold $\mathbb{R}^{3|3}$ with local coordinates $\{z^0, z^1, z^2 ; \zeta^0, \zeta^1, \zeta^2\}$ given by the same expressions for W_k ($k = 1, 2, 3$) and Z_ℓ ($\ell = 1, 2, 3$) in the proposition above (corresponding to the parameter values $d = 1$ and $e = 0$), together with,*

$$W_0 = i\left(z^0 \frac{\partial}{\partial z^0} + \zeta^0 \frac{\partial}{\partial \zeta^0}\right)$$

and

$$Z_0 = i\left(\zeta^0 \frac{\partial}{\partial z^0} + \frac{\lambda}{2} z^0 \frac{\partial}{\partial \zeta^0}\right).$$

In order to find the multiplication law for representations given in Proposition 5.1, we use following facts

$$
\begin{array}{lll}
W_0^k z^j = i^k z^j & W_3^k z^1 = i^k z^1 & W_3^k z^2 = (-i)^k z^2 \\
W_2^{2k} z^j = (-1)^k z^j & W_2^{2k+1} z^1 = -(-1)^k z^2 & W_2^{2k+1} z^2 = (-1)^k z^1 \\
W_1^{2k} z^j = (-1)^k z^j & W_1^{2k+1} z^1 = (-1)^k i z^2 & W_1^{2k+1} z^2 = (-1)^k i z^1.
\end{array}
$$

It is easy to see that,

$$\Gamma_{w_0}{}^* = \mathrm{Exp}(t_0 W_0) : \begin{cases} z^1 \mapsto e^{it_0} z^1 \\ z^2 \mapsto e^{it_0} z^2 \\ \zeta^1 \mapsto e^{it_0} \zeta^1 \\ \zeta^2 \mapsto e^{it_0} \zeta^2 \end{cases} \qquad \Gamma_{w_3}{}^* = \mathrm{Exp}(t_3 W_3) : \begin{cases} z^1 \mapsto e^{it_3} z^1 \\ z^2 \mapsto e^{-it_3} z^2 \\ \zeta^1 \mapsto e^{it_3} \zeta^1 \\ \zeta^2 \mapsto e^{-it_3} \zeta^2 \end{cases}$$

$$\Gamma_{w_2}{}^* = \mathrm{Exp}(t_2 W_2) : \begin{cases} z^1 \mapsto \cos t_2 z^1 - \sin t_2 z^2 \\ z^2 \mapsto \sin t_2 z^1 + \cos t_2 z^2 \\ \zeta^1 \mapsto \cos t_2 \zeta^1 - \sin t_2 \zeta^2 \\ \zeta^2 \mapsto \sin t_2 \zeta^1 + \cos t_2 \zeta^2 \end{cases}$$

$$\Gamma_{w_1}{}^* = \mathrm{Exp}(t_1 W_1) : \begin{cases} z^1 \mapsto \cos t_1 z^1 + i \sin t_1 z^2 \\ z^2 \mapsto i \sin t_1 z^1 + \cos t_1 z^2 \\ \zeta^1 \mapsto \cos t_1 \zeta^1 + i \sin t_1 \zeta^2 \\ \zeta^2 \mapsto i \sin t_1 \zeta^1 + \cos t_1 \zeta^2 \end{cases}$$

then, if $\Gamma_g^* = \mathrm{Exp}(t_0 W_0) \circ \mathrm{Exp}(t_3 W_3) \circ \mathrm{Exp}(t_2 W_2) \circ \mathrm{Exp}(t_1 W_1)$, we obtain

$$\Gamma_{\mathbf{g}}^* : \begin{cases} z^1 \mapsto \alpha z^1 + \gamma z^2 \\ z^2 \mapsto \beta z^1 + \delta z^2 \\ \zeta^1 \mapsto \alpha \zeta^1 + \gamma \zeta^2 \\ \zeta^2 \mapsto \beta \zeta^1 + \delta \zeta^2, \end{cases}$$

where

$$\alpha = (\cos t_1 \cos t_2 + i \sin t_1 \sin t_2)\, e^{i(t_0 + t_3)},$$
$$\beta = (i \sin t_1 \cos t_2 + \cos t_1 \sin t_2)\, e^{i(t_0 + t_3)},$$
$$\gamma = (-\cos t_1 \sin t_2 + i \sin t_1 \cos t_2)\, e^{i(t_0 - t_3)},$$
$$\delta = (\cos t_1 \cos t_2 - i \sin t_1 \sin t_2)\, e^{i(t_0 - t_3)}$$

and we can see that, up to e^{it_0}, $\delta = \overline{\alpha}$ and $\gamma = -\overline{\beta}$. On the other hand,

$$\Gamma_{Z_0}{}^* = \mathrm{Exp}(\tau_0 Z_0) : \begin{cases} z^1 \mapsto z^1 + ik\tau_0 \zeta^1 \\ z^2 \mapsto z^2 + ik\tau_0 \zeta^2 \\ \zeta^1 \mapsto \zeta^1 + ig\tau_0 z^1 \\ \zeta^2 \mapsto \zeta^2 + ig\tau_0 z^2 \end{cases} \qquad \Gamma_{Z_3}{}^* = \mathrm{Exp}(\tau_3 Z_3) : \begin{cases} z^1 \mapsto z^1 + ie\tau_3 \zeta^1 \\ z^2 \mapsto z^2 - ie\tau_3 \zeta^2 \\ \zeta^1 \mapsto \zeta^1 + id\tau_3 z^1 \\ \zeta^2 \mapsto \zeta^2 - id\tau_3 z^2 \end{cases}$$

$$\Gamma_{Z_2}{}^* = \mathrm{Exp}(\tau_2 Z_2) : \begin{cases} z^1 \mapsto z^1 - e\tau_2 \zeta^2 \\ z^2 \mapsto z^2 + e\tau_2 \zeta^1 \\ \zeta^1 \mapsto \zeta^1 - d\tau_2 z^2 \\ \zeta^2 \mapsto \zeta^2 + d\tau_2 z^1 \end{cases} \qquad \Gamma_{Z_1}{}^* = \mathrm{Exp}(\tau_1 Z_1) : \begin{cases} z^1 \mapsto z^1 + ie\tau_1 \zeta^2 \\ z^2 \mapsto z^2 + ie\tau_1 \zeta^1 \\ \zeta^1 \mapsto \zeta^1 + id\tau_1 z^2 \\ \zeta^2 \mapsto \zeta^2 + id\tau_1 z^1 \end{cases}$$

and, from $\Gamma^* = \mathrm{Exp}(\tau_0 Z_0) \circ \mathrm{Exp}(\tau_3 Z_3) \circ \mathrm{Exp}(\tau_2 Z_2) \circ \mathrm{Exp}(\tau_1 Z_1)$, we know that

$$
\begin{aligned}
z^1 \mapsto{} & (1 + eg\tau_0\tau_3)(1 - ied\tau_2\tau_1)z^1 + ie(-g\tau_0 + d\tau_3)(i\tau_1 - \tau_2)z^2 \\
& + i(k\tau_0 + e\tau_3)(1 - ied\tau_2\tau_1)\zeta^1 + e(1 - dk\tau_0\tau_3)(i\tau_1 - \tau_2)\zeta^2 \\
z^2 \mapsto{} & -ie(g\tau_0 + d\tau_3)(i\tau_1 + \tau_2)z^1 + (1 - eg\tau_0\tau_3)(1 + ied\tau_2\tau_1)z^2 \\
& + e(1 + dk\tau_0\tau_3)(i\tau_1 + \tau_2)\zeta^1 + i(k\tau_0 - e\tau_3)(1 + ied\tau_2\tau_1)\zeta^2 \\
\zeta^1 \mapsto{} & i(g\tau_0 + d\tau_3)(1 - ied\tau_2\tau_1)z^1 + d(1 - eg\tau_0\tau_3)(i\tau_1 - \tau_2)z^2 \\
& + (1 + dk\tau_0\tau_3)(1 - ied\tau_2\tau_1)\zeta^1 + id(-k\tau_0 + e\tau_3)(i\tau_1 - \tau_2)\zeta^2 \\
\zeta^2 \mapsto{} & d(1 + eg\tau_0\tau_3)(i\tau_1 + \tau_2)z^1 + i(g\tau_0 - d\tau_3)(1 + ied\tau_2\tau_1)z^2 \\
& - id(k\tau_0 + e\tau_3)(i\tau_1 + \tau_2)\zeta^1 + (1 - dk\tau_0\tau_3)(1 + ied\tau_2\tau_1)\zeta^2.
\end{aligned}
$$

5.1. CASE $[\lambda, \mu, \nu = 0]$

By considering $e = 0$,

$$\Gamma^* : \begin{cases} z^1 \mapsto z^1 + ik\tau_0 \zeta^1 \\ z^2 \mapsto z^2 + ik\tau_0 \zeta^2 \\ \zeta^1 \mapsto i(g\tau_0 + d\tau_3)z^1 + d(i\tau_1 - \tau_2)z^2 \\ \qquad + (1 + dk\tau_0\tau_3)\zeta^1 - idk\tau_0(i\tau_1 - \tau_2)\zeta^2 \\ \zeta^2 \mapsto d(i\tau_1 + \tau_2)z^1 + i(g\tau_0 - d\tau_3)z^2 \\ \qquad - idk\tau_0(i\tau_1 + \tau_2)\zeta^1 + (1 - dk\tau_0\tau_3)\zeta^2. \end{cases}$$

Let $\Psi = \Gamma_{\mathbf{g}}^* \circ \Gamma^*$, then

$$\Psi \begin{pmatrix} z^1 \\ z^2 \end{pmatrix} = \mathbf{g} \begin{pmatrix} z^1 \\ z^2 \end{pmatrix} + k\rho_0 \, \mathbf{g} \begin{pmatrix} \zeta^1 \\ \zeta^2 \end{pmatrix},$$

$$\Psi \begin{pmatrix} \zeta^1 \\ \zeta^2 \end{pmatrix} = \mathbf{g}(g\rho_0 \mathbb{1} + d\boldsymbol{\rho}) \begin{pmatrix} z^1 \\ z^2 \end{pmatrix} + \mathbf{g}(\mathbb{1} - dk\rho_0\boldsymbol{\rho}) \begin{pmatrix} \zeta^1 \\ \zeta^2 \end{pmatrix},$$

which is actually as in section §3.1, but $\rho_0 = i\tau_0$ and $\boldsymbol{\rho} = \begin{pmatrix} i\tau_3 & i\tau_1+\tau_2 \\ i\tau_1-\tau_2 & -i\tau_3 \end{pmatrix}$, then we know that

$$\mathbf{g}'' = \mathbf{g}' \, \mathbf{g} + k\rho_0 \, \mathbf{g}' (g\rho_0' \mathbb{1} + d\boldsymbol{\rho}') \, \mathbf{g},$$
$$\rho_0'' = \rho_0' + \rho_0,$$
$$\boldsymbol{\rho}'' = \boldsymbol{\rho} + \mathbf{g}^{-1} \, \boldsymbol{\rho}' \, \mathbf{g} - dk\rho_0(\mathbf{g}^{-1} \, \boldsymbol{\rho}' \, \mathbf{g})^2$$

and it is easy to find the multiplication law $(\mathbf{g}', i\tau_0', i\boldsymbol{\tau}') \cdot (\mathbf{g}, i\tau_0, i\boldsymbol{\tau})$, which is

$$\left(\mathbf{g}' \, \mathbf{g} - \frac{\lambda}{2} i\tau_0' i\tau_0 \, \mathbf{g}' \, \mathbf{g} + \mu i\tau_0 \, \mathbf{g}' \, i\boldsymbol{\tau}' \, \mathbf{g} \, , i\tau_0' + i\tau_0 \, , i\boldsymbol{\tau} + \mathbf{g}^{-1} \, i\boldsymbol{\tau}' \, \mathbf{g} - \mu i\tau_0(\mathbf{g}^{-1} \, i\boldsymbol{\tau}' \, \mathbf{g})^2 \right),$$

where $i\boldsymbol{\tau}' = \begin{pmatrix} i\tau_3' & i\tau_1'+\tau_2' \\ i\tau_1'-\tau_2' & -i\tau_3' \end{pmatrix}$ and $i\boldsymbol{\tau} = \begin{pmatrix} i\tau_3 & i\tau_1+\tau_2 \\ i\tau_1-\tau_2 & -i\tau_3 \end{pmatrix}$.

Simple computations show that left-invariant supervector fields associated to this multiplication law are the same as those found in §4.1, associated to the multiplication morphism

$$\mathbf{m}^* x_{ij} = \left\{ 1 + \left(\mu - \frac{\lambda}{2}\right) \left(\frac{p_1^*\xi_{11} + p_1^*\xi_{22}}{2} \right) \left(\frac{p_2^*\xi_{11} + p_2^*\xi_{22}}{2} \right) \right\} \sum_{k=1}^{2} p_1^* x_{ik} p_2^* x_{kj}$$

$$+ \mu \left(\frac{p_2^*\xi_{11} + p_2^*\xi_{22}}{2} \right) \sum_{k=1}^{2} p_1^* x_{ik} p_1^* \xi_{k\ell} p_2^* x_{\ell j},$$

$$\mathbf{m}^* \xi_{ij} = p_2^* \xi_{ij} + \sum_{k,\ell=1}^{2} p_2^* y_{ik} p_1^* \xi_{k\ell} p_2^* x_{\ell j}$$

$$- \mu \left(\frac{p_2^*\xi_{11} + p_2^*\xi_{22}}{2} \right) \left(\sum_{k,\ell=1}^{2} p_2^* y_{ik} p_1^* \xi_{k\ell} p_2^* x_{\ell j} \right)^2,$$

where $(y_{ij}) = (x_{11}x_{22} - x_{12}x_{21})^{-1} \begin{pmatrix} x_{22} & -x_{12} \\ -x_{21} & x_{11} \end{pmatrix}$, that is

$$X_{pq} = \sum_{k=1}^{2} x_{kp} \frac{\partial}{\partial x_{kq}} + \xi_{kp} \frac{\partial}{\partial \xi_{kq}} - \xi_{qk} \frac{\partial}{\partial \xi_{pk}},$$

$$Y_{pq} = \frac{\partial}{\partial \xi_{pq}} + \frac{\delta_{pq}}{2} \left\{ \sum_{i,j=1}^{2} \left(\left(\frac{\lambda}{2} - \mu\right) \left(\frac{\xi_{11} + \xi_{22}}{2} \right) x_{ij} + \mu \sum_{k=1}^{2} x_{ik}\xi_{kj} \right) \frac{\partial}{\partial x_{ij}} \right.$$

$$\left. - \mu \sum_{i,j,k=1}^{2} \xi_{ik}\xi_{kj} \frac{\partial}{\partial \xi_{ij}} \right\},$$

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so that they correspond to the equivalence class of $[\lambda, \mu, \nu = 0]$.

5.2. CASE $[\lambda \neq 0, \mu = 0, \nu = 0]$

In the case $[\lambda \neq 0, \mu = 0, \nu = 0]$, which comes from Proposition 5.3, we have the integral flows for even supervector fields

$$\Gamma_{w_0}{}^* = \text{Exp}(t_0 W_0) : \begin{cases} z^0 \mapsto e^{it_0} z^0 \\ z^1 \mapsto z^1 \\ z^2 \mapsto z^2 \\ \zeta^0 \mapsto e^{it_0} \zeta^0 \\ \zeta^1 \mapsto \zeta^1 \\ \zeta^2 \mapsto \zeta^2 \end{cases} \qquad \Gamma_{w_3}{}^* = \text{Exp}(t_3 W_3) : \begin{cases} z^0 \mapsto z^0 \\ z^1 \mapsto e^{it_3} z^1 \\ z^2 \mapsto e^{-it_3} z^2 \\ \zeta^0 \mapsto \zeta^0 \\ \zeta^1 \mapsto e^{it_3} \zeta^1 \\ \zeta^2 \mapsto e^{-it_3} \zeta^2 \end{cases}$$

$$\Gamma_{w_2}{}^* = \text{Exp}(t_2 W_2) : \begin{cases} z^0 \mapsto z^0 \\ z^1 \mapsto \cos t_2 z^1 - \sin t_2 z^2 \\ z^2 \mapsto \sin t_2 z^1 + \cos t_2 z^2 \\ \zeta^0 \mapsto \zeta^0 \\ \zeta^1 \mapsto \cos t_2 \zeta^1 - \sin t_2 \zeta^2 \\ \zeta^2 \mapsto \sin t_2 \zeta^1 + \cos t_2 \zeta^2 \end{cases}$$

$$\Gamma_{w_1}{}^* = \text{Exp}(t_1 W_1) : \begin{cases} z^0 \mapsto z^0 \\ z^1 \mapsto \cos t_1 z^1 + i \sin t_1 z^2 \\ z^2 \mapsto i \sin t_1 z^1 + \cos t_1 z^2 \\ \zeta^0 \mapsto \zeta^0 \\ \zeta^1 \mapsto \cos t_1 \zeta^1 + i \sin t_1 \zeta^2 \\ \zeta^2 \mapsto i \sin t_1 \zeta^1 + \cos t_1 \zeta^2, \end{cases}$$

then, if $\Gamma_{\mathbf{g}}^* = \text{Exp}(t_0 W_0) \circ \text{Exp}(t_3 W_3) \circ \text{Exp}(t_2 W_2) \circ \text{Exp}(t_1 W_1)$, we have that

$$\Gamma_{\mathbf{g}}^* : \begin{cases} z^0 \mapsto \epsilon z^0 \\ z^1 \mapsto \alpha z^1 + \gamma z^2 \\ z^2 \mapsto \beta z^1 + \delta z^2 \\ \zeta^0 \mapsto \epsilon \zeta^0 \\ \zeta^1 \mapsto \alpha \zeta^1 + \gamma \zeta^2 \\ \zeta^2 \mapsto \beta \zeta^1 + \delta \zeta^2, \end{cases}$$

where

$$\epsilon = e^{it_0},$$
$$\alpha = (\cos t_1 \cos t_2 + i \sin t_1 \sin t_2) e^{it_3},$$
$$\beta = (i \sin t_1 \cos t_2 + \cos t_1 \sin t_2) e^{it_3},$$
$$\gamma = (-\cos t_1 \sin t_2 + i \sin t_1 \cos t_2) e^{-it_3},$$
$$\delta = (\cos t_1 \cos t_2 - i \sin t_1 \sin t_2) e^{-it_3}$$

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and we can see that $\delta = \overline{\alpha}$, $\gamma = -\overline{\beta}$ and $\alpha\delta - \beta\gamma = 1$. By computing the integral flows for the odd supervector fields

$$\Gamma_{Z_0}{}^* = \mathrm{Exp}(\tau_0 Z_0) : \begin{cases} z^0 \mapsto z^0 + i\tau_0\zeta^0 \\ z^1 \mapsto z^1 \\ z^2 \mapsto z^2 \\ \zeta^0 \mapsto \zeta^0 + i\frac{\lambda}{2}\tau_0 z^0 \\ \zeta^1 \mapsto \zeta^1 \\ \zeta^2 \mapsto \zeta^2 \end{cases} \qquad \Gamma_{Z_3}{}^* = \mathrm{Exp}(\tau_3 Z_3) : \begin{cases} z^0 \mapsto z^0 \\ z^1 \mapsto z^1 \\ z^2 \mapsto z^2 \\ \zeta^0 \mapsto \zeta^0 \\ \zeta^1 \mapsto \zeta^1 + i\tau_3 z^1 \\ \zeta^2 \mapsto \zeta^2 - i\tau_3 z^2 \end{cases}$$

$$\Gamma_{Z_2}{}^* = \mathrm{Exp}(\tau_2 Z_2) : \begin{cases} z^0 \mapsto z^0 \\ z^1 \mapsto z^1 \\ z^2 \mapsto z^2 \\ \zeta^0 \mapsto \zeta^0 \\ \zeta^1 \mapsto \zeta^1 - \tau_2 z^2 \\ \zeta^2 \mapsto \zeta^2 + \tau_2 z^1 \end{cases} \qquad \Gamma_{Z_1}{}^* = \mathrm{Exp}(\tau_1 Z_1) : \begin{cases} z^0 \mapsto z^0 \\ z^1 \mapsto z^1 \\ z^2 \mapsto z^2 \\ \zeta^0 \mapsto \zeta^0 \\ \zeta^1 \mapsto \zeta^1 + i\tau_1 z^2 \\ \zeta^2 \mapsto \zeta^2 + i\tau_1 z^1 \end{cases}$$

and, from $\Gamma^* = \mathrm{Exp}(\tau_0 Z_0) \circ \mathrm{Exp}(\tau_3 Z_3) \circ \mathrm{Exp}(\tau_2 Z_2) \circ \mathrm{Exp}(\tau_1 Z_1)$, we know that

$$\Gamma^* : \begin{cases} z^0 \mapsto z^0 + i\tau_0\zeta^0 \\ z^1 \mapsto z^1 \\ z^2 \mapsto z^2 \\ \zeta^0 \mapsto \zeta^0 + i\frac{\lambda}{2}\tau_0 z^0 \\ \zeta^1 \mapsto i\tau_3 z^1 + (i\tau_1 - \tau_2)z^2 + \zeta^1 \\ \zeta^2 \mapsto (i\tau_1 + \tau_2)z^1 - i\tau_3 z^2 + \zeta^2. \end{cases}$$

Putting $\Psi = \Gamma_{\mathbf{g}}^* \circ \Gamma^*$ we find

$$\Psi(\mathbf{z}) = \begin{pmatrix} \epsilon & 0 \\ 0 & \mathbf{a} \end{pmatrix} \mathbf{z} + \begin{pmatrix} \epsilon\rho_0 & 0 \\ 0 & 0 \end{pmatrix} \zeta,$$

$$\Psi(\zeta) = \begin{pmatrix} \frac{\lambda}{2}\epsilon\rho_0 & 0 \\ 0 & \mathbf{a}\boldsymbol{\rho} \end{pmatrix} \mathbf{z} + \begin{pmatrix} \epsilon & 0 \\ 0 & \mathbf{a} \end{pmatrix} \zeta,$$

where

$$\mathbf{z} = \begin{pmatrix} z^0 \\ z^1 \\ z^2 \end{pmatrix}, \ \zeta = \begin{pmatrix} \zeta^0 \\ \zeta^1 \\ \zeta^2 \end{pmatrix}, \ \rho_0 = i\tau_0, \ \boldsymbol{\rho} = \begin{pmatrix} i\tau_3 & i\tau_1 + \tau_2 \\ i\tau_1 - \tau_2 & -i\tau_3 \end{pmatrix}, \ \mathbf{a} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

In order to find the solution to the problem $\Psi'' = \Psi' \circ \Psi$ we use the one found in §3.2. We know that the solution is given by

$$\epsilon'' = \epsilon'\epsilon(1 - \frac{\lambda}{2}\rho_0'\rho_0) \qquad \mathbf{a}'' = \mathbf{a}'\mathbf{a}$$

$$\rho_0'' = \rho_0' + \rho_0 \qquad\qquad \boldsymbol{\rho}'' = \boldsymbol{\rho} + \mathbf{a}^{-1}\boldsymbol{\rho}'\mathbf{a}$$

and that the multiplication law for elements $(\mathbf{g}', \gamma') = \left(\begin{pmatrix} \epsilon' & 0 \\ 0 & \mathbf{a}' \end{pmatrix}, \begin{pmatrix} i\tau_0' & 0 \\ 0 & i\tau' \end{pmatrix} \right)$ and
$(\mathbf{g}, \gamma) = \left(\begin{pmatrix} \epsilon & 0 \\ 0 & \mathbf{a} \end{pmatrix}, \begin{pmatrix} i\tau_0 & 0 \\ 0 & i\tau \end{pmatrix} \right)$, is given by

$$(\mathbf{g}', \gamma') \cdot (\mathbf{g}, \gamma) = \left(\begin{pmatrix} \epsilon'\epsilon(1 - \frac{\lambda}{2} i\tau_0' i\tau_0) & 0 \\ 0 & \mathbf{a}'\mathbf{a} \end{pmatrix}, \begin{pmatrix} i\tau_0' + i\tau_0 & 0 \\ 0 & i\tau + \mathbf{a}^{-1} i\tau' \mathbf{a} \end{pmatrix} \right),$$

where $i\tau' = \begin{pmatrix} i\tau_3' & i\tau_1' + \tau_2' \\ i\tau_1' - \tau_2' & -i\tau_3' \end{pmatrix}$ and $i\tau = \begin{pmatrix} i\tau_3 & i\tau_1 + \tau_2 \\ i\tau_1 - \tau_2 & -i\tau_3 \end{pmatrix}$.

Once more, simple computations shows that the left-invariant supervector fields associated to the multiplication morphism (see §3.2)

$$\mathbf{m}^* x_{00} = p_1^* x_{00} p_2^* x_{00} - \frac{\lambda}{2} p_1^* x_{00} p_2^* x_{00} p_1^* \xi_{00} p_2^* \xi_{00},$$

$$\mathbf{m}^* x_{ij} = \sum_{k=1}^{2} p_1^* x_{ik} p_2^* x_{kj},$$

$$\mathbf{m}^* \xi_{00} = p_1^* \xi_{00} + p_2^* \xi_{00},$$

$$\mathbf{m}^* \xi_{ij} = p_2^* \xi_{ij} + \sum_{k,\ell=1}^{2} p_2^* (x^{-1})_{ik} p_1^* \xi_{k\ell} p_2^* x_{\ell j},$$

are

$$X_{00} = x_{00} \frac{\partial}{\partial x_{00}},$$

$$X_{ij} = \sum_{k=1}^{2} x_{ki} \frac{\partial}{\partial x_{kj}} + \xi_{ki} \frac{\partial}{\partial \xi_{kj}} - \xi_{jk} \frac{\partial}{\partial \xi_{ik}},$$

$$Y_{00} = \frac{\partial}{\partial \xi_{00}} - \frac{\lambda}{2} x_{00} \xi_{00} \frac{\partial}{\partial x_{00}},$$

$$Y_{ij} = \frac{\partial}{\partial \xi_{ij}}$$

and that they define a Lie superalgebra which belongs to the equivalence class of $[\lambda \neq 0, \mu = 0, \nu = 0]$.

5.3. Case $[\lambda = 0, \mu = 0, \nu \neq 0]$

In the case $[\lambda = 0, \mu = 0, \nu \neq 0]$ we know, from Proposition 5.2, that the integral flows for the even supervector fields are

$$\Gamma_{w_0}{}^* = \mathrm{Exp}(t_0 W_0) : \begin{cases} z^0 \mapsto z^0 \\ z^1 \mapsto e^{it_0} z^1 \\ z^2 \mapsto e^{it_0} z^2 \\ \zeta^0 \mapsto \zeta^0 \\ \zeta^1 \mapsto e^{it_0} \zeta^1 \\ \zeta^2 \mapsto e^{it_0} \zeta^2 \end{cases} \qquad \Gamma_{w_3}{}^* = \mathrm{Exp}(t_3 W_3) : \begin{cases} z^0 \mapsto z^0 \\ z^1 \mapsto e^{it_3} z^1 \\ z^2 \mapsto e^{-it_3} z^2 \\ \zeta^0 \mapsto \zeta^0 \\ \zeta^1 \mapsto e^{it_3} \zeta^1 \\ \zeta^2 \mapsto e^{-it_3} \zeta^2 \end{cases}$$

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$$\Gamma_{w_2}{}^* = \mathrm{Exp}(t_2 W_2) : \begin{cases} z^0 \mapsto z^0 \\ z^1 \mapsto \cos t_2 z^1 - \sin t_2 z^2 \\ z^2 \mapsto \sin t_2 z^1 + \cos t_2 z^2 \\ \zeta^0 \mapsto \zeta^0 \\ \zeta^1 \mapsto \cos t_2 \zeta^1 - \sin t_2 \zeta^2 \\ \zeta^2 \mapsto \sin t_2 \zeta^1 + \cos t_2 \zeta^2 \end{cases}$$

$$\Gamma_{w_1}{}^* = \mathrm{Exp}(t_1 W_1) : \begin{cases} z^0 \mapsto z^0 \\ z^1 \mapsto \cos t_1 z^1 + i \sin t_1 z^2 \\ z^2 \mapsto i \sin t_1 z^1 + \cos t_1 z^2 \\ \zeta^0 \mapsto \zeta^0 \\ \zeta^1 \mapsto \cos t_1 \zeta^1 + i \sin t_1 \zeta^2 \\ \zeta^2 \mapsto i \sin t_1 \zeta^1 + \cos t_1 \zeta^2. \end{cases}$$

Then, if $\Gamma_{\mathbf{g}}^* = \mathrm{Exp}(t_0 W_0) \circ \mathrm{Exp}(t_3 W_3) \circ \mathrm{Exp}(t_2 W_2) \circ \mathrm{Exp}(t_1 W_1)$, we have that

$$\Gamma_{\mathbf{g}}^* : \begin{cases} z^0 \mapsto z^0 \\ z^1 \mapsto \alpha z^1 + \gamma z^2 \\ z^2 \mapsto \beta z^1 + \delta z^2 \\ \zeta^0 \mapsto \zeta^0 \\ \zeta^1 \mapsto \alpha \zeta^1 + \gamma \zeta^2 \\ \zeta^2 \mapsto \beta \zeta^1 + \delta \zeta^2, \end{cases}$$

where

$$\alpha = (\cos t_1 \cos t_2 + i \sin t_1 \sin t_2)\, e^{i(t_0 + t_3)},$$
$$\beta = (i \sin t_1 \cos t_2 + \cos t_1 \sin t_2)\, e^{i(t_0 + t_3)},$$
$$\gamma = (-\cos t_1 \sin t_2 + i \sin t_1 \cos t_2)\, e^{i(t_0 - t_3)},$$
$$\delta = (\cos t_1 \cos t_2 - i \sin t_1 \sin t_2)\, e^{i(t_0 - t_3)},$$

as before. The integral flows for the odd supervector fields are

$$\Gamma_{Z_0}{}^* = \mathrm{Exp}(\tau_0 Z_0) : \begin{cases} z^0 \mapsto z^0 \\ z^1 \mapsto z^1 \\ z^2 \mapsto z^2 \\ \zeta^0 \mapsto \zeta^0 + i\,\tau_0 z^0 \\ \zeta^1 \mapsto \zeta^1 \\ \zeta^2 \mapsto \zeta^2 \end{cases} \qquad \Gamma_{Z_3}{}^* = \mathrm{Exp}(\tau_3 Z_3) : \begin{cases} z^0 \mapsto z^0 \\ z^1 \mapsto z^1 + i\nu\tau_3 \zeta^1 \\ z^2 \mapsto z^2 - i\nu\tau_3 \zeta^2 \\ \zeta^0 \mapsto \zeta^0 \\ \zeta^1 \mapsto \zeta^1 + i\tau_3 z^1 \\ \zeta^2 \mapsto \zeta^2 - i\tau_3 z^2 \end{cases}$$

$$\Gamma_{Z_2}{}^* = \mathrm{Exp}(\tau_2 Z_2) : \begin{cases} z^0 \mapsto z^0 \\ z^1 \mapsto z^1 - \nu\tau_2 \zeta^2 \\ z^2 \mapsto z^2 + \nu\tau_2 \zeta^1 \\ \zeta^0 \mapsto \zeta^0 \\ \zeta^1 \mapsto \zeta^1 - \tau_2 z^2 \\ \zeta^2 \mapsto \zeta^2 + \tau_2 z^1 \end{cases} \qquad \Gamma_{Z_1}{}^* = \mathrm{Exp}(\tau_1 Z_1) : \begin{cases} z^0 \mapsto z^0 \\ z^1 \mapsto z^1 + i\nu\tau_1 \zeta^2 \\ z^2 \mapsto z^2 + i\nu\tau_1 \zeta^1 \\ \zeta^0 \mapsto \zeta^0 \\ \zeta^1 \mapsto \zeta^1 + i\tau_1 z^2 \\ \zeta^2 \mapsto \zeta^2 + i\tau_1 z^1 \end{cases}$$

and computing $\Gamma^* = \text{Exp}(\tau_0 Z_0) \circ \text{Exp}(\tau_3 Z_3) \circ \text{Exp}(\tau_2 Z_2) \circ \text{Exp}(\tau_1 Z_1)$ on the local coordinates we get

$$\Gamma^* : \begin{cases}
z^0 \mapsto z^0 \\
z^1 \mapsto (1 - \nu\tau_2 i\tau_1)z^1 + \nu i\tau_3(i\tau_1 - \tau_2)z^2 \\
\qquad + \nu i\tau_3(1 - \nu\tau_2 i\tau_1)\zeta^1 + \nu(i\tau_1 - \tau_2)\zeta^2 \\
z^2 \mapsto -\nu i\tau_3(i\tau_1 + \tau_2)z^1 + (1 + \nu\tau_2 i\tau_1)z^2 \\
\qquad + \nu(i\tau_1 + \tau_2)\zeta^1 - \nu i\tau_3(1 + \nu\tau_2 i\tau_1)\zeta^2 \\
\zeta^0 \mapsto \zeta^0 + i\tau_0 z^0 \\
\zeta^1 \mapsto i\tau_3(1 - \nu\tau_2 i\tau_1)z^1 + (i\tau_1 - \tau_2)z^2 \\
\qquad + (1 - \nu\tau_2 i\tau_1)\zeta^1 + \nu i\tau_3(i\tau_1 - \tau_2)\zeta^2 \\
\zeta^2 \mapsto (i\tau_1 + \tau_2)z^1 - i\tau_3(1 - \nu\tau_2 i\tau_1)z^2 \\
\qquad - \nu i\tau_3(i\tau_1 + \tau_2)\zeta^1 + (1 - \nu\tau_2 i\tau_1)\zeta^2.
\end{cases}$$

Putting $\Psi = \Gamma_{\mathbf{g}}^* \circ \Gamma^*$ we finally obtain:

$$\Psi : \begin{cases}
z^0 \mapsto & z^0 \\
z^1 \mapsto & [\alpha(1 - \nu\tau_2 i\tau_1) + \nu\beta i\tau_3(i\tau_1 - \tau_2)]z^1 \\
& + [\gamma(1 - \nu\tau_2 i\tau_1) + \nu\delta i\tau_3(i\tau_1 - \tau_2)]z^2 \\
& + [\nu\alpha i\tau_3(1 - \nu\tau_2 i\tau_1) + \nu\beta(i\tau_1 - \tau_2)]\zeta^1 \\
& + [\nu\gamma i\tau_3(1 - \nu\tau_2 i\tau_1) + \nu\delta(i\tau_1 - \tau_2)]\zeta^2 \\
z^2 \mapsto & [-\nu\alpha i\tau_3(i\tau_1 + \tau_2) + \beta(1 + \nu\tau_2 i\tau_1)]z^1 \\
& + [-\nu\gamma i\tau_3(i\tau_1 + \tau_2) + \delta(1 + \nu\tau_2 i\tau_1)]z^2 \\
& + [\nu\alpha(i\tau_1 + \tau_2) - \nu\beta i\tau_3(1 + \nu\tau_2 i\tau_1)]\zeta^1 \\
& + [\nu\gamma(i\tau_1 + \tau_2) - \nu\delta i\tau_3(1 + \nu\tau_2 i\tau_1)]\zeta^2 \\
\zeta^0 \mapsto & \zeta^0 + i\tau_0 z^0 \\
\zeta^1 \mapsto & [\alpha i\tau_3(1 - \nu\tau_2 i\tau_1) + \beta(i\tau_1 - \tau_2)]z^1 \\
& + [\gamma i\tau_3(1 - \nu\tau_2 i\tau_1) + \delta(i\tau_1 - \tau_2)]z^2 \\
& + [\alpha(1 - \nu\tau_2 i\tau_1) + \nu\beta i\tau_3(i\tau_1 - \tau_2)]\zeta^1 \\
& + [\gamma(1 - \nu\tau_2 i\tau_1) + \nu\delta i\tau_3(i\tau_1 - \tau_2)]\zeta^2 \\
\zeta^2 \mapsto & [\alpha(i\tau_1 + \tau_2) - \beta i\tau_3(1 - \nu\tau_2 i\tau_1)]z^1 \\
& + [\gamma(i\tau_1 + \tau_2) - \delta i\tau_3(1 - \nu\tau_2 i\tau_1)]z^2 \\
& + [-\nu\alpha i\tau_3(i\tau_1 + \tau_2) + \beta(1 - \nu\tau_2 i\tau_1)]\zeta^1 \\
& + [-\nu\gamma i\tau_3(i\tau_1 + \tau_2) + \delta(1 - \nu\tau_2 i\tau_1)]\zeta^2.
\end{cases}$$

Defining

$$\mathbf{p} = \begin{pmatrix} 1 - \nu\tau_2 i\tau_1 & -\nu i\tau_3(i\tau_1 + \tau_2) \\ \nu i\tau_3(i\tau_1 - \tau_2) & 1 + \nu\tau_2 i\tau_1 \end{pmatrix} \quad \text{and} \quad \mathbf{q} = \begin{pmatrix} i\tau_3(1 - \nu\tau_2 i\tau_1) & i\tau_1 + \tau_2 \\ i\tau_1 - \tau_2 & -i\tau_3(1 - \nu\tau_2 i\tau_1) \end{pmatrix}$$

we have

$$\Psi(\mathbf{z}) = \begin{pmatrix} 1 & 0 \\ 0 & \mathbf{g}\,\mathbf{p} \end{pmatrix} \mathbf{z} + \begin{pmatrix} 0 & 0 \\ 0 & \nu\mathbf{g}\,\mathbf{q} \end{pmatrix} \zeta,$$

$$\Psi(\zeta) = \begin{pmatrix} i\tau_0 & 0 \\ 0 & \mathbf{g}\,\mathbf{q} \end{pmatrix} \mathbf{z} + \begin{pmatrix} 1 & 0 \\ 0 & \mathbf{g}\,\mathbf{p} \end{pmatrix} \zeta,$$

where $\mathbf{z} = \begin{pmatrix} z^0 \\ z^1 \\ z^2 \end{pmatrix}$ and $\zeta = \begin{pmatrix} \zeta^0 \\ \zeta^1 \\ \zeta^2 \end{pmatrix}$, but this problem is similar to the one found in §3.3, and then it is so difficult like the one found there.

From the classification in §5 we know that

$$\Gamma(w_0, w_0) = i\lambda w_0 \,, \quad \Gamma(w_0, w_3) = i\mu w_3 \,, \quad \Gamma(w_3, w_3) = 2i\nu w_0$$

and we have realizations in supervector fields in $\mathbb{R}^{2|2}$ and $\mathbb{R}^{3|3}$ supermanifolds given by the appropriate restrictions. We now want to compute a general composition law in terms of the arbitrary parameter values $[\lambda, \mu, \nu]$.

6.1 Proposition. *Lie superalgebras in the equivalence classes $[\lambda, \mu, \nu]$ admit an explicit realization in terms of supervector fields in the supermanifold $\mathbb{R}^{2|2}$ with local coordinates $\{z^1, z^2 ; \zeta^1, \zeta^2\}$ given by,*

$$W_0 = i\left(z^1 \frac{\partial}{\partial z^1} + z^2 \frac{\partial}{\partial z^2} + \zeta^1 \frac{\partial}{\partial \zeta^1} + \zeta^2 \frac{\partial}{\partial \zeta^2}\right)$$

$$W_3 = i\left(z^1 \frac{\partial}{\partial z^1} - z^2 \frac{\partial}{\partial z^2} + \zeta^1 \frac{\partial}{\partial \zeta^1} - \zeta^2 \frac{\partial}{\partial \zeta^2}\right)$$

$$Z_0 = ik\left(\zeta^1 \frac{\partial}{\partial z^1} + \zeta^2 \frac{\partial}{\partial z^2}\right) + ig\left(z^1 \frac{\partial}{\partial \zeta^1} + z^2 \frac{\partial}{\partial \zeta^2}\right)$$

$$Z_3 = ie\left(\zeta^1 \frac{\partial}{\partial z^1} - \zeta^2 \frac{\partial}{\partial z^2}\right) + id\left(z^1 \frac{\partial}{\partial \zeta^1} - z^2 \frac{\partial}{\partial \zeta^2}\right),$$

where $\lambda = 2gk$, $\mu = eg + dk$ and $\nu = ed$.

The integral flows for the even supervector fields in the Proposition above are

$$\Gamma_{w_0}{}^* = \mathrm{Exp}(t_0 W_0) : \begin{cases} z^1 \mapsto e^{it_0} z^1 \\ z^2 \mapsto e^{it_0} z^2 \\ \zeta^1 \mapsto e^{it_0} \zeta^1 \\ \zeta^2 \mapsto e^{it_0} \zeta^2 \end{cases} \quad \Gamma_{w_3}{}^* = \mathrm{Exp}(t_3 W_3) : \begin{cases} z^1 \mapsto e^{it_3} z^1 \\ z^2 \mapsto e^{-it_3} z^2 \\ \zeta^1 \mapsto e^{it_3} \zeta^1 \\ \zeta^2 \mapsto e^{-it_3} \zeta^2, \end{cases}$$

whereas for the odd supervector fields we have

$$\Gamma_{Z_0}{}^* = \mathrm{Exp}(\tau_0 Z_0) : \begin{cases} z^1 \mapsto z^1 + ik\tau_0 \zeta^1 \\ z^2 \mapsto z^2 + ik\tau_0 \zeta^2 \\ \zeta^1 \mapsto \zeta^1 + ig\tau_0 z^1 \\ \zeta^2 \mapsto \zeta^2 + ig\tau_0 z^2 \end{cases} \quad \Gamma_{Z_3}{}^* = \mathrm{Exp}(\tau_3 Z_3) : \begin{cases} z^1 \mapsto z^1 + ie\tau_3 \zeta^1 \\ z^2 \mapsto z^2 - ie\tau_3 \zeta^2 \\ \zeta^1 \mapsto \zeta^1 + id\tau_3 z^1 \\ \zeta^2 \mapsto \zeta^2 - id\tau_3 z^2. \end{cases}$$

Note that

$$\Gamma_g^* = \Gamma_{w_0}^* \circ \Gamma_{w_3}^* : \begin{cases} z^1 \mapsto e^{i(t_0 + t_3)} z^1 \\ z^2 \mapsto e^{i(t_0 - t_3)} z^2 \\ \zeta^1 \mapsto e^{i(t_0 + t_3)} \zeta^1 \\ \zeta^2 \mapsto e^{i(t_0 - t_3)} \zeta^2. \end{cases}$$

By setting $\alpha = e^{i(t_0 + t_3)}$ and $\delta = e^{i(t_0 - t_3)}$ we have a correspondence

$$\Gamma_g^* \leftrightarrow \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}.$$

Furthermore,

$$\Gamma^*_{Z_0} \circ \Gamma^*_{Z_3} : \begin{cases} z^1 \mapsto (1 - egi\tau_0 i\tau_3)z^1 + (ki\tau_0 + ei\tau_3)\zeta^1 \\ z^2 \mapsto (1 + egi\tau_0 i\tau_3)z^2 + (ki\tau_0 - ei\tau_3)\zeta^2 \\ \zeta^1 \mapsto (gi\tau_0 + di\tau_3)z^1 + (1 - dki\tau_0 i\tau_3)\zeta^1 \\ \zeta^2 \mapsto (gi\tau_0 - di\tau_3)z^2 + (1 + dki\tau_0 i\tau_3)\zeta^2 \end{cases}$$

and, by choosing $\Psi = \Gamma^*_g \circ \Gamma^*_{Z_0} \circ \Gamma^*_{Z_3}$, we can write

$$\Psi(z) = g(\mathbb{1} - eg\rho_0\rho_3)z + g(k\rho_0 + e\rho_3)\zeta,$$
$$\Psi(\zeta) = g(g\rho_0 + d\rho_3)z + g(\mathbb{1} - dk\rho_0\rho_3)\zeta,$$

where

$$\rho_0 = \begin{pmatrix} i\tau_0 & 0 \\ 0 & i\tau_0 \end{pmatrix}, \rho_3 = \begin{pmatrix} i\tau_3 & 0 \\ 0 & -i\tau_3 \end{pmatrix}, z = \begin{pmatrix} z^1 \\ z^2 \end{pmatrix}, \zeta = \begin{pmatrix} \zeta^1 \\ \zeta^2 \end{pmatrix}.$$

From $\Psi'' = \Psi' \circ \Psi$ find the following:

(1)
$$g''(\mathbb{1} - eg\rho''_0\rho''_3) = g'(\mathbb{1} - eg\rho'_0\rho'_3)\,g(\mathbb{1} - eg\rho_0\rho_3) - g'(g\rho'_0 + d\rho'_3)\,g(k\rho_0 + e\rho_3)$$
(2)
$$g''(k\rho''_0 + e\rho''_3) = g'(k\rho'_0 + e\rho'_3)\,g(\mathbb{1} - eg\rho_0\rho_3) + g'(\mathbb{1} - dk\rho'_0\rho'_3)\,g(k\rho_0 + e\rho_3)$$
(3)
$$g''(g\rho''_0 + d\rho''_3) = g'(\mathbb{1} - eg\rho'_0\rho'_3)\,g(g\rho_0 + d\rho_3) + g'(g\rho'_0 + d\rho'_3)\,g(\mathbb{1} - dk\rho_0\rho_3)$$
(4)
$$g''(\mathbb{1} - dk\rho''_0\rho''_3) = g'(\mathbb{1} - dk\rho'_0\rho'_3)\,g(\mathbb{1} - dk\rho_0\rho_3) - g'(k\rho'_0 + e\rho'_3)\,g(g\rho_0 + d\rho_3).$$

Now, from (1), we have

(5)
$$g'' = \{g'(\mathbb{1} - eg\rho'_0\rho'_3)\,g(\mathbb{1} - eg\rho_0\rho_3) - g'(g\rho'_0 + d\rho'_3)\,g(k\rho_0 + e\rho_3)\}\,(\mathbb{1} + eg\rho''_0\rho''_3).$$

Using this result in (2) and (3), we get

$$k\rho''_0 + e\rho''_3 = \{g'(\mathbb{1} - eg\rho'_0\rho'_3)\,g(\mathbb{1} - eg\rho_0\rho_3) - g'(g\rho'_0 + d\rho'_3)\,g(k\rho_0 + e\rho_3)\}^{-1} \cdot$$
$$\{g'(k\rho'_0 + e\rho'_3)\,g(\mathbb{1} - eg\rho_0\rho_3) + g'(\mathbb{1} - dk\rho'_0\rho'_3)\,g(k\rho_0 + e\rho_3)\}$$
$$g\rho''_0 + d\rho''_3 = \{g'(\mathbb{1} - eg\rho'_0\rho'_3)\,g(\mathbb{1} - eg\rho_0\rho_3) - g'(g\rho'_0 + d\rho'_3)\,g(k\rho_0 + e\rho_3)\}^{-1} \cdot$$
$$\{g'(\mathbb{1} - eg\rho'_0\rho'_3)\,g(g\rho_0 + d\rho_3) + g'(g\rho'_0 + d\rho'_3)\,g(\mathbb{1} - dk\rho_0\rho_3)\}.$$

But, from fact that $\rho_0\rho_3 = -\rho_3\rho_0$ and g commutes with all elements, we have that

$$\{g'(\mathbb{1} - eg\rho'_0\rho'_3)\,g(\mathbb{1} - eg\rho_0\rho_3) - g'(g\rho'_0 + d\rho'_3)\,g(k\rho_0 + e\rho_3)\}^{-1} =$$
$$(g'\,g)^{-1}\{(\mathbb{1} + eg\rho'_0\rho'_3)(\mathbb{1} + eg\rho_0\rho_3) + (g\rho'_0 + d\rho'_3)(k\rho_0 + e\rho_3)\}$$

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and then,

$$kp_0'' + e\rho_3'' = k(\rho_0' + \rho_0) + e(\rho_3' + \rho_3),$$
$$g\rho_0'' + d\rho_3'' = g(\rho_0' + \rho_0) + d(\rho_3' + \rho_3),$$

which implies

$$\rho_0'' = \rho_0' + \rho_0 \qquad \text{and} \qquad \rho_3'' = \rho_3' + \rho_3.$$

It follows, from (5), that

$$\mathbf{g}'' = \mathbf{g}' \, \mathbf{g}\{(\mathbb{1} - gk\rho_0'\rho_0)(\mathbb{1} - ed\rho_3'\rho_3) + (eg + dk)\rho_0\rho_3'\}.$$

Simple computations show that (4) is satisfied with the results above. We can then write the multiplication law for the elements $(\mathbf{g}', i\tau_0', i\tau_3')$ and $(\mathbf{g}, i\tau_0, i\tau_3)$ as

$$\left(\mathbf{g}' \, \mathbf{g} \left\{ (\mathbb{1} - \frac{\lambda}{2} i\tau_0' i\tau_0)(\mathbb{1} - \nu i\tau_3' i\tau_3) + \mu i\tau_0' i\tau_3 \right\}, i\tau_0' + i\tau_0, i\tau_3' + i\tau_3 \right).$$

This multiplication law exhibits the λ, μ, ν parameters in general. In order to find the left-invariant supervector fields, we use the techniques from §4 defining the projection maps $x_{11}(\mathbf{g}, i\tau_0, i\tau_3) = \mathbf{g}_{11}$, $x_{22}(\mathbf{g}, i\tau_0, i\tau_3) = \mathbf{g}_{22}$, $\xi_{11}(\mathbf{g}, i\tau_0, i\tau_3) = i\tau_0$ and $\xi_{22}(\mathbf{g}, i\tau_0, i\tau_3) = i\tau_3$. In terms of them, the multiplication morphism is given by

$$m^* x_{jj} = p_1^* x_{jj} p_2^* x_{jj} \left\{ (1 - \frac{\lambda}{2} p_1^* \xi_{11} p_2^* \xi_{11})(1 - \nu p_1^* \xi_{22} p_2^* \xi_{22}) + (-1)^{j+1} \mu p_2^* \xi_{11} p_1^* \xi_{22} \right\}$$
$$m^* \xi_{jj} = p_1^* \xi_{jj} + p_2^* \xi_{jj}.$$

The identity-element morphism ε satisfying $m \circ (\varepsilon, id) = id = m \circ (id, \varepsilon)$ is given by $\varepsilon^*(x_{jj}) = 1$ and $\varepsilon^*(\xi_{jj}) = 0$. With these results we find that left-invariant supervector fields are

$$X_1 = x_{11} \frac{\partial}{\partial x_{11}},$$
$$X_2 = x_{22} \frac{\partial}{\partial x_{22}},$$
$$Y_1 = \frac{\partial}{\partial \xi_{11}} + \sum_{j=1}^{2} x_{jj} \left(\frac{\lambda}{2} \xi_{11} + (-1)^{j+1} \mu \xi_{22} \right) \frac{\partial}{\partial x_{jj}},$$
$$Y_2 = \frac{\partial}{\partial \xi_{22}} + \nu \sum_{j=1}^{2} x_{jj} \xi_{22} \frac{\partial}{\partial x_{jj}}.$$

Putting $w_0 = X_1 + X_2$, $w_3 = X_1 - X_2$, $z_0 = Y_1$ and $z_3 = Y_2$ we can find the $[\lambda, \mu, \nu]$ equivalence class they belong to.

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In the last section we worked with the maximal torus associated to the classification in §5. Thus, by choosing the Abelian Lie superalgebra generated by I and H as even Lie subalgebra and $\pi(I)$ and $\pi(H)$ as odd generators, we found the corresponding maximal torus inside the ten real Lie superalgebra structures on $\mathfrak{gl}_2$ obtained by restriction to the real forms $\mathfrak{u}_2$. However, there is a related problem to understand within the spirit that has guided us throughout this work: Namely, *to classify all Lie superalgebras whose underlying 2-dimensional Lie algebra is Abelian under the assumption that the action of the even Lie algebra into the odd module is given via the adjoint representation.* These Lie superalgebras are classified by symmetric bilinear maps $\Gamma : \mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathfrak{g}_0$ with no restrictions, since the Jacobi identities are trivially satisfied.

Let $\mathfrak{g}_0 = \mathrm{Span}_{\mathbb{R}}\{w_1, w_2\}$ be the Abelian 2-dimensional Lie algebra and let $\mathfrak{g}_1 = \{\pi w_1, \pi w_2\}$ be the $\mathfrak{g}_0$-module defined by the adjoint representation. Then

$$\Gamma(\pi w_j, \pi w_j) = \theta_{ij}^1 w_1 + \theta_{ij}^2 w_2$$

defines a Lie superalgebra structure for arbitrary parameters θ_{ij}^k in $\mathbb{R}$. A different symmetric bilinear map $\Gamma' : \mathfrak{g}_1' \times \mathfrak{g}_1' \to \mathfrak{g}_0'$ would yield a different set of parameters $(\theta')_{ij}^k$. The Lie superalgebras generated by θ^k and $(\theta')^k$ will be isomorphic if and only if there is a Lie algebra isomorphism $T : \mathfrak{g}_0 \to \mathfrak{g}_0'$ of the Abelian Lie algebra (actually, any linear isomorphism $T \in \mathrm{GL}_2$ will do it) and a linear isomorphism $S : \mathfrak{g}_1 \to \mathfrak{g}_1'$ such that $\Gamma'(S(x), S(y)) = T(\Gamma(x, y))$ for any $x, y \in \mathfrak{g}_1$. This condition can be written in terms of matrices as

$$S^{\mathrm{t}} (\theta')^1 S = T_{11}\theta^1 + T_{12}\theta^2,$$
$$S^{\mathrm{t}} (\theta')^2 S = T_{21}\theta^1 + T_{22}\theta^2.$$

Therefore, we can approach the corresponding classification problem, whose solution is stated in the following proposition. Its corollary, on the other hand, shows what the relationship is between the maximal tori found in the last section and the supertori given by the classification problem just posed.

Proposition. *The group* $\mathrm{GL}_2(\mathbb{R}) \times \mathrm{GL}_2(\mathbb{R})$ *acts on the left of* $\mathrm{Sym}_2(\mathbb{R}) \times \mathrm{Sym}_2(\mathbb{R})$ *via*

$$(T, S) \cdot (\theta^1, \theta^2) = \left(T_{11}S^{-1} \cdot \theta^1 + T_{12}S^{-1} \cdot \theta^2, \; T_{21}S^{-1} \cdot \theta^1 + T_{22}S^{-1} \cdot \theta^2 \right),$$

where $\mathrm{Sym}_2(\mathbb{R})$ *is the set of symmetric* 2×2 *matrices over* $\mathbb{R}$ *and* $S^{-1} \cdot \theta = S^{-1}\theta(S^{-1})^t$ *is the natural left action of* $\mathrm{GL}_2(\mathbb{R})$ *on* $\mathrm{Sym}_2(\mathbb{R})$. *This action defines seven different orbits whose representatives* $\widetilde{\theta^1}$ *and* $\widetilde{\theta^2}$ *are listed in the following table*

Type	$\widetilde{\theta^1}$	$\widetilde{\theta^2}$
1	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
2	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
3	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
4	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$
5	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
6	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
7	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$

Corollary. *There is a surjection from maximal tori in §6 , onto the tori obtained from the action just defined.*

Proof of corollary. For real cases in λ, μ and ν, we know that $\theta^1 = \begin{pmatrix} \lambda & 0 \\ 0 & \nu \end{pmatrix}$ and $\theta^2 = \begin{pmatrix} 0 & \mu \\ \mu & 0 \end{pmatrix}$. We can see these cases in terms of the above Type as follows:

Type	$[\lambda, \mu, \nu]$
1	$[0, 0, 0]$
2	$[0, 0, 1]$, $[1, 0, 0]$
3	$[1, 0, 1]$
4	$[1, 1, 1]$
5	$[1, 0, -1]$, $[0, 1, 0]$
6	$[1, 1, -1]$
7	$[0, 1, 1]$, $[1, 1, 0]$.

Proof of proposition. Let us explain what the philosophy of the proof is. By means of the action $(\theta^1, \theta^2) \mapsto (S^{-1}\theta^1 (S^{-1})^t , S^{-1}\theta^2 (S^{-1})^t)$, we try first to see under what conditions can both $S^{-1}\theta^1 (S^{-1})^t$ and $S^{-1}\theta^2 (S^{-1})^t$ be brought to a diagonal form. Once they are both diagonal, we can further act with an appropriate group element $T \in \mathrm{GL}_2(\mathbb{R})$ so as to simplify each $\widetilde{\theta^i} = T_{i1} S^{-1}\theta^1 (S^{-1})^t + T_{i2} S^{-1}\theta^2 (S^{-1})^t$ $(i = 1, 2)$ as much as possible. There are some cases in which it is impossible to simultaneously have $S^{-1}\theta^1 (S^{-1})^t$ and $S^{-1}\theta^2 (S^{-1})^t$ in diagonal form. These cases are then treated separately. At the end, one only needs to check that with the chosen representatives one really reaches any pair of symmetric matrices under the given $\mathrm{GL}_2(\mathbb{R}) \times \mathrm{GL}_2(\mathbb{R})$-action and that the representatives really belong to different orbits.

There are a few simple cases where one immediately knows that both, θ^1 and θ^2, can be simultaneously diagonalized. Say, if from the outset, θ^1 is proportional to θ^2, then both can be diagonalized at once with the same $S \in \mathrm{GL}_2(\mathbb{R})$. If this is the

case (say $\theta^2 = a\theta^1$, with $a \neq 0$), several subcases have to be considered: Namely, either θ^1 is positive definite; or θ^1 is negative definite; or θ^1 is nondegenerate but nondefinite; or θ^1 has rank-one with a positive eigenvalue; or θ^1 has rank-one with a negative eigenvalue; or θ^1 is identically zero.

In all these cases, by choosing an appropriate $T \in \mathrm{GL}_2(\mathbb{R})$ one can easily see that if the eigenvalues of θ^1 either have equal signs, or one of them is zero, then $\widetilde{\theta}^1 = T_{11}S^{-1}\theta^1(S^{-1})^t + T_{12}S^{-1}\theta^2(S^{-1})^t$ can be chosen so as to be either the identity matrix, or the diagonal matrix with diagonal entries $(1,0)$ if θ^1 was rank-one, or diagonal entries $(0,0)$ if θ^1 was identically zero. In any case, the choice of T can also be adjusted so as to have $\widetilde{\theta}^2 = T_{21}S^{-1}\theta^1(S^{-1})^t + T_{22}S^{-1}\theta^2(S^{-1})^t$ identically zero. This accounts for the first three types in the statement of the Proposition, plus Type 5.

There are other less obvious cases where one can simultaneously diagonalize θ^1 and θ^2: Namely, we use the well-known result that this is the case, provided one of the two bilinear forms —say, θ^1— is invertible and the product $(\theta^1)^{-1}\theta^2$ is diagonalizable (see for example, [Horn, R. and Johnson, C., *Matrix Analysis*, pp. 228-234]).

So, if θ^1 and θ^2 are not proportional to each other and θ^1 is positive definite, then an appropriate choice of S will bring $S^{-1}\theta^1(S^{-1})^t$ into diagonal form with diagonal entries $(1,1)$. Whence, the identity matrix. On the other hand, regardless of what form $S^{-1}\theta^2(S^{-1})^t$ might have achieved with this choice of S, it is still a symmetric matrix and hence diagonalizable. Actually, by means of a rotation $S = \begin{pmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{pmatrix}$, which is an element of the isotropy group at $S^{-1}\theta^1(S^{-1})^t = \mathrm{diag}(1,1)$, we can bring $S^{-1}\theta^2(S^{-1})^t$ into diagonal form which, under the assumption that θ^1 and θ^2 were not proportional at the outset, have different diagonal entries. Therefore, the theorem we have just quoted applies and we can see that the new diagonal entries of the matrices $\widetilde{\theta}^i = T_{i1}S^{-1}\theta^1(S^{-1})^t + T_{i2}S^{-1}\theta^2(S^{-1})^t$ ($i = 1, 2$) can be chosen so that the product of T with the matrix M whose columns are the diagonal entries $(1,1)$ of $S^{-1}\theta^1(S^{-1})^t$ and (a,d) of $S^{-1}\theta^2(S^{-1})^t$, is equal to the identity matrix. Whence, the representative pair for this orbit is that listed under Type 4 in the statement. Besides, it is easy to see that the same argument applies if θ^1 was negative definite, since the isotropy group is still the same in this case.

The case that remains to be analyzed is that when θ^1 is nondegenerate, but nondefinite and θ^2 was not proportional to θ^1. With an appropriate $S \in \mathrm{GL}_2(\mathbb{R})$ we may assume that $S^{-1}\theta^1(S^{-1})^t$ is diagonal with diagonal entries $(1, -1)$. The isotropy group of this element is formed by the matrices of the Lorentz group and, by choosing $S = \begin{pmatrix} \cosh\omega & -\sinh\omega \\ -\sinh\omega & \cosh\omega \end{pmatrix}$, it is easy to see that $S^{-1}\theta^2(S^{-1})^t$ will be diagonalizable by means of such a Lorentz transformation if and only if $\tanh 2\omega = \frac{2b}{a+c}$, where we originally had $S^{-1}\theta^2(S^{-1})^t = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$. This will obviously be the case if and only if the absolute value of $\frac{2b}{a+c}$ is strictly less than 1. But if this condition is fulfilled, then a T can be chosen as in the previous paragraph and therefore fall into Type 4.

Problems in the Lorentz-transformation argument arise when the absolute value of $\frac{2b}{a+c}$ is either strictly bigger than 1, or exactly equal to 1. In the first case we have a typical situation of two symmetric matrices that cannot be simultaneously diagonalized, but still have the chance of bringing the pair $S^{-1}\theta^1(S^{-1})^t$ and $S^{-1}\theta^2(S^{-1})^t$

into the representatives given in Type 6 of the statement. The condition that is definitely different, on the other hand, is that when $\frac{2b}{a+c}$ is equal to either $+1$ or to -1. In this case, $S^{-1}\theta^1(S^{-1})^t$ and $S^{-1}\theta^2(S^{-1})^t$ can only be brought into the representatives given in Type 7 of the statement.

LIE SUPERGROUPS AND LIE SUPERALGEBRAS

A.1 Supervector spaces. Let $\mathbb{F}$ be $\mathbb{R}$ or $\mathbb{C}$. A supervector space V over $\mathbb{F}$ is a vector space over $\mathbb{F}$, together with a direct sum decomposition $V = V_0 \oplus V_1$ and a *parity function* $|\cdot| : V_0 - \{0\} \cup V_1 - \{0\} \to \mathbb{Z}_2 = \{0, 1\}$, in such way that $|v| = \mu$ whenever $v \in V_\mu - \{0\}$. The supervector space $V = V_0 \oplus V_1$ over $\mathbb{F}$ is also called a *supervector space* when the field is clear. Elements in $V_0 - \{0\}$ are called *even* and those in $V_1 - \{0\}$ are called *odd*. The even and odd elements are also called *homogeneous*. The supervector space $V = V_0 \oplus V_1$ is finite-dimensional of dimension (m, n) if $\dim V_0 = m$ and $\dim V_1 = n$ (see [8], [10], [11]).

A.2 Associative superalgebras. An associative superalgebra A over $\mathbb{F}$ is an associative $\mathbb{F}$–algebra having the structure of a supervector space over $\mathbb{F}$, $A = A_0 \oplus A_1$, and such that its multiplication map $m_A : A \times A \to A$ satisfies $m_A(A_\mu, A_\nu) \subset A_{\mu+\nu}$ for all μ, ν in $\mathbb{Z}_2$. In particular, the unit element 1_A lies in the subspace A_0. An associative superalgebra A is called $\mathbb{Z}_2$-*commutative* if $m_A(a, b) = (-1)^{|a||b|} m_A(b, a)$, for a and b homogeneous. A morphism between the associative superalgebras A and B is an $\mathbb{F}$–linear map $T : A \to B$ of the underlying vector spaces such that: (1) $T(A_\mu) \subset B_\mu$ for $\mu = 0, 1$, (2) $T(m_A(x, y)) = m_B(T(x), T(y))$ and (3) $T(1_A) = 1_B$ (see [8], [10], [11]).

A.3 Lie superalgebras. A Lie superalgebra $\mathfrak{g}$ over $\mathbb{F}$ is a supervector space $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with an $\mathbb{F}$-bilinear map $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying: $[x, y] = -(-1)^{|x||y|}[y, x]$ and

$$(-1)^{|x||z|}[x, [y, z]] + (-1)^{|z||y|}[z, [x, y]] + (-1)^{|y||x|}[y, [z, x]] = 0,$$

for any homogeneous x, y and z. An $\mathbb{F}$-bilinear map $[\cdot, \cdot]$ as above is also called *Lie bracket*. A morphism between the Lie superalgebras $\mathfrak{g}$ and $\mathfrak{h}$ is a morphism $T : \mathfrak{g} \to \mathfrak{h}$ of vector spaces such that: (1) $T(\mathfrak{g}_\mu) \subset \mathfrak{h}_\mu$ for $\mu = 0, 1$ and (2) $T[x, y]_\mathfrak{g} = [T(x), T(y)]_\mathfrak{h}$ (see [8], [16]).

Remark. Any associative superalgebra A gives rise to a Lie superalgebra by defining the Lie bracket $[\cdot, \cdot] : A \times A \to A$ on homogeneous elements a and b, by letting $[a, b] = m_A(a, b) - (-1)^{|a||b|} m_A(b, a)$ and extending it bilinearly.

A.4 Example. If $V = V_0 \oplus V_1$ is a supervector space over $\mathbb{F}$ then $\mathrm{End}_\mathbb{F} V$, the space of $\mathbb{F}$-linear maps $T : V \to V$, has a natural associative superalgebra structure for the multiplication map $m_{\mathrm{End}_\mathbb{F} V}(T, S) = T \circ S$, with respect to the $\mathbb{Z}_2$–grading,

$$(\mathrm{End}_\mathbb{F} V)_0 = \{T \in \mathrm{End}_\mathbb{F} V \,|\, T(V_0) \subset V_0 \text{ and } T(V_1) \subset V_1\},$$
$$(\mathrm{End}_\mathbb{F} V)_1 = \{T \in \mathrm{End}_\mathbb{F} V \,|\, T(V_0) \subset V_1 \text{ and } T(V_1) \subset V_0\}.$$

Therefore, $\mathrm{End}_\mathbb{F} V$ has a Lie superalgebra structure $[\cdot, \cdot] : \mathrm{End}_\mathbb{F} V \times \mathrm{End}_\mathbb{F} V \to \mathrm{End}_\mathbb{F} V$ defined on homogeneous elements by $[S, T] = S \circ T - (-1)^{|S||T|} T \circ S$ and extending it $\mathbb{F}$-bilinearly to $\mathrm{End}_\mathbb{F} V$.

By choosing a basis of $V = V_0 \oplus V_1$ ordered in such a way that the first $\dim V_0 = m$ elements of it form a basis of V_0 and the last $\dim V_1 = n$ form a basis of V_1, the

Lie superalgebra $\operatorname{End}_{\mathbb{F}} V$ can be realized in terms of $(m+n, m+n)$ matrices with entries in $\mathbb{F}$, so that

$$(\operatorname{End}_{\mathbb{F}} V)_0 \simeq \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \;\middle|\; \begin{array}{l} A \in \mathfrak{gl}_m(\mathbb{F}) \\ D \in \mathfrak{gl}_n(\mathbb{F}) \end{array} \right\}$$

$$(\operatorname{End}_{\mathbb{F}} V)_1 \simeq \left\{ \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \;\middle|\; \begin{array}{l} B \in \mathfrak{mat}_{m \times n}(\mathbb{F}) \\ C \in \mathfrak{mat}_{n \times m}(\mathbb{F}) \end{array} \right\}$$

and the Lie superalgebra structure is given by

$$\left[\begin{pmatrix} A_1 & 0 \\ 0 & D_1 \end{pmatrix}, \begin{pmatrix} A_2 & 0 \\ 0 & D_2 \end{pmatrix} \right] = \begin{pmatrix} A_1 A_2 - A_2 A_1 & 0 \\ 0 & D_1 D_2 - D_2 D_1 \end{pmatrix}$$

$$\left[\begin{pmatrix} A_1 & 0 \\ 0 & D_1 \end{pmatrix}, \begin{pmatrix} 0 & B_2 \\ C_2 & 0 \end{pmatrix} \right] = \begin{pmatrix} 0 & A_1 B_2 - B_2 D_1 \\ D_1 C_2 - C_2 A_1 & 0 \end{pmatrix}$$

$$\left[\begin{pmatrix} 0 & B_1 \\ C_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & B_2 \\ C_2 & 0 \end{pmatrix} \right] = \begin{pmatrix} B_1 C_2 + B_2 C_1 & 0 \\ 0 & C_1 B_2 + C_2 B_1 \end{pmatrix}.$$

Remark. The definition of a Lie superalgebra can also be rephrased as follows: Let V be a vector space and $\mathfrak{h}$ a Lie algebra. Given a representation of Lie algebras $\rho : \mathfrak{h} \to \operatorname{End} V$ and a bilinear symmetric map $\Gamma : V \times V \to \mathfrak{h}$ satisfying

(1) $$[\rho(x), \Gamma(u,v)] = \Gamma(\rho(x)u, v) + \Gamma(u, \rho(x)v)$$
(2) $$\rho(\Gamma(u,v))w + \rho(\Gamma(w,u))v + \rho(\Gamma(v,w))u = 0$$

for $x \in \mathfrak{h}$ and u, v and w in V, we define a Lie superalgebra $\mathfrak{g} = \mathfrak{h} \oplus V$ by setting $\mathfrak{g}_0 = \mathfrak{h}$, $\mathfrak{g}_1 = V$ and the Lie bracket

$$[x+u, y+v]_{\mathfrak{g}} = [x,y]_{\mathfrak{h}} + \rho(x)v - \rho(y)v + \Gamma(u,v).$$

Conversely, for a given Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with Lie bracket $[\cdot, \cdot]_{\mathfrak{g}}$ we already know that $\mathfrak{g}_0$ is a Lie algebra using the restriction of $[\cdot, \cdot]_{\mathfrak{g}}$ to $\mathfrak{g}_0 \times \mathfrak{g}_0$; we also have a representation $\rho : \mathfrak{g}_0 \to \operatorname{End} \mathfrak{g}_1$ given by $\rho(x)(v) = [x, v]_{\mathfrak{g}}$, for $x \in \mathfrak{g}_0$ and $v \in \mathfrak{g}_1$; finally, we also have a bilinear symmetric map $\Gamma : \mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathfrak{g}_0$ defined by $\Gamma(u,v) = [u,v]_{\mathfrak{g}}$ that satisfies (1) and (2) (as it follows from the Jacobi identities).

In a sense, therefore, a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is some 'superstructure' to be built over the Lie algebra $\mathfrak{g}_0$. This point of view is quite useful for both, the general theory and the explicit construction of Lie superalgebras (see [8], [16]).

A.5 Supermanifolds. A smooth (resp. holomorphic) (m, n)-dimensional supermanifold $(M, \mathcal{A})$ is a smooth (resp. holomorphic) m-dimensional manifold M together with a sheaf of associative $\mathbb{Z}_2$-commutative superalgebras $\mathcal{A}$ over M having the following properties: Let $\mathcal{N}$ be the ideal generated by the sections of $\mathcal{A}_1$ in the decomposition $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$. Note the $\mathbb{Z}_2$-commutative property implies that the sections from $\mathcal{A}_1$ are nilpotent. Therefore, all the sections in $\mathcal{N}$ are nilpotent. It is then required that $\mathcal{A}/\mathcal{N}$ be isomorphic to the sheaf C_M^∞ (or $\mathcal{O}_M$ depending on whether the supermanifold is smooth or holomorphic, respectively) and that there exists some $n \in \mathbb{N}$ such that $\mathcal{N}^n \neq \{0\}$ but $\mathcal{N}^{n+1} = \{0\}$. Then, n is called *the odd dimension of the supermanifold.*

Now, $\mathcal{N}/\mathcal{N}^2$ has the structure of a locally free sheaf of $\mathcal{A}/\mathcal{N}$—modules; that is, a vector bundle (the so called Batchelor bundle) whose rank is equal to the odd dimension. We shall write $\mathcal{E}$ instead of $\mathcal{N}/\mathcal{N}^2$ when we think of it as the sheaf of sections of the corresponding vector bundle $E \to M$. The nilpotent ideal defines a filtration $\mathcal{N}^i \supset \mathcal{N}^{i+1}$ and gives rise to its associated $\mathbb{Z}$—graded algebra $\operatorname{Gr}\mathcal{A} = \oplus \mathcal{N}^i/\mathcal{N}^{i+1} \simeq \wedge(\mathcal{N}/\mathcal{N}^2) \simeq \wedge\mathcal{E}$. The natural surjection $\mathcal{A} \to \operatorname{Gr}\mathcal{A}$ gives rise to a local isomorphism of $\mathbb{Z}_2$—graded algebras $\mathcal{A}(U) \to \Gamma(U, \wedge E)$. The question of whether this extends to a global isomorphism has been settled by several authors under different assumptions. The first such result is due to Batchelor (see [2]) and it states that for any smooth supermanifold, its structure sheaf $\mathcal{A}$ is globally isomorphic to $\Gamma(\wedge E)$. Supermanifolds having this global property have been called *split* or *Batchelor trivial* ever since. The main issue in Batchelor's proof is the existence of a partition of unity. Examples of non-split supermanifolds have been given by P. Green [5], M. Rothstein [14] and Y. Manin [11]. By the mid 90's Koszul proved that a supermanifold splits if and only if a *superconnection* can be globally defined on it. This amounts to have globally defined connections on TM and E (see [9]) which can always be done in the smooth category, but not always in the holomorphic setting.

The simplest examples –and local models– of supermanifolds are the so called (m,n)-dimensional *superdomains*. They are defined by letting U be an open domain in $\mathbb{R}^m$ (or $\mathbb{C}^m$) and taking E to be the trivial rank-n vector bundle over U —say, generated by some set $\{\xi^1, \xi^2, \ldots, \xi^n\}$ of linearly independent sections, so that $\xi^1\xi^2\cdots\xi^n \neq 0^1$. A standard notation for such a superdomain is $(U, C^\infty(U) \otimes \wedge[\xi^1, \ldots, \xi^n])$. A set of m sections $x^1, x^2, \ldots, x^m$ in $\bigoplus_{k\geq 0} C^\infty(U) \otimes \wedge^{2k}[\xi^1, \ldots, \xi^n]$ are called *even coordinates for the superdomain* if their projections $\widetilde{x^1}, \widetilde{x^2}, \ldots, \widetilde{x^m}$ under the natural map $\mathcal{A} \to \mathcal{A}/\mathcal{N}$ (which in this case is just $C^\infty(U) \otimes \wedge[\xi^1, \ldots, \xi^n] \to C^\infty(U) \otimes \wedge^0[\xi^1, \ldots, \xi^n] \simeq C^\infty(U)$) form an ordinary set of smooth coordinates on U. Thus a *coordinate system for a superdomain* $(U, C^\infty(U) \otimes \wedge[\xi^1, \ldots, \xi^n])$ is a set $\{y^1, y^2, \ldots, y^m; \zeta^1, \zeta^2, \ldots, \zeta^n\}$ such that (1) $\{y^1, y^2, \ldots, y^m\}$ is an even coordinate system and (2) $\zeta^1, \zeta^2, \ldots, \zeta^n$ are sections in $\bigoplus_{k\geq 0} C^\infty(U) \otimes \wedge^{2k+1}[\xi^1, \ldots, \xi^n]$ satisfying $\zeta^1\zeta^2\cdots\zeta^n \neq 0$ (see [8], [10]).

A morphism $\varphi : (M, \mathcal{A}) \to (N, \mathcal{B})$ between the supermanifolds $(M, \mathcal{A})$ and $(N, \mathcal{B})$ is a pair $(\widetilde{\varphi}, \varphi^*)$ consisting of a morphism of sheafs of algebras $\varphi^* : \mathcal{B} \to \widetilde{\varphi}_* \mathcal{A}$ over N along the smooth (resp. holomorphic) map $\widetilde{\varphi} : M \to N$, where for any open subset $W \subset N, \widetilde{\varphi}_*\mathcal{A}(W) = \mathcal{A}\left(\widetilde{\varphi}^{-1}(W)\right)$. A very useful theorem (due to Leites) states that morphisms between superdomains are completely determined by their effect on a set of local coordinates. More precisely:

Theorem. *Let $(U, \mathcal{A})$ and $(W, \mathcal{B})$ be superdomains and let $\{x^1, \ldots, x^m; \xi^1, \ldots, \xi^n\}$ be a coordinate system on $(W, \mathcal{B})$.*

    a) *Let $\varphi^* : \mathcal{B} \to \mathcal{A}$ be any morphism of superalgebras. We consider the set of sections $y^i = \varphi^*(x^i)$ $(i = 1, \ldots, m)$ and $\eta^i = \varphi^*(\xi^i)$ $(i = 1, \ldots, n)$. Then, the set of sections $\{y^1, \ldots, y^m; \eta^1, \ldots, \eta^n\}$ satisfies the following conditions: (1) $y^i \in \bigoplus_{k\geq 0} C^\infty(U) \otimes \wedge^{2k}[\zeta^1, \ldots, \zeta^n]$, (2) $\eta^i \in \bigoplus_{k\geq 0} C^\infty(U) \otimes \wedge^{2k+1}[\zeta^1, \ldots, \zeta^n]$ and (3) if $u \in U$, then $(\widetilde{y^1}(u), \ldots, \widetilde{y^m}(u))$ belongs to W.*

---

[1]We shall write $\xi^1\xi^2$ instead of $\xi^1 \wedge \xi^2$, etc.

b) Let $\{y^1, \ldots, y^m; \eta^1, \ldots, \eta^n\}$ be a set of sections in $\mathcal{A}$ satisfying (1), (2) and (3) as above. Then there is one and only one homomorphism of superalgebras $\varphi^* : \mathcal{B} \to \mathcal{A}$ for which $\varphi^*(x^i) = y^i$ and $\varphi^*(\xi^i) = \eta^i$.

c) To every homomorphism of superalgebras $\psi^* : \mathcal{B} \to \mathcal{A}$ there corresponds one and only one morphism of superdomains $(\widetilde{\varphi}, \varphi^*) : (U, \mathcal{A}) \to (W, \mathcal{B})$ such that $\varphi^* : \mathcal{B} \to \widetilde{\varphi}_* \mathcal{A}$ is the same as ψ^*.

A.6 Example. Let $\mathbb{R}^{1|1}$ be the $(1, 1)$-dimensional supermanifold $(\mathbb{R}, \mathcal{A}(\mathbb{R}))$, where $\mathcal{A}(\mathbb{R}) = C^\infty(\mathbb{R}) \otimes \wedge[\tau]$, so that $\mathbb{R}^{1|1}$ is Batchelor trivial for the trivial rank-1 vector bundle $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and τ can be thought of as the global section $t \mapsto (t, 1)$ of this bundle. An element f in the associative superalgebra $\mathcal{A}(\mathbb{R})$ can be written as $f = f_0 + f_1 \tau$, where f_0 and f_1 in $C^\infty(\mathbb{R})$. This is a very important example because, for any given supermanifold $(M, \mathcal{A}_M)$, we have the one-to-one correspondence (see [17])

$$\mathcal{A}_M \longleftrightarrow \mathrm{Mor}\left((M, \mathcal{A}_M), \mathbb{R}^{1|1}\right),$$

where $\mathrm{Mor}\left((M, \mathcal{A}_M), \mathbb{R}^{1|1}\right)$ is the sheaf of all morphisms from the supermanifold $(M, \mathcal{A}_M)$ into $\mathbb{R}^{1|1}$. The correspondence is given as follows: If $\xi \in \mathcal{A}_M(U) = \mathcal{A}_M(U)_0 \oplus \mathcal{A}_M(U)_1$ so that $\xi = \xi_0 + \xi_1$, with $\xi_\mu \in \mathcal{A}_M(U)_\mu$ $(\mu = 0, 1)$, then $\varphi_\xi : (M, \mathcal{A}_M) \to \mathbb{R}^{1|1}$ is defined —according to the theorem above— by letting $\varphi_\xi^* t = \xi_0$ and $\varphi_\xi^* \tau = \xi_1$.

This correspondence can be turned into an abstract algebra isomorphism. The algebra structure on $\mathrm{Mor}\left((M, \mathcal{A}_M), \mathbb{R}^{1|1}\right)$ can be given in terms of an abstract algebra structure defined on $\mathbb{R}^{1|1}$ by means of the 'sum' morphism $s : \mathbb{R}^{1|1} \times \mathbb{R}^{1|1} \to \mathbb{R}^{1|1}$, defined by

$$s^* t = p_1^* t + p_2^* t \qquad \text{and} \qquad s^* \tau = p_1^* \tau + p_2^* \tau$$

and the 'product' morphism $\mathrm{m} : \mathbb{R}^{1|1} \times \mathbb{R}^{1|1} \to \mathbb{R}^{1|1}$, defined by

$$\mathrm{m}^* t = p_1^* t\, p_2^* t + p_1^* \tau\, p_2^* \tau \qquad \text{and} \qquad \mathrm{m}^* \tau = p_1^* t\, p_2^* \tau + p_1^* \tau\, p_2^* t,$$

where $p_i : \mathbb{R}^{1|1} \times \mathbb{R}^{1|1} \to \mathbb{R}^{1|1}$ are the projections. One can check that the morphisms s and m satisfy the appropriate commutative diagrams that state the associativity and distributivity laws (see [17] for further details).

A.7 Superderivations. Let A be a $\mathbb{Z}_2$−commutative associative superalgebra with multiplication map $m_A : A \times A \to A$. Define $\mathrm{Der}\, A = (\mathrm{Der}\, A)_0 \oplus (\mathrm{Der}\, A)_1$ where

$$(\mathrm{Der}\, A)_\mu = \{X \in \mathrm{End}\, A \,|\, X(m_A(a, b)) = m_A(X(a), b) + (-1)^{\mu|a|} m_A(a, X(b))\}.$$

It can be proved that $\mathrm{Der}\, A$ has a Lie superalgebra structure by defining the Lie bracket $[X, Y] = X \circ Y - (-1)^{|X||Y|} Y \circ X$, on homogeneous elements X and Y, and extending this definition bilinearly to the whole $\mathrm{Der}\, A$. The sections in $\mathrm{Der}\, A$ are called *superderivations* of A; hence, $\mathrm{Der}\, A$ is called the Lie superalgebra of superderivations of A.

A.8 Ordinary differential equations in supermanifolds. Any vector field X on a smooth manifold M can be identified with a derivation $X \in \operatorname{Der} C^\infty(M)$ [18]. Similarly, vector fields on a supermanifold $(M, \mathcal{A})$ are identified with elements of $\operatorname{Der} \mathcal{A}(M)$. It is well known that any vector field on a smooth manifold gives rise to an ordinary differential equation (ODE) on it that has a unique (local) integral flow. Likewise, it has been proved in [13] that *given a supervector field $X \in \operatorname{Der} \mathcal{A}(M)$ there is a unique integral flow* $\Phi : \mathbb{R}^{1|1} \times (U, \mathcal{A}_M(U)) \to (U, \mathcal{A}_M(U))$ —that may be defined only locally in a neighborhood I of $t_0 = 0 \in \mathbb{R}$ and in a neighborhood U of a given point in M— such that,

$$(3) \qquad \operatorname{ev}|_{t_0} \circ \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \tau} \right) \circ \Phi^* = \operatorname{ev}|_{t_0} \circ \Phi^* \circ X$$

restricted by the condition

$$(4) \qquad \operatorname{ev}|_{t_0} \circ \Phi^* = id^*.$$

Here, both equations are understood as maps $\mathcal{A}_M(U) \to \mathcal{A}_M(U)$. Actually, the morphism $\operatorname{ev}|_{t_0} : \mathcal{A}_{\mathbb{R} \times M}(I \times U) \to \mathcal{A}_M(U)$ is defined by the conditions: If $f = \widetilde{f} + f_\tau \tau + \sum_\mu f_\mu \xi^\mu + \sum_\mu f_{\tau,\mu} \tau \xi^\mu + \sum_{\mu,\nu} f_{\mu,\nu} \xi^\mu \xi^\nu + \cdots$, then

$$\operatorname{ev}|_{t_0}(f) = \widetilde{f}(t_0, \cdot) + \sum_\mu f_\mu(t_0, \cdot) \xi^\mu + \sum_{\mu,\nu} f_{\mu,\nu}(t_0, \cdot) \xi^\mu \xi^\nu + \cdots.$$

It was also proved in [13] that under special circumstances the ODE associated to $X \in \operatorname{Der} \mathcal{A}(M)$ in (3) can be posed with no $\operatorname{ev}|_{t_0}$ on it. Those special circumstances are precisely the conditions needed for the homogeneous components X_0 and X_1 of X to define a Lie superalgebra over $\mathbb{R}$. Therefore it is important to understand the different Lie superalgebras that can be defined with one even and one odd generators. It is easy to see that, up to isomorphism, there are only three of them:

$$
(5) \qquad
\begin{array}{llll}
\text{Type 1} & [X_0, X_1] = 0 & \text{and} & [X_1, X_1] = 0 \\
\text{Type 2} & [X_0, X_1] = 0 & \text{and} & [X_1, X_1] = X_0 \\
\text{Type 3} & [X_0, X_1] = X_1 & \text{and} & [X_1, X_1] = 0.
\end{array}
$$

These give rise to the observation that the superderivations of $\mathbb{R}^{1|1}$ given by

$$
(6) \qquad
\begin{array}{lll}
\text{Type 1} & D_0 = \dfrac{\partial}{\partial t} & \text{and} \quad D_1 = \dfrac{\partial}{\partial \tau} \\[2ex]
\text{Type 2} & D_0 = \dfrac{\partial}{\partial t} & \text{and} \quad D_1 = \dfrac{\partial}{\partial \tau} + \tau \dfrac{\partial}{\partial t} \\[2ex]
\text{Type 3} & D_0 = \dfrac{\partial}{\partial t} + \tau \dfrac{\partial}{\partial \tau} & \text{and} \quad D_1 = \dfrac{\partial}{\partial \tau}
\end{array}
$$

faithfully realize these three non isomorphic Lie superalgebras. Therefore, one would expect three different Lie supergroup structures on $\mathbb{R}^{1|1}$ and this is indeed the case (see [13]).

A.9 Lie supergroups. A *Lie supergroup* $(G, \mathcal{A}_G)$ is (see [3]) a supermanifold with morphisms $m : (G, \mathcal{A}_G) \times (G, \mathcal{A}_G) \to (G, \mathcal{A}_G)$, $\varepsilon : (G, \mathcal{A}_G) \to (G, \mathcal{A}_G)$ and $\alpha : (G, \mathcal{A}_G) \to (G, \mathcal{A}_G)$ satisfying the conditions

$$m \circ (p_1, m \circ (p_2, p_3)) = m \circ (m \circ (p_1, p_2), p_3) \qquad \text{(associativity property)}$$

$$m \circ (id, \varepsilon) = id = m \circ (\varepsilon, id) \qquad \text{(identity property)}$$

$$m \circ (id, \alpha) = \varepsilon = m \circ (\alpha, id) \qquad \text{(inverse property)}$$

where $p_i : (G, \mathcal{A}_G) \times (G, \mathcal{A}_G) \times (G, \mathcal{A}_G) \to (G, \mathcal{A}_G)$ stands for the i-th projection.

A *left action* from the Lie supergroup $(G, \mathcal{A}_G)$ into the supermanifold $(M, \mathcal{A}_M)$ is a morphism $\Psi : (G, \mathcal{A}_G) \times (M, \mathcal{A}_M) \to (M, \mathcal{A}_M)$ such that $\Psi \circ (\varepsilon \circ p_1, p_2) = p_2$ and $\Psi \circ (p_1, \Psi \circ (p_2, p_3)) = \Psi \circ (m \circ (p_1, p_2), p_3)$, where again, p_i stands for the appropriate projection onto the i-th factor.

Let $(G, \mathcal{A}_G)$ be a Lie supergroup and let X be a vector field on it. We use X to produce a vector field $\widehat{X}$ on $(G, \mathcal{A}_G) \times (G, \mathcal{A}_G) = (G \times G, \mathcal{A}_{G \times G})$. It is defined as the unique element in $\mathrm{Der}\, \mathcal{A}_{G \times G}(G \times G)$ that satisfies the conditions $\widehat{X} p_1^* f = 0$ and $\widehat{X} p_2^* f = p_2^* X f$, for every $f \in \mathcal{A}_G(G)$. Now, X is called a *left invariant vector field* if $\widehat{X}$ satisfies

$$\varepsilon^{(2)^*} \circ \widehat{X} \circ (p_1, \mathrm{m})^* = \varepsilon^{(2)^*} \circ (p_1, \mathrm{m})^* \circ \widehat{X},$$

where $\varepsilon^{(2)^*} : \mathcal{A}(G \times G) \to \mathcal{A}(G)$ is the morphism of sheaves of algebras associated to the morphism $\varepsilon^{(2)} : (G, \mathcal{A}_G) \to (G, \mathcal{A}_G) \times (G, \mathcal{A}_G)$ given by $\varepsilon^{(2)} = (id, \varepsilon)$.

It can be proved that the set of left invariant vector fields on the Lie supergroup $(G, \mathcal{A}_G)$ has, under the commutator of derivations, the structure of a finite-dimensional Lie superalgebra over the ground field $\mathbb{F}$. Its dimension is precisely that of the Lie supergroup $(G, \mathcal{A}_G)$. It is therefore called *the Lie superalgebra associated to the Lie supergroup* $(G, \mathcal{A}_G)$. It is a theorem due to Kostant [8] that if the Lie superalgebra of $(G, \mathcal{A}_G)$ is written in terms of its even and odd subspaces as $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, then $\mathrm{Lie}(G) = \mathfrak{g}_0$ and $\mathcal{A}_G(G) = C^\infty(G) \otimes \wedge \mathfrak{g}_1$. Thus, all Lie supergroups are Batchelor trivial.

Remark. Given a Lie superalgebra, one can in principle find the associated Lie supergroup —up to coverings of the underlying Lie group, as in the smooth category— by following Lie's techniques. That is, by realizing first the Lie superalgebra generators as superderivations (ie, supervector fields) on some supermanifold and finding their integral flows. These integral flows, which depend on the even and odd parameters coming from $\mathbb{R}^{1|1}$, can be composed in some prescribed order and then composed with another similar set (in the same order) depending on different integration parameter values. The result of this full composition on two different set of integration parameters is then interpreted as a set of a similar sort, from which one can in principle obtain the composition law for the integration parameters themselves; hence the composition law for the supergroup. This was done first in [13] as an application of the integration of ODE's on supermanifolds and it was found that, for the supervector fields (6), the composition morphisms for the $(1, 1)$-dimensional Lie supergroups associated to the $(1, 1)$-dimensional Lie superalgebras realized in (6) are respectively given by

Type 1	$\mathrm{m}\left((t_1, \tau_1), (t_2, \tau_2)\right) = (t_1 + t_2, \tau_1 + \tau_2)$
Type 2	$\mathrm{m}\left((t_1, \tau_1), (t_2, \tau_2)\right) = (t_1 + t_2 + \tau_1\tau_2, \tau_1 + \tau_2)$
Type 3	$\mathrm{m}\left((t_1, \tau_1), (t_2, \tau_2)\right) = (t_1 + t_2, \tau_2 + \mathrm{e}^{t_2}\,\tau_1).$

Thus, with these three Lie supergroup structures on $\mathbb{R}^{1|1}$ in sight, it has been proved (see [13, Thm 3.6]) that the integral flow Φ of a vector field $X = X_0 + X_1$ *defines a Type $j - \mathbb{R}^{1|1} - action$* (for $j = 1, 2, 3$) on a supermanifold $(M, \mathcal{A}_M)$ if and only if X_0 and X_1 generate the following (1,1)-dimensional Lie superalgebra:

$$[X_0, X_1] = \delta_{j3} X_1 \qquad \text{and} \qquad [X_1, X_1] = \delta_{j2} X_0 \quad (j = 1, 2, 3).$$

In that case, the integral flow Φ of X satisfies the equation

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \tau} + \tau D_j\right) \circ \Phi^* = \Phi^* \circ X,$$

without the $\mathrm{ev}\,|_{t_0}$−morphism and $D_j = \delta_{j2}\frac{\partial}{\partial t} + \delta_{j3}\frac{\partial}{\partial \tau}$.

A main goal in this Thesis. What we do for the main part of this work is to solve Lie's problem of finding the Lie supergroups associated to the Lie superalgebras whose underlying Lie algebra is $\mathfrak{gl}_2(\mathbb{F})$ and whose odd $\mathfrak{gl}_2(\mathbb{F})$-module is $\mathfrak{gl}_2(\mathbb{F})$ under the adjoint representation.

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