CIMAT

Centro de Investigación en Matemáticas, A.C.

SCHWARZ METHOD AND BALAYAGE.

T H E S I S

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Diego Alexander Castro Guevara

Advisor:

Dr. Hector Andrés Chang Lara

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Chapter 1

Introduction

The Dirichlet problem or Boundary problem for the Laplace equation consists of finding a harmonic function in a domain (open and connected) $\Omega \subseteq \mathbb{R}^n$, with boundary conditions given by a continuous function $\varphi(x)$. For some domains as are the circle or the rectangle it is possible to build a solution to the Dirichlet problem. In the first one, using the Poisson's representation for the ball. In the rectangle, we can write a series solution by Fourier's method.

What happens in an arbitrary domain Ω? Is it possible to solve a Dirichlet problem for any continuous boundary data φ? In Chapter 2, we will answer these questions and in addition we will present two methods for solving the

Dirichlet problem: The first one, it was developed by O. Perron [15] known as Perron's method for subharmonic functions. This method consists in given the family of functions

$$S_\varphi = \{v : v \text{ is subharmonic and } v \leq \varphi \text{ on } \partial\Omega\}$$

taking the pointwise supremum over this family

$$u(x) = \sup_{v \in S_\varphi} v(x)$$

and proving that u is harmonic in Ω. We will present as in [3] the definition and properties of subharmonic functions, and also we will give geometric conditions over the domain to guarantee that Perron's solution solves the boundary problem.

The second method that will be presented was devised by H. Schwarz in 1869-1870, known as the Schwarz Alternating Method [6], [5]. Its idea is to solve a Dirichlet problem in a domain that can be decomposed as the union of two or more domains for which the Dirichlet problem is solvable.

The idea was that given two domains with nonempty intersection, to solve in the first domain. Then, with boundary data given by this result, solving in the second domain. Preceeding in the same way with the first domain, and doing this process to build an iterative method for solving the Dirichlet problem in their union.

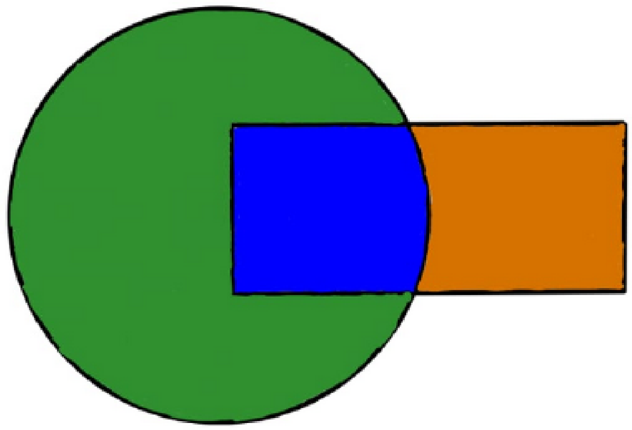


Figure 1.1: representation of the problem considered by H. A. Schwarz

The original problem considered by Schwarz was a Dirichlet problem on a domain consisting of a circle and a rectangle [13], such that their intersection was nonempty. The idea was to solve first in the circle, then, with boundary conditions given by the previous solution to solve over the rectangle. We return again over the circle and we solve with boundary conditions given by the previous result over the rectangle. Thus, continuing of this form, Schwarz proved that in the limit, this sequence converges to the solution of the Dirichlet problem in the union of the circle and rectangle.

We will present this method for two arbitrary domains $\Omega_i \subseteq \mathbb{R}^n$, $i = \{1, 2\}$ such that the Dirichlet problem is solvable in each domain and $\Omega_1 \cap \Omega_2 \neq \emptyset$.

Using the same recursive method described by Schwarz, we can build a sequence of solutions in each domain Ω_i, such that it converges to the solution of the Dirichlet problem in the whole Ω.

Chapter 2 will be distributed in the following form. In Section 2.1 we will present the maximum and minimum principle for subharmonic and superharmonic functions, then we will give an important inequality known as Harnack's inequality in Section 2.2 that will be useful for proving the Perron's method in Section 2.3. In Section 2.4 we will submit a variation of the Perron's method known as the obstacle problem and in Section 2.5 we will present the main theorem that will be the Schwarz's method.

A variation of the previous methods for reconstructing a harmonic function in a domain, from its values on the boundary was devised by H. Poincaré [7], [8] known as balayage method. Let Ω be a bounded domain in $\mathbb{R}^n$. Let μ a measure with support contained in Ω. The balayage for the measure μ is a measure ν, such that $\nu = 0$ in Ω and $\Phi^\mu = \Phi^\nu$ outside Ω. Where Φ^μ represents the Newtonian potential for the measure μ.

Since μ can be recovered by Φ^μ via $-\Delta\Phi^\mu = \mu$, another way of constructing the balayage measure is by solving a Dirichlet problem with boundary data Φ^μ, and extend the solution by Φ^μ outside Ω. This procedure is called balayage (from French "sweeping"), since the mass given by μ is "swept out" from Ω onto the boundary.

In his original publication on the balayage method, Poincaré began the construction of the balayage measure for a ball. Poincaré solved the following Poisson's problem. Given a measure μ in B, find u such that $-\Delta u = \mu$ in B and $u = 0$ on the boundary of B. Then, define the potential V such that $u = \Phi^{\mu} - V$, and so the measure given by $\nu = -\Delta V$ is the balayage for the measure μ.

The idea of Poincaré for an arbitrary domain was covered Ω with a numerable set of balls, and then apply the Schwarz's method in such a way that each ball gets visited an infinite number of times.

Balayage [11] is useful in different models as: *Internal Diffusion Limits Aggregation* or *Internal DLA*, *The Rotor-Router model* and *Divisible Sandpile*. On the first one, we take particles and these are repeatedly dropped at the origin of the lattice $\mathbb{Z}^d$. Each successive particle, then performs independent simple random walk in $\mathbb{Z}^d$ until reaching an unoccupied site. The next model was studied the first time by Priezzhev [4] under the name "Eurelian walkers". In this model, at each site in the lattice $\mathbb{Z}^2$ is a rotor pointing toward one of the four cardinal points. A particle starts at the origin; during each step, the rotor at the particle's current location is rotated clockwise by 90 degrees, and the particle takes a step in the direction of the newly rotated rotor until reaches an unoccupied site. For the *divisible sandpile*, suppose that at each vertex, we have a certain amount of mass (or sand piles). We wish that at each vertex the amount of mass is determined by a measure μ defined in $\mathbb{Z}^d$. Thus, in this model, each site distributes its excess mass equally among its neighbors if the amount of mass in the vertex is more than the measure given by μ in the vertex.

The goal in Chapter 3 is to present two main theorems about classic and partial balayage in graphs. In the first one, the idea will be, given a subset D of a graph G, and a measure μ_0 with support contained in the D, to build a sequence of measures μ_i in such a way that in each iteration, we sweep the masses given by the measure μ_0 in each vertex, until cleaning completely the subset D. The sequences μ_i converges to measure μ such that $\mu = 0$ in D.

In partial balayage, the idea will be similar to the divisible sandpile. Given a measure μ with support in D, or a sand among in each vertex, redistribute the excess of sand in relation to other measure λ, in such a way that if in a vertex, the initial amount of sand given by μ is more than the given by λ, then this vertex redistributes the sand excess to adjacent vertices, if the amount is less than λ, the amount remains equal. These results will be proved in Section 3.7 where we will prove that the process for classic and partial balayage converges.

Therefore, in Chapter 3 we will focus in to present analogue results to those in Chapter 2, now over graphs. Thus, this chapter is distributed as follows. In Section 3.2 we will introduce the divergence theorem in graphs and the formula of integration by parts. In Sections 3.3 and 3.4 we will prove the maximum principle in graphs and the Dirichlet problem respectively. In Section 3.6 similarly as in the Dirichlet problem for the continuous case, we will present the Perron's method and we will prove that this defines a harmonic function in the graph.

Chapter 2

Schwarz's Method

Now, we will present the necessary definitions and theorems for proving the main result of this chapter, which is the Schwarz's Theorem. This result can be enunciated as follows.

Theorem 1. *Let $\Omega_1, \Omega_2 \subseteq \mathbb{R}^n$ be open and bounded domain with $\Omega_1 \cap \Omega_2 \neq \emptyset$, such that for each one of them the classical Dirichlet problem is solvable for arbitrary continuous boundary values. Given $\Omega = \Omega_1 \cup \Omega_2$ and $g \in C^0(\overline{\Omega})$ a subharmonic function in Ω. Consider the sequence $\{u_i\}$ such that $u_0 = g$, and $\{v_i\}$ where v_i is the unique solution for*

$$\begin{cases} \Delta v_i = 0 \ \textit{in } \Omega_1 \\ v_i = u_{i-1} \ \textit{on } \overline{\Omega} \setminus \Omega_1 \end{cases}$$

and u_i is the unique solution for

$$\begin{cases} \Delta u_i = 0 \ \textit{in} \ \Omega_2 \\ u_i = v_i \ \textit{on} \ \overline{\Omega} \setminus \Omega_2 \end{cases}$$

then, $\{u_i\}$ converges uniformly to $u \in C^2(\Omega) \cap C(\overline{\Omega})$ the unique solution to

$$\begin{cases} \Delta u = 0 \ \textit{in} \ \Omega \\ u = g \ \textit{on} \ \partial\Omega \end{cases} \tag{2.1}$$

Perron's method will be the principal tool for proving Theorem 1. Before, it is necessary to present some previous results.

In this chapter, we will study the main properties about subharmonic (resp. superharmonic) functions in a domain Ω, that are C^2-functions, such that its Laplacian is nonnegative (resp. nonpositive) in Ω. We will present a weak definition of subharmonic (resp. superharmonic) function, only requesting continuity, and we will prove some equivalences between the two definitions.

More specifically, we will prove the Mean Values Formula and the Maximum Principle, using similar ideas as in [12]. In Section 2.2 we will prove the Harnack's Inequality for harmonic functions, that will be fundamental in the proof of Perron's method.

In Section 3.3, we will present the Perron's method, we will give the notion of barrier function as in [3, Pag. 120], and we will show that if we can build a barrier for all points $\xi \in \partial\Omega$, that is a superharmonic function such that is positive in Ω and zero in the point ξ, then the Dirichlet problem can be solved by the Perron's solution. In Section 3.4 we will give an analogue form for solving the Dirichlet problem, formulating an obstacle problem.

The first important property of subharmonic (resp. superharmonic) functions is the mean value property that we will mention in the following lemma.

Lemma 1 (The Mean Values Inequalities)**.** *Let $u \in C^2(\Omega)$ satisfying $\Delta u = 0\,(\geq\, 0, \leq\, 0\,)$ in Ω. Then, for any ball $B = B_R(y) \subset\subset \Omega$ (compactly contained), we have*

$$u(y) = (\geq, \leq) \fint_{\partial B} u(x)dS(x) \tag{2.2}$$

$$u(y) = (\geq, \leq) \fint_{B} u(x)dx \tag{2.3}$$

Proof. Let $r \in (0, R)$ and define the function

$$\phi(r) := \fint_{\partial B_r(y)} u(x) dS(x)$$

taking $x = y + rz$ we obtain

$$\phi(r) := \fint_{\partial B_1(0)} u(y + rz) dS(z)$$

thus

$$\phi'(r) = \fint_{\partial B_1(0)} Du(y + rz) \cdot z dS(z) = \fint_{\partial B_r(y)} Du(y) \cdot \underbrace{\frac{x - y}{r}}_{\nu} dS$$

therefore

$$\phi'(r) = \fint_{\partial B_r(y)} Du \cdot \nu(x) dS(x) = \frac{r}{n} \fint_{B_r(y)} \Delta u dx$$

from the divergence theorem. Now, if $\Delta u = 0$ we get $\phi'(r) = 0$ hence, ϕ is constant. In particular $\phi(R) = \lim_{r \to 0} \phi(r)$ therefore

$$u(y) = \lim_{r \to 0} \phi(r) = \phi(R) = \fint_{\partial B_R(y)} u dS.$$

In the case $\Delta u \geq 0$ it means that ϕ is increasing monotone therefore

$$u(y) = \lim_{r \to 0} \phi(r) \leq \phi(R) = \fint_{\partial B_R(y)} u dS$$

and similar for $\Delta u \leq 0$. Finally, For getting the mean value inequality, over $B_R(y)$ we use polar coordinates

$$\int_{B_R(y)} u dx = \int_0^R \left(\int_{\partial B_s(y)} u dS \right) ds = u(y) \int_0^R n\alpha(n) s^{n-1} ds = \alpha(n) R^n u(y)$$

and equivalently for subharmonic and superharmonic functions. $\square$

Example 1. *Let f be a holomorphic function nonconstant in a domain $\Omega \subset \mathbb{C}$. By the Cauchy -Riemann equations $\mathcal{R}(f)$ and $\mathcal{I}m(f)$ are harmonics. Let $\overline{B}(z, r)$ be contained in the domain Ω, thereby*

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(z + re^{i\theta}) d\theta$$

hence,

$$|f(z)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(z + re^{i\theta})| d\theta = \fint_{B(r,z)} |f(\zeta)| d\zeta$$

this means that the function defined by $u(x) = |f(x)|$ is subharmonic in Ω.

The previous result allows us to characterize the harmonic functions in terms of the average in each ball contained in Ω.

Corollary 1. *If $u \in C^2(\Omega)$ satisfies*

$$u(x) = \fint_{\partial B_r(x)} uds$$

for each ball $B_r(x) \subset \Omega$ then, u is harmonic.

Proof. If $\Delta u \not\equiv 0$ there exist some a ball $B_r(x) \subset \Omega$ such that, $\Delta u > 0$ within $B_r(x)$. Then, for ϕ as in Lemma 1

$$0 = \phi'(r) = \frac{r}{n} \fint_{B_r(x)} \Delta u(y) dy > 0$$

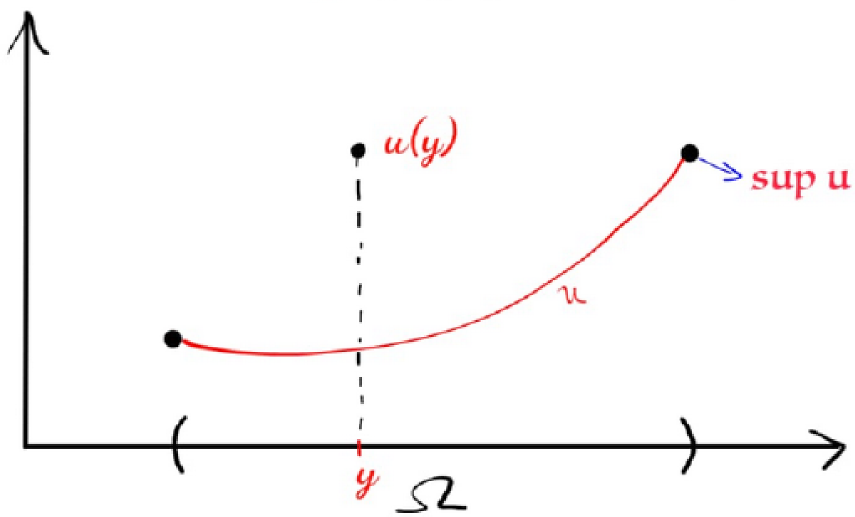
which is a contradiction. □

2.1 Maximum and Minimun Principle

Another important property from subharmonic and superharmonic functions, is the Maximum and Minimum Principle. It establishes that if $\Omega \subseteq \mathbb{R}^n$ is a domain, u is subharmonic in Ω and attains its maximum in a point in the interior of Ω, then u is constant. In the case when u is superharmonic we will obtain the same result if u attains its minimum in Ω. This principle is equivalent to the maximum modulus principle for holomorphic functions.

Theorem 2. *Let* $\Delta u\,(\geq,\,\leq)\,0$ *in the domain* Ω*. Suppose there exists a point* $y \in \Omega$ *for which* $u(y) = \sup_{\Omega} u (\inf_{\Omega} u)$*. Then,* u *is constant.*

Consequently, a harmonic function cannot assume an interior maximum or minimum values unless it is constant.



Proof. Let $\Delta u \geq 0$ in Ω, $M = \sup_{\Omega} u$ and define $\Omega_M = \{x \in \Omega \,|\, u(x) = M\}$. By assumption Ω_M is not empty. Furthermore, since u is continuous, Ω_M is closed relative to Ω.

Let z be any point in Ω_M and apply the mean value inequality to the subharmonic function $u - M$ in the ball $B = B_R(z) \subset\subset \Omega$. We obtain

$$0 = u(z) - M \leq \fint_B (u - M)dx \leq 0$$

thus, $u = M$ in $B_R(z)$. Consequently, Ω_M is also open relative to Ω. Since Ω is connected we have that $\Omega_M = \Omega$. The case of superharmonic functions follows by replacement of u by $-u$. □

The strong maximum principle implies a result of uniqueness.

Corollary 2. *Let $u, v \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfy $\Delta u = \Delta v$ in Ω, and $u = v$ in $\partial\Omega$. Then, $u = v$ in Ω*

Proof. Define $h = u - v$, then $\Delta h = 0$ in Ω and $h = 0$ in $\partial\Omega$. By the Maximum Principle $h \equiv 0$ in Ω. □

In the proof of Theorem 2 we only employ the mean value inequalities. Thus, the maximum and minimum principle only depend on the mean value inequalities, not in the second order differentiability of the function.

Now, we will give a new definition of subharmonic, superharmonic function for $u \in C^0(\Omega)$ such that, it implies one more time the mean value formula. In addition, if $u \in C^2(\Omega)$ then, $\Delta u \leq 0$, $\Delta u \geq 0$ respectively.

The simplest domain in which the Dirichlet problem could be represented is the ball in $\mathbb{R}^n$. S. Poisson [14] proved that

$$\overline{u}(x) := \begin{cases} \frac{R^2-|x-z|^2}{n\alpha(n)R} \int\limits_{\partial B_R} \frac{u(y)}{|x-y|^n} dS, \text{ if } x \in B \\ \\ u(x), \text{ if } x \in \partial B \end{cases}$$

belong to $C^2(B) \cap C^0(\overline{B})$ and $\Delta\overline{u} = 0$ in B for $u \in C^0(\partial B)$. This representation is known as Poisson's integral.

Definition 1. *A $C^0(\Omega)$ function u will be called subharmonic (resp. superharmonic) in Ω, if for every ball $B \subset\subset \Omega$ and harmonic function h in B satisfying $u \leq h$ (resp. $u \geq h$) on ∂B, we also have $u \leq h$ (resp. $u \geq h$) in B.*

The idea will be that this definition is consistent with the given when u is twice differentiability, that is, if u is C^2 and subharmonic, then $\Delta u \geq 0$. Before proving the consistency, we need to introduce a new concept.

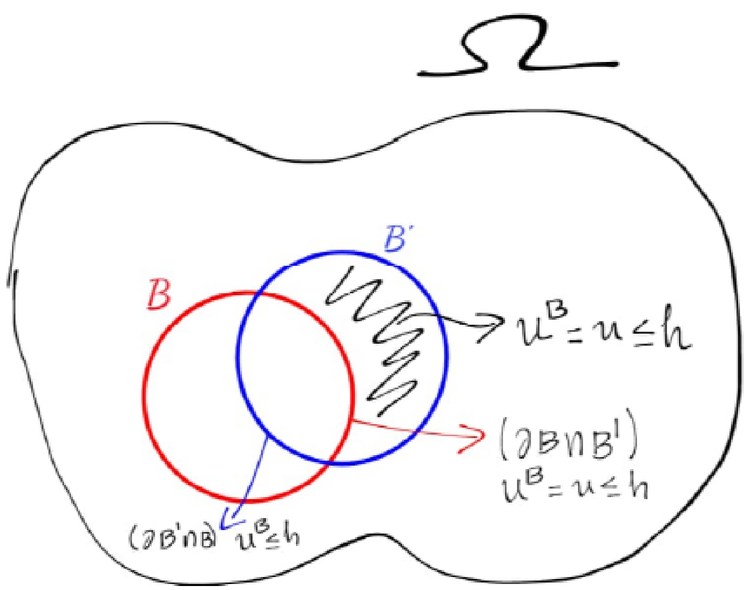
Let $u \in C^0(\Omega)$ be a subharmonic function in Ω and B be a ball strictly contained in Ω. Denoted by $\overline{u}$ the harmonic function in B (given by the Poisson integral of u on ∂B) satisfying $\overline{u} = u$ on ∂B. We define in Ω the

Harmonic Lifting of u in B by

$$u^B(x) := \begin{cases} \overline{u}(x) \text{ for all } x \in B, \\ \\ u(x) \text{ for all } x \in \Omega \setminus B. \end{cases}$$

Lemma 2. *The Harmonic Lifting of a subharmonic function in Ω is subharmonic in Ω.*

Proof. Let $B' \subset\subset \Omega$ be an arbitrary ball and let h be a harmonic function in B' satisfying $u^B \leq h$ on $\partial B'$. Since, $u \leq u^B$ in Ω we have $u \leq h$ in $\partial B'$ thus, $u \leq h$ in B' since u is subharmonic and hence, $u^B \leq h$ in $B' \setminus B$.



Note that u^B and h are harmonics in $B \cap B'$ and $u^B = u \leq h$ on $\partial B \cap B'$.

Now, on $\partial B' \cap B$ we have that $u^B \leq h$ by assumption. Thus, $u^B \leq h$ on $\partial(B \cap B')$ and by the maximum principle $u^B \leq h$ in $B \cap B'$. Consequently, $u^B \leq h$ in B' and therefore, u^B is subharmonic in Ω. $\square$

Harmonic lifting will be the key to prove one of the main results in this section, that is Perron's method. Also, we will establish the connection between the mean value formula and subharmonic functions in the weak sense. The next lemma establishes this relationship.

Lemma 3. *Let $u \in C^0(\Omega)$ be bounded. Then, the following assertions are equivalent:*

(i) u is subharmonic in Ω.

(ii) u satisfies

$$u(x) \leq \fint_{\partial B_r(x)} u(y)dS$$

for all $x \in \Omega$, $r > 0$ and $B_r(x) \subset\subset \Omega$.

In addition, if $u \in C^2(\Omega)$ and satisfies (ii), then $\Delta u(x) \geq 0$ for all x in Ω.

Proof. (i)$\Rightarrow$(ii)

Let $\overline{u}$ be the harmonic lifting of u in $B_r(x) \subset\subset \Omega$. Then

$$u(x) \leq \overline{u}(x) = \fint_{\partial B_r(x)} \overline{u}(y)dS = \fint_{\partial B_r(x)} u(y)dS$$

(ii)⇒(i)

Note that u satisfies the mean value formula by assumption. Since ψ is harmonic, then the function $h = u - \psi$ satisfies the mean value formula. Now, $h \leq 0$ on ∂B, thus by the maximum principle $h \leq 0$ in B.

If (ii) holds and $u \in C^2(\Omega)$ assume there exists $x_0 \in B_r(x) \subset\subset \Omega$ such that, $\Delta u(x_0) < 0$. Then, if $u \in C^2(\Omega)$ we have Δu is continuous in Ω and there exists a radius $r_0 \in (0, dist(\partial B_r(x)))$ such that, $\Delta u < 0$ in $B_R(x_0)$ for all $R \in (0, r_0)$. We compute as in Lemma 1 the function ϕ which is decreasing and hence

$$u(x_0) > \fint_{\partial B_r(x_0)} udS$$

which shows that u cannot be subharmonic in Ω. □

Example 2. *Define the function*

$$u(x, y) = \sqrt{x^2 + y^2} \text{ in the ball } B(0, R)$$

note that u is continuous in B, but is not differentiable. Now, u can be *written as*

$$u(x,y) = f(z) = |z| \text{ in the complex plane,}$$

the function $h(z) = z$ is holomorphic in B, therefore satisfies the mean value formula

$$z = \fint_{\partial B(z,r)} \xi dS(\xi)$$

and

$$|z| \leq \fint_{\partial B(z,r)} |\xi| dS(\xi)$$

thereby

$$u(x,y) \leq \fint_{\partial B(z,r)} u(\xi_1, \xi_2) dS(\xi)$$

or equivalently u is C^0-subharmonic.

Corollary 3. *A $C^0(\Omega)$ function u is harmonic, if and only if, for every ball $B = B_R(y) \subset\subset \Omega$ it satisfies the mean value property*

$$u(y) := \fint_{\partial B} u dS = \fint_{B} u dx$$

Proof. If u is harmonic we have already proved that it must satisfy the mean value formula. We must focus on the opposite implication. Since, $u \in C^0(\Omega)$, for any ball $B \subset\subset \Omega$ there exists a harmonic function h in B such that $h = u$ on ∂B. We define $f = u - h$, since u satisfies the mean value property, then f satisfies the same property. Using the same proof as in Theorem 2, we see that f attains its maximum and minimum on ∂B. But, $f = 0$ on ∂B therefore, $f = 0$ in B. This implies that $u = h$ in B and u is harmonic in B for any ball $B \subset\subset \Omega$. Thus, u is harmonic in Ω. □

For proving the Schwarz's Theorem, we will need to prove that a sequence of harmonic functions is convergent.

Corollary 4. *The limit of a uniformly convergent sequence of harmonic functions is harmonic.*

Proof. Suppose $\{u_n\}_{n=1}^{+\infty}$ is a sequence of harmonic functions such that $u_n \to u$ uniformly, since each u_n is harmonic it satisfies

$$u_n(x) = \fint_{\partial B_r(x)} u_n(y)dy$$

therefore

$$\lim_{n\to+\infty} u_n(x) = \fint_{\partial B_R(x)} \lim_{n\to+\infty} u_n(y)dy$$

thereby

$$u(x) = \fint_{\partial B_r(x)} u(y)dy$$

the previous is due to the limit uniformly of continuous functions is continuous. Thus, u is continuous and satisfies the mean value property, therefore, u is harmonic by Corollary 3. $\square$

If u is a C^2-subharmonic function and v is a C^2-superharmonic function, such that $u \leq v$ on $\partial\Omega$, we see that $u \leq v$ in Ω by the maximum principle, but in the proof of the maximum principle, we only use the mean value formulas, not requiring the C^2 regularity, and by Lemma 3 the weak definition of subharmonic (resp. superharmonic) function implies the mean value formulas and therefore the maximum principle. We will present the analogous comparison principle for C^0-subharmonic and superharmonic functions.

Theorem 3 (Comparison)**.** *Let u be C^0-subharmonic and v be C^0-superharmonic functions in Ω such that, $u \leq v$ on $\partial\Omega$, then $u \leq v$ in Ω.*

Proof. Define the function $w := u - v$, then for each $B_r(x) \subset\subset \Omega$ w satisfies

$$w(x) \leq \fint_{\partial B_r(x)} w ds$$

this implies that w is subharmonic. Thus, w satisfies the maximum principle, since $w \leq 0$ on $\partial\Omega$, then $w \leq 0$ in Ω therefore, $u \leq v$ in Ω. $\square$

The next lemma will be essentially for proving the Perron's Theorem.

Lemma 4. *Let* $u_1, u_2, ..., u_N$ *be subharmonic in* Ω. *Then, the function* $u(x) := \max\{u_1(x), ..., u_N(x)\}$ *is also, subharmonic in* Ω.

Proof. Let $B \subset\subset \Omega$ and let ψ be a harmonic function in B such that $\max\{u_1(x), ..., u_N(x)\} \leq \psi$ in ∂B. By definition of the maximum,

$$u_i(x) \leq \psi \text{ for all } i = \{1, 2, ..., N\} \text{ on } \partial B$$

since u_i is subharmonic, then $u_i(x) \leq \psi$ in B for each $i \in \{1, ..., N\}$ therefore

$$\max\{u_1(x), ..., u_N(x)\} \leq \psi \text{ in } \Omega$$

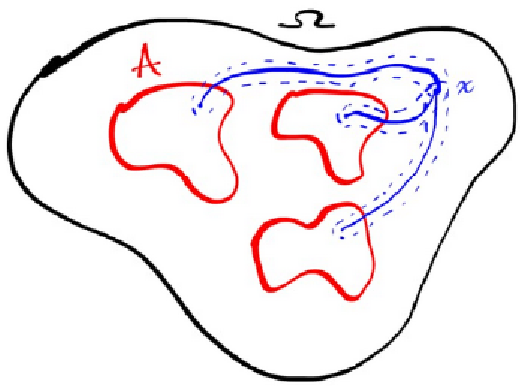
□

2.2 Harnack's Inequality

In this section we will see an important inequality introduced by A. Harnack [1] that will be essential for proving some results in uniform convergence of harmonic functions.

Lemma 5. *Suppose* Ω *be a bounded, open and connected set in* $\mathbb{R}^n$. *Then, for each* $A \subset\subset \Omega$ *there exists an open connected* V *such that* $A \subset\subset V \subset\subset \Omega$.

Proof. We will give an idea of the proof. Let us take a point $x \in \Omega \setminus A$. Due to Ω is arch-connected, for each $a \in A$ there exists a curve $\gamma \subset \Omega$ joining x to a.



Since γ is compact there exist a finite family $\{B_i\}$ of balls such that is a finite cover of γ that is

$$\gamma \subset \bigcup_{i=1}^{N} B_i \subset \Omega$$

the cover for γ is open and connected. We will name U_a the cover associated to the curve γ_{xa}.

Thereby, if we consider the set

$$V = \bigcup_{a \in A} U_a$$

since x is a common point for each U_a this set is open and connected, and

$A \subset V \subset\subset \Omega$. $\square$

Theorem 4. *Let u be non-negative harmonic function in Ω. Then, for any subset $\Omega' \subset\subset \Omega$ there exists a constant C depending only on n, Ω' and Ω such that*

$$\sup_{\Omega'} u \leq C \inf_{\Omega'} u$$

Proof. Without loss of generality, we will suppose that Ω' is connected. Indeed, by Lemma 5 for each subset A with $A \subset\subset \Omega$ there exists a domain V such that, $A \subset\subset V$. If the result is valid for V that is connected, then

$$\sup_{A} u \leq \sup_{V} u \leq C \inf_{V} u \leq C \inf_{A} u$$

thus, it is only necessary to make the proof in the case when Ω' is connected.

Let $y \in \Omega$ and $B_{4R}(y) \subset \Omega$. Then, for any two points $x_1, x_2 \in B_R(y)$ we have by the mean value inequalities

$$u(x_1) = \frac{1}{\alpha(n) R^n} \int\limits_{B_R(x_1)} u dx \leq \frac{1}{\alpha(n) R^n} \int\limits_{B_{2R}(y)} u dx,$$

$$u(x_2) = \frac{1}{\alpha(n)(3R)^n} \int\limits_{B_{3R}(x_2)} u dx \geq \frac{1}{\alpha(n)(3R)^n} \int\limits_{B_{2R}(y)} u dx,$$

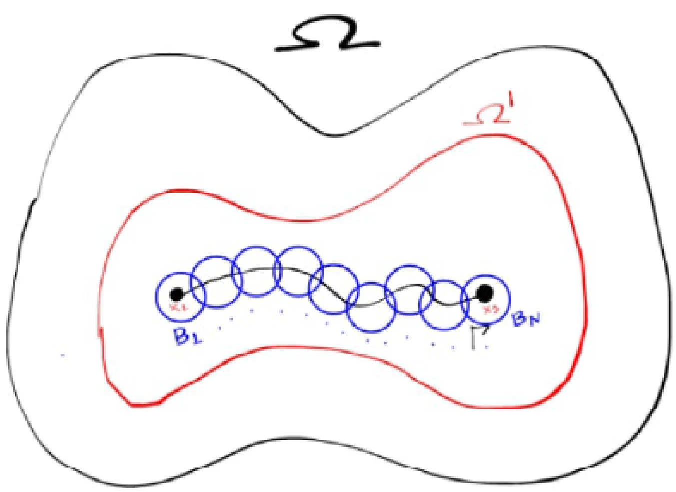
consequently, we obtain

$$\sup_{B_R(y)} u \leq 3^n \inf_{B_R(y)} u. \tag{2.4}$$

Let $\Omega' \subset\subset \Omega$ and choose $x_1, x_2 \in \overline{\Omega'}$ such that, $u(x_1) = \sup_{\Omega'} u$, $u(x_2) = \inf_{\Omega'} u$.

Let $\Gamma \subset \overline{\Omega'}$ be a curve joining x_1 and x_2 and choose R such that $4R < dist(\Gamma, \partial\Omega)$. By virtue of the Heine-Borel Theorem, Γ can be covered by a finite number N (depending only on Ω', Ω because $\overline{\Omega}$ is compact, therefore, has a finite cover, and the cover for Γ is contained in the cover of $\overline{\Omega}$) of balls of radius R. Applying the estimative (2.4) in each ball and combining the resulting inequalities, we get

$$u(x_1) \leq 3^{nN} u(x_2)$$



To see this, without loss of generality, suppose that $x_1 \in B_1$, $x_2 \in B_N$ and $B_i \cap B_{i+1} \neq \emptyset$ with $B_i = B_R(\zeta_i)$ for $\zeta_i \in \Gamma$. Note that

$$u(x_1) \leq \sup_{B_1} u.$$

and by assumption

$$\sup_{B_1} u \leq 3^n \inf_{B_1} u$$

and it holds the inequalities

$$\inf_{B_1} u \leq \inf_{B_1 \cap B_2} u \leq \sup_{B_1 \cap B_2} u \leq \sup_{B_2} u \leq 3^n \inf_{B_2} u$$

thereby

$$u(x_1) \leq 3^n (3^n \inf_{B_2} u) = 3^{2n} \inf_{B_2} u,$$

making this process over each ball, we see that

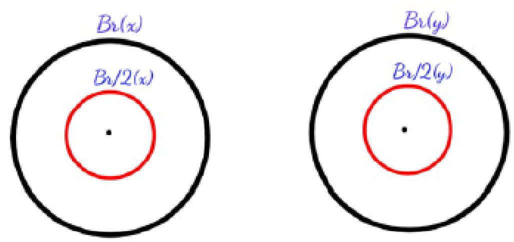
$$u(x_1) \leq 3^{nN} \inf_{B_N} u$$

and $\inf_{B_N} u \leq u(x_2)$ because by assumption $x_2 \in B_N$ hence,

$$u(x_1) \leq 3^{nN} u(x_2)$$

then, the estimate holds with $C = 3^{nN}$. $\square$

Remark 1. *The assumption Ω be connected is requisite because, if Ω is disconnected then, taking for example*



$$\Omega = B_r(x) \cup B_r(y)$$

with the condition $B_r(x) \cap B_r(y) = \emptyset$. If we take $A \subset\subset \Omega$ as $A = B_{r/2}(x) \cup B_{r/2}(y)$ and define

$$u(z) = \chi_{B_r(y)}(z)$$

thus, if there exists a constant C such that

$$\sup_A u \leq C \inf_A u$$

then, $1 \leq C * 0$.

Lemma 6. *Let $\{u_n\}$ be a monotone increasing sequence of harmonic functions in a domain Ω. Suppose that for some point $y \in \Omega$ the sequence*

$\{u_n(y)\}$ is bounded. Then, the sequence converges uniformly on any bounded subdomain $\Omega' \subset\subset \Omega$ to a harmonic function.

Proof. The sequence $\{u_n(y)\}$ will converge so that, for arbitrary $\epsilon > 0$ there is a number N such that, $0 \leq u_m(y) - u_n(y) < \epsilon$ for all $m \geq n > N$. Now, suppose that $y \in \Omega'$. By the Harnack's inequality we have

$$\sup_{\Omega'} |u_m(x) - u_n(x)| \leq C \inf_{\Omega'} |u_m(x) - u_n(x)| < C\epsilon$$

for some constant C depending on Ω' and Ω. Consequently, $\{u_n\}$ converges uniformly to harmonic function in Ω'. If $y \notin \Omega'$, we take a connected set V containing to $\Omega' \cup \{y\}$ and we apply the same proof. $\square$

2.3 The Dirichlet Problem: Perron's Method

Let Ω be bounded and φ be a bounded function on $\partial\Omega$. A $C^0(\overline{\Omega})$ subharmonic function u is called a *subfunction* relative to φ, if it satisfies $u \leq \varphi$ on $\partial\Omega$ (The analogous definition for superfunction relative to φ).

By the maximum principle every subfunction is less than or equal to every superfunction, because if $\underline{u}$ is a subfunction relative to φ and $\overline{u}$ superfunction relative to φ, then

$$\underline{u} \le \varphi \le \overline{u} \text{ on } \partial\Omega$$

by maximum principle

$$\underline{u} \le \overline{u} \text{ in } \Omega.$$

In particular, constant functions less than $\inf\limits_{\partial\Omega} \varphi$ (resp. more than $\sup\limits_{\partial\Omega} \varphi$) are subfunctions (resp. superfunctions).

Let S_φ denote the set of subfunctions relative to φ. The basic result of Perron's method is contained in the following theorem.

Theorem 5 (Perron's Method)**.** *The function*

$$u(x) = \sup_{v \in S_\varphi} v(x)$$

is harmonic in Ω*.*

Proof. Given $x_0 \in \Omega$, by definition of $u(x)$ there exists $\{v_k\} \subset S_\varphi$ such that, $v_k(x_0) \to u(x_0)$. The functions

$$w_k = \max\{v_1, ..., v_k\} \quad k \ge 1$$

are in S_φ and $w_k \le w_{k+1}$.

By definition of u and w_k we have that

$$v_k(x_0) \leq w_k(x_0) \leq u(x_0)$$

therefore, $\lim\limits_{k\to+\infty} w_k(x_0) = u(x_0)$. Let B be a ball such that $B \subset\subset \Omega$. For each $k \geq 1$ we have that $w_k \leq w_k^B \leq u$ because $w_k^B \in S_\varphi$ and hence,

$$\lim_{k\to+\infty} w_k^B(x_0) = u(x_0)$$

since, each w_k^B is harmonic in B, increasing monotonic and bounded it converges punctually. By Harnack's inequality this convergence is uniform. Thus, $\lim\limits_{k\to+\infty} w_k^B(x) = w(x)$ is harmonic in B and $w(x_0) = u(x_0)$. We must show that $w = u$ in B.

By definition of u we have $w \leq u$. Suppose that there exists $x_1 \in B$ such that, $w(x_1) < u(x_1)$. Let $\{\beta_k\} \subset S_\varphi$ such that, $\beta_k(x_1) \to u(x_1)$. We define for $k \geq 1$, $z_k = \max\{\beta_1, ..., \beta_k, w_k\}$ then, $z_k^B \in S_\varphi$ and $w_k \leq z_k^B \leq u$, $v_k \leq z_k^B \leq u$ in $\overline{\Omega}$ thus,

$$\lim z_k^B(x_1) = u(x_1) \text{ and } \lim z_k^B(x_0) = u(x_0)$$

again $\{z_k^B\}$ converges in B to a harmonic function z with $z(x_1) = u(x_1)$. By construction $w \leq z$ in B and $w(x_0) = z(x_0) = u(x_0)$. Then, the function $z - w$ is harmonic nonnegative with an interior minimum equal to zero at x_0. The maximum principle establishes $z - w \equiv 0$ in B that leads to the contradiction

$$w(x_1) = z(x_1) = u(x_1) > w(x_1)$$

Thus, $u = w$ in B and u is harmonic in B and since x_0 is arbitrary, we conclude that u is harmonic in Ω. $\square$

In the preceding method still we have not guaranteed the existence of a solution of the Dirichlet problem with boundary condition given by φ. It is because in the Perron's method the study of boundary behavior of the solution is essentially separate from the existence problem. The assumption of boundary values is connected to the geometric properties of the boundary through the concept of barrier function.

Definition 2. *Let ξ be a point in $\partial\Omega$. Then, a $C^0(\overline{\Omega})$-function $w = w_\xi$ is called a barrier at ξ relative to Ω if*

(i) w is superharmonic in Ω.

(ii) $w > 0$ in $\overline{\Omega} \setminus \{\xi\}$ and $w(\xi) = 0$.

A more general definition of barrier requires only that the superharmonic function w is continuous and positive in Ω and that

$$w(x) \to 0 \text{ as } x \to \xi.$$

The barrier concept is a local property of the boundary $\partial\Omega$. Let us define w be a *local barrier* at $\xi \in \partial\Omega$ if there is a neighborhood N of ξ such that w satisfies the Definition 2 in $\Omega \cap N$.

Lemma 7. *If $\xi \in \partial\Omega$ has a local barrier, then, there exists a global barrier at ξ.*

Proof. Let N a neighborhood of ξ and w a barrier in $N \cap \Omega$. Let B be a ball satisfying $\xi \in B \subset\subset N$ and

$$m = \inf_{N \setminus B} w > 0$$

the function

$$\overline{w}(x) := \begin{cases} \min(m, w(x)) \text{ if } x \in \overline{\Omega} \cap B \\ m \text{ if } x \in \overline{\Omega} \setminus B \end{cases}$$

is continuous and $m, w(x)$ are superharmonic in $\overline{\Omega} \cap B$. Therefore, $\min(m, w(x))$

is superharmonic. Thus, $\overline{w}(x)$ satisfies the property (i). The property (ii) it is obtained because w is a local barrier. $\square$

A boundary point will be called *regular* (with respect to the Laplacian) if there exists a barrier at the point.

Theorem 6. *Let u be the harmonic function defined in Ω by the Perron's method. If ξ is a regular boundary point of Ω and φ is continuous at ξ, then*

$$u(x) \to \varphi(\xi) \ \ as \ \ x \to \xi$$

Proof. Choose $\epsilon > 0$, and let $M = \sup|\varphi|$. Since ξ is a regular boundary point, there is a barrier w at ξ and by virtue of the continuity of φ, there are constants δ and k such that

$$|\varphi(x) - \varphi(\xi)| < \epsilon \text{ as } \ |x - \xi| < \delta$$

and

$$kw(x) \geq 2M \text{ if } \ |x - \xi| \geq \delta.$$

We can verify that the functions $\varphi(\xi) + \epsilon + kw$, $\varphi(\xi) - \epsilon - kw$ are respectively superfunction and subfunction relative to φ. Hence, from definition of u and the fact that every superfunction dominates every subfunction we have in Ω

that

$$\varphi(\xi) - \epsilon - kw(x) \leq u(x) \leq \varphi(\xi) + \epsilon + kw(x)$$

or equivalently

$$|u(x) - \varphi(\xi)| \leq \epsilon + kw(x)$$

since $w(x) \to 0$ as $x \to \xi$, we obtain $u(x) \to \varphi(\xi)$ as $x \to \xi$. $\square$

Theorem 7. *The classical Dirichlet problem in a bounded domain is solvable for arbitrary continuous boundary values if and only if the boundary points are all regular.*

Proof. If the boundary values φ are continuous and the boundary $\partial\Omega$ consists of regular points, the preceding theorem states that the harmonic function provided by the Perron's method solves the Dirichlet problem. Conversely, suppose that the Dirichlet problem can be solved for all continuous boundary values. Let $\xi \in \partial\Omega$ then, the function

$$\varphi(x) = |x - \xi|$$

is continuous on $\partial\Omega$ and the harmonic function solving the Dirichlet problem in Ω with boundary values φ is a barrier at ξ. Hence, ξ is regular as are all points of $\partial\Omega$. $\square$

Remark 2. *A simple sufficient condition for solvability in a bounded domain $\Omega \subset \mathbb{R}^n$ is that Ω satisfies the exterior sphere condition, that is, for every point $\xi \in \partial\Omega$, there exist a ball $B = B_R(y)$ satisfying $\overline{B} \cap \overline{\Omega} = \{\xi\}$. If such a condition is fulfilled then, the function w given by*

$$w(x) := \begin{cases} R^{2-n} - |x-y|^{2-n} \text{ for } n \geq 3 \\ \\ \log \frac{|x-y|}{R} \text{ for } n = 2 \end{cases}$$

will be a barrier at ξ.

2.4 Obstacle Problem

One equivalent way to solve the Dirichlet problem is minimizing the Dirichlet Energy

$$E[u] = \frac{1}{2} \int_{\Omega} |Du|^2$$

over the set where u is equal to boundary data. The obstacle problem [10] consists in studying the properties of minimizers of the Dirichlet energy under the constraints that u less than φ on the boundary of Ω. It arises in the mathematical study of variational inequalities and free boundary problems. The idea is to find the largest subharmonic function that is less than the obstacle φ.

Theorem 8. *Let $\Omega \subseteq \mathbb{R}^n$, $\varphi \in C(\overline{\Omega})$ and S_φ the collection of subharmonic functions in $C(\overline{\Omega})$ that are below φ in $\overline{\Omega}$. Then, $u\colon \overline{\Omega} \to \mathbb{R}$ defined as*

$$u(x) = \sup_{v \in S_\varphi} v(x)$$

is continuous, subharmonic over Ω and harmonic over set $\{u < \varphi\}$.

Proof.

i) Define the function

$$u^*(x) = \limsup_{y \to x} u(y)$$

where u^* is the upper semicontinous envelope of the function u. The function u^* is the smallest upper semicontinuous function that is pointwise greater than or equal to u. Let us see that u^* is a subsolution for the problem

$$\begin{cases} \Delta u = 0 \text{ in } \Omega \\ u = \varphi \text{ on } \partial\Omega \end{cases}$$

Let $B \subset\subset \Omega$ and h be harmonic in B such that $u^* \leq h$ on ∂B. By definition $u \leq u^*$, thus $u \leq h$ on ∂B.

By definition of u, for each $v \in S_\varphi$ we have that $v \leq u$ thus, $v \leq h$ on

∂B but each v is subharmonic in Ω therefore, $v \leq h$ in B. By definition of u for each $\delta \geq 0$ there exists $v \in S_\varphi$ such that

$$u(x) - \delta \leq v(x) \text{ for fix } x \in B$$

hence, $u(x) - \delta \leq v(x) \leq h(x)$ but δ is arbitrary, therefore, $u(x) \leq h(x)$ in B.

Using this fact, by contradiction we suppose that there exists $x_0 \in B$ such that

$$u^*(x_0) > h(x_0)$$

by definition of u^* there exists a sequence $(x_k)_k$ such that, $u(x_k) \to u^*(x_0)$ as $x_k \to x_0$. Owing to that $u \leq h$ in B and $x_k \to x_0$ there exists a ball $B_\delta(x_0) \subset\subset B$ such that $x_k \in B_\delta(x_0)$ for all $k \geq M_0$ but,

$$u(x_k) \leq h(x_k), \quad k \geq M_0$$

thus

$$u^*(x_0) \leq h(x_0) \text{ as } k \to +\infty$$

therefore, u^* is subharmonic in Ω. Owing to u^* is the smallest upper semicontinuous function that is greater than or equal to u, and φ is upper semicontinuous and $\varphi \leq u$, then $\varphi \leq u^*$ therefore $u^* \in S_\varphi$ and $u^* = u$ is subsolution.

ii) Let us see that u is harmonic in the set $\{u < \varphi\}$.

Let us suppose that u is not harmonic in $\{u < \varphi\}$. Then, by Corollary 3 there exists $x_0 \in \{u < \varphi\}$ and a sequence of radius $r_i \to 0$ such that

$$u(x_0) \neq \fint_{B_{r_i}(x_0)} u dy$$

since u is subharmonic, we have

$$u(x_0) < \fint_{B_{r_i}(x_0)} u dy = \fint_{B_{r_i}(x_0)} u^{B_i} = u^{B_i}(x_0)$$

thus, $u(x_0) < u^{B_i}(x_0)$. Now, $u \leq \varphi$, $u(x_0) < \varphi(x_0)$ and since $u^{B_i} = u$ on $\Omega \setminus B_i$, then, $u^{B_i} \leq \varphi$ in $\Omega \setminus B_i$.

Owing to $(\varphi - u)(x_0) > 0$ there exists $\delta > 0$ with the property that $(\varphi - u)(x_0) > \delta$ and by virtue of the continuity of φ we have that there exists r_j such that

$$\inf_{B_{r_j}(x_0)} \varphi \geq \varphi(x_0) - \delta/2$$

also, by upper semicontinuity of u there exists r_k such that

$$u(x_0) \geq \sup_{\partial B_{r_k}(x_0)} u - \delta/2.$$

Taking $r = \min\{r_k, r_j\}$ we have that

$$\begin{aligned}\inf_{B_r(x_0)} \varphi \geq \varphi(x_0) - \delta/2 &> u(x_0) - \delta/2 + \delta \\ &\geq \sup_{\partial B_r(x_0)} u - \delta/2 - \delta/2 + \delta = \sup_{\partial B_r(x_0)} u\end{aligned}$$

therefore

$$\inf_{B_r(x_0)} \varphi \geq \sup_{\partial B_r(x_0)} u = \sup_{\overline{B}_r(x_0)} u^{B_r}$$

hence, $u^{B_r} \leq \varphi$ in $\overline{\Omega}$ therefore, $u^{B_r} \in S_\varphi$. This contradicts the fact $u^{B_r}(x_0) > u(x_0)$, thus, u is harmonic in the set $\{u < \varphi\}$.

iii) Finally, as u harmonic in $\{u < \varphi\}$ then, support$(\Delta u) \subseteq \{u = \varphi\}$ and as φ is continuous then, u is continuous relative to set $\{u = \varphi\}$ therefore by Evans' Lemma [10, Theorem 1] u is continuous in $\overline{\Omega}$.

$\square$

2.5 Schwarz Method

We will present the main result of this chapter and we will discuss its implementation.

Theorem 9 (Schwarz Method). *Let $\Omega_1, \Omega_2 \subseteq \mathbb{R}^n$ open and bounded domains with $\Omega_1 \cap \Omega_2 \neq \emptyset$ such that, for each one of them the classical Dirichlet problem is solvable for arbitrary continuous boundary values. Given $\Omega = \Omega_1 \cup \Omega_2$ and $g \in C^0(\overline{\Omega})$ be a subharmonic function in Ω, consider the sequences $(u_i)_{i\in\mathbb{N}}$ such that, $u_0 = g$ and $(v_i)_{i\in\mathbb{N}}$ where v_i is the unique solution for*

$$\begin{cases} \Delta v_i = 0 \ \textit{in} \ \Omega_1 \\ v_i = u_{i-1} \ \textit{on} \ \overline{\Omega} \setminus \Omega_1 \end{cases}$$

and u_i is the unique solution for

$$\begin{cases} \Delta u_i = 0 \ \textit{in} \ \Omega_2 \\ u_i = v_i \ \textit{on} \ \overline{\Omega} \setminus \Omega_2, \end{cases}$$

then $(u_i)_{i\in\mathbb{N}}$ converges uniformly to $u \in C^2(\Omega) \cap C(\overline{\Omega})$ the unique solution to

$$\begin{cases} \Delta u = 0 \ \textit{in} \ \Omega \\ u = g \ \textit{on} \ \partial\Omega. \end{cases} \tag{2.5}$$

Proof.

i) $(u_n)_{n\in\mathbb{N}}$ is increasing monotone:

We will do the proof by induction. For $n = 1$ let us see that $g = u_0 \leq u_1$ in $\overline{\Omega}$. In $\Omega_1 \setminus \Omega_2$ we have that $u_0 = g \leq v_1 = u_1$ (maximum principle) and in Ω_2, u_1 is harmonic and $u_1 = v_1 \geq g$ on $\overline{\Omega}\setminus\Omega_2$. Then, $u_0 = g \leq u_1$ in $\overline{\Omega_2}$ once more from the maximum principle. Thus, $u_0 \leq u_1$ in $\overline{\Omega}$. Suppose that

$$u_{i-1} \leq u_i \text{ in } \overline{\Omega}.$$

Define the function $h = u_i - u_{i+1}$ then,

$$\begin{cases} \Delta h = 0 \text{ in } \Omega_2 \\ h = v_i - v_{i+1} \text{ on } \overline{\Omega} \setminus \Omega_2. \end{cases}$$

Now, take $k = v_i - v_{i+1}$ hence, it satisfies

$$\begin{cases} \Delta k = 0 \text{ in } \Omega_1 \\ k = u_{i-1} - u_i \leq 0 \text{ on } \overline{\Omega} \setminus \Omega_1 \end{cases}$$

by virtue of the maximum principle $k = v_i - v_{i+1} \leq 0$ in $\overline{\Omega}$ therefore, $h \leq 0$ in $\overline{\Omega}$ then, $u_i \leq u_{i+1}$.

ii) Each u_n is subharmonic in Ω:

Note that $u_0 = g$ is subharmonic by assumption. Suppose that u_{n-1} is subharmonic in Ω. Let $B \subset\subset \Omega$ and ψ be harmonic in B such that $u_n \leq \psi$ in ∂B, as $u_{n-1} \leq u_n$ then, $u_{n-1} \leq u_n \leq \psi$ on ∂B and so $u_{n-1} \leq \psi$ in B.

Now, in $\Omega_1 \cap \Omega_2$ u_n, v_n are harmonics, and if we define $h_n = u_n - v_n$, then h_n is harmonic in $\Omega_1 \cap \Omega_2$. Also, $h_n = 0$ on $\partial\Omega_2 \cap \Omega_1$ and $h_n = u_n - u_{n-1} \geq 0$ on $\partial\Omega_1 \cap \Omega_2$ therefore $h_n \geq 0$ on $\partial(\Omega_1 \cap \Omega_2)$ by the maximum principle $h_n = u_n - v_n \geq 0$ in $\Omega_1 \cap \Omega_2$.

Using this fact, we see that $v_n \leq u_n \leq \psi$ on $\partial B \cap (\Omega_1 \cap \Omega_2)$, on $\partial B \cap (\Omega \setminus \Omega_2)$ $v_n = u_n \leq \psi$ and on $B \cap \partial\Omega_1$ $v_n = u_{n-1} \leq \psi$. Therefore, $v_n \leq \psi$ on $\partial(B \cap \Omega_1)$ but v_n is harmonic in Ω_1 thus, $v_n \leq \psi$ in $B \cap \Omega_1$. In particular, $u_n = v_n \leq \psi$ on $\partial\Omega_2 \cap B$ then, $u_n \leq \psi$ on $\partial(B \cap \Omega_2)$ and u_n harmonic in $B \cap \Omega_2$ therefore, $u_n \leq \psi$ in $B \cap \Omega_2$ and $u_n = v_n \leq \psi$ in $B \cap (\Omega_1 \setminus \Omega_2)$ thus, $u_n \leq \psi$ in B.

iii) Finally, note that v_n is harmonic in Ω_1 and $v_n \geq u_{n-1}$ on $\overline{\Omega} \setminus \Omega_1$. Since, u_{n-1} is subharmonic then, $u_{n-1} \leq v_n$ in Ω. Using this fact and the same process as above for u_n, we show that v_n is subharmonic in Ω and again as u_n is harmonic in Ω_2 and $v_n \leq u_n$ on $\overline{\Omega} \setminus \Omega_2$ we have that $v_n \leq u_n$. Thus,

$$u_{n-1} \leq v_n \leq u_n$$

the previous estimate allows us to conclude

$$\lim_{n \to +\infty} u_n = \lim_{n \to +\infty} v_n$$

and since $u_n, v_n \leq \max_{\overline{\Omega}} g$ we have that u_n, v_n converge pointwise and by Harnack's inequality and the fact u_n, v_n are increasing monotone, then u_n converges locally uniformly in Ω_2, v_n converges locally uniformly in Ω_1 and this limits are harmonics in Ω_1, Ω_2 respectively. Now, by condition $u_{n-1} \leq v_n \leq u_n$

$$\lim_{n \to +\infty} u_n(x) = \lim_{n \to +\infty} v_n(x) = u(x)$$

where $u(x)$ is harmonic in $\Omega = \Omega_1 \cup \Omega_2$ and solves Problem 2.5. The uniqueness is immediately from Theorem 2.

$\square$

2.5.1 Implementation Schwarz Method

In this section, we will show an idea about the implementation of the Schwarz's method. The idea was take some region, that can be described as the union of sets more simples for which the Dirichlet problem is solvable and this solution can be gotten by an algorithm.

First, we will present as solving the Dirichlet problem over a rectangle in $\mathbb{R}^2$, using the Fourier's method. Based on these ideas, to build a discrete version of the Fourier's method in order to implement this algorithm.

Consider the set

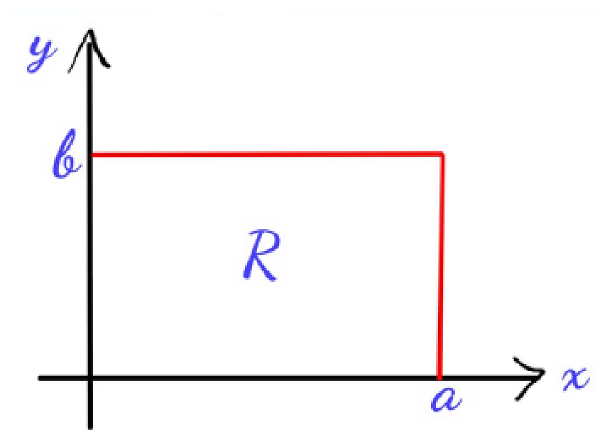


Figure 2.1: $R = \{(x, y)|\, 0 \leq x \leq a,\ 0 \leq y \leq b\}$

and

$$\begin{cases} \Delta u = f & \text{in } R \\ u = g & \text{on } \partial R \end{cases} \tag{2.6}$$

we know that the convolution $p = \Phi * f$ with Φ fundamental solution for $\Delta u = 0$ is a particular solution for $\Delta u = f$ thus, it is only necessary to solve

$$\begin{cases} \Delta u = 0 & \text{in } R \\ u = g - p & \text{on } \partial R. \end{cases} \tag{2.7}$$

If h is solution for (2.7) then, $u = h + p$ will be solution for the problem (2.6). Now, we establish the conditions:

$$(C) := \begin{cases} u(x,0) = g_1^*(x); \quad u(x,b) = g_2^*(x) \\ u(0,y) = g_3^*(y); \quad u(a,y) = g_4^*(y) \end{cases}$$

where $g_i^* = g_i - p_i$, g_i, p_i are the values that are taking g and p on ∂R_i and ∂R_i represent either side of R and $g_j^* = 0$ if $j \neq i$.

Due to the lineality of the Laplace's operator, it is possible to define the problem:

$$\begin{cases} \Delta u_i = 0 & \text{in } R \\ u_i = g_i^* & \text{on } \partial R. \end{cases} \tag{2.8}$$

The solution for (2.7) can be written as

$$u = \sum_1^4 u_i$$

where each u_i is the solution for (2.8). The general solution for (2.6) will be

$$u = \sum_{1}^{4} u_i + p.$$

Hence, we will compute the solution for the problem

$$\begin{cases} \Delta u = 0 & \text{in } R \\ u(0,y) = u(a,y) = 0, \quad u(x,b) = 0 \\ u(x,0) = g_1. \end{cases} \tag{2.9}$$

Let us suppose that there exists a solution of the form $u(x,y) = X(x)Y(y)$. We will employ the method of separation of variables for finding $X(x)$ and $Y(y)$ such that,

$$u(x,y) := X(x)Y(y)$$

be a solution for (2.9). The Laplacian for u will be

$$\Delta u = \partial_x^2 u + \partial_y^2 u = X''(x)Y(y) + X(x)Y''(y) = 0$$

dividing by $X(x)Y(y)$, we get

$$\frac{X''}{X} + \frac{Y''}{Y} = 0 \text{ then } \frac{X''}{X} = -\frac{Y''}{Y} = -\lambda^2.$$

Or equivalently

$$\begin{cases} X'' = -\lambda^2 X \\ X(0) = X(a) = 0, \end{cases} \tag{2.10}$$

$$\begin{cases} Y'' = \lambda^2 Y \\ Y(b) = 0. \end{cases} \tag{2.11}$$

In the first problem, a simple computation shows that

$$X(x) = A_1 \sin(\lambda x) + A_2 \cos(\lambda x),$$

using the initial condition $X(0) = 0 = A_2$ and $X(a) = 0 = \sin(\lambda a)$ we can get that

$$\lambda_n = \frac{n\pi}{a},$$

then

$$X_n(x) = \sin(\frac{n\pi}{a}x)$$

and using the identity

$$2\sin(mx)\sin(nx) = \cos((m-n)x) - \cos((m+n)x),$$

we can verify that $< X_n, X_k >= 0$. Furthermore, for the problem (2.11)

$$Y'' = \lambda^2 Y \text{ then } Y'' = \left(\frac{n\pi}{a}\right)^2 Y,$$

hence

$$Y_n(y) = A_n \frac{\sinh(\frac{n\pi}{a}(b-y))}{\sinh(\frac{n\pi b}{a})}.$$

Therefore

$$u_n(x,y) = Y_n(y)X_n(x) = \left[A_n \frac{\sinh(\frac{n\pi}{a}(b-y))}{\sinh(\frac{n\pi b}{b})}\right] \sin(\frac{n\pi}{a}x).$$

The family of functions $\{u_n(x,y)\}$ is an orthogonal set, because of the inner product

$$< u_n, u_m >= \int_a^b \int_a^b X_n(x)Y_n(y)X_m(x)Y_m(y)dxdy = \Big(\int_a^b X_n(x)X_m(x)\Big)\Big(\int_a^b Y_n(y)Y_m(y)\Big) = 0$$

for $n \neq m$. Each function u_n is a solution of the Laplace's equation in Ω which satisfies the boundary conditions $u(x,b) = 0$, $u(0,y) = 0$, and $u(a,y) = 0$. The Laplace's equation is linear, therefore we can take any combination of solutions $\{u_n\}$ and get a solution of the Laplace's equation which satisfies these three boundary conditions. In this case

$$u(x,y) = \sum_{n=1}^{\infty} u_n(x,y) = \sum_{n=1}^{\infty} \Big[A_n \frac{\sinh(\frac{n\pi}{a}(b-y))}{\sinh(\frac{n\pi b}{b})}\Big] \sin(\frac{n\pi}{a}x)$$

will be a solution. This solution should satisfy the condition $u(x,0) = g_1(x)$, then

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin(\frac{n\pi}{a}x) = g_1(x)$$

that is, we want to be able to express g_1 in terms of its Fourier sine series on $[0,a]$ hence, the coefficients A_n are given by

$$A_n = \frac{\langle g_1 | \sin(\frac{n\pi}{a}x)\rangle}{\langle \sin(\frac{n\pi}{a}x) | \sin(\frac{n\pi}{a}x)\rangle},$$

where the $\langle , \rangle$ is the inner product in $L^2(0,a)$, thus

$$\langle \sin(\frac{n\pi}{a}x) | \sin(\frac{n\pi}{a}x) \rangle := \int_0^a \sin^2(n\pi x/a)dx = \frac{a}{2}$$

therefore

$$A_n = \frac{2}{a} \langle g_1 | \sin(\frac{n\pi}{a}x) \rangle .$$

Suppose that we want to write a code to solve the problem

$$\begin{cases} \Delta u = 0 \text{ in } R \\ u = g \text{ on } \partial R, \end{cases} \tag{2.12}$$

with $R = \{(x,y) |\ 0 < x < a,\ 0 < y < b\}$. In the first example we saw how to solve this problem employing the method of separation of variables. In order to do a numerical implementation we will proceed to give a discretization of the problem and to present a discrete version of the Fourier's method.

Define g in R and $N \in \mathbb{Z}^+$ such that, $\frac{bN}{a} \in \mathbb{Z}^+$ (N size of the discretization). The discretization of g in R is

$$g_{ij} = g(\frac{ia}{N}, \frac{jb}{N}) \text{ for } i \in \{0, ..., N\},\ j \in \{0, ..., M = \frac{Nb}{a}\},$$

where g_{ij} satisfies

$$g_{ij} \leq \frac{g_{i+1,j} + g_{i-1,j} + g_{i,j+1} + g_{i,j-1}}{4}.$$

The idea for solving 2.12 is the same as in the continuous case. We will employ the method of separation of variables and we will solve in each side of rectangle R with their respective boundary conditions. If we write $u_{ij} = x_i y_j$, then

$$0 = \Delta u_{ij} = \frac{(x_{i+1} - 2x_i + x_{i-1})y_j + x_i(y_{j+1} - 2y_j + y_{j-1})}{h^2}$$

where $h = 1/N$ is the step. Dividing by u_{ij} we get

$$x_{i+1} - 2x_i + x_{i-1} = -2\lambda^2 x_i,$$

and

$$y_{i+1} - 2y_i + y_{i-1} = 2\lambda^2 y_j.$$

The first problem has conditions

$$x_0 = 0, \ x_N = 0,$$

and the second one problem $y_M = 0$, in the case when we have boundary data $g_{i,0}$.

The solution for the first problem is

$$(x_i)_n = \sin\left(\frac{n\pi i}{N}\right) \ (n \in \mathbb{N}),$$

with $\lambda_n^2 = 2(1 - \cos\left(\frac{n\pi}{N}\right))$. For the second one problem the solution is

$$y_j = A_n\left(r_+^j - \left(\frac{r_+}{r_-}\right)^M r_-^j\right)$$

where r_+, r_- are the roots of the characteristic polynomial associated to

$$y_{j+1} - (2 + 2\lambda_n^2)y_j + y_{j-1} = 0.$$

The solution u_{ij} can be written as

$$u_{ij} = \sum_{n=1}^{N-1} A_n\left(r_+^j - \left(\frac{r_+}{r_-}\right)^M r_-^j\right) \sin(\frac{n\pi i}{N})$$

where A_n it is the Fourier discrete coefficient given by

$$A_n = \frac{\sum_{i=1}^{N-1} g_{i,0} \sin(\frac{n\pi i}{N})}{\left(1 - \left(\frac{r_+}{r_-}\right)^M \sum_{i=1}^{N-1} \sin^2(\frac{n\pi i}{N})\right)}$$

u_{ij} solves when $g_{i,M} = g_{0,j} = g_{N,j} = 0$. Note that if we change in the coefficient A_n, $g_{i,0}$ by $g_{i,M}$ then, $u_{i,j} = u_{i,M-j}$ solves for $g_{i,0} = g_{0,j} = g_{N,j} = 0$ and the same for other sides of the boundary. Now, the superposition of these solutions solves the problem 2.12.

Consider the problem

$$\begin{cases} \Delta u = 0 \text{ in } R \\ u = g \text{ on } \partial R \end{cases} \tag{2.13}$$

for $R = R_1 \cup R_2$ where

$$R_1 = \{(x, y) \,|\, 0 < x < a,\, 0 < y < b\}$$

,

$$R_2 = \{(x, y) \,|\, 0 < x < b,\, 0 < y < a\}.$$

Note that R is a domain for which to build a solution is difficult, but if we take each R_1, R_2 separately, using the discrete method described previously it is possible to solve for each rectangle. But, we need a solution for R. Thus, the idea is to employ the Iterarive Schwarz Method for calculating in a step finite number an approximate solution in R.

Also, we can take other union of rectangles (or domains) more sophisticated and employ the same idea.

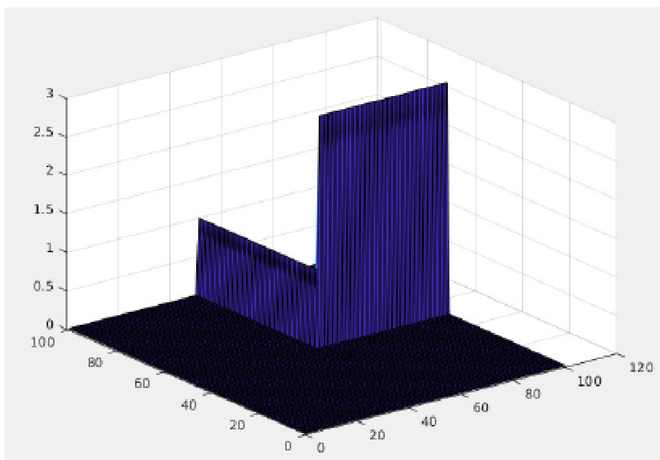
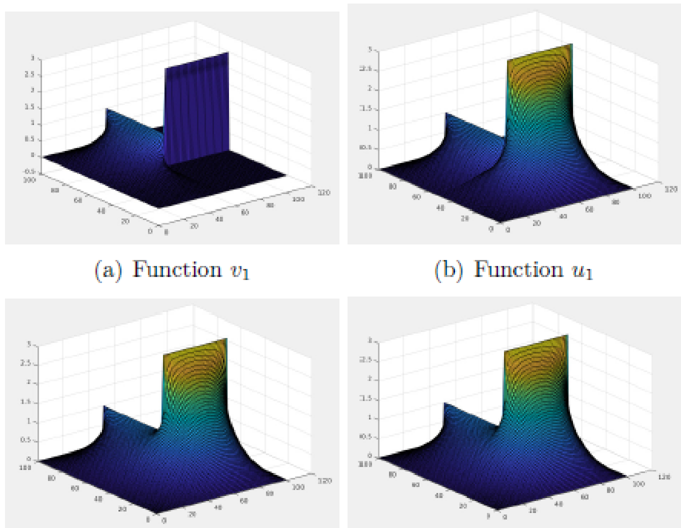


Figure 2.2: Initial condition $u_0 = g$

where

$$g(x,y) := \begin{cases} 1, & \text{if } x = 1 \text{ and } 1 \leq y \leq 2 \\ 3, & \text{if } y = 1 \text{ and } 1 \leq x \leq 2 \\ 0, & \text{In the rest of } [0,2]^2. \end{cases}$$



(a) Function v_1 (b) Function u_1

(c) Function v_2 (d) Function u_2

Figure 2.3: First four iteration using the Schwarz's method.

The figure above shows the first four iterations, using Schwarz's method on the region $R = R_1 \cup R_2$ where

$$R_1 = \{(x, y) \,|\, 0 < x < 2 \text{ and } 0 < y < 1\}$$

$$R_2 = \{(x, y) \,|\, 0 < x < 1 \text{ and } 0 < y < 2\}$$

with boundary data $u_0 = g$. Note that between the third and fourth iteration the difference is no longer noticeable.

Remark 3. *If we take the discretization of g as a matrix "$n \times n$", it is necessary to solve n^2 equations with n^2 unknow variables. Therefore, This* *requires a number of computations of order* n^6. *But, through the Fourier's method we only need to compute* n^2 *coefficients and it is only necessary* n^2 *steps hence, the Fourier's method is more efficient.*

Chapter 3

Balayage In Graphs

In this chapter, we will introduce the concept of balayage (Classic and Partial) and we will prove the main result about the balayage process.

We will present some important results about harmonic and subharmonic functions as in Chapter 2, but in graphs. Principally, it will be related to the maximum principle and Perron's method. Also, we establish through some examples the relationship between harmonic functions in graphs, electrical networks and random walks as in [9].

3.1 Preliminary definitions

A graph consists of a finite number of points (called vertices) and a finite number of lines (called edges) joining some of them. We will denote a graph by $G = (V, E)$ where V denotes the vertices and E the edges, and assume that our graph is oriented. In the notation, given an $e \in E$ we denote by $e_- \in V$ its starting point and by $e_+ \in V$ its final point. We also denote a special vertex by ∞ (which is an analogue of a point at ∞ in $\mathbb{R}^n$) such that, every vertex can be reached from ∞ by a path that ignores the orientation of the edges. Let $x, y \in V$, if there exist an edge e such that $x = e_-$ and $y = e_+$ we say that x and y are adjacent, and we denote it by $x \sim y$.

For a subset $D \subseteq V \setminus \{\infty\}$, we define $\partial D = \partial_+ D \cup \partial_- D$ by

$$\partial_\pm D = \{v \in V \setminus D : \text{ There exists } e \in E \text{ such that } e_\pm = v \text{ and } e_\mp \in D\}$$

and $\nu : \partial D \to \mathbb{R}$ such that $\nu_D(e) = \pm 1$, if $e \in \partial_\pm D$ analogue to the normal vector.

We will introduce the basic definitions about subharmonic, superharmonic and harmonic functions and their basic properties in networks.

Definition 3. *Given a scalar field $U \colon V \to \mathbb{R}$, we define the gradient field* *by*

$$DU(e) = U(e_+) - U(e_-)$$

Definition 4. *Given a measure μ in $D \subseteq V$, we define the measure of D with respect to μ by*

$$\mu(D) = \sum_{v \in D} \mu(v)$$

and the integral of U with respect to the measure μ by

$$\int U d\mu = \sum_{v \in V} U(v)\mu(v).$$

For a vector field $i : \mathbb{R}^n \to \mathbb{R}^n$

$$\begin{aligned}\nabla \cdot i(x) &= \lim_{\epsilon \to 0} \frac{1}{|B_\epsilon(x)|} \int\limits_{B_\epsilon(x)} \nabla \cdot i(y) dy \\ &= \lim_{\epsilon \to 0} \frac{n}{|\partial B_\epsilon(x)|} \int\limits_{\partial B_\epsilon(x)} i(y) \cdot \nu(y) dS(y),\end{aligned}$$

where $\nu(y) = (y - x)/\epsilon$ is the exterior normal vector at $y \in \partial B_\epsilon(x)$. Thus, for $i\colon E \to \mathbb{R}$ we define

$$\nabla \cdot i(v) = \frac{1}{|\{e \in E : v \in \{e_+, e_-\}|} \left(\sum_{e \in E | e_- = v} i(e) - \sum_{e \in E | e_+ = v} i(e) \right).$$

Under these constructions

$$\nabla \cdot i(v) = \frac{1}{\#\partial\{v\}} \sum_{e\in\partial\{v\}} i(e)\nu_v(e).$$

3.2 Divergence's Theorem

Remember that for a field $i : \mathbb{R}^n \to \mathbb{R}^n$ and $\Omega \subseteq \mathbb{R}^n$, if $i \in C^1(\Omega)$ and Ω is simply connected with $\partial\Omega$ regular, then

$$\int\limits_{\Omega} \nabla \cdot i = \int\limits_{\partial\Omega} i \cdot \nu dS.$$

It is known as the Divergence Theorem.

Theorem 10 (Divergence's Theorem on graphs)**.** *Let $i : E \to R$ and D, ∂D as above, then*

$$\int\limits_{D} \nabla \cdot i = \int\limits_{\partial D} i \cdot \nu dS$$

Proof. We will define

$$\int\limits_{D} \nabla \cdot i(v)dv = \sum_{v\in D} \nabla \cdot i(v)\#\partial\{v\}.$$

We will verify that if $e \in E$ such that, $e_\pm \in D$, then, the contribution of $i(e)$

in the integral $\int\limits_D \nabla \cdot i(v)$ is zero (i.e, the contribution is only by part of $e \in E$ such that if $e_\pm \in D$, then $e_\mp \in \partial D$). Let us see this. Let $v, w \in D$ such that $v \sim w$ (v is adjacent to w) and define the edge joining v with w by $e_{vw} = e_{wv}$. Without loss of generality, suppose that $(e_{vw})_- = v$ and $(e_{vw})_+ = w$, then

$$\begin{aligned}\nabla \cdot i(v)\#\partial\{v\} &= \sum_{\{e\,:\,e_-=v\}} i(e) - \sum_{\{e\,:\,e_+=v\}} i(e)\\ &= i(e_{vw}) + \sum_{\substack{\{e\,:\,e_-=v,\}\\ e_+\neq w}} i(e) - \sum_{\{e\,:\,e_+=v\}} i(e)\end{aligned}$$

and

$$\begin{aligned}\nabla \cdot i(w)\#\partial\{w\} &= \sum_{\{e\,:\,e_-=w\}} i(e) - \sum_{\{e\,:\,e_+=w\}} i(e)\\ &= \sum_{\{e\,:\,e_-=w\}} i(e) - \sum_{\substack{\{e\,:\,e_+=w,\}\\ e_-\neq v}} i(e) - i(e_{wv}).\end{aligned}$$

Thus, in the term $(\nabla \cdot i(v)\#\partial\{v\} + \nabla \cdot i(w)\#\partial\{w\})$ the contribution by $i(e_{vw}) = 0$. This proves that the contribution is only by part of $e \in E$ such that if $e_\pm \in D$ then, $e_\mp \in \partial D$. It means that

$$\begin{aligned}\sum_{x\in D} \nabla \cdot i(v)\#\partial\{v\} &= \sum_{\substack{\{e:e_-\in D,\}\\ e_+\in\partial D}} i(e) - \sum_{\substack{\{e:e_+\in D,\}\\ e_-\in\partial D}} i(e)\\ &= \sum_{\{e\in\partial D\}} i(e)\nu(e)\\ &= \int\limits_{\partial D} i\cdot\nu.\end{aligned}$$

$\square$

Definition 5. *For $i, j : E \to \mathbb{R}$ the product $i \cdot j(v)$ is defined by*

$$i\cdot j(v) := \frac{1}{2\#\partial\{v\}} \sum_{e\in\partial\{v\}} i(e)j(e),$$

where $\#\partial\{v\}$ denotes the number of elements that are adjacent to the vertex v.

Now, we will prove an analogue result of the formula of integration by parts in the graph which will be very useful.

Theorem 11 (Integration by Parts)**.** *Let $U : V \to R$ and $i : E \to R$ then, for $D \subseteq V \setminus \{\infty\}$, we have the identity*

$$\int\limits_D U\nabla\cdot i = \int\limits_{\partial D} (Ui\cdot\nu) - \int\limits_D i\cdot DU.$$

Proof. First, it is necessary to verify the product rule

$$\nabla \cdot (\overline{U}i)(v) = U(v)\nabla \cdot i(v) + (i \cdot DU)(v)$$

where

$$\overline{U} : E \to \mathbb{R}, \text{ with } \overline{U}(e) = \frac{1}{2}(U(e_+) + U(e_-)).$$

Note that

$$\nabla \cdot (\overline{U}i)(v) = \frac{1}{\#\partial\{v\}}\left(\sum_{e_-=v} \frac{(U(v) + U(e_+))}{2} i(e) - \sum_{e_+=v} \frac{(U(v) + U(e_-))}{2} i(e)\right)$$
$$= \frac{1}{\#\partial\{v\}}\frac{U(v)}{2}\left(\sum_{e_-=v} i(e) - \sum_{e_+=v} i(e)\right) + \frac{1}{\#\partial\{v\}}\left(\sum_{e_-=v} \frac{U(e_+)i(e)}{2} - \sum_{e_+=v} \frac{U(e_-)i(e)}{2}\right)$$
$$= \frac{U(v)}{2}\nabla \cdot i(v) + \frac{1}{\#\partial\{v\}}\left(\sum_{e_-=v} \frac{U(e_+)i(e)}{2} - \sum_{e_+=v} \frac{U(e_-)i(e)}{2}\right).$$

Adding and substracting the term

$$\frac{U(v)\nabla \cdot i(v)}{2},$$

in the above equation, we get

$$\nabla \cdot (\overline{U}i)(v) = U(v)\nabla \cdot i(v) + \frac{1}{\#\partial\{v\}} \sum_{e\in\partial\{v\}} \frac{DU(e)i(e)}{2}$$

$$= U(v)\nabla \cdot i(v) + (i \cdot DU)(v).$$

Thus, applying the Gauss formula to the function $(\overline{U}i)$ we get

$$\int_D \nabla \cdot (\overline{U}i) = \int_{\partial D} Ui \cdot \nu$$

and applying the product rule for $\nabla \cdot (\overline{U}i)$ we deduce

$$\int_D U\nabla \cdot i = \int_{\partial D} Ui \cdot \nu - \int_D i \cdot DU.$$

$\square$

3.3 Maximum Principle

We will define the Laplacian in the graph and then we will prove analogues for the mean value formula and the maximum principle.

Definition 6. *Let $G = (V, E)$ a graph, we define $\Delta U : V \to \mathbb{R}$ by*

$$\Delta U(v) = \frac{1}{\#\partial\{v\}} \sum_{w \sim v} (U(w) - U(v)).$$

Definition 7. *We say that the function U is subharmonic (resp. superharmonic) if $\Delta U \geq 0$ (resp. $\Delta U \leq 0$).*

Remark 4. *If U be subharmonic in $D \subseteq V \setminus \{\infty\}$ then, for each $v \in D$ we have*

$$U(v) \leq \frac{1}{\#\partial\{v\}} \sum_{w \sim v} U(w).$$

Now, we will present the main result in this section and then, we will show a small relationship with other fields.

Theorem 12 (Maximum Principle)**.** *Let U be subharmonic in D. Then, U attains its maximum in ∂D.*

Proof. Suppose that there exists $v \in D$ such that,

$$U(v) = M = \max_{\overline{D}} U$$

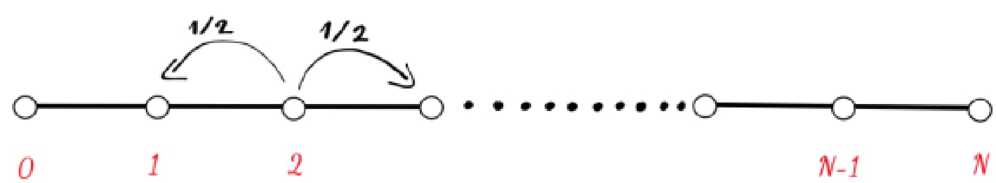
since U subharmonic, then

$$U(v) \leq \frac{1}{\#\partial\{v\}} \sum_{w \sim v} U(w)$$

but $U(v) \geq U(w)$ for all $w \in D$. Thus, $U(w) = M$ for $w \sim v$. Continuing in this form over any $w \sim v$ we can conclude that $U(z) = M$ for $z \sim w$ and proceeding similarly, we have that $U = M$ on ∂D. $\square$

Remark 5. *If U is superharmonic, the same argument for $-U$ prove that U attains its minimum on ∂D.*

Example 3. *Suppose that we have the next graph*



with labels $\{0, 1, 2, ..., N\}$ for each vertex. Suppose that the vertices $0, N$ are fixed points in the sense that if we are in these points, we stay there always.

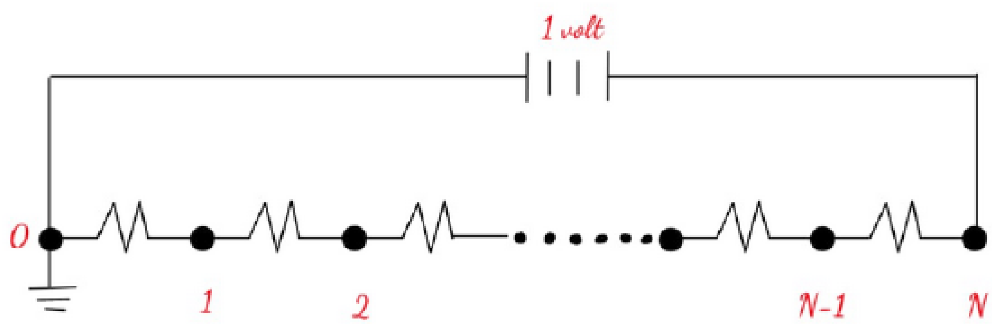
The problem consist in to find the probability $p(x)$ that if a walker, starting a random walk at the vertex $x \neq 0, N$, will reach the vertex N before reaching the vertex 0.

Suppose that, in each step the walker only can go to the right (vertex $x+1$) or to the left (vertex $x-1$), with the same probability. If we think the event E as "the walker ends at the vertex N", F is the event "the first step is to the *left", and G the event "the first step is to the right", then, the walker starts in the vertex x, $p(E) = p(x)$, $p(F) = p(G) = 1/2$ and $p(E \ givenF) = p(x-1)$, $p(E \ given \ G) = p(x+1)$ so, by conditional probability we have that*

$$p(x) = \frac{1}{2}p(x-1) + \frac{1}{2}p(x+1)$$

that is, the function p is harmonic for $x \in \{1, ..., N-1\}$.

Example 4. *Now, let us considere the next electrical network*



The idea is to determine the voltages $v(x)$ for each $x = 1, 2, ..., N-1$. Similar as the previous example in the random walk, we suppose that we connected the node $x = 0$ with $x = N$ putting a unit voltage, and all resistances are equal. By Ohm's Law, if the nodes x and y are adjacent ($x \sim y$) by a resistance of magnitude R, then the current i_{xy} that flows from x to y is equal to

$$i_{xy} = \frac{v(x) - v(y)}{R}$$

now, by Kirchhoff's law, the current flowing into x must be equal to the current flowing out, thus

$$\begin{aligned} i_{x-1,x} &= i_{x,x+1}, \text{ that is} \\ \frac{v(x-1)-v(x)}{R} &= \frac{v(x)-v(x+1)}{R}, \end{aligned}$$

therefore

$$v(x) = \frac{1}{2}(v(x-1)+v(x+1)),$$

this implies that $v(x)$ is harmonic for $x = 1, ..., N-1$. Hence, both $v(x)$ and $p(x)$ are harmonics.

3.4 Dirichlet Problem

Given a domain $D \subseteq V \setminus \{\infty\}$ and a function $g : \partial D \to \mathbb{R}$ the Dirichlet problem consists in finding a function u satisfying

$$\begin{cases} \Delta u = 0 \text{ in } D \\ u = g \text{ on } \partial D \end{cases}$$

Note that if u and w satisfy this problem, and we define $h = u - w$, then h satisfies

$$\begin{cases} \Delta h = 0 \text{ in } D \\ h = 0 \text{ on } \partial D \end{cases}$$

and by the maximum principle $u = w$ in D. In the graph, it is equivalent to solve a system of linear equations, but another method is to find the minimum for the Dirichlet energy, that is to minimize

$$\min \left\{ E[u] = \frac{1}{2} \sum_{v,w} (Du(e_{vw}))^2 \text{ such that } u = g \text{ on } \partial D \right\}$$

where e_{vw} is the edge joining v with w.

Note that for us definition of Laplacian and Divergence in the graph, we have

$$\begin{aligned} \nabla \cdot Du(v) &= \frac{1}{\#\partial\{v\}} \left(\sum_{e_- = v} Du(e) - \sum_{e_+ = v} Du(e) \right) \\ &= \frac{1}{\#\partial\{v\}} \left(\sum_{e_- = v} (u(e_+) - u(v)) - \sum_{e_+ = v} (u(v) - u(e_-)) \right) \\ &= \frac{1}{\#\partial\{v\}} \sum_{w \sim v} (u(w) - u(v)) = \Delta u(v). \end{aligned}$$

Theorem 13. *If u minimizes the energy $E[u]$ such that, $u = g$ on ∂D, then*

u solves the Dirichlet problem

$$\begin{cases} \Delta u = 0 \ \textit{in} \ D, \\ \\ u = g \ \textit{on} \ \partial D. \end{cases}$$

Proof. Suppose that u minimizes the energy E, thus, for all φ such that, $\varphi|_{\partial D} = 0$ the function

$$h(t) = E[u + t\varphi] \text{ it is minimized for } t = 0$$

thus

$$\frac{d}{dt}h(t) = 0 \text{ for } t = 0,$$

therefore

$$\frac{d}{dt}E[u + t\varphi]|_{t=0} = \sum_{v,w}(Du + tD\varphi)D\varphi|_{t=0} = \sum_{e\in D} Du(e)D\varphi(e),$$

now, remember that

$$\frac{1}{2\#\partial\{v\}} \sum_{e_\pm = v} Du(e)D\varphi(e) = Du \cdot D\varphi(v).$$

Therefore

$$\sum_{e \in D} Du(e) D\varphi(e) = \sum_{v \in D} (Du \cdot D\varphi)(v) \# \partial\{v\}$$
$$\int\limits_{D} Du \cdot D\varphi = \int \varphi Du \cdot \nu dS - \int\limits_{D} \varphi \nabla \cdot (Du)$$

by integration by parts. Thus

$$\frac{d}{dt} h(0) = 0 = \int\limits_{D} (\nabla \cdot Du)\varphi = \int\limits_{D} (\Delta u)\varphi$$

for each φ. Therefore, $\Delta u = 0$ for $u = g$ on ∂D. $\square$

Example 5. *Recall the two functions $p(x)$ and $v(x)$ given in the examples in the Section 3.3. Note that the function $p(x)$ satisfies*

$$p(0) = 0 \text{ and } p(N) = 1$$

because if $x = 0$ the probability to reach $x = N$ is zero. But, if we are in the vertex N the probability to reach this vertex is one. Thus, p satisfies a Dirichlet problem.

On the other hand, if we establish a unit voltage to the node $x = N$ and voltage zero to the node $x = 0$, we note that $v(x)$ satisfies the same Dirichlet problem that $p(x)$. By the uniqueness result that we obtain by the maximum *principle, we can conclude that $p(x) = v(x)$.*

Finally, if we take the function $f(x) = x/N$ note that

$$\frac{1}{2}(\frac{x+1}{N} + \frac{x-1}{N}) = \frac{x}{N},$$

that is

$$f(x) = \frac{1}{2}(f(x+1) + f(x-1))$$

and $f(0) = 0$, $f(N) = 1$. Thus, f also satisfies the same Dirichlet problem. Therefore

$$p(x) = v(x) = \frac{x}{N}.$$

3.5 Newtonian Potential

Remember that in $\mathbb{R}^n$ there exists a fundamental solution Φ for the Laplace equation, such that

$$-\Delta\Phi^x(y) = \delta(y)$$

and satisfies that $\lim \Phi(x) \to 0$ as $|x| \to +\infty$, where $\delta(y)$ is the Dirac measure. We will define the analogues for the fundamental solution in the graph.

Definition 8. *The fundamental solution for the Laplacian in the graph, will be the function $\Phi_x(v)$ such that,*

$$\begin{cases} -\Delta\Phi_x(v) = 1_{x(v)} \\ \Phi_x(\infty) = 0 \end{cases}$$

where

$$1_{x(v)} = \begin{cases} 1 \ \textit{if} \ v = x \\ 0 \ \textit{if} \ v \neq x \end{cases}$$

The vertex "∞" in the graph G is any vertex that we choose, and we name it of this form, these vertex was analogue to infinity in $\mathbb{R}^n$.

Now, to solve $-\Delta\Phi_x(v) = 1_{x(v)}$ in the graph is equivalent to solve a system of linear equations, which has solution if, in the homogeneous problem the unique solution is the zero solution.

By the maximum principle $-\Delta\Phi_x(v) = 0$ with boundary equal zero, the unique solution will be zero. This guarantee that the fundamental solution there exist.

Theorem 14. *The fundamental solution $\Phi_x(v)$ satisfies*

$$\Phi_x(v) = \Phi_v(x) \textit{ for } (x \neq v).$$

Proof. From the integration by parts formula, we have

$$\int\limits_V \Phi_x(z)\Delta\Phi_v(z) = \int\limits_{\partial V=\infty} \Phi_x(z)D\Phi_v(z)\cdot\nu - \int\limits_V D\Phi_x(z)\cdot D\Phi_v(z),$$

but $\Phi_x(\infty) = 0$ and Φ_v is harmonic for $z \neq v$, thus

$$\int\limits_V \Phi_x(z)\Delta\Phi_v(z) = -\Phi_x(v) = -\sum_z D\Phi_x(z)D\Phi_v(z),$$

similarly

$$\int\limits_V \Phi_v(z)\Delta\Phi_x(z) = \int\limits_{\partial V=\infty} \Phi_v(z)D\Phi_x(z)\cdot\nu - \int\limits_V D\Phi_v(z)\cdot D\Phi_x(z),$$

thus

$$-\Phi_v(x) = -\sum_z D\Phi_x(z)D\Phi_v(z).$$

Therefore

$$\Phi_x(v) = \Phi_v(x).$$

$\square$

Note that in the continuous case the proof of this result is more complicated, because we need to prove convergence for this integrals due to the fundamental solution at the singularity is "∞".

Another important function is the Green's function for some domain $\Omega \subseteq \mathbb{R}^n$. The Green's function is useful for building one representation's formula for Dirichlet's problem. In the case of the graph, we say that G^x will be the Green's function for $D \subseteq V \setminus \{\infty\}$ if it satisfies:

$$\begin{cases} -\Delta G_x(v) = 1_x \text{ in } D \\ G = 0 \text{ on } \partial D \end{cases}$$

The same argument that in the construction for fundamental solution, is applied for proving the existence of Green function.

If μ is a positive measure with compact support, we define its Newtonian potential as the convolution

$$\Phi^\mu(x) = \Phi_x * \mu = \sum_{v \in D} \Phi_x(v)\mu(v).$$

Thus, for measures μ, σ we can define the inner product

$$
\begin{aligned}
(\mu, \sigma)_e &= \sum_v \Phi_x^\mu(v)\sigma(v) \\
&= \sum_x \sum_v \Phi_x(v)\mu(x)\sigma(v)
\end{aligned}
$$

as $\Phi_x(v) = \Phi_v(x)$ we see that $(\mu, \sigma)_e = (\sigma, \mu)_e$ and we can define the energy for μ by

$$
||\mu||_e^2 = \sum_x \sum_v \Phi_x(v)\mu(x)\mu(v).
$$

Definition 9. *Let $D \subseteq V \setminus \{\infty\}$ be a finite graph. Let g be a function defined on the boundary of D. This function determines a unique harmonic function ψ_g, such that solves the Dirichlet problem*

$$
\begin{cases}
-\Delta\psi = 0 \;\; in\ D, \\
\psi = g \;\; on\ \partial D.
\end{cases}
$$

We say that μ_x is the harmonic measure of the domain D with respect to the point $x \in D$ if satisfies

$$
\sum_{v \in \partial D} \psi(v)\mu_x(v) = \psi(x).
$$

In the probabilistic sense $\mu_x(S)$ for $S \subseteq \partial D$ measure the probability that if a brownian motion started inside the domain ,at the point $x \in D$ hits S.

On the other hand, if we think in a conductive medium, $\mu_x(S)$ for $S \subseteq \partial D$ measure the amount of current passing through S.

3.6 Perron's Method

We have seen that it is possible to solve the Dirichlet problem in alternative ways. In Chapter 2, we showed a useful method that is known as Perron's method for subharmonic functions.

Let $D \subseteq V \setminus \{\infty\}$ be bounded and let g defined on ∂D. Set

$$S_g = \{\phi : \phi \text{ subharmonic in } D \text{ and } \phi \leq g \text{ on } \partial D\}.$$

Theorem 15. *The function defined by*

$$u(x) = \sup_{\phi \in S_g} \phi(x)$$

is harmonic in D.

Proof.

i) Let us see that u is subharmonic. Fix $x_0 \in D$. By definition of u for all $\epsilon > 0$ there exists $\phi \in S_g$ such that $u(x_0) \leq \phi(x_0) + \epsilon$, thus

$$u(x_0) \leq \frac{1}{\#\partial\{x_0\}} \sum_{v \sim x_0} \phi(v) + \epsilon.$$

but $\phi(v) \leq u(v)$, therefore

$$u(x_0) \leq \frac{1}{\#\partial\{x_0\}} \sum_{v \sim x_0} u(v) + \epsilon$$

and since ϵ is arbitrary we have

$$u(x_0) \leq \frac{1}{\#\partial\{x_0\}} \sum_{v \sim x_0} u(v)$$

implying that, u is subharmonic in D.

ii) Now, let us define $u^{B_{x_0}}(x)$ by

$$u^{B_{x_0}}(x) = \begin{cases} \frac{1}{\#\partial\{x\}} \sum_{v \sim x_0} u(v) \ \text{ if } x = x_0, \\ \\ u(x), \ x \neq x_0, \end{cases}$$

known as the "harmonic lifting" of u. This definition implies that

$u(x) \leq u^{B_{x_0}}(x)$ and

$$\begin{aligned} u(x) \leq & \frac{1}{\#\partial\{x\}} \sum_{v\sim x} u(v) \\ & \leq \frac{1}{\#\partial\{x\}} \sum_{v\sim x} u^{B_{x_0}}(v). \end{aligned}$$

Hence, for $x \neq x_0$ we have $u(x) = u^{B_{x_0}}(x)$, and then

$$u^{B_{x_0}}(x) \leq \frac{1}{\#\partial\{x\}} \sum_{v\sim x} u^{B_{x_0}}(v)$$

where the equality holds when $x = x_0$. Thus, $u^{B_{x_0}}(x)$ is subharmonic in D, therefore, $u^{B_{x_0}} \in S_g$.

iii) Note that $u^{B_{x_0}}(x_0) \geq u(x_0)$ but by definition of u, $u(x_0) \geq u^{B_{x_0}}(x_0)$ which implies that $u(x_0) = u^{B_{x_0}}(x_0)$. The latter guarantees that u is harmonic in x_0 and since x_0 is arbitrary we have that u is harmonic in D.

$\square$

Remark 6. *Note that in the proof of Perron's method in the continuous case we had to be careful because the proof was more constructive, and we needed to prove uniform convergence using Harnack's inequality. However, the proof in the graph is easier due to punctual and uniform convergence are* *equivalent.*

3.7 Balayage

In this section, we will build by a recursive method the balayage measure for a domain $D \subseteq V \setminus \{\infty\}$. Also, we will prove that this sequence of measures converges, and that this is equivalent to solve a Dirichlet problem. This result, it will be presented in two cases. The first one, for classical balayage and the second one for partial balayage.

3.7.1 Classical Balayage

Here, we will present the ideas about classical balayage as in [2]. Recall that for a measure μ, we defined its Newtonian potential by Φ^{μ}. In the classical potential theory it is studied the change that in some way happens to the potential when the measure μ changes, because μ can be recovered from Φ^{μ} via

$$-\Delta \Phi^{\mu} = \mu.$$

The principal idea in the balayage process is that given a measure μ in a domain D, to build a measure ν such that $\nu = 0$ in D, and $\Phi^{\mu} = \Phi^{\nu}$ on D^c.

It is the process of cleaning D from any mass of μ in such a way that the potential remains unchanged outside D. The process of to change $(\mu \to \nu)$ more explicit is the requirements

$$\nu = 0 \text{ in } D;$$

$$\Phi^{\nu} = \Phi^{\mu} \text{ on } D^c$$

we remark that it is allowed that μ has mass also outside D. Part of μ will be unchanged, while $\mu|_D$ will redistribute on ∂D.

The most intuitive way of producing ν from μ is by minimizing the energy for the change:

$$\min ||\mu - \nu||_e^2 : \quad \nu = 0 \text{ in } D.$$

Another way of constructing the balayage measure is by solving a Dirichlet problem: let V be the solution of

$$\begin{cases} -\Delta V = 0 \text{ in } D, \\ V = \Phi^{\mu} \text{ on } \partial D, \end{cases}$$

and extend V by $V = \Phi^{\mu}$ outside D. Thus, V will be the potential of a measure ν, $V = \Phi^{\nu}$ and this measure satisfies the required properties. Now, in terms of the difference $u = \Phi^\mu - V$ we have $\Delta u = \nu - \mu$, hence

$$\begin{cases} -\Delta u = \mu \text{ in } D, \\ u = 0 \text{ on } D^c, \end{cases}$$

the notation for this process in general is

$$\nu = \mathrm{Bal}(\mu, D^c) = \mu + \Delta u.$$

Theorem 16. *Let $\Omega \subseteq \mathbb{R}^n$ and $\{B_i\}$ a collection of balls such that, for every $N \in \mathbb{N}$*

$$\bigcup_{i \geq N} B_i = \Omega.$$

Let u_0 be a subharmonic function in $C(\overline{\Omega})$ and define u_i recursively by taking the harmonic lifting of u_{i-1} over B_i. Then $u_i \to u$ locally uniformly in Ω for some u harmonic.

Proof. Similar to the proof of the Schwarz method we will prove that the sequence $(u_i)_{i\in\mathbb{N}}$ is monotone and bounded. Let $x \in \Omega$ thus, there exists $B_j \in \{B_i\}$ such that, $x \in B_j$. Moreover, there exists $\epsilon > 0$ such that, $B(x,\epsilon) \subset B_j$. Note that u_j is harmonic in B_j thus, is harmonic in $B(x,\epsilon)$. Now, u_{j+1} is harmonic in B_{j+1} and $u_{j+1} = u_j$ on $\Omega \setminus B_{j+1}$ thus, u_{j+1} is harmonic in $B(x,\epsilon)$. Then, u_k is harmonic in $B(x,\epsilon)$ for all $k \geq j$. Since

$\{u_k\}$ is monotone and bounded, it converges puntually and by Harnack inequality, this convergence is uniform, $u_k \to u$ which is harmonic in $B(x, \epsilon)$ thus, u_i converges locally uniformly in Ω for some u harmonic. $\square$

Now, we will present the main result in the Classic Balayage Process for graphs.

Theorem 17. *Let $D \subseteq V \backslash \{\infty\}$, $\mu_0 : V \to [0, +\infty)$ such that support $\mu_0 \subseteq D$ and $\{x_i\}$ a collection of vertices such that for every $N \in \mathbb{N}$*

$$\bigcup_{i \geq N} \{x_i\} = D.$$

Let $u_0 = 0$ (the zero function) and define recursively

$$\mu_i = \mu_{i-1} + \mu_{i-1}(x_i)\Delta\delta_{x_i},$$

$$u_i = u_{i-1} + \mu_{i-1}(x_i)\delta_{x_i},$$

then, $(u_i)_{i \in \mathbb{N}}$ converges to the unique solution of

$$\begin{cases} -\Delta u = \mu_0 \ \textit{in } D, \\ u = 0 \ \textit{on } \partial D. \end{cases}$$

Proof. By the definition of u_i and μ_i, it is satisfied

$$\begin{cases} -\Delta u_i = \mu_0 - \mu_i \text{ in } D, \\ u_i = 0 \text{ on } \partial D. \end{cases} \tag{3.1}$$

Now, let us see that u_i converges. First, note that $u_i = u_{i-1} + \mu_{i-1}(x_i)\delta_{x_i}$ thus, $u_i \geq u_{i-1}$ implying that, u_i is increasing monotone. We just need to prove that u_i is bounded in D. To see this, since u_i satisfies (3.1) we define $\phi_i = \Phi^{\mu_0} - \Phi^{\mu_i}$ where Φ is the fundamental solution and Φ^{μ_0}, Φ^{μ_i} are the Newtonian potential associated to μ_0, μ_i respectively. Thus,

$$\Delta\phi_i = \mu_i - \mu_0.$$

By Perron's method, we can solve

$$\begin{cases} \Delta\psi_i = 0 \text{ in } D, \\ \psi_i = -\phi_i \text{ on } \partial D, \end{cases}$$

and u_i can be written as $u_i = \phi_i + \psi_i$. Now, it is only necessary to verify that ϕ_i is bounded in D.

$$\phi_i = \Phi^{\mu_i} - \Phi^{\mu_0} \leq \Phi^{\mu_0} < +\infty \text{ in } D \text{ therefore, } u_i \to u \text{ for some } u.$$

This implies that $\Delta u_i \to \Delta u$ thus, μ_i converges. Note that for each $x \in D$ there exists $x_{k_1} \in \{x_i\}$ such that, $x = x_{k_1}$ and $\mu_{k_1}(x) = 0$. By definition from D

$$D = \bigcup_{i>k_1} \{x_i\},$$

therefore, there exists x_{k_2} such that $x = x_{k_2}$ and $\mu_{k_2}(x) = 0$ for $k_1 > k_2$. Continuining in the same way we build a subsequence of μ_i such that,

$$\mu_{i_k}(x) \to 0.$$

therefore, $\mu_i(x) \to 0$ and u satisfies

$$\begin{cases} -\Delta u = \mu_0 \text{ in } D \\ u = 0 \text{ on } \partial D \end{cases}$$

□

In the continuous case minimizing the energy for the measures we can build the balayage measure, in this sense of the graph will be

$$\min ||\nu - \mu||_e^2 : \nu = 0 \text{ in } D \text{ and } \sum \nu = \sum \mu = 1$$

where

$$||\mu||_e^2 = \sum_x \sum_y \mu(x)\mu(y)\Phi_x(y)$$

hence,

$$||\nu - \mu||_e^2 = \sum_x \sum_y (\nu - \mu)(x)(\nu - \mu)(y)\Phi_x(y)$$

$$= \sum_{x \in D} \sum_{y \in D} (\nu - \mu)(x)(\nu - \mu)(y)\Phi_x(y) + \sum_{x \notin D} \sum_{y \notin D} (\nu - \mu)(x)(\nu - \mu)(y)\Phi_x(y)$$

$$+2 \sum_{x \in D} \sum_{y \notin D} (\nu - \mu)(x)(\nu - \mu)(y)\Phi_x(y)$$

$$= F_1(\nu) + F_2(\nu) + F_3(\nu)$$

note that $D_\nu F_1(\nu) = 0$ because $\nu = 0$ in D now,

$$D_\nu F_2(\nu) = 2 \sum_{y \in V} \Phi_x(y)(\nu - \mu)(y)$$

and F_3 is linear in ν, so $D_\nu F_3(\nu) = 1$, hence

$$2 \sum_{y \in V} \Phi_x(y)(\nu - \mu)(y) = \lambda$$

or equivalently

$$\Phi^\nu - \Phi^\mu = \frac{\lambda}{2} \text{ if } x \notin D$$

but $\Phi(\infty) = 0$ because $\infty \notin)$ thus, $\lambda = 0$ therefore,

$$\sum_{y \in V} \Phi_x(y)(\nu - \mu)(y) = 0$$

it defines $\text{Bal}(\mu, D^c)$ and satisfies $\Phi^\nu = \Phi^\mu$ on D^c.

3.7.2 Partial Balayage

In this section, analogous as in the classical balayage, we will see the process for a "partial sweeping" of a domain D where the distribution of mass depending to another measure. Thus, partial balayage means that we only make some partial cleaning. The role of the domain D is then, taken over by a measure λ which tells how much mass is allowed to be left.

Theorem 18. *Let $D \subseteq V \setminus \{\infty\}$*

$$\lambda : V \to [0, +\infty]$$

$$\mu_0 : V \to [0, +\infty)$$

such that spt $\mu_0 \subseteq D$ and $\lambda(\infty) = +\infty$. Let $\{x_i\}$ a collection of vertices such that for every $N \in \mathbb{N}$

$$\bigcup_{i \geq N} \{x_i\} = D$$

define recursively μ_i *and* u_i *as:*

$$\mu_i = \mu_{i-1} + (\mu_{i-1} - \lambda)^+(x_i)\Delta\delta_{x_i}$$

$$u_i = u_{i-1} + (\mu_{i-1} - \lambda)^+(x_i)\delta_{x_i}$$

for $u_0 = 0$. *Then, for every* $x \in V$

$$\lim u_i(x) = u(x) = \inf\{v(x) : v \geq 0 \textit{ and } \Delta v \leq \lambda - \mu_0\}$$

moreover, u *is the unique solution of*

$$\begin{cases} \min(-\Delta u + \lambda - \mu_0, u) = 0 \textit{ in } D \\ u = 0 \textit{ on } \partial D \end{cases}$$

Here, the idea unlike the classical balayage is to do a “partial sweeping” of μ_0, in which a certain density of measure is allowed to remain in D. The amount or density of measure allowed is less than the density of measure given by λ. Thus, the measure λ says how is sweeping the domain D, it is, λ is an upper bound for the process.

Proof. Note that each u_i satisfies

$$\begin{cases} \Delta u_i = \mu_i - \mu_0 \text{ in } D \\ u_i = 0 \text{ on } \partial D \end{cases}$$

Then, using the same argument as in Theorem 17, we can prove that the sequence u_i converges and let $\overline{u}$ this limit.

Now, let us see that $\overline{u}$ satisfies $\min(-\Delta\overline{u} + \lambda - \mu_0, \overline{u}) = 0$ in D.

Suppose that $x \in \{\overline{u} > 0\}$ since $u_i(x) \to \overline{u}(x) > 0$ then, there exists N such that, $u_i(x) > 0$ for all $i \geq N$. This condition implies that $\mu_{k_1-1}(x) > \lambda(x)$ and $x = x_{k_1}$ for some $k_1 \in \mathbb{N}$ therefore, $\mu_{k_1}(x) = \lambda(x)$. By the fact that

$$\bigcup_{i>k_1} \{x_i\} = D$$

there exists a $k_2 > k_1$ such that, $x = x_{k_2}$ and $\mu_{k_2}(x) = \lambda(x)$. In the same way we can obtain a subsequence $\{x_{k_j}\}$ such that, $\mu_{k_j}(x) \to \lambda(x)$, but we know μ_i converges thus, $u_i(x) \to \lambda(x)$.

This implies that for $x \in \{\overline{u} > 0\}$ we have that $-\Delta\overline{u}(x) + \lambda(x) - \mu_0(x) = 0$ Thus, $\min(-\Delta\overline{u} + \lambda - \mu_0, \overline{u}) = 0$.

Now, let us see that $\overline{u} \leq v$ for each $v \in \{w : w \geq 0 \text{ and } \Delta w \leq \lambda - \mu_0\}$.

- If $x \in \{\overline{u} = 0\}$ the inequality it holds by the fact that each $v \geq 0$.

- Now, let us take $D \setminus \{\overline{u} = 0\}$ then, $\overline{u} > 0$ and $\Delta\overline{u} = \lambda - \mu_0$ thus, $\Delta v \leq \Delta\overline{u}$ for all $v \in \{w : w \geq 0 \text{ and } \Delta w \leq \lambda - \mu_0\}$. If we define $h = v - \overline{u}$ then, $h|_{\partial(D\setminus\{\overline{u}=0\})} = v \geq 0$ and

$$\Delta h = \Delta v - \Delta\overline{u} \leq 0 \text{ then, } h \text{ is superharmonic}$$

therefore, $h \geq 0$ in $D \setminus \{\overline{u} = 0\}$ and $\overline{u} \leq v$ in D. Moreover, $\overline{u} \in \{w : w \geq 0 \text{ and } \Delta w \leq \lambda - \mu_0\} = A$ then,

$$\overline{u}(x) = \inf_{v\in A} v(x) = u(x)$$

$\square$

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