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# NUMERICAL INVARIANTS IN PRIME CHARACTERISTIC

**T H E S I S**

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Wágner Badilla Céspedes

**Advisor:**

Professor Luis Núñez Betancourt

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A mis padres y hermanos por su apoyo durante todos estos años.

As long as algebra and geometry proceeded along separate paths, their advance was slow and their applications limited. But when these sciences joined company, they drew from each other fresh vitality and thenceforward marched on at a rapid pace toward perfection.

Count Joseph-Louis de Lagrange
1736-1813.

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# Abstract

In prime characteristic there are important numerical invariants that allow us to detect and measure singularities. For certain cases, it is known that they are rational numbers. In the first part of this work, we show that certain $F$-thresholds, $F$-pure thresholds, and Cartier thresholds are rational numbers for Stanley-Reisner rings. In addition, we give conditions of the regularity in Stanley-Reisner rings modulo Frobenius power of ideals. The methods obtain these results rely in singularity theory in prime characteristic and the combinatorial structure of Stanley-Reisner rings.

In the second part of this work, we introduce a numerical invariant called $F$-volume. This is motivated by the mixed test ideals associated to a sequence of ideals, and their constancy regions. The $F$-volume extends the definition of $F$-threshold of an ideal to a sequence of ideals. We obtain several properties that emulate those of the $F$-threshold. In particular, the $F$-volume detects $F$-pure complete intersections. In addition, we relate this invariant to the Hilbert-Kunz multiplicity, and provide support for a conjecture of Núñez-Betancourt and Smirnov.

## Key Words

Stanley-Reisner rings, $F$-thresholds, $F$-pure thresholds, Cartier thresholds, $a$-invariants, Castelnuovo-Mumford regularity, $F$-volumes, $F$-pure complete intersections, Hilbert-Kunz multiplicity.

# Resumen

En característica prima existen invariantes numéricos importantes que nos permiten detectar y medir singularidades. Para ciertos casos, es conocido que ellos son números racionales. En la primera parte de este trabajo, mostramos que ciertos $F$-umbrales, umbrales $F$-puros y umbrales de Cartier son números racionales para anillos de Stanley-Reisner. Además, damos condiciones de la regularidad en anillos de Stanley-Reisner modulo potencias de Frobenius de ideales. Los métodos que obtienen estos resultados se basan en la teoría de la singularidad en característica prima y la estructura combinatoria de los anillos de Stanley-Reisner.

En la segunda parte de este trabajo, introducimos un invariante numérico llamado $F$-volumen. Este es motivado por los ideales de prueba mixtos asociados a una sucesión de ideales y sus regiones de constancia. El $F$-volumen extiende la definición de $F$-umbral de un ideal a una sucesión de ideales. Obtenemos varias propiedades que emulan las del $F$-umbral. En particular, el $F$-volumen detecta intersecciones completas $F$-puras. Además, relacionamos este invariante a la multiplicidad de Hilbert-Kunz y proporcionamos soporte para una conjetura de Núñez-Betancourt and Smirnov.

## Palabras Claves

Anillos de Stanley-Reisner, $F$-umbrales, umbrales $F$-puros, umbrales de Cartier, $a$-invariantes, regularidad de Castelnuovo-Mumford, $F$-volúmenes, intersecciones completas $F$-puras, multiplicidad de Hilbert-Kunz.

# Contents

# CHAPTER 1

# Introduction

In prime characteristic there are important numerical invariants that allow us to measure singularities. Among them are $F$-thresholds, $F$-pure thresholds, $F$-signature, and Hilbert-Kunz multiplicity. The main goal in this thesis is to study these invariants. This thesis is divided in two parts. In the first part, we show that the $F$-thresholds and $F$-pure thresholds are rational numbers in several cases for Stanley-Reisner rings. These results are contained in a paper by the author [BC20]. For the second part, we take the general case, extend the concept of $F$-threshold to a sequence of ideals, and called it $F$-volume. We also give the relation between this number with the $F$-pure complete intersections and Hilbert-Kunz multiplicity. This part contains results obtained in a joint work with Luis Núñez-Betancourt and Sandra Rodríguez-Villalobos [BCNnBRV19]. Throughout this work, all rings are Noetherian and commutative with one.

## 1.1 Invariants in Characteristic Zero

Throughout this work we study the mixed generalized test ideals, $F$-jumping numbers, $F$-thresholds, and $F$-pure thresholds. Our motivation comes from birational geometry in characteristic zero.

Given an ideal $\mathfrak{a}$ on a smooth variety $X$ and a real positive number $c$, the multiplier ideal $\mathcal{J}(\mathfrak{a}^c)$ is described via a log resolution $\pi : X' \longrightarrow X$ of the pair $(X, \mathfrak{a})$, that is, a proper birational map with $X'$ smooth and such that $\mathfrak{a}\mathcal{O}_{X'} = \mathcal{O}_{X'}(-E)$, where $E$ is a simple normal crossing divisor. The multiplier ideal is defined as

$$\mathcal{J}(\mathfrak{a}^c) := \pi_*\mathcal{O}_{X'}(K_{X'/X} - \lceil cE \rceil),$$

where $K_{X'/X}$ is the relative canonical divisor. These ideals help us to describe and measure the singularities of the algebraic variety $V(\mathfrak{a})$ defined by the ideal $\mathfrak{a}$.

The definition can be extended to a sequence of ideals. For ideals $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ and positive real numbers $c_1, \ldots, c_n$ we take a log resolution for the pair $(X, \mathfrak{a}_1 \cdots \mathfrak{a}_n)$. The mixed multiplier ideal is defined as

$$\mathcal{J}(\mathfrak{a}_1^{c_1} \cdots \mathfrak{a}_n^{c_n}) := \pi_* \mathcal{O}_{X'}(K_{X'/X} - \lceil c_1 E_1 + \cdots + c_n E_n \rceil),$$

with $\mathfrak{a}_i \mathcal{O}_{X'} = \mathcal{O}_{X'}(-E_i)$.

A constancy region for $\mathcal{J}(\mathfrak{a}_1^{c_1} \cdots \mathfrak{a}_n^{c_n})$ is the set of $c' \in \mathbb{R}^n_{\geq 0}$ where $\mathcal{J}(\mathfrak{a}_1^{c_1} \cdots \mathfrak{a}_n^{c_n}) = \mathcal{J}(\mathfrak{a}_1^{c'_1} \cdots \mathfrak{a}_n^{c'_n})$. If we take the exponent $c$ in a cube, then the number of constancy regions is finite.

A jumping number of $\mathfrak{a}$ is a positive real number $c$ such that $\mathcal{J}(\mathfrak{a}^c) \neq \mathcal{J}(\mathfrak{a}^{c-\varepsilon})$ for every $\varepsilon > 0$. For each $\mathfrak{a}$ the set of jumping numbers is a discrete subset of the rational numbers.

In characteristic zero, the log canonical threshold, $\operatorname{lct}(f)$, of a polynomial $f$ with coefficients in a field, is an important invariant in birational geometry [BFS13]. From an analytical point of view, we take $f \in \mathbb{C}[x_1, \ldots, x_n]$ with $f(0) = 0$ and with a singularity at zero, we have the function

$$\begin{aligned} \frac{1}{|f|} : \mathbb{C}^n \backslash V(f) &\longrightarrow \mathbb{R} \\ z &\longrightarrow \frac{1}{|f(z)|}. \end{aligned}$$

The log canonical threshold of $f$ is defined as

$$\operatorname{lct}(f) = \sup \left\{ \lambda \in \mathbb{R}_+ \mid \text{there exist } \varepsilon > 0 \text{ such that } \int_{B_\varepsilon(0)} \frac{1}{|f(z)|^{2\lambda}} < \infty \right\}.$$

It turns out that $\operatorname{lct}(f) = \sup\{\lambda \in \mathbb{R}_+ \mid \mathcal{J}(f^\lambda) = (1)\}$. Therefore, the $\operatorname{lct}(f)$ is the first jumping number of $f$. This number measures the singularities of $f$ near to zero.

## 1.2 Invariants in Characteristic $p > 0$

In this manuscript, we discuss the analogous to the mixed multiplier ideals in prime characteristic, denoted by $\tau(\mathfrak{a}_1^{c_1} \cdots \mathfrak{a}_n^{c_n})$. These ideals are called the mixed generalized test ideals. They were originally introduced by Hochster and Huneke [HH90], and later generalized by Hara and Yoshida [HY03]. The mixed generalized test ideal of a sequence of ideals $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ in $R$, with exponents $c_1, \ldots, c_n$, is defined as

$$\tau(\mathfrak{a}^c) = \bigcup_{e>0} (\mathfrak{a}^{\lceil cp^e \rceil})^{[1/p^e]},$$

where $\mathfrak{a}^c = \mathfrak{a}_1^{c_1} \cdots \mathfrak{a}_n^{c_n}$.

The region where $\tau(\mathfrak{a}^c)$ is constant is called the constancy region of the mixed test ideal [HY03, BMS08]. In a regular ring, it has been an object of interest [Tak04, Pér13]. It is a natural and still open question whether the number of constancy regions of a mixed test ideal ($n > 1$) in the cube is finite. We discuss a few of their properties, specifically the conditions to have a finite number of constancy regions.

We also have an analogous for the jumping numbers, which are called $F$-jumping exponents. These also form a discrete set of rational numbers for $F$-finite regular rings [BMS08].

In positive characteristic, the $F$-pure threshold of an ideal $\mathfrak{a} \subseteq R$, denoted $\operatorname{fpt}(\mathfrak{a})$, was defined by Takagi and Watanabe [TW04]. Roughly speaking, this measures the splitting order of $\mathfrak{a}$. It is defined as

$$\operatorname{fpt}(f) = \sup\left\{\frac{a}{p^e} \mid \text{the inclusion } Rf^{\frac{a}{p^e}} \subseteq R^{1/p^e} \text{ is a split}\right\}$$

for $f \in R$.

The $F$-pure threshold is considered the analogue to the log canonical threshold, and they share similar properties [TW04, MTW05]. In particular, if $f$ is an element in $\mathbb{Z}[x_1, \dots, x_n]$, then $\lim\limits_{p\to\infty} \operatorname{fpt}(f \bmod p) = \operatorname{lct}(f_{\mathbb{Q}})$ [HY03, MTW05], where $f_{\mathbb{Q}}$ is seen in the set of the rational numbers.

In this work, we study a general form of the $F$-pure threshold called the Cartier threshold of $\mathfrak{a}$ with respect to $J$. This is defined as $\operatorname{ct}_J(\mathfrak{a}) = \lim\limits_{e\to\infty} \frac{b_{\mathfrak{a}}^J(p^e)}{p^e}$, where

$$b_{\mathfrak{a}}^J(p^e) = \max\{t \in \mathbb{N} \mid \mathfrak{a}^t \not\subseteq J_e\}$$

and

$$J_e = \{f \in R \mid \varphi(f^{1/p^e}) \in J, \text{ for all } \varphi \in \operatorname{Hom}_R(R^{1/p^e}, R)\}.$$

These numbers are studied in more depth in an upcoming work [DSHNnBW]. If we consider $(R, \mathfrak{m}, K)$ a local ring or a standard graded $K$-algebra which is $F$-finite and $F$-pure, the $\operatorname{ct}_{\mathfrak{m}}(\mathfrak{a}) = \operatorname{fpt}(\mathfrak{a})$.

We now recall the definition of the $F$-thresholds. They are numbers obtained by comparing ordinary powers with Frobenius powers. The $F$-thresholds were introduced in regular rings by Mustaţă, Takagi and Watanabe [MTW05], and their existence, in the general case, was proved by De Stefani, Núñez-Betancourt and Pérez [DSNnBP18]. These are defined as $c^J(\mathfrak{a}) = \lim\limits_{e\to\infty} \frac{\nu_{\mathfrak{a}}^J(p^e)}{p^e}$, where $\nu_{\mathfrak{a}}^J(p^e) = \max\{m \in \mathbb{N} \mid \mathfrak{a}^m \not\subseteq J^{[p^e]}\}$, and $\mathfrak{a}$, $J \subseteq R$ are ideals.

A recent line of research consists in understanding under which conditions the set of $F$-thresholds is a discrete subset of the rational numbers. This was proved by Blickle, Mustaţă, and Smith [BMS08] for an $F$-finite regular ring. Although the $F$-threshold is a rational number in regular case, this situation is unknown in general Noetherian rings. Trivedi [Vij18] showed that, in general, the $F$-thresholds of the a maximal ideal are not necessarily discrete.

Let $I, J \subseteq R$ be ideals such that $I \subseteq \sqrt{J}$. For regular rings, the $F$-thresholds are related to test ideals. Specifically, the $F$-thresholds measure the length of a region where test ideal are contained in a given ideal. Then, the set of $F$-thresholds of $I$ is the same as the set of $F$-jumping numbers of the test ideals of $I$ [BMS08]. If the ring is not regular, these two sets of invariants differ (e.g. [MOY10]). However, the $F$-thresholds give extra information about $I, J$, and $R$. For instance, one can use $F$-thresholds to study integral closure, tight closure, Hilbert-Samuel multiplicities [HMTW08], and $a$-invariants [DSNnB18, DSNnBP18].

## 1.3 Stanley-Reisner

In this section, we focus on Stanley-Reisner rings. The combinatorial nature of these rings has been useful to study their structures in prime characteristic. For instance, in this case one can describe their algebras of Frobenius and Cartier operators [ÀlMBZ12, BZ19]. In this work, we show that the Cartier threshold of $\mathfrak{a}$ with respect to $J$ in Stanley-Reisner is a rational number in certain cases.

**Theorem A** (see Theorem 3.4.14 and Corollaries 3.4.15, 3.4.16)**.** *Let $\mathfrak{a}$, $J$ be two ideals in a Stanley-Reisner ring $R$, such that $\mathfrak{a} \subseteq J$, and $J$ is a radical ideal. Then, the Cartier threshold of $\mathfrak{a}$ with respect to $J$ is a rational number.*

In order to obtain Theorem A, we need to reduce the computation of $\mathrm{ct}_J(\mathfrak{a})$ to the case where $J$ is a monomial ideal. For this trick, we need to replace $R$ by the completion of a suitable localization. Then, the problem is reduced to the regular case by taking a quotient with respect to the Cartier core (see Definition 3.3.10).

In this work, we study the rationality of $F$-thresholds for Stanley-Reisner rings.

**Theorem B** (see Theorem 3.2.1)**.** *Let $\mathfrak{a}$, $J$ two ideals in a Stanley-Reisner ring $R$, such that $\mathfrak{a} \subseteq \sqrt{J}$, and $J$ is a monomial ideal. Then, the $F$-threshold of $\mathfrak{a}$ with respect to $J$ is a rational number.*

The key idea to prove Theorem B is to work modulo the minimal primes, which yields a regular ring. The result follows from comparing the $F$-thresholds of $R$ with these quotients. We point out that Theorem B is a key component of the proof of Theorem A.

The Castelnuovo-Mumford regularity is an invariant that measures the complexity of the free resolution of a standard graded $K$-algebra $(R, \mathfrak{m}, K)$. The growth of $\mathrm{reg}(R/J^{[p^e]})$ has been intensively studied due to its relation to discreteness of $F$- jumping coefficients [KZ14, KSSZ14, Zha15], localization of tight closure [Kat98, Hun00], and existence of the generalized Hilbert-Kunz multiplicity [DS13, Vra16]. We recall that the Castelnuovo-Mumford regularity can be computed in terms of the $a$-invariants introduced by Goto and Watanabe [GW78]. In this manuscript, we provide a formula for the limits of $\mathrm{reg}(R/J^{[p^e]})$.

**Theorem C** (see Theorem 3.5.3)**.** *Let $J$ be a homogeneous ideal in a Stanley-Reisner ring $R$. Then,*

$$\lim_{e\to\infty} \frac{\operatorname{reg}(R/J^{[p^e]})}{p^e} = \max_{\substack{1\le i\le d\\ \alpha\in\mathcal{A}'}} \{a_i(S/(J_\alpha + J)) + |\alpha|\},$$

*where $\mathcal{A}' = \{\alpha \in \mathbb{N}^n \mid 0 \le \alpha_i \le 1$ for $i = 1,\dots,n\}$, $J_\alpha = (I : x^\alpha)$, and $d = \max\{\dim(S/(J_\alpha + J)) \mid \alpha \in \mathcal{A}'\}$. In particular, this limit is an integer number.*

## 1.4 $F$-Volumes

In this section $R$ denotes a ring, not necessarily a Stanley-Reisner ring. Motivated by the mixed test ideals associated to a sequence of ideals $I_1,\dots,I_t$ and their constancy regions, we define an analogue of the $F$-threshold for a sequence of ideals. In our main result, we describe this invariant as a limit of a convergent sequence.

**Theorem D** (see Theorem 4.1.13 and Definition 4.1.2)**.** *Let $\underline{I} = I_1,\dots,I_t \subseteq R$ be a sequence of ideals, and $J \subseteq R$ be an ideal such that $I_1,\dots,I_t \subseteq \sqrt{J}$. Let*

$$\mathrm{V}_{\underline{I}}^J(p^e) = \{(a_1,\dots,a_t) \in \mathbb{N}^t \mid I_1^{a_1}\cdots I_t^{a_t} \not\subseteq J^{[p^e]}\}.$$

*Then,*

$$\lim_{e\to\infty} \frac{|\,\mathrm{V}_{\underline{I}}^J(p^e)|}{p^{et}}$$

*converges. This limit is called the $F$-volume of $\underline{I}$ with respect to $J$, and it is denoted by $\mathrm{Vol}_F^J(\underline{I})$.*

In the regular case, this limit is the sum of the volumes of the constancy regions where $\tau(I_1^{a_1}\cdots I_t^{a_t}) \not\subseteq J$.

The proof of Theorem D is based on a technical extension of the case of a single ideal [DSNnBP18]. However, the case of multiple ideals is not a simple consequence of this case. We devote Section 4.1 to this proof. We also show a few properties of the $F$-volume that extend those of the $F$-thresholds.

If $R$ is an $F$-pure ring, the $F$-volume is the measure of a set in $\mathbb{R}^\ell$ (see Propositions 4.2.5 and 4.3.5). In Section 4.3, we present this and other results for $F$-pure rings. In particular, we show that $F$-volumes detect $F$-pure complete intersections.

**Theorem E** (see Theorem 4.3.13)**.** *Suppose that $(R,\mathfrak{m},K)$ is a local regular ring. Let $I \subseteq \mathfrak{m}$ be an ideal in $R$, and $\underline{f} = f_1,\dots,f_t$ be minimal generators of $I$. Then, $\mathrm{Vol}_F^{\mathfrak{m}}(\underline{f}) = 1$ if and only if $I$ is an $F$-pure complete intersection.*

In Section 4.4, we relate the $F$-volume and the Hilbert-Kunz multiplicity. Specifically, we obtain the following result.

**Theorem F** (see Theorem 4.4.1)**.** *Suppose that* $(R, \mathfrak{m}, K)$ *is a local ring. Let* $\underline{f} = f_1, \ldots, f_t$ *be part of a system of parameters for* $R$*,* $I = (\underline{f})$*, and* $S = R/I$*. Then,*

$$\mathrm{e}_{HK}(JS; S) \geq \frac{\mathrm{e}_{HK}(J; R)}{\mathrm{Vol}_F^J(\underline{f})}$$

*for any* $\mathfrak{m}$*-primary ideal* $J$*, such that* $I \subseteq J$*.*

In Remark 4.4.3, we relate Theorem F with a conjecture regarding the $F$-thresholds and the Hilbert-Kunz multiplicity [NnBS20]. In particular, we improve an estimate given in previous results [NnBS20].

# CHAPTER 2

# Background

In this chapter we introduce basic properties regarding the local cohomology, integral closure of ideals, Frobenius map, test ideals, $F$-thresholds, $F$-purity, standard graded $K$-algebras, $F$-pure thresholds, $a$-invariants, and Hilbert-Kunz multiplicity. These are concepts and tools that play an important role in this work.

## 2.1 Local Cohomology

Suppose that $R$ is a ring. Let $I \subseteq R$ be an ideal generated by the elements $f_1, \ldots, f_\ell \in R$. Consider the following complex that is called Čech complex, and it denoted by $\check{\mathrm{C}}^\bullet(\underline{f}; R)$

$$0 \to R \to \bigoplus_i R_{f_i} \to \bigoplus_{i<j} R_{f_i f_j} \to \ldots \to R_{f_1 \cdots f_\ell} \to 0,$$

where $\check{\mathrm{C}}^i(\underline{f}; R) = \bigoplus_{1 \leq j_1 < \cdots < j_i \leq \ell} R_{f_{j_1} \cdots f_{j_i}}$ and the homomorphism in each summand is a localization map with an appropriate sign. For an $R$-module $M$, we consider the complex $\check{\mathrm{C}}^\bullet(\underline{f}; R) \otimes M$, and we define the $i$-th local cohomology module of $M$ with support in $I$ $H^i_I(M)$ as $H^i(\check{\mathrm{C}}^\bullet(\underline{f}; R) \otimes M)$. The local cohomology module $H^i_I(M)$ does not depend on the choice of generators of $I$.

These modules capture several algebraic and geometric properties of the ring $R$, ideal $I$, and $R$-module $M$; for instance, Cohen-Macaulayness of $R$, depth of $I$, and dimension of $M$. In addition, there are strong connections between local cohomology of modules and cohomology of sheaves. Let $V(I)$ be the algebraic set defined by the vanishing of elements in $I$. Then elements of $H^1_I(M)$ give the obstruction to extending sections of $M$ supported off $V(I)$ to all $\mathrm{Spec}(R)$. The study of the structure of the local cohomology modules gives us an understanding of $R, I$ and $M$.

## 2.2 Integral Closure of Ideals

In this section, we introduce properties about integral closure. For details we refer to the book of Huneke and Swanson [HS06].

**Definition 2.2.1.** Let $I$ be an ideal in $R$. An element $r \in R$ is integral over $I$ if there exist a positive integer $n$ and elements $a_i \in I^i$, $i = 1, \dots, n$ such that

$$r^n + a_1 r^{n-1} + a_2 r^{n-2} + \cdots + a_{n-1} r + a_n = 0.$$

The set of all elements of $R$ that are integral over $I$ is called the integral closure of $I$, and it is denoted by $\overline{I}$. Then, we say that $I$ is integrally closed if $I = \overline{I}$.

**Remark 2.2.2.** Let $I$, $J$ be two ideals in $R$. Then,

(1) $I \subseteq \overline{I} \subseteq \sqrt{I}$.

(2) If $I \subseteq J$, then $\overline{I} \subseteq \overline{J}$.

**Definition 2.2.3.** Let $J \subseteq I$ be two ideals in $R$. We say that $J$ is a reduction of $I$ if there exists a nonnegative integer $n$ such that $I^{n+1} = JI^n$.

The following propositions give relations between the integral closure of an ideal and its reductions.

**Proposition 2.2.4.** *Let $J \subseteq I$ be two ideals in $R$. Then, $J$ is a reduction of $I$ if and only if $\overline{I} = \overline{J}$.*

**Proposition 2.2.5.** *Let $I$ be an ideal in $R$. Then, $\overline{I}$ is also an ideal of $R$, and $\overline{\overline{I}} = \overline{I}$.*

**Proposition 2.2.6.** *Let $(R, \mathfrak{m}, K)$ be a local ring, and $I$ be an $\mathfrak{m}$-primary ideal of $R$. The following sentences hold.*

*(1) There exists an integer $n$ such that $I^n$ has a reduction generated by a system of parameters.*

*(2) If $K$ is infinite, then $I$ has a reduction generated by a system of parameters.*

## 2.3 Frobenius Morphism

In this section we review concepts and properties that are needed in our study of rings in prime characteristic. We omit details and refer to Huneke's book [Hun96] to the interested reader.

Given a ring $R$ of prime characteristic $p$, the Frobenius map is the function $F : R \longrightarrow R$ given by $F(x) = x^p$ for $x \in R$. The Frobenius map is a ring homomorphism. For a nonnegative integer $e$, we can consider the iterated Frobenius map $F^e : R \longrightarrow R$ given by $F^e(x) = x^{p^e}$ for $x \in R$. In this way, $R$ has an $R$-module structure by restriction of scalars via $F^e$, and we denoted this module action on $R$ by $F^e_* R$. Equivalently, $R$ is an $R^{p^e}$-module, with $R^{p^e} = F^e(R)$.

Suppose that $R$ is a reduced ring. We take $\operatorname{Frac}(R)$ the total ring of fractions of $R$. We note that $\operatorname{Frac}(R) = \bigoplus_i^t K_i$, where each $K_i$ is a field. Let $\overline{K} = \bigoplus_i^t \overline{K_i}$ with $\overline{K_i}$ the algebraic closure of $K_i$. Then, there are inclusions $R \subseteq \operatorname{Frac}(R) \subseteq \overline{K}$ and we let

$$R^{1/p^e} = \{s \in \overline{K} \mid s^{p^e} \in R\}.$$

In other words, $R^{1/p^e}$ is the ring of $p^e$-th roots of elements of $R$. Again $R^{1/p^e}$ is a ring abstractly isomorphic to $R$ via the map $R^{1/p^e} \longrightarrow R$ which sends $x \longrightarrow x^{p^e}$. In particular, the map $F^e$ can be identified with the $R$-module inclusion $R \subseteq R^{1/p^e}$. Hence, there is an $R$-module isomorphism between $F^e_* R$ and $R^{1/p^e}$.

We take $q = p^e$ for some integer $e > 0$ and an ideal $I$ in $R$, we denote the extension of $I$ through $F^e$ by $I^{[q]}$. The ideal $I^{[q]}$ is called the Frobenius power of $I$. If $I = (r_1, \ldots, r_s)$, then $I^{[q]} = ({r_i}^q \mid i = 1, \ldots, s)$. Furthermore, we note that

$$IR^{1/p^e} = (I^{[p^e]})^{1/p^e}, \text{ and } IF^e_* R = F^e_* I^{[p^e]}.$$

**Definition 2.3.1.** The ring $R$ is called $F$-finite if the Frobenius morphism $F^e$ is finite for some (equivalently, for all) integer $e \geq 1$.

Most of the rings we encounter in algebraic geometry are $F$-finite.

**Remark 2.3.2.** Let $R$ be an $F$-finite ring. Then, the following hold.

(1) Any homomorphic image of $R$ is $F$-finite.

(2) Given an ideal $I$ in $R$, we have that $R/I$ is $F$-finite.

(3) The polynomial ring $R[x_1, \ldots, x_n]$ is $F$-finite.

(4) The power series ring $R[[x_1, \ldots, x_n]]$ is $F$-finite.

(5) Any finitely generated $R$-algebra is $F$-finite.

(6) If $S$ is a multiplicative system in $R$, then $S^{-1}R$ is also an $F$-finite ring.

(7) If $R$ is a local ring, then $\widehat{R}$ is $F$-finite.

**Remark 2.3.3.** Given $(R, \mathfrak{m}, K)$ complete local ring, $R$ is $F$-finite is equivalent to that $K$ is $F$-finite.

We now state a result that characterizes regularity in terms Frobenius morphism.

**Theorem 2.3.4** ([Kun69])**.** *Let $R$ be a ring of prime characteristic $p$. Then, $R$ is a regular ring if and only if $F$ is faithfully flat.*

The previous result motivates the study of singularities in prime characteristic via the Frobenius morphism.

**Corollary 2.3.5.** *Let $R$ be an $F$-finite ring of prime characteristic $p$. Then, $R$ is a regular ring if and only if $R$ is a projective $R^p$-module.*

Suppose that $R$ is an $F$-finite regular ring of prime characteristic $p$. By Theorem 2.3.4 the Frobenius map $F$ is faithfully flat. If $I$ is an ideal in $R$ and $x \in R$, then $x^q \in I^{[q]}$ if and only if $x \in I$.

Regular rings correspond to smooth varieties. In order to discuss this, we recall a few definitions.

**Definition 2.3.6.** Let $X \subseteq \mathbb{A}^n$ be an irreducible affine variety, and $f_1, \ldots, f_t \in K[x_1, \ldots, x_n]$ be such that $\mathcal{I}(X) = (f_1, \ldots, f_t)$. Then, $X$ is nonsingular at a point $x_0$ if the rank of the Jacobian matrix $(\partial f_i / \partial x_j(x_0))$ is $n - r$, where $r = \dim(X)$.

We now see a characterization of nonsingular points in terms of regular rings.

**Theorem 2.3.7.** *Let $X \subseteq \mathbb{A}^n$ be an irreducible affine variety, and let $x_0$ be a point in $X$. Then, $X$ is nonsingular at $x_0$ if and only if the local ring $\mathcal{O}_{X,x_0}$ is a regular local ring.*

## 2.4 Mixed Generalized Test Ideals

Through this section $R$ denotes an $F$-finite regular ring of prime characteristic $p$.

### 2.4.1 The Ideal $\mathfrak{b}^{[1/q]}$

We introduce the definition of ideals $\mathfrak{b}^{[1/q]}$, which can be seen with more detail in the work of Bickle, Mustaţă, and Smith [BMS08]. These ideals are the building blocks in the definition of test ideals.

**Definition 2.4.1** ([BMS08, Definition 2.2])**.** If $\mathfrak{b}$ is an ideal of $R$ and $q = p^e$, where $e$ is a positive integer, we define $\mathfrak{b}^{[1/q]}$ as the unique and smallest ideal $I$ of $R$ such that $\mathfrak{b} \subseteq I^{[q]}$.

The ideal $\mathfrak{b}^{[1/q]}$ is well defined. By Corollary 2.3.5, $R$ is a proyective $R^q$-module. Therefore, we have $(\bigcap_i I_i)^{[q]} = \bigcap_i I_i^{[q]}$ for every family of ideals $\{I_i\}_i$ in $R$.

If $R$ is free over $R^q$, we can find an explicit description of the ideal $\mathfrak{b}^{[1/q]}$.

**Theorem 2.4.2** ([BMS08, Proposition 2.5]). *Suppose that $R$ is free module over $R^q$, and let $e_1, \dots, e_N$ be a basis of $R$ over $R^q$. Let $h_1, \dots, h_s$ be generators of an ideal $\mathfrak{b}$ of $R$, and for every $i = 1, \dots, s$ we write*

$$h_i = \sum_{j=1}^{N} a_{i,j}^q e_j,$$

*with $a_{i,j} \in R$. Then, $\mathfrak{b}^{[1/q]} = (a_{i,j} \mid i \leq s,\ j \leq N)$.*

### 2.4.2 Test Ideals

Test ideals were introduced and studied by Hochster and Huneke [HH90, HH94], and were later generalized to the context of pairs by Hara and Yoshida [HY03]. In this subsection we follow the concrete description of these ideals given by Bickle, Mustaţă, and Smith [BMS08].

**Definition 2.4.3.** Let $x$ be a real number. We define

$$\lceil x \rceil = \min\{m \in \mathbb{Z} \mid x \leq m\}.$$

**Notation 2.4.4.** Let $\mathfrak{a}_1, \dots, \mathfrak{a}_n$ be ideals in $R$, and $c = (c_1, \dots, c_n) \in \mathbb{R}^n_{\geq 0}$. We denote $\mathfrak{a}_1^{c_1} \cdots \mathfrak{a}_n^{c_n}$ by $\mathfrak{a}^c$. If $r = (r_1, \dots, r_n) \in \mathbb{R}^n_{\geq 0}$, we denote $(\lceil r_1 \rceil, \dots, \lceil r_n \rceil)$ by $\lceil r \rceil$.

Given $c \in \mathbb{R}^n_{\geq 0}$, and $e \in \mathbb{N}_{>0}$, we have $\lceil c_i p^e \rceil / p^e \geq \lceil c_i p^{e+1} \rceil / p^{e+1}$. Then, it follows that

$$(\mathfrak{a}^{\lceil c p^e \rceil})^{[1/p^e]} \subseteq (\mathfrak{a}^{\lceil c p^{e+1} \rceil})^{[1/p^{e+1}]}.$$

Now, we define the mixed generalized test ideals.

**Definition 2.4.5.** Given $\mathfrak{a}_1, \dots, \mathfrak{a}_n$ ideals in $R$ and $c = (c_1, \dots, c_n) \in \mathbb{R}^n_{\geq 0}$, we define the mixed generalized test ideal with exponents $c_1, \dots, c_n$ as

$$\tau(\mathfrak{a}^c) = \bigcup_{e>0} (\mathfrak{a}^{\lceil c p^e \rceil})^{[1/p^e]}.$$

Since $R$ is a Noetherian ring, the family of ideals $\{(\mathfrak{a}^{\lceil c p^e \rceil})^{[1/p^e]}\}_{e>0}$ stabilizes. Therefore, $\tau(\mathfrak{a}^c) = (\mathfrak{a}^{\lceil c p^e \rceil})^{[1/p^e]}$ for all $e \gg 0$. When $n = 1$, $\tau(\mathfrak{a}^c)$ is called the generalized test ideal of $\mathfrak{a}$ with exponent $c$.

**Theorem 2.4.6** ([Pér13, Theorem 3.10]). *Let $\mathfrak{a}_1, \dots, \mathfrak{a}_n$ be ideals inside $R$, and $c = (c_1, \dots, c_n) \in \mathbb{R}^n_{\geq 0}$. There exists $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in \mathbb{R}^n_{>0}$ such that for every $r = (r_1, \dots, r_n)$, with $0 < r_i < \varepsilon_i$, we have $\tau(\mathfrak{a}^c) = \tau(\mathfrak{a}^{c+r})$.*

By Theorem 2.4.6, we obtain the following definition in the case $n = 1$.

**Definition 2.4.7.** Let $\mathfrak{a}$ be an ideal in $R$, and $c$ be a positive real number. We say that $c$ is an $F$-jumping exponent ($F$-jumping number) for $\mathfrak{a}$ if $\tau(\mathfrak{a}^c) \neq \tau(\mathfrak{a}^{c-\varepsilon})$ for every positive $\varepsilon$. We also consider 0 as an $F$-jumping exponent.

**Theorem 2.4.8** ([BMS08])**.** *The set of $F$-jumping exponent of all ideals in $R$ is discrete, and is contained in the set of rational numbers.*

This implies that, the set of test ideals with exponents in a bounded interval is finite. Our goal is to study the points in $\mathbb{R}^n$ where the mixed test ideals change. Give $l = (l_1, \ldots, l_n) \in \mathbb{R}^n$, we denote $[0, l]$ by the set $[0, l_1] \times \ldots \times [0, l_n]$. If $n > 1$, one can not expect discreteness. However, the regions where $\tau(\mathfrak{a}^c)$ is constant are finite in specific cases.

**Theorem 2.4.9** ([Pér13, Theorem 3.16])**.** *Given nonzero ideals $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ in $R$, where $R$ is essentially of finite type over a finite field $K$, with $K$ of prime characteristic $p$, then $\{\tau(\mathfrak{a}^c) \mid c \in [0, l]\}$ is finite.*

If we omit the condition of finite field, it is still an open problem.

**Conjecture 2.4.10.** Given nonzero ideals $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ in $R$, where $R$ is essentially of finite type over any field $K$, with $K$ of prime characteristic $p$, then $\{\tau(\mathfrak{a}^c) \mid c \in [0, l]\}$ is finite.

The analogous problem is known for mixed multiplier ideals. In particular, if $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ are nonzero ideals, the set $\{\mathcal{J}(\mathfrak{a}^c) \mid c \in [0, l]\}$ is finite. Furthermore, the constancy region in $[0, l]$ can be decomposed into a finite set of rational polytopes with nonoverlapping interiors such that on the interior of each face of such a polytope.

We end this subsection giving a proposition, which says that the test ideal of $\mathfrak{a}$ depend only on its integral closure.

**Proposition 2.4.11** ( [HY03, HT04])**.** *Let $R$ be a regular ring of prime characteristic $p$, $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ be ideals in $R$, and $c = (c_1, \ldots, c_n) \in \mathbb{R}^n_{\geq 0}$. Then, $\tau(\mathfrak{a}^c) = \tau(\overline{\mathfrak{a}}^c)$, where $\overline{\mathfrak{a}}^c = \overline{\mathfrak{a}_1}^{c_1} \cdots \overline{\mathfrak{a}_n}^{c_n}$.*

## 2.5 $F$-Thresholds

The $F$-thresholds were introduced by Mustaţă, Takagi and Watanabe [MTW05] for $F$-finite regular local rings of prime characteristic. In a subsequent work together with Huneke [HMTW08], they defined the $F$-thresholds in general rings of positive characteristic, through upper limits and lower limits, provided they exist. The existence of these invariants in the general case is described in the work of De Stefani, Núñez-Betancourt and Pérez [DSNnBP18].

### 2.5.1 Definition and First Properties

In this subsection $R$ denotes a ring of prime characteristic $p$. We discuss properties related to $F$-thresholds.

**Definition 2.5.1.** Let $R$ be a ring. Given $\mathfrak{a}$, $J$ ideals in $R$ such that $\mathfrak{a} \subseteq \sqrt{J}$, we define

$$\nu_{\mathfrak{a}}^{J}(p^e) = \max\{m \in \mathbb{N} \mid \mathfrak{a}^m \not\subseteq J^{[p^e]}\}.$$

**Notation 2.5.2.** Let $R$ be a ring, and $\mathfrak{a}$ be an ideal in $R$. The minimal number of generators of $\mathfrak{a}$ is denoted by $\mu(\mathfrak{a})$.

**Lemma 2.5.3** ([DSNnBP18, Lemma 3.3])**.** *Let $R$ be a ring, and $\mathfrak{a}, J$ be two ideals in $R$ such that $\mathfrak{a} \subseteq \sqrt{J}$. Then,*

$$\frac{\nu_{\mathfrak{a}}^{J}(p^{e_1+e_2})}{p^{e_1+e_2}} - \frac{\nu_{\mathfrak{a}}^{J}(p^{e_1})}{p^{e_1}} \leq \frac{\mu(\mathfrak{a})}{p^{e_1}}$$

*for every $e_1,\ e_2 \in \mathbb{N}$.*

We now state a key theorem regarding $F$-thresholds.

**Theorem 2.5.4** ([DSNnBP18, Theorem 3.4])**.** *Let $R$ be a ring, and $\mathfrak{a}, J$ be two ideals in $R$ such that $\mathfrak{a} \subseteq \sqrt{J}$. Then, $\lim\limits_{e\to\infty} \frac{\nu_{\mathfrak{a}}^{J}(p^e)}{p^e}$ exists.*

The previous theorem gives existence to the $F$-thresholds and we may define them.

**Definition 2.5.5** ([DSNnBP18])**.** Let $R$ be a ring. Given $\mathfrak{a}, J$ ideals of $R$ such that $\mathfrak{a} \subseteq \sqrt{J}$, we define the $F$-threshold of $\mathfrak{a}$ with respect to $J$ by

$$c^J(\mathfrak{a}) = \lim_{e\to\infty} \frac{\nu_{\mathfrak{a}}^{J}(p^e)}{p^e}.$$

**Proposition 2.5.6** ([MTW05, Proposition 2.7] & [HMTW08, Proposition 2.2])**.** *Let $R$ be a ring, and let $\mathfrak{a}, I, J$ be ideals in $R$. Then, the following statements hold.*

*(1) If $J \subseteq I$, and $\mathfrak{a} \subseteq \sqrt{J}$, then $c^I(\mathfrak{a}) \leq c^J(\mathfrak{a})$.*

*(2) If $\mathfrak{a} \subseteq \sqrt{J}$, then $c^{J^{[p]}}(\mathfrak{a}) = p \cdot c^J(\mathfrak{a})$.*

The following proposition describes the behavior of $F$-thresholds under integral closure.

**Proposition 2.5.7** ([MTW05, HMTW08])**.** *Let $R$ be ring of prime characteristic $p$, and $\mathfrak{a}, J$ be two ideals of $R$ such that $\mathfrak{a} \subseteq \sqrt{J}$. Then, $c^J(\overline{\mathfrak{a}}) = c^J(\mathfrak{a})$.*

### 2.5.2 $F$-Thresholds in Regular Rings

In this subsection we give a few properties of the $F$-thresholds in the case of a regular ring. In particular, we assume that $R$ is an $F$-finite regular ring.

The next theorem gives the relation between the test ideals and $F$-thresholds. It also is a key result for the comparison of the $F$-thresholds and the $F$-jumping numbers.

**Theorem 2.5.8** ([BMS08, Proposition 2.29]). *Let $\mathfrak{a}, J$ be two ideals of $R$. Then, the following statements hold.*

*(1) If $\mathfrak{a} \subseteq \sqrt{J}$, then $\tau(\mathfrak{a}^{c^J(\mathfrak{a})}) \subseteq J$.*

*(2) If $c$ is a nonnegative real number, then $\mathfrak{a} \subseteq \sqrt{\tau(\mathfrak{a}^c)}$ and $c^{\tau(\mathfrak{a}^c)}(\mathfrak{a}) \leq c$.*

**Corollary 2.5.9.** *For any ideal $\mathfrak{a}$ in $R$, the set of $F$-jumping numbers of $\mathfrak{a}$ is equal to the set of $F$-thresholds for $\mathfrak{a}$.*

Since $R$ is an $F$-finite regular ring, by Theorem 2.4.8 and Corollary 2.5.9, we know that the set of $F$-thresholds is contained in the set of rational numbers. This gets us to the next question.

**Question 2.5.10.** Let $R$ be any Noetherian ring, not necessarily regular. Is it true that $c^J(\mathfrak{a}) \in \mathbb{Q}$ for all ideals $\mathfrak{a}$, $J$ in $R$, such that $\mathfrak{a} \subseteq \sqrt{J}$?

## 2.6 Frobenius Splitting and $F$-Purity

The theory of $F$-purity was introduced by Hochster and Roberts [HR76]. This plays a pivotal role in the theory of singularities of rings of positive characteristic. Throughout this section we work with rings of prime characteristic $p$.

**Definition 2.6.1.** A ring $R$ is called $F$-pure if the Frobenius endomorphism $F$ is a pure morphism, that is, $F \otimes 1 : R \otimes M \longrightarrow R \otimes M$ is injective for every $R$-module $M$.

If $R$ is $F$-pure, then $R$ is reduced. Then, for every $e \in \mathbb{N}$, we identify the map $F^e$ with the $R$-module inclusion $R \subseteq R^{1/p^e}$, where $R^{1/p^e}$ denotes the ring of $p^e$-th roots of $R$.

**Remark 2.6.2.** As a consequence of the Kunz' Theorem, all regular rings are $F$-pure.

The following well known proposition relates $F$-pure rings and Frobenius powers.

**Proposition 2.6.3.** *Let $R$ be an $F$-pure ring, and $e$ be a nonnegative integer. Let $I$ be an ideal in $R$, and $x \in R$. Then, $x \in I$ if and only if $x^{p^e} \in I^{[p^e]}$.*

We now define another $F$-singularity called Frobenius splitting.

**Definition 2.6.4.** A ring $R$ is called $F$-split if $F$ is a split monomorphism.

Frobenius splitting is related with $F$-purity. In general, every $F$-split ring is $F$-pure. However, the reciprocal is not necessarily true. And so, $F$-purity can be viewed as a weakening of Frobenius splitting. Remark 2.6.5 gives us the condition where these two concepts are equivalent.

**Remark 2.6.5** ([HR76, Corollary 5.3])**.** Let $R$ be an $F$-finite ring. Then, $R$ is $F$-split if and only if $R$ is $F$-pure. Consequently, the natural inclusion $R \subseteq R^{1/p^e}$ of $R$-modules splits for some (equivalently, for all) $e \geq 1$.

In Remark 2.6.5, the characterization is still true if we substitute the condition of $F$-finite by $(R, \mathfrak{m}, K)$ complete local ring.

Another test for Frobenius splitting is given in the work of Fedder [Fed83]. This is called Fedder's Criterion, which characterizes $F$-purity rings that are quotients of regular rings.

**Theorem 2.6.6** ([Fed83, Theorem 1.12])**.** *Let $(S, \mathfrak{m})$ be a regular local ring of prime characteristic $p$, and let $R = S/I$. Then, $R$ is $F$-pure if and only if $(I^{[p]} : I) \not\subseteq \mathfrak{m}^{[p]}$.*

Fedder's Criterion reduces the $F$-purity of complete intersection to the case of hypersurfaces.

**Proposition 2.6.7** ([Fed83, Proposition 2.1])**.** *If $(S, \mathfrak{m})$ is a regular local ring of prime characteristic $p$, $f_1, \ldots, f_t$ is a regular sequence, and $\underline{f} = f_1 \cdots f_t$, then the following are equivalent:*

*(1) $S/(f_1, \ldots, f_t)$ is $F$-pure,*

*(2) $S/(\underline{f})$ is $F$-pure,*

*(3) $\underline{f}^{p-1} \notin \mathfrak{m}^{[p]}$.*

## 2.7 Standard Graded $K$-Algebras

We begin this section defining standard graded $K$-algebras. Given a field $K$, a $K$-algebra $R$ is $\mathbb{N}$-graded if there exist vector subspaces $R_i \subseteq R$ such that $R = \bigoplus_{i \in \mathbb{N}} R_i$, and $R_i \cdot R_j \subseteq R_{i+j}$ for every $i$ and $j$. We say that $R$ is a standard graded $K$-algebra if it is an $\mathbb{N}$-graded ring such that $R_0 = K$, and $R$ is a finitely generated $K$-algebra, generated in degree one. We use $(R, \mathfrak{m}, K)$ to denote a standard graded $K$-algebra, where $\mathfrak{m} = \bigoplus_{i \geq 1} R_i$ is the irrelevant maximal ideal.

Let $R$ be a standard graded $K$-algebra. A graded module is an $R$-module $M = \bigoplus_{i \in \mathbb{Z}} M_i$ such that $R_i M_j \subseteq M_{i+j}$. In addition, an $R$-homomorphism $\varphi : M \longrightarrow N$ between graded $R$-modules is called homogeneous of degree $c$ if $\varphi(M_i) \subseteq N_{i+c}$ for every $i \in \mathbb{Z}$.

Suppose that $R$ is a standard graded $K$-algebra. Let $I \subseteq R$ be homogeneous ideal, and $M$ be graded $R$-module. Then, the $i$-th local cohomology $H^i_I(M)$ is graded as well.

Moreover, if $\varphi : M \longrightarrow N$ is a homogeneous $R$-module homomorphism of degree $c$, the induced map $H_I^i(M) \longrightarrow H_I^i(N)$ is also homogeneous of degree $c$.

Let $(R, \mathfrak{m}, K)$ be a reduced standard graded $K$-algebra of positive characteristic $p$. We view $R^{1/p^e}$ as a $\frac{1}{p^e}\mathbb{N}$-graded $R$-module. In fact, if $r^{1/p^e} \in R^{1/p^e}$, then $r \in R$ and we can write $r = r_{d_1} + \cdots + r_{d_n}$, with $r_{d_j} \in R_{d_j}$. Then, $r^{1/p^e} = r_{d_1}^{1/p^e} + \cdots + r_{d_n}^{1/p^e}$, and each $r_{d_j}^{1/p^e}$ has degree $d_j/p^e$. In the same way, let $M$ be a $\mathbb{Z}$-graded $R$-module. We have that $M^{1/p^e}$ is a $\frac{1}{p^e}\mathbb{Z}$-graded $R$-module, where $M^{1/p^e}$ denotes the $R$-module which has the same additive structure of abelian group as $M$, and multiplication defined by $r \cdot m^{1/p^e} = (r^{p^e} m)^{1/p^e}$ for all $r \in R$ and $m^{1/p^e} \in M^{1/p^e}$.

**Remark 2.7.1.** As a submodule of $R^{1/p^e}$, $R$ inherits a natural $\frac{1}{p^e}\mathbb{N}$ grading, which is compatible with the standard grading.

## 2.8 $F$-Pure Thresholds

In this section we focus on working with $R$ an $F$-finite $F$-pure ring. The $F$-pure threshold of an ideal $\mathfrak{a} \subseteq R$ was introduced by Takagi and Watanabe [TW04]. In positive characteristic this is considered as analogous to the log canonical threshold, an important invariant of singularities of hypersurfaces in characteristic zero. In particular, the log canonical threshold is the first jumping number of $\mathcal{J}(f^c)$. The study of the $F$-pure threshold is motivated by the log canonical threshold, because both have similar proprieties. Roughly speaking, the $F$-pure threshold measures the splitting order of $\mathfrak{a}$.

Now, let us introduce an ideal that allows the study of homomorphisms that do not give splittings.

**Definition 2.8.1** ([AE05])**.** Let $(R, \mathfrak{m}, K)$ be a local ring or a standard graded $K$-algebra, which is $F$-finite and $F$-pure. We define

$$I_e(R) = \{f \in R \mid \varphi(f^{1/p^e}) \in \mathfrak{m}, \text{ for all } \varphi \in \operatorname{Hom}_R(R^{1/p^e}, R)\},$$

where $e \in \mathbb{N}$.

**Remark 2.8.2.** The set $I_e(R)$ is an ideal of $R$, and is called the $e$-th splitting ideal of $R$. Then, $f \notin I_e(R)$ if and only if $\varphi(f^{1/p^e}) = 1$ for some map $\varphi \in \operatorname{Hom}_R(R^{1/p^e}, R)$.

The ideal $I_e(R)$ is used to define the $F$-signature. Smith and Van den Bergh in their work [SVdB97] showed the existence of this invariant when the ring $R$ is strongly $F$-regular and has finite Frobenius representation type. Later, in the work of Huneke and Leuschke [HL02], they showed that this invariant exists if $R$ is a complete local Gorenstein domain. For Gorenstein Rings on the punctured spectrum, its existence was given by Yao [Yao06]. Subsequently, Tucker [Tuc12], showed existence of the $F$-signature in $R$ with full generality.

Finally, let us state Tucker's Theorem.

**Theorem 2.8.3** ([Tuc12, Theorem 4.9]). *Let $(R, \mathfrak{m}, K)$ be a $d$-dimensional $F$-finite $F$-pure local ring. Then,*

$$\lim_{e\to\infty} \frac{\lambda(R/I_e(R))}{p^{ed}}$$

*exists, where $\lambda(-)$ denotes the length of an $R$-module.*

The previous theorem motivates the following definition.

**Definition 2.8.4.** Suppose that $(R, \mathfrak{m}, K)$ is a $F$-finite $F$-pure local ring of dimension $d$. The $F$-signature of $R$ is defined by

$$s(R) = \lim_{e\to\infty} \frac{\lambda(R/I_e(R))}{p^{ed}},$$

where $\lambda(-)$ denotes the length of an $R$-module.

We now concentrate on properties of $I_e(R)$.

**Proposition 2.8.5** ([AE05]). *Let $(R, \mathfrak{m}, K)$ be a local ring or a standard graded $K$-algebra, which is $F$-finite and $F$-pure. Then, the following statements hold.*

*(1) If $f \notin I_e(R)$, then $f$ is a nonzerodivisor;*

*(2) $\mathfrak{m}^{[p^e]} \subseteq I_e(R)$.*

**Proposition 2.8.6** ([AE05]). *Let $(R, \mathfrak{m}, K)$ be a local ring or a standard graded $K$-algebra, which is $F$-finite and $F$-pure. Then, $\bigcap_{e\in\mathbb{N}} I_e(R)$ is a prime ideal of $R$.*

**Definition 2.8.7** ([AE05]). Let $(R, \mathfrak{m}, K)$ be a local ring or a standard graded $K$-algebra, which is $F$-finite and $F$-pure. We define the splitting prime of $R$ as $\mathcal{P}(R) = \bigcap_{e\in\mathbb{N}} I_e(R)$.

With the help of the splitting ideal, we can define the $F$-pure threshold.

**Definition-Theorem 2.8.8** ([DSNnB18]). Let $(R, \mathfrak{m}, K)$ be either a local ring or a standard graded $K$-algebra, which is $F$-finite and $F$-pure ring. Given $\mathfrak{a}$ a proper ideal in $R$, we define

$$b_{\mathfrak{a}}(p^e) = \max\{t \in \mathbb{N} \mid \mathfrak{a}^t \not\subseteq I_e(R)\}.$$

We define the $F$-pure threshold of $\mathfrak{a}$ in $R$ as

$$\operatorname{fpt}(\mathfrak{a}) = \lim_{e\to\infty} \frac{b_{\mathfrak{a}}(p^e)}{p^e}.$$

When $\mathfrak{a} = \mathfrak{m}$, the $F$-pure threshold $\operatorname{fpt}(\mathfrak{m})$ is denoted by $\operatorname{fpt}(R)$.

The next proposition gives us a relation of the $F$-pure threshold and the height of an ideal.

**Proposition 2.8.9** ([TW04, Proposition 2.6])**.** *Let $(R, \mathfrak{m}, K)$ be an $F$-finite $F$-pure local ring with infinite residue field, and let $\mathfrak{a} \subsetneq R$ be a proper ideal of positive height. Then,* $\operatorname{fpt}(\mathfrak{a}) \leq \operatorname{ht}(\mathfrak{a})$

We end this section with a property of the $F$-pure thresholds that is similar to the log canonical thresholds.

**Remark 2.8.10** ([BMS08])**.** Suppose that $(R, \mathfrak{m}, K)$ is an $F$-finite regular local ring. Let $f \in R \backslash \{0\}$. Then, $\operatorname{fpt}(f) = \sup\{c > 0 \mid \tau(f^c) = R\}$. We note that the $\operatorname{fpt}(f)$ is the first $F$-jumping number of $f$.

## 2.9 $a$-Invariants and Regularity

In this section we focus on standard graded $K$-algebras. We study the $a$-invariants and regularity in rings modulo Frobenius powers of an ideal.

Suppose that $(R, \mathfrak{m}, K)$ is a standard graded $K$-algebra. Let $I$ be a homogeneous ideal of $R$. We recall that if $M$ is a graded $R$-module, its $i$-th local cohomology $H^i_I(M)$ is a graded module. Moreover, if $M$ is finitely generated, the module $H^i_{\mathfrak{m}}(M)$ is Artinian. Therefore, one can define the following number.

**Definition 2.9.1** ([GW78])**.** Let $(R, \mathfrak{m}, K)$ be a standard graded $K$-algebra. Let $M$ be an $\frac{1}{p^e}\mathbb{N}$-graded finitely generated $R$-module. If $H^i_{\mathfrak{m}}(M) \neq 0$, we define the $i$-th $a$-invariant of $M$ by

$$a_i(M) = \max\left\{ s \in \frac{1}{p^e}\mathbb{Z} \mid H^i_{\mathfrak{m}}(M)_s \neq 0 \right\}.$$

If $H^i_{\mathfrak{m}}(M) = 0$, we set $a_i(M) = -\infty$.

**Remark 2.9.2.** Suppose that $M$ is a finitely generated graded $R$-module. Since $(-)^{1/p^e}$ is exact, then $H^i_{\mathfrak{m}}(M^{1/p^e}) \cong H^i_{\mathfrak{m}}(M)^{1/p^e}$. As a consequence, $a_i(M^{1/p^e}) = \frac{a_i(M)}{p^e}$.

**Definition 2.9.3.** Let $(R, \mathfrak{m}, K)$ be a standard graded $K$-algebra. Let $M$ be an $\frac{1}{p^e}\mathbb{N}$-graded finitely generated $R$-module. We define the regularity of $M$ by

$$\operatorname{reg}(M) = \max_{i \in \mathbb{Z}}\{a_i(M) + i\}.$$

Next theorem gives us conditions for the regularity in rings modulo Frobenius power of ideals.

**Theorem 2.9.4** ([DSNnBP18, Theorem 5.4])**.** *Let $(R, \mathfrak{m}, K)$ be a standard graded $K$-algebra that is $F$-finite and $F$-pure. Suppose that $J$ is a homogeneous ideal of $R$. If there exists a constant $C$ such that $\operatorname{reg}(R/J^{[p^e]}) \leq Cp^e$ for all $e \gg 0$, then*

$$\lim_{e \to \infty} \frac{\operatorname{reg}(R/J^{[p^e]})}{p^e}$$

*exists, and it is bounded below by $\max_{i \in \mathbb{N}}\{a_i(R/J)\} + \operatorname{fpt}(R)$.*

## 2.10 Hilbert-Kunz Multiplicity

In this section we give an abbreviated collection of properties of the Hilbert-Kunz multiplicity that are necessary to study Theorem F. For details we refer to the work of Huneke [Hun13].

Throughout this section $(R, \mathfrak{m}, K)$ denotes a local ring of prime characteristic $p$. We use $\lambda(-)$ to denote the length of a $R$-module. We focus on an important numerical invariant that measures the failure of flatness for the iterated Frobenius, the Hilbert-Kunz multiplicity [Mon83, PT18].

Let $M$ be an $R$-module, $I \subseteq R$ be an ideal, $e$ be nonnegative integer, and $q = p^e$. The study of the behavior of $\lambda(M/I^{[q]}M)$ as a function on $q$ was introduced by Kunz [Kun69], as a way to measure how close the ring $R$ is to being regular.

**Theorem 2.10.1** ([Kun69, Proposition 3.2])**.** *Let $(R, \mathfrak{m}, K)$ be a local ring of dimension $d$. For every $e \in \mathbb{N}$, and $q = p^e$, $\lambda(R/\mathfrak{m}^{[q]}) \geq q^d$. If $R$ is a regular ring and $I$ is an $\mathfrak{m}$-primary ideal of $R$, then $\lambda(R/I^{[q]}) = q^d \lambda(R/I)$.*

Next result was proved by Monsky [Mon83], and shows the existence of the main invariant of this section.

**Theorem 2.10.2** ([Mon83, Theorem 1.8])**.** *Let $(R, \mathfrak{m}, K)$ be a local ring of dimension $d$. Let $M$ be a finitely generated $R$-module, and $I$ be an $\mathfrak{m}$-primary ideal of $R$. Then,*

$$\lim_{e \to \infty} \frac{\lambda(M/I^{[p^e]}M)}{p^{ed}}$$

*exists.*

The previous theorem motivates the following definition.

**Definition 2.10.3.** Suppose that $(R, \mathfrak{m}, K)$ is a local ring of dimension $d$. Let $M$ be a finitely generated $R$-module, and $I$ be an $\mathfrak{m}$-primary ideal of $R$. The Hilbert-Kunz multiplicity of $M$ along $I$ is defined by

$$\mathrm{e}_{HK}(I, M) = \lim_{e \to \infty} \frac{\lambda(M/I^{[p^e]}M)}{p^{ed}}.$$

We often remove the $R$ in $\mathrm{e}_{HK}(I, R)$ and write $\mathrm{e}_{HK}(I)$. When $I = \mathfrak{m}$, we set $\mathrm{e}_{HK}(M) = \mathrm{e}_{HK}(I, M)$, and refer to this value as the Hilbert-Kunz multiplicity of $M$ along $\mathfrak{m}$.

The following proposition presents properties of the Hilbert-Kunz multiplicity that are also satisfied by the Hilbert-Samuel multiplicity.

**Proposition 2.10.4.** *Let $(R, \mathfrak{m}, K)$ be a local ring, $M$ be a finitely generated $R$-module, and $I$ be an $\mathfrak{m}$-primary ideal of $R$. The following sentences hold.*

(1) *Let* $\Lambda$ *be the set of minimal prime ideals* $P$ *of* $R$ *such that* $\dim(R/P) = \dim(R)$. *Then,*

$$\mathrm{e}_{HK}(I, M) = \sum_{P \in \Lambda} \mathrm{e}_{HK}(I, R/P)\lambda(M_P).$$

(2) *If* $R$ *is a domain, then* $\mathrm{e}_{HK}(I, M) = \mathrm{e}_{HK}(I, R)\,\mathrm{rank}_R(M)$.

(3) *If* $R$ *is equidimensional and* $\mathrm{e}_{HK}(R) = 1$, *then* $R$ *is a domain.*

(4) *If* $f$ *is a nonzerodivisor of* $R$, *then* $\mathrm{e}_{HK}(R/fR) \geq \mathrm{e}_{HK}(R)$.

We end this section with a result on the relation between Hilbert-Kunz multiplicity and regular rings.

**Definition 2.10.5.** Let $(R, \mathfrak{m}, K)$ be a local ring. We say that $R$ is unmixed if $\dim(\widehat{R}) = \dim(\widehat{R}/P)$ for every associated prime $P$ of $\widehat{R}$.

**Theorem 2.10.6** ([WY00, Theorem 1.5])**.** *Let* $(R, \mathfrak{m}, K)$ *be an unmixed local ring. Then,* $\mathrm{e}_{HK}(R) = 1$ *if and only if* $R$ *is regular.*

**Remark 2.10.7.** In Theorem 2.10.6, the direction $R$ regular implies $\mathrm{e}_{HK}(R) = 1$, does not need the condition unmixed.

# CHAPTER 3

# $F$-Invariants of Stanley-Reisner Rings

In this chapter we study a general form of the $F$-pure thresholds, called the Cartier thresholds. These and the $F$-thresholds are important invariants in prime characteristic. In certain cases, it is known that they are rational numbers. We show this property for Stanley-Reisner rings in several cases (see Theorems A and B). Moreover, we conclude this chapter with conditions of the regularity in Stanley-Reisner rings modulo Frobenius power of ideals (see Theorem C).

The results presented in this chapter are contained in a paper by the author [BC20].

## 3.1 Stanley-Reisner Rings

Throughout this section we use the following notation.

**Notation 3.1.1.** We denote $S = K[x_1, \ldots, x_n]$ with $K$ an $F$-finite field of prime characteristic $p$. Let $I$ be a squarefree monomial ideal of $S$. Let $I = \bigcap_{i=1}^{l} \mathfrak{p}_i$ such that $\mathfrak{p}_i \not\subseteq \mathfrak{p}_j$ for $i \neq j$ and $\mathfrak{p}_1, \ldots, \mathfrak{p}_l$ are generated by variables. We take $R = S/I$.

These rings have mild singularities, for instance, they are $F$-pure. They also have combinatorial structure given by simplicial complexes.

Suppose that $\mathfrak{a} \subseteq R$ is an ideal. We abuse the notation and denote the inverse image of $\mathfrak{a} \subseteq R$ under the natural projection $S \longrightarrow S/I$ by $\mathfrak{a} \subseteq S$.

We now characterize the ring of $p$-th roots of $R$ in terms of quotient ideals.

**Proposition 3.1.2.** *If $q = p^e$, where $e$ is a nonnegative integer, then*

$$R^{1/q} = S^{1/q}/I^{1/q} \cong \bigoplus_{\substack{1 \leq i \leq s \\ \alpha \in \mathcal{A}}} S/J_{i,\alpha}(a_i x^\alpha)^{1/q}$$

*as $R$-modules, with $\mathcal{A} = \{\alpha \in \mathbb{N}^n \mid 0 \le \alpha_i \le q-1$ for $i = 1, \dots, n\}$, $\mathcal{B} = \{{a_i}^{1/q} \mid i = 1, \dots, s\}$ is a base of $K^{1/q}$ as $K$-vector space, and $J_{i,\alpha} = (I : a_i x^\alpha)$.*

*Proof.* Each element $r^{1/q} \in S^{1/q}$ can be written uniquely as

$$r^{1/q} = \bigoplus_{\substack{1 \le i \le s \\ \alpha \in \mathcal{A}}} r_{i,\alpha} (a_i x^\alpha)^{1/q},$$

where $r_{i,\alpha} \in S$. We take

$$\varphi : S^{1/q} \longrightarrow \bigoplus_{\substack{1 \le i \le s \\ \alpha \in \mathcal{A}}} S/J_{i,\alpha} (a_i x^\alpha)^{1/q},$$

defined by

$$\varphi(r^{1/q}) = \bigoplus_{\substack{1 \le i \le s \\ \alpha \in \mathcal{A}}} (r_{i,\alpha} + J_{i,\alpha})(a_i x^\alpha)^{1/q}.$$

We have that $\varphi$ is a surjective $S$-linear morphism.

We claim that $\ker \varphi = I^{1/q}$. Let $r^{1/q} \in \ker \varphi$. It is sufficient to consider $r$ a monomial. Then, $r^{1/q} = x^\theta (a_i x^\alpha)^{1/q}$ for some $\theta \in \mathbb{N}^n$, $\alpha \in \mathcal{A}$, and $i \in \{1, \dots, s\}$. Hence, $0 = \varphi(r^{1/q}) = (x^\theta + J_{i,\alpha})(a_i x^\alpha)^{1/q}$. Thus, $x^\theta \in J_{i,\alpha}$. This implies that $a_i x^{\alpha+\theta} \in I$, and so, $x^{\theta/q} (a_i x^\alpha)^{1/q} \in I^{1/q}$. It follows that $r^{1/q} = x^\theta (a_i x^\alpha)^{1/q} = (x^{\theta/q})^q (a_i x^\alpha)^{1/q} \in I^{1/q}$.

To show the other inclusion, it is enough to consider $r^{1/q} = x^\theta (a_i x^\alpha)^{1/q} x^{\beta/q} \in I^{1/q}$ with $\theta \in \mathbb{N}^n$, $\alpha \in \mathcal{A}$, $i \in \{1, \dots, s\}$, and $x^\beta$ a generator of $I$. Since $0 \le \alpha_j \le q-1$ and $0 \le \beta_j \le 1$ for every $1 \le j \le n$, there exists $\gamma \in \mathbb{N}^n$ with $0 \le \gamma_j \le 1$ such that $\alpha + \beta - q\gamma \in \mathcal{A}$. Let $\alpha' = \alpha + \beta - q\gamma$. We note that $x^{\theta+\gamma}(a_i x^{\alpha'}) \in I$. As a consequence, $x^{\theta+\gamma} \in J_{i,\alpha'}$. Furthermore, $r^{1/q} = x^{\theta+\gamma} (a_i x^{\alpha'})^{1/q}$. Subsequently, $\varphi(r^{1/q}) = (x^{\theta+\gamma} + J_{i,\alpha'})(a_i x^{\alpha'})^{1/q} = 0$. Thus, $r^{1/q} \in \ker \varphi$.

It follows that

$$R^{1/q} \cong \bigoplus_{\substack{1 \le i \le s \\ \alpha \in \mathcal{A}}} S/J_{i,\alpha} (a_i x^\alpha)^{1/q}$$

as $S$-module. Therefore, they are isomorphic as $R$-modules. □

**Remark 3.1.3.** As in Notation 3.1.1, let $\mathfrak{q}$ be a prime ideal of $S$. Suppose that $\mathfrak{p}_1, \dots, \mathfrak{p}_r \subseteq \mathfrak{q}$ with $r \le l$, $\mathfrak{p}_j \not\subseteq \mathfrak{q}$ for $r < j$, and $(x_1, \dots, x_u) = \sum_{i=1}^r \mathfrak{p}_i$.

Let $\widetilde{\mathfrak{q}}_0, \dots, \widetilde{\mathfrak{q}}_t \in \operatorname{Spec} S_\mathfrak{q}$ be such that $(x_1, \dots, x_u)S_\mathfrak{q} = \widetilde{\mathfrak{q}}_0 \subsetneqq \widetilde{\mathfrak{q}}_1 \subsetneqq \dots \subsetneqq \widetilde{\mathfrak{q}}_t$. There exist $\mathfrak{q}_0, \dots, \mathfrak{q}_t \in \operatorname{Spec} S$, where $\mathfrak{q}_i \subseteq \mathfrak{q}$ and $\mathfrak{q}_i = \widetilde{\mathfrak{q}}_i \cap S$. We have that

$$(0) \subsetneqq (x_1) \subsetneqq (x_1, x_2) \subsetneqq, \dots, \subsetneqq (x_1, \dots, x_u) = \mathfrak{q}_0 \subsetneqq \mathfrak{q}_1 \subsetneqq \dots \subsetneqq \mathfrak{q}_t \subseteq \mathfrak{q}$$

is a chain of prime ideals in $S$ with length $u+t$, thus $u+t \le \operatorname{ht}(\mathfrak{q})$. Hence, $t \le \operatorname{ht}(\mathfrak{q}) - u$. Then, $\dim S_\mathfrak{q}/(x_1, \dots, x_u)S_\mathfrak{q} \le \operatorname{ht}(\mathfrak{q}) - u$. Therefore,

$$\operatorname{ht}(\mathfrak{q}) - u = \dim S_\mathfrak{q}/(x_1, \dots, x_u)S_\mathfrak{q}.$$

In particular, if we take $A = \widehat{S_{\mathfrak{q}}}$, we have that

$$\dim A - u = \dim A/(x_1, \ldots, x_u)A.$$

Since $A$ is a complete local regular ring, $A \cong K[[x_1, \ldots, x_u, y_1, \ldots, y_t]]$. Moreover, we have that $IA = \bigcap_{i=1}^{l} \mathfrak{p}_i A$ is squarefree monomial ideal of $A$ in variables $x_1, \ldots, x_u$. We denote $\underline{x}^{\theta} = x_1{}^{\theta_1} \cdots x_u{}^{\theta_u} y_1{}^{\theta_{u+1}} \cdots y_t{}^{\theta_{u+t}}$. We take $B = A/IA$ and $\mathfrak{m}$ its maximal ideal.

**Proposition 3.1.4.** *If $q = p^e$, where $e$ is a nonnegative integer, then*

$$B^{1/q} \cong \bigoplus_{\substack{1 \leq i \leq s \\ \alpha \in \mathcal{A}}} A/J_{i,\alpha}(a_i \underline{x}^{\alpha})^{1/q}$$

*as $B$-modules, with $\mathcal{A} = \{\alpha \in \mathbb{N}^{u+t} \mid 0 \leq \alpha_i \leq q-1$ for $i = 1, \ldots, u+t\}$, $\mathcal{B} = \{a_i{}^{1/q} \mid i = 1, \ldots, s\}$ is a base of $K^{1/q}$ as $K$-vector space, and $J_{i,\alpha} = (IA : a_i \underline{x}^{\alpha})$.*

*Proof.* The proof is analogous to Proposition 3.1.2. □

## 3.2 $F$-Thresholds in Stanley-Reisner Rings

In this section, we focus on Stanley-Reisner rings. We denote $S = K[x_1, \ldots, x_n]$ with $K$ an $F$-finite field of prime characteristic $p$. Let $I$ be a squarefree monomial ideal of $S$, and $R = S/I$.

The following proposition is one of the main results of this work, Theorem B. Using the fact that the quotient of $R$ with each of its minimal prime ideals is a regular ring, we obtain a case where the $F$-threshold is a rational number.

**Theorem 3.2.1.** *Let $\mathfrak{a}, J$ be two ideals of $R$, with $\mathfrak{a} \subseteq \sqrt{J}$, and $J$ monomial. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_l$ be the minimal prime ideals of $R$. Then,*

$$c_R^J(\mathfrak{a}) = \max \left\{ c_{S/\mathfrak{p}_i}^J(\mathfrak{a}) \right\}.$$

*In particular, $c_R^J(\mathfrak{a}) \in \mathbb{Q}$.*

*Proof.* We know that $I = \bigcap_{i=1}^{l} \mathfrak{p}_i$. Moreover, each $\mathfrak{p}_i$ is generated by variables. We claim that $c_R^J(\mathfrak{a}) \geq \max \left\{ c_{S/\mathfrak{p}_i}^J(\mathfrak{a}) \right\}$. Let $e$ be a nonnegative integer. We take $t_i = \nu_{S/\mathfrak{p}_i}^J(\mathfrak{a}, p^e)$. Then, $\mathfrak{a}^{t_i}/\mathfrak{p}_i \not\subseteq J^{[p^e]}/\mathfrak{p}_i$. Hence, there exists $r \in \mathfrak{a}^{t_i}$ such that $r - c \notin \mathfrak{p}_i$ for every $c \in J^{[p^e]}$. Thus, $r - c \notin I$, and so $r \notin J^{[p^e]}$. As a consequence $\mathfrak{a}^{t_i} \not\subseteq J^{[p^e]}$.

We have that $t_i \leq \nu_R^J(\mathfrak{a}, p^e)$ for all $i$. Then, $\frac{\nu_{S/\mathfrak{p}_i}^J(\mathfrak{a}, p^e)}{p^e} \leq \frac{\nu_R^J(\mathfrak{a}, p^e)}{p^e}$. Thus, $c_{S/\mathfrak{p}_i}^J(\mathfrak{a}) \leq c_R^J(\mathfrak{a})$. Therefore, $c_R^J(\mathfrak{a}) \geq \max \left\{ c_{S/\mathfrak{p}_i}^J(\mathfrak{a}) \right\}$.

We now show that $\bigcap_{i=1}^{l} (J^{[p^e]} + \mathfrak{p}_i) \subseteq J^{[p^e]} + I$. We proceed by contradiction. Let $s$ be a generator of $\bigcap_{i=1}^{l} (J^{[p^e]} + \mathfrak{p}_i)$ such that $s \notin J^{[p^e]} + I$. Since $J^{[p^e]}$ and each $\mathfrak{p}_i$ are monomial ideals, we have that every $J^{[p^e]} + \mathfrak{p}_i$ is a monomial ideal too. Hence,

$\bigcap_{i=1}^{l}(J^{[p^e]}+\mathfrak{p}_i)$ is a monomial ideal. We can take $s$ as a monomial. Furthermore, $s\notin J^{[p^e]}$ and $s\notin I$. Thus, there exists an $i$ such that $s\notin \mathfrak{p}_i$. However, $s\in J^{[p^e]}+\mathfrak{p}_i$. Since $s$ is a monomial and $\mathfrak{p}_i$ is generated by variables, we conclude that $s\in J^{[p^e]}$, we get a contradiction. Thus, $s\in J^{[p^e]}+I$.

We prove that $c_R^J(\mathfrak{a})\leq \max\left\{c^J_{S/\mathfrak{p}_i}(\mathfrak{a})\right\}$. Let $e$ be a nonnegative integer. We take $t=\nu_R^J(\mathfrak{a},p^e)$. Then, $\mathfrak{a}^t\not\subseteq J^{[p^e]}$. Hence, there exists $r\in\mathfrak{a}^t$ such that $r-c\notin I$ for every $c\in J^{[p^e]}$. As a consequence, $r\notin J^{[p^e]}+I$, and so $r\notin\bigcap_{i=1}^{l}(J^{[p^e]}+\mathfrak{p}_i)$. Hence, $r\notin J^{[p^e]}+\mathfrak{p}_i$ for some $i$. It follows that $\mathfrak{a}^t/\mathfrak{p}_i\not\subseteq J^{[p^e]}/\mathfrak{p}_i$.

Consequently, we have $t\leq \nu^J_{S/\mathfrak{p}_i}(\mathfrak{a},p^e)$ for some $i$. Then, $\frac{\nu_R^J(\mathfrak{a},p^e)}{p^e}\leq\max\left\{\frac{\nu^J_{S/\mathfrak{p}_i}(\mathfrak{a},p^e)}{p^e}\right\}$. Therefore, $c_R^J(\mathfrak{a})\leq\max\left\{c^J_{S/\mathfrak{p}_i}(\mathfrak{a})\right\}$.

Now, each $S/\mathfrak{p}_i$ is a regular ring, then $c^J_{S/\mathfrak{p}_i}(\mathfrak{a})$ is a rational number by Theorem 2.4.8 and Corollary 2.5.9. Since

$$c_R^J(\mathfrak{a})=\max\left\{c^J_{S/\mathfrak{p}_i}(\mathfrak{a})\right\},$$

$c_R^J(\mathfrak{a})$ is a rational number. □

**Remark 3.2.2.** Given $\widetilde{S}=K[[x_1,\dots,x_n]]$ with $K$ an $F$-finite field of prime characteristic $p$. We take $\widetilde{I}$ as a squarefree monomial ideal of $\widetilde{S}$, and $\widetilde{R}=\widetilde{S}/\widetilde{I}$, same as in Theorem 3.2.1. Let $\widetilde{\mathfrak{a}}$, $\widetilde{J}$ be two ideals of $\widetilde{R}$, with $\widetilde{\mathfrak{a}}\subseteq\sqrt{\widetilde{J}}$, and $\widetilde{J}$ monomial. Then, $c^{\widetilde{J}}_{\widetilde{R}}(\widetilde{\mathfrak{a}})\in\mathbb{Q}$.

## 3.3 The Ideal $J_e$

In this section we present an ideal, which is related to the Cartier operators. We study the Cartier core and we give properties of both ideals. We also see the behavior of them in the Stanley-Reisner rings for monomial prime ideals.

### 3.3.1 Cartier Contraction

We begin this subsection with a definition given by De Stefani, Hernández, Núñez-Betancourt and Witt [DSHNnBW].

**Definition 3.3.1** ([DSHNnBW])**.** Let $R$ be an $F$-finite $F$-pure ring, and $J$ be an ideal in $R$. We define

$$J_e=\{f\in R\mid \varphi(f^{1/p^e})\in J,\ \text{for all}\ \varphi\in\operatorname{Hom}_R(R^{1/p^e},R)\},$$

for $e\in\mathbb{N}$.

**Remark 3.3.2.** The set $J_e$ is an ideal of $R$. If $(R,\mathfrak{m},K)$ is a local ring or a standard graded $K$-algebra, and $\mathfrak{m}=J$, we have $I_e(R)=J_e$.

**Proposition 3.3.3.** *Let $R$ be an $F$-finite $F$-pure ring, and $J$ be an ideal of $R$. Then, for every $e$ nonnegative integer $J^{[p^e]} \subseteq J_e \subseteq J$.*

*Proof.* First, we show the inclusion $J^{[p^e]} \subseteq J_e$. Let $x$ be an element of $J$. For every $\varphi \in \mathrm{Hom}_R(R^{1/p^e}, R)$, $\varphi((x^{p^e})^{1/p^e}) = \varphi(x \cdot 1) = x\varphi(1) \in J$. Therefore, $x^{p^e} \in J_e$.

To show the other inclusion, we proceed by contrapositive. We suppose that there exists $r \notin J$. Since $R \subseteq R^{1/p^e}$ is an $R$-module split, we can take $\beta \in \mathrm{Hom}_R(R^{1/p^e}, R)$ such that $\beta|_R = 1_R$. It follows that $\beta((r^{p^e})^{1/p^e}) = \beta(r) = r \notin J$. Hence, $r^{p^e} \notin J_e$, and so, $r \notin J_e$. □

The equality $J_e = J$ holds under certain conditions. This is done in Proposition 3.3.7 below.

The following proposition shows that the formation of the ideals $J_e$ commutes with arbitrary intersections.

**Proposition 3.3.4.** *Let $R$ be an $F$-finite $F$-pure ring, and $\{J_i\}_i$ be a family of ideals in $R$. Then, $\left(\bigcap_i J_i\right)_e = \bigcap_i (J_i)_e$ for every $e$ nonnegative integer.*

*Proof.* For every $\varphi \in \mathrm{Hom}_R(R^{1/p^e}, R)$, we have that

$$\begin{aligned} x \in \left(\bigcap_i J_i\right)_e &\Leftrightarrow \varphi(x^{1/p^e}) \in \bigcap_i J_i \\ &\Leftrightarrow \varphi(x^{1/p^e}) \in J_i \text{ for every } i \\ &\Leftrightarrow x \in (J_i)_e \text{ for every } i \\ &\Leftrightarrow x \in \bigcap_i (J_i)_e. \end{aligned}$$

□

**Proposition 3.3.5.** *Let $R$ be an $F$-finite $F$-pure ring, and $\mathfrak{q}$ be a prime ideal of $R$. Then, $\mathfrak{q}_e$ is a $\mathfrak{q}$-primary ideal for every $e \in \mathbb{N}$.*

*Proof.* We show that $\sqrt{\mathfrak{q}_e} = \mathfrak{q}$. By Proposition 3.3.3, $\mathfrak{q}^{[p^e]} \subseteq \mathfrak{q}_e \subseteq \mathfrak{q}$, and so,

$$\mathfrak{q} = \sqrt{\mathfrak{q}} = \sqrt{\mathfrak{q}^{[p^e]}} \subseteq \sqrt{\mathfrak{q}_e} \subseteq \sqrt{\mathfrak{q}} = \mathfrak{q}.$$

We now show that $\mathfrak{q}_e$ is primary. Suppose that there exist $a, b \in R$ such that $a \notin \mathfrak{q}_e$ and $b \notin \mathfrak{q}$. There is $\varphi \in \mathrm{Hom}_R(R^{1/p^e}, R)$ satisfying $\varphi(a^{1/p^e}) \notin \mathfrak{q}$. As $\mathfrak{q}$ is a prime ideal, $\varphi((b^{p^e}a)^{1/p^e}) = \varphi(ba^{1/p^e}) = b\varphi(a^{1/p^e}) \notin \mathfrak{q}$. Hence, $b^{p^e}a \notin \mathfrak{q}_e$, and so, $ab \notin \mathfrak{q}_e$. Therefore, $\mathfrak{q}_e$ is a $\mathfrak{q}$-primary ideal of R. □

We now recall the definition of uniformly compatible. Our goal is to study the biggest uniformly compatible ideal contained in other given ideal.

**Definition 3.3.6** ([Sch10])**.** Let $R$ be an $F$-finite ring, and $J$ be an ideal of $R$. We say that $J$ is uniformly $F$-compatible if $\varphi(J^{1/p^e}) \subseteq J$ for every $e > 0$ and every $\varphi \in \mathrm{Hom}_R(R^{1/p^e}, R)$.

**Proposition 3.3.7.** *Let $R$ be an $F$-finite $F$-pure ring. Let $J$ be an ideal of $R$. Then, $J_e = J$ for every $e$ nonnegative integer if and only if $J$ is uniformly $F$-compatible.*

*Proof.* We suppose that $J_e = J$ for every $e \geq 0$. We have that $\varphi(J^{1/p^e}) \subseteq J$ for every $\varphi \in \operatorname{Hom}_R(R^{1/p^e}, R)$ by Definition 3.3.1.

For the other direction, it is enough to see that $J \subseteq J_e$ for every $e > 0$. In fact, by Definition 3.3.6, $\varphi(J^{1/p^e}) \subseteq J$ for all $\varphi \in \operatorname{Hom}_R(R^{1/p^e}, R)$. Therefore, $J \subseteq J_e$. $\square$

**Lemma 3.3.8.** *Let $R$ be an $F$-finite $F$-pure ring. Let $J$ be an ideal of $R$. Then, $\bigcap_{s\in\mathbb{N}} J_s$ is uniformly $F$-compatible.*

*Proof.* We proceed by contradiction. We suppose that $\varphi\left(\left(\bigcap_{s\in\mathbb{N}} J_s\right)^{1/p^e}\right) \not\subseteq \bigcap_{s\in\mathbb{N}} J_s$ for some $e > 0$ and $\varphi \in \operatorname{Hom}_R(R^{1/p^e}, R)$, and so, we have an $f \in \bigcap_{s\in\mathbb{N}} J_s$ such that $\varphi(f^{1/p^e}) \notin \bigcap_{s\in\mathbb{N}} J_s$. Thus, $\varphi(f^{1/p^e}) \notin J_s$ for some $s \in \mathbb{N}$. Consequently, there exists $\phi \in \operatorname{Hom}_R(R^{1/p^s}, R)$ such that $\phi(\varphi(f^{1/p^e})^{1/p^s}) \notin J$.

If we take $\psi : R^{1/p^{e+s}} \longrightarrow R^{1/p^s}$ such that $\psi(r^{1/p^{e+s}}) = \varphi(r^{1/p^e})^{1/p^s}$, we have that $\psi$ is $R$-linear. As a consequence, $\sigma = \phi \circ \psi \in \operatorname{Hom}_R(R^{1/p^{e+s}}, R)$. Then,

$$\sigma(f^{1/p^{e+s}}) = \phi \circ \psi(f^{1/p^{e+s}}) = \phi(\varphi(f^{1/p^e})^{1/p^s}) \notin J.$$

Therefore, $f \notin J_{e+s}$, and we reach a contradiction. $\square$

**Proposition 3.3.9.** *Let $R$ be an $F$-finite $F$-pure ring. Let $J$ be an ideal of $R$. Then, $\bigcap_{s\in\mathbb{N}} J_s$ is the biggest uniformly $F$-compatible ideal contained in $J$.*

*Proof.* Let $I \subseteq J$ be an uniformly $F$-compatible ideal. By Proposition 3.3.7, $I = I_e \subseteq J_e$ for every $e \geq 0$. Therefore, $I \subseteq \bigcap_{s\in\mathbb{N}} J_s$. $\square$

Motivated by the splitting prime ideal [AE05] and differential core [BJNnB19], we introduce the Cartier core.

**Definition 3.3.10.** Let $R$ be an $F$-finite $F$-pure ring. Given $J$ an ideal of $R$, we define the Cartier core of $J$ as $\mathcal{P}(J) = \bigcap_{s\in\mathbb{N}} J_s$.

**Remark 3.3.11.** Let $(R, \mathfrak{m}, K)$ be a local ring or a standard graded $K$-algebra, and $\mathfrak{m} = J$. Then, the ideal $\mathcal{P}(J)$ coincides with the splitting prime of $R$, denoted $\mathcal{P}(R)$, and introduced by Aberbach and Enescu [AE05].

In Proposition 3.3.13, we see a characterization of the Cartier core. This plays an important role in Subsection 3.3.2 to describe the ideal $\mathfrak{q}_e$ for Stanley-Reisner rings.

**Remark 3.3.12.** Let $R$ be an $F$-finite $F$-pure ring, and $J$ be an ideal of $R$. For every $r \in \sqrt{\mathcal{P}(J)}$, $r^{p^e} \in \mathcal{P}(J)$ for some $e \in \mathbb{N}$. Since $R \subseteq R^{1/p^e}$ is an $R$-module split, there exists $\beta \in \operatorname{Hom}_R(R^{1/p^e}, R)$ such that $\beta|_R = 1_R$. Moreover, $r = (r^{p^e})^{1/p^e} \in (\mathcal{P}(J))^{1/p^e}$, thus $r = \beta(r) \in \mathcal{P}(J)$ by Lemma 3.3.8. Therefore, the Cartier core of $J$ is a radical ideal.

Since $J_{s+1}$ is not necessarily contained in $J_s$, we need to show that $\bigcap_{s\geq e} J_s$ is the Cartier core for any $e$.

**Proposition 3.3.13.** *Let $R$ be an $F$-finite $F$-pure ring, and $J$ be an ideal of $R$. Then, $\mathcal{P}(J) = \bigcap_{s\geq e} J_s$ for every nonnegative integer $e$.*

*Proof.* We must show that $\bigcap_{s\geq e} J_s \subseteq \mathcal{P}(J)$. Let $x \in \bigcap_{s\geq e} J_s$. Thus $x \in J$ by Proposition 3.3.3. Hence, $x^{p^s} \in J^{[p^s]}$ for every $s \leq e$. As a consequence, $x^{p^e} \in J^{[p^s]}$. As $x^{p^e} \in \bigcap_{s\geq e} J_s$, we have that $x^{p^e} \in \mathcal{P}(J)$. Thus, $x \in \sqrt{\mathcal{P}(J)}$. Therefore, $x \in \mathcal{P}(J)$ by Remark 3.3.12. $\square$

### 3.3.2 The Ideal $\mathfrak{q}_e$ in Stanley-Reisner Rings

Throughout this subsection, we denote $S = K[x_1, \ldots, x_n]$ with $K$ an $F$-finite field of prime characteristic $p$. Let $I$ be a squarefree monomial ideal of $S$, $R = S/I$, and $\mathfrak{p}_1, \ldots, \mathfrak{p}_l$ are the minimal prime ideals of $R$. We want to compute the ideal $\mathfrak{q}_e$, when $\mathfrak{q}$ is a monomial prime ideal of $R$.

**Lemma 3.3.14.** *Let $J$ be a monomial ideal in $R$, and $e$ be a nonnegative integer. Then, $J_e$ and $\mathcal{P}(J)$ are monomial ideals.*

*Proof.* We note that $R^{1/p^e}$ and $R$ are $\mathbb{N}^n$-graded. To show that $J_e$ is a monomial ideal, it suffices to prove that $J_e$ is a homogeneous ideal with the $\mathbb{N}^n$ grading. Let $r = r_{\alpha_1} + \cdots + r_{\alpha_t} \in J_e$, with $r_{\alpha_i}$ of degree $\alpha_i$. Let $\varphi \in \operatorname{Hom}_R(R^{1/p^e}, R)$. Since $R^{1/p^e}$ is a finitely generated $R$-module, every homomorphism $R^{1/p^e} \longrightarrow R$ is a sum of graded homomorphisms. Thus, we can take $\varphi$ homogeneous of degree $\beta$. Then,

$$\varphi(r) = \varphi(r_{\alpha_1}^{1/p^e}) + \cdots + \varphi(r_{\alpha_t}^{1/p^e}) \in J,$$

and each $\varphi(r_{\alpha_i}^{1/p^e})$ has degree $\alpha_i + \beta$. As $J$ is homogeneous, we get $\varphi(r_{\alpha_i}^{1/p^e}) \in J$ for all $i \in \{1, \ldots, t\}$, showing that $r_{\alpha_i} \in J_e$. Then, $J_e$ is a homogeneous ideal.

Since $J_e$ is a monomial ideal, $\mathcal{P}(J)$ is a monomial ideal by its definition. $\square$

**Proposition 3.3.15.** *Given $\mathfrak{q}$ a monomial prime ideal of $R$, then for $e \in \mathbb{N}$, and $q = p^e$, $\mathfrak{q}_e = \mathfrak{q}^{[q]} + \mathcal{P}(\mathfrak{q})$.*

*Proof.* We must show $\mathfrak{q}_e \subseteq \mathfrak{q}^{[q]} + \mathcal{P}(\mathfrak{q})$. We proceed by contradiction. Let $r$ be an element in $\mathfrak{q}_e$. We suppose that $r \notin \mathfrak{q}^{[q]} + \mathcal{P}(\mathfrak{q})$. From Lemma 3.3.14, $\mathfrak{q}_e$ is a monomial ideal of $R$. Then, we can take $r = x^\beta$, with $\beta \in \mathbb{N}^n$.

Thus, $x^\beta \notin \mathfrak{q}^{[q]}$, and $x^\beta \notin \mathcal{P}(\mathfrak{q})$. By Proposition 3.3.13, $x^\beta \notin \bigcap_{s\geq e} \mathfrak{q}_s$, and so, there exists $e' \geq e$ such that $x^\beta \notin \mathfrak{q}_{e'}$.

Let $\mathcal{A} = \{\alpha \in \mathbb{N}^n \mid 0 \leq \alpha_i \leq q-1 \text{ for } i = 1, \ldots, n\}$, $\mathcal{A}' = \{\alpha' \in \mathbb{N}^n \mid 0 \leq \alpha'_i \leq p^{e'} - 1 \text{ for } i = 1, \ldots, n\}$, $\mathcal{B} = \{{a_i}^{1/q} \mid i = 1, \ldots, s\}$ be a base of $K^{1/q}$ as $K$-vector space, and $\mathcal{B}' = \{(a'_i)^{1/p^{e'}} \mid i = 1, \ldots, s'\}$ be a base of $K^{1/p^{e'}}$ as $K$-vector space. We may suppose that $a_1 = a'_1 = 1$.

Moreover, $x^{\beta/p^e} = x^{\theta}x^{\alpha/p^e}$ and $x^{\beta/p^{e'}} = x^{\theta'}x^{\alpha'/p^{e'}}$, with $\theta,\ \theta' \in \mathbb{N}^n$, $\alpha \in \mathcal{A}$, and $\alpha' \in \mathcal{A}'$. As $p^{e'} \geq p^e$, then $\alpha_i \leq \alpha'_i$ and $\theta_i \geq \theta'_i$ for every $i$. Thus, there exists $\tau_i \in \mathbb{N}$ such that $\theta_i = \theta'_i + \tau_i$.

Furthermore, $J_{1,\alpha} = (I : x^{\alpha}) \subseteq (I : x^{\alpha'}) = J_{1,\alpha'}$. Hence, we take a morphism

$$\phi \in \operatorname{Hom}_R((S/J_{1,\alpha})x^{\alpha/p^e}, (S/J_{1,\alpha'})x^{\alpha'/p^{e'}})$$

such that $\phi(x^{\alpha/p^e}) = x^{\alpha'/p^{e'}}$.

Since $x^{\beta} \notin \mathfrak{q}_{e'}$, there exists $\psi \in \operatorname{Hom}_R((S/J_{1,\alpha'})x^{\alpha'/p^{e'}}, R)$ such that $\psi(x^{\theta'}x^{\alpha'/p^{e'}}) \notin \mathfrak{q}$ by Proposition 3.1.2.

We have an $R$-linear map

$$\varphi : R^{1/q} \longrightarrow \bigoplus_{\substack{1\leq i\leq s\\ \alpha\in\mathcal{A}}} S/J_{i,\alpha}(a_i x^{\alpha})^{1/q}$$

such that

$$\varphi(r^{1/q}) = \bigoplus_{\substack{1\leq i\leq s\\ \alpha\in\mathcal{A}}} (r_{i,\alpha} + J_{i,\alpha})(a_i x^{\alpha})^{1/q},$$

where

$$r^{1/q} = \bigoplus_{\substack{1\leq i\leq s\\ \alpha\in\mathcal{A}}} r_{i,\alpha}(a_i x^{\alpha})^{1/q}.$$

Taking $\gamma = \psi \circ \phi \circ \pi_{1,\alpha} \circ \varphi$, we have $\gamma \in \operatorname{Hom}_R(R^{1/q}, R)$, and $\gamma(x^{\beta/p^e}) = \psi(x^{\theta}x^{\alpha'/p^{e'}}) = \psi(x^{\tau}x^{\theta'}x^{\alpha'/p^{e'}}) = x^{\tau}\psi(x^{\theta'}x^{\alpha'/p^{e'}})$.

In addition, $x^{\beta} = x^{q\theta}x^{\alpha} = x^{q\tau}x^{q\theta'}x^{\alpha}$. As $x^{\beta} \notin \mathfrak{q}^{[q]}$, we get that $x^{\tau} \notin \mathfrak{q}$. Since $x^{\beta} \in \mathfrak{q}_e$, it follows that $x^{\tau}\psi(x^{\theta'}x^{\alpha'/p^{e'}}) = \gamma(x^{\beta/p^e}) \in \mathfrak{q}$. We get a contradiction, because $\mathfrak{q}$ is a prime ideal in $R$, and $x^{\tau},\ \psi(x^{\theta'}x^{\alpha'/p^{e'}}) \notin \mathfrak{q}$. □

**Proposition 3.3.16.** *Let $e$ be a nonnegative integer, $q = p^e$, $\overline{R} = R/\mathcal{P}(\mathfrak{q})$ with $\mathfrak{q}$ a monomial prime ideal in $R$, and $f \in R$. Then, the following hold.*

*(1) If $f \in \mathfrak{q}_e$, then $\overline{f} \in (\overline{\mathfrak{q}})_e$;*

*(2) $\overline{f} \in \overline{\mathfrak{q}}^{[q]}$ if and only if $f \in \mathfrak{q}_e$.*

*Proof.* We show Part (1). We can assume that $f$ a monomial, because $\mathfrak{q}_e$ and $(\overline{\mathfrak{q}})_e$ are monomial ideals by Lemma 3.3.14.

We have that $f \in \mathfrak{q}_e = \mathfrak{q}^{[q]} + \mathcal{P}(\mathfrak{q})$ by Proposition 3.3.15. Since $f$ is a monomial, it follows that $f \in \mathfrak{q}^{[q]}$ or $f \in \mathcal{P}(\mathfrak{q})$. If $f \in \mathcal{P}(\mathfrak{q})$, then $\overline{f} = 0 \in (\overline{\mathfrak{q}})_e$. Moreover, if $f \in \mathfrak{q}^{[q]}$, then $\overline{f} \in \overline{\mathfrak{q}}^{[q]} \subseteq (\overline{\mathfrak{q}})_e$.

Now, we show Part (2). From Proposition 3.3.15, we see that

$$\begin{aligned}\overline{f} \in \overline{\mathfrak{q}}^{[q]} = \overline{\mathfrak{q}^{[q]}} &\Leftrightarrow f - g \in \mathcal{P}(\mathfrak{q}) \text{ for some } g \in \mathfrak{q}^{[q]}\\ &\Leftrightarrow f \in \mathfrak{q}^{[q]} + \mathcal{P}(\mathfrak{q}) = \mathfrak{q}_e.\end{aligned}$$

□

**Proposition 3.3.17.** *Suppose $A$ as in Remark 3.1.3 and $B = A/IA$. Given $\mathfrak{q}$ a monomial prime ideal of $B$, then for $e \in \mathbb{N}$, and $q = p^e$, $\mathfrak{q}_e = \mathfrak{q}^{[q]} + \mathcal{P}(\mathfrak{q})$.*

*Proof.* The proof is analogous to Proposition 3.3.15. □

**Proposition 3.3.18.** *Suppose $A$ as in Remark 3.1.3 and $B = A/IA$. Let $e$ be a nonnegative integer, $q = p^e$, $\overline{B} = B/\mathcal{P}(\mathfrak{q})$ with $\mathfrak{q}$ a monomial prime ideal in $B$, and $f \in B$. Then, the following hold.*

*(1) If $f \in \mathfrak{q}_e$, then $\overline{f} \in (\overline{\mathfrak{q}})_e$;*

*(2) $\overline{f} \in \overline{\mathfrak{q}}^{[q]}$ if and only if $f \in \mathfrak{q}_e$.*

*Proof.* The proof is analogous to Proposition 3.3.16. □

## 3.4 Cartier Threshold of $\mathfrak{a}$ with Respect to $J$

In this section we prove other of our main results, Theorem A. In order to obtain this, we define the Cartier threshold of $\mathfrak{a}$ with respect to $J$. We give some properties of this and show that it is preserved under localization and completion. We study its relation with the $F$-thresholds. We also compare this number with its corresponding in $\overline{R} = R/\mathcal{P}(J)$.

**Definition-Theorem 3.4.1** ([DSHNnBW])**.** Let $R$ be an $F$-finite $F$-pure ring. Given $\mathfrak{a}$, $J$ two ideals in $R$ such that $\mathfrak{a} \subseteq \sqrt{J}$, we define

$$b_{\mathfrak{a}}^{J}(p^e) = \max\{t \in \mathbb{N} \mid \mathfrak{a}^t \not\subseteq J_e\}.$$

We define the Cartier threshold of $\mathfrak{a}$ in $R$ with respect to $J$ by

$$\mathrm{ct}_J(\mathfrak{a}) = \lim_{e \to \infty} \frac{b_{\mathfrak{a}}^{J}(p^e)}{p^e}.$$

If $(R, \mathfrak{m}, K)$ is a local ring or a standard graded $K$-algebra and $\mathfrak{m} = J$, the Cartier threshold $\mathrm{ct}_J(\mathfrak{a})$ coincides with the $F$-pure threshold $\mathrm{fpt}(\mathfrak{a})$. When $\mathfrak{a} = \mathfrak{m}$, $\mathrm{fpt}(\mathfrak{m})$ is denoted by $\mathrm{fpt}(R)$.

Using the Proposition 3.3.4, it follows that $\mathrm{ct}_J(\mathfrak{a})$ also commutes with arbitrary intersections.

**Proposition 3.4.2.** *Let $R$ be an $F$-finite $F$-pure ring. Let $\{\mathfrak{q}_i\}_i$ be a family of ideals in $R$, and $\mathfrak{a}$, $J$ be ideals inside $R$ such that $\mathfrak{a} \subseteq \sqrt{J}$, and $J = \bigcap_i \mathfrak{q}_i$. Then, $\mathrm{ct}_J(\mathfrak{a}) = \sup\{\mathrm{ct}_{\mathfrak{q}_i}(\mathfrak{a})\}$.*

*Proof.* By Proposition 3.3.4, we have that $J_e = \bigcap_i (\mathfrak{q}_i)_e$ for every nonnegative integer $e$. Then,

$$\begin{aligned} t \geq b_{\mathfrak{a}}^J(p^e) &\Leftrightarrow \mathfrak{a}^{t+1} \subseteq J_e \\ &\Leftrightarrow \mathfrak{a}^{t+1} \subseteq (\mathfrak{q}_i)_e \text{ for every } i \\ &\Leftrightarrow t \geq b_{\mathfrak{a}}^{\mathfrak{q}_i}(p^e) \text{ for every } i \\ &\Leftrightarrow t \geq \sup\{b_{\mathfrak{a}}^{\mathfrak{q}_i}(p^e)\}. \end{aligned}$$

Hence, $\frac{b_{\mathfrak{a}}^J(p^e)}{p^e} = \sup\left\{\frac{b_{\mathfrak{a}}^{\mathfrak{q}_i}(p^e)}{p^e}\right\}$. Therefore, $\mathrm{ct}_J(\mathfrak{a}) = \sup\{\mathrm{ct}_{\mathfrak{q}_i}(\mathfrak{a})\}$. □

Since $\mathfrak{q}_e$ is a $\mathfrak{q}$-primary ideal by Proposition 3.3.5, we have that $\mathrm{ct}_J(\mathfrak{a})$ is preserved under localization. This fact, we prove it in Proposition 3.4.4 below.

**Lemma 3.4.3.** *Let $R$ be an $F$-finite $F$-pure ring, $\mathfrak{q}$ be a prime ideal of $R$, and $f \in R$. Then, $\frac{f}{1} \in I_e(R_{\mathfrak{q}})$ if and only if $f \in \mathfrak{q}_e$.*

*Proof.* We focus on the first direction. Let $\psi \in \mathrm{Hom}_R(R^{1/p^e}, R)$. Since $(R^{1/p^e})_{\mathfrak{q}} \cong {R_{\mathfrak{q}}}^{1/p^e}$ as $R_{\mathfrak{q}}$-module, $\psi_{\mathfrak{q}} \in \mathrm{Hom}_{R_{\mathfrak{q}}}({R_{\mathfrak{q}}}^{1/p^e}, R_{\mathfrak{q}})$, and so, $\frac{\psi(f^{1/p^e})}{1} = \psi_{\mathfrak{q}}(\frac{f^{1/p^e}}{1}) = \psi_{\mathfrak{q}}((\frac{f}{1})^{1/p^e}) \in \mathfrak{q}R_{\mathfrak{q}}$. Hence, as $\mathfrak{q}$ is a prime ideal, $\psi(f^{1/p^e}) \in \mathfrak{q}$. Therefore, $f \in \mathfrak{q}_e$.

We now show the other direction. Let $\psi \in \mathrm{Hom}_{R_{\mathfrak{q}}}({R_{\mathfrak{q}}}^{1/p^e}, R_{\mathfrak{q}})$. Since

$$\mathrm{Hom}_{R_{\mathfrak{q}}}({R_{\mathfrak{q}}}^{1/p^e}, R_{\mathfrak{q}}) \cong \mathrm{Hom}_R(R^{1/p^e}, R)_{\mathfrak{q}},$$

we have that $\psi = \varphi_{\mathfrak{q}}$ for some $\varphi \in \mathrm{Hom}_R(R^{1/p^e}, R)$. As a consequence, $\psi((\frac{f}{1})^{1/p^e}) = \psi(\frac{f^{1/p^e}}{1}) = \frac{\varphi(f^{1/p^e})}{1} \in \mathfrak{q}R_{\mathfrak{q}}$. Therefore, $\frac{f}{1} \in I_e(R_{\mathfrak{q}})$.

□

**Proposition 3.4.4.** *Let $R$ be an $F$-finite $F$-pure ring. Let $\mathfrak{a}$, $\mathfrak{q}$ be two ideals of $R$ with $\mathfrak{q}$ a prime ideal, and $\mathfrak{a} \subseteq \mathfrak{q}$. Then, $\mathrm{ct}_{\mathfrak{q}}(\mathfrak{a}) = \mathrm{fpt}(\mathfrak{a}R_{\mathfrak{q}})$.*

*Proof.* By Lemma 3.4.3, we observe that,

$$\begin{aligned} b_{\mathfrak{a}}^{\mathfrak{q}}(p^e) &= \max\{t \in \mathbb{N} \mid \mathfrak{a}^t \not\subseteq \mathfrak{q}_e\} \\ &= \max\{t \in \mathbb{N} \mid \mathfrak{a}^t R_{\mathfrak{q}} \not\subseteq I_e(R_{\mathfrak{q}})\} \\ &= \max\{t \in \mathbb{N} \mid (\mathfrak{a}R_{\mathfrak{q}})^t \not\subseteq I_e(R_{\mathfrak{q}})\} \\ &= b_{\mathfrak{a}R_{\mathfrak{q}}}^{\mathfrak{q}R_{\mathfrak{q}}}(p^e). \end{aligned}$$

Therefore, $\mathrm{ct}_{\mathfrak{q}}(\mathfrak{a}) = \mathrm{fpt}(\mathfrak{a}R_{\mathfrak{q}})$. □

Consider a local ring $(R, \mathfrak{m}, K)$. Let $\mathfrak{a} \subseteq \sqrt{J}$ be two ideals of $R$. We claim that the Cartier threshold of $\mathfrak{a}$ with respect to $J$ does not vary under completion. To show this, we compare the ideal $J_e$ versus $(J\widehat{R})_e$.

**Lemma 3.4.5.** *Let $(R, \mathfrak{m}, K)$ be an $F$-finite $F$-pure local ring, $f \in R$, and $J$ be an ideal in $R$. Then, $f \in J_e$ if and only if $f \in (J\widehat{R})_e$.*

*Proof.* We suppose that $f \in J_e$. Let $\varphi \in \operatorname{Hom}_{\widehat{R}}(\widehat{R}^{1/p^e}, \widehat{R})$. Since $R$ is an $F$-finite ring and $\widehat{R}^{1/p^e} \cong \widehat{R^{1/p^e}}$ as $\widehat{R}$-module, we have

$$\begin{aligned}\operatorname{Hom}_{\widehat{R}}(\widehat{R}^{1/p^e}, \widehat{R}) &\cong \widehat{\operatorname{Hom}_R(R^{1/p^e}, R)}\\ &\cong \operatorname{Hom}_R(R^{1/p^e}, R) \otimes_R \widehat{R}.\end{aligned}$$

Hence, $\varphi = \sum_{i=1}^{n} \varphi_i \otimes r_i$ with $\varphi_i \in \operatorname{Hom}_R(R^{1/p^e}, R)$ and $r_i \in \widehat{R}$. Then, $\varphi(f^{1/p^e}) = \sum_{i=1}^{n} r_i \varphi_i(f^{1/p^e})$. However, $f \in J_e$, in consequence $\varphi_i(f^{1/p^e}) \in J$, thus $\varphi(f^{1/p^e}) \in J\widehat{R}$. Therefore, $f \in (J\widehat{R})_e$.

We now suppose that $f \in (J\widehat{R})_e$. Let $\varphi \in \operatorname{Hom}_R(R^{1/p^e}, R)$. Since $\widehat{R}^{1/p^e} \cong \widehat{R^{1/p^e}}$ as $\widehat{R}$-module, we have $\widehat{\varphi} \in \operatorname{Hom}_{\widehat{R}}(\widehat{R}^{1/p^e}, \widehat{R})$. Then, $\widehat{\varphi}(f^{1/p^e}) \in J\widehat{R}$, and so, $\varphi(f^{1/p^e}) \in J$. Therefore, $f \in J_e$. $\square$

**Proposition 3.4.6.** *Suppose that $(R, \mathfrak{m}, K)$ is an $F$-finite $F$-pure local ring. Let $\mathfrak{a}$, $J$ be two ideals in $R$ such that $\mathfrak{a} \subseteq \sqrt{J}$. Then, $\operatorname{ct}_J(\mathfrak{a}) = \operatorname{ct}_{J\widehat{R}}(\mathfrak{a}\widehat{R})$.*

*Proof.* By Lemma 3.4.5, we observe that

$$\begin{aligned}b_{\mathfrak{a}}^{J}(p^e) &= \max\{t \in \mathbb{N} \mid \mathfrak{a}^t \not\subseteq J_e\}\\ &= \max\{t \in \mathbb{N} \mid \mathfrak{a}^t\widehat{R} \not\subseteq (J\widehat{R})_e\}\\ &= \max\{t \in \mathbb{N} \mid (\mathfrak{a}\widehat{R})^t \not\subseteq (J\widehat{R})_e\}\\ &= b_{\mathfrak{a}\widehat{R}}^{J\widehat{R}}(p^e).\end{aligned}$$

Therefore, $\operatorname{ct}_J(\mathfrak{a}) = \operatorname{ct}_{J\widehat{R}}(\mathfrak{a}\widehat{R})$. $\square$

Given $J$ an ideal in $R$, we consider the ring $\overline{R} = R/\mathcal{P}(J)$. Let $\mathfrak{a}$ be an ideal in $R$ such that $\mathfrak{a} \subseteq \sqrt{J}$. Our goal is to compare the Cartier threshold of $\mathfrak{a}$ with respect to $J$ versus the Cartier threshold of $\overline{\mathfrak{a}}$ with respect to $\overline{J}$.

**Lemma 3.4.7.** *Let $R$ be an $F$-finite $F$-pure ring, $J$ be an ideal of $R$, $\overline{R} = R/\mathcal{P}(J)$, and $f \in R$. Then, $\overline{f} \in (\overline{J})_e$ implies that $f \in J_e$.*

*Proof.* For every $\varphi \in \operatorname{Hom}_R(R^{1/p^e}, R)$, we take $\overline{\varphi} : \overline{R}^{1/p^e} \longrightarrow \overline{R}$ such that $\overline{\varphi}(\overline{x}^{1/p^e}) = \overline{\varphi(x^{1/p^e})}$. By Lemma 3.3.8, it follows that $\overline{\varphi}$ is well defined.

Since $\varphi \in \operatorname{Hom}_R(R^{1/p^e}, R)$, it follows that $\overline{\varphi} \in \operatorname{Hom}_{\overline{R}}(\overline{R}^{1/p^e}, \overline{R})$. As $\overline{f} \in (\overline{J})_e$, then $\overline{\varphi(f^{1/p^e})} = \overline{\varphi}(\overline{f}^{1/p^e}) \in \overline{J}$. Hence, there exists $y \in J$ such that $\varphi(f^{1/p^e}) - y \in \mathcal{P}(J) \subseteq J$, and so $\varphi(f^{1/p^e}) \in J$. Therefore, $f \in J_e$. $\square$

**Proposition 3.4.8.** *Let $R$ be an $F$-finite $F$-pure ring. Let $\mathfrak{a}$, $J$ be two ideals in $R$ such that $\mathfrak{a} \subseteq \sqrt{J}$, and $\overline{R} = R/\mathcal{P}(J)$. Then, $\operatorname{ct}_J(\mathfrak{a}) \leq \operatorname{ct}_{\overline{J}}(\overline{\mathfrak{a}})$. In particular, if $(R, \mathfrak{m}, K)$ is a local ring or a standard graded $K$-algebra, then* $\operatorname{fpt}(\mathfrak{a}) \leq \operatorname{fpt}(\overline{\mathfrak{a}})$.

*Proof.* From Lemma 3.4.7, we have that

$$\begin{aligned} b_{\mathfrak{a}}^{J}(p^e) &= \max\{t \in \mathbb{N} \mid \mathfrak{a}^t \not\subseteq J_e\} \\ &\leq \max\{t \in \mathbb{N} \mid \overline{\mathfrak{a}}^t \not\subseteq (\overline{J})_e\} \\ &= b_{\overline{\mathfrak{a}}}^{\overline{J}}(p^e). \end{aligned}$$

Therefore, $\operatorname{ct}_J(\mathfrak{a}) = \lim\limits_{e\to\infty} \frac{b_{\mathfrak{a}}^{J}(p^e)}{p^e} \leq \lim\limits_{e\to\infty} \frac{b_{\overline{\mathfrak{a}}}^{\overline{J}}(p^e)}{p^e} = \operatorname{ct}_{\overline{J}}(\overline{\mathfrak{a}})$. ☐

### 3.4.1 Relation Between $c^J(\mathfrak{a})$ and $\operatorname{ct}_J(\mathfrak{a})$

In this subsection we give a characterization of $\operatorname{ct}_J(\mathfrak{a})$ using $F$-thresholds.

**Remark 3.4.9.** Suppose that $R$ is an $F$-finite $F$-pure ring. Let $\mathfrak{a}$, $J$ be two ideals in $R$ such that $\mathfrak{a} \subseteq \sqrt{J}$. Since $J^{[p^e]} \subseteq J_e$, we have that

$$\begin{aligned} b_{\mathfrak{a}}^{J}(p^e) &= \max\{t \in \mathbb{N} \mid \mathfrak{a}^t \not\subseteq J_e\} \\ &\leq \max\{t \in \mathbb{N} \mid \mathfrak{a}^t \not\subseteq J^{[p^e]}\} \\ &= \nu_{\mathfrak{a}}^{J}(p^e) \end{aligned}$$

Therefore, $\operatorname{ct}_J(\mathfrak{a}) \leq c^J(\mathfrak{a})$.

The following propositions are an extension of the work done by De Stefani, Núñez-Betancourt and Pérez [DSNnBP18, Theorem 4.6].

**Proposition 3.4.10.** *Let $R$ be an $F$-finite $F$-pure ring. Let $J$ be an ideal in $R$. Then, $J_e^{[p]} \subseteq J_{e+1}$ for every $e \in \mathbb{N}$.*

*Proof.* Let $f$ be an element in $J_e$. Let $\varphi \in \operatorname{Hom}_R(R^{1/p^{e+1}}, R)$. As $R^{1/p^e} \subseteq R^{1/p^{e+1}}$, we have that $\varphi|_{R^{1/p^e}} \in \operatorname{Hom}_R(R^{1/p^e}, R)$. Thus, $\varphi((f^p)^{1/p^{e+1}}) = \varphi|_{R^{1/p^e}}(f^{1/p^e}) \in J$. Hence, $f^p \in J_{e+1}$, and so, $J_e^{[p]} \subseteq J_{e+1}$. ☐

**Proposition 3.4.11.** *Let $R$ be an $F$-finite $F$-pure ring, and $\mathfrak{a}$, $J$ be two ideals in $R$ such that $\mathfrak{a} \subseteq \sqrt{J}$. The sequence $\left\{\frac{c^{J_e}(\mathfrak{a})}{p^e}\right\}_{e\geq 0}$ is decreasing and bounded by zero. In particular, its limit exists.*

*Proof.* Let $e$ be nonnegative integer, $J_e^{[p]} \subseteq J_{e+1}$. Thus, $c^{J_{e+1}}(\mathfrak{a}) \leq c^{J_e^{[p]}}(\mathfrak{a}) = p \cdot c^{J_e}(\mathfrak{a})$ by Proposition 2.5.6. Therefore, $\frac{c^{J_{e+1}}(\mathfrak{a})}{p^{e+1}} \leq \frac{c^{J_e}(\mathfrak{a})}{p^e}$. ☐

The following proposition gives us a relation between the Cartier thresholds and $F$-thresholds. Specifically, we can obtain the Cartier threshold as a limit $F$-thresholds.

**Proposition 3.4.12.** *Let $R$ be an $F$-finite $F$-pure ring. Let $\mathfrak{a}$, $J$ be two ideals in $R$ such that $\mathfrak{a} \subseteq \sqrt{J}$. Then, $\operatorname{ct}_J(\mathfrak{a}) = \lim\limits_{e\to\infty} \frac{c^{J_e}(\mathfrak{a})}{p^e}$.*

*Proof.* Let $e$ be nonnegative integer. We note that

$$\begin{aligned} b_{\mathfrak{a}}^{J}(p^e) &= \max\{t \in \mathbb{N} \mid \mathfrak{a} \not\subseteq J_e\} \\ &= \max\{t \in \mathbb{N} \mid \mathfrak{a} \not\subseteq J_e^{[p^0]}\} \\ &= \nu_{\mathfrak{a}}^{J_e}(p^0). \end{aligned}$$

For every nonnegative integer $s$, we have

$$\frac{\nu_{\mathfrak{a}}^{J_e}(p^s)}{p^s} - \frac{\nu_{\mathfrak{a}}^{J_e}(p^0)}{p^0} \leq \frac{\mu(\mathfrak{a})}{p^0}$$

by Lemma 2.5.3.

The sequence $\left\{\frac{\nu_{\mathfrak{a}}^{J_e}(p^s)}{p^s}\right\}_{s\geq 0}$ is increasing, because $R$ is a $F$-pure ring. As a consequence,

$$0 \leq \frac{\nu_{\mathfrak{a}}^{J_e}(p^s)}{p^s} - \nu_{\mathfrak{a}}^{J_e}(p^0) \leq \mu(\mathfrak{a}).$$

Thus,

$$0 \leq \frac{\nu_{\mathfrak{a}}^{J_e}(p^s)}{p^s} - b_{\mathfrak{a}}^{J}(p^e) \leq \mu(\mathfrak{a}).$$

We take limit over $s$ to get

$$0 \leq c^{J_e}(\mathfrak{a}) - b_{\mathfrak{a}}^{J}(p^e) \leq \mu(\mathfrak{a}),$$

dividing by $p^e$ gives

$$0 \leq \frac{c^{J_e}(\mathfrak{a})}{p^e} - \frac{b_{\mathfrak{a}}^{J}(p^e)}{p^e} \leq \frac{\mu(\mathfrak{a})}{p^e}.$$

Taking limit over $e$ we conclude that

$$\operatorname{ct}_J(\mathfrak{a}) = \lim_{e\to\infty} \frac{c^{J_e}(\mathfrak{a})}{p^e}.$$

$\square$

**Corollary 3.4.13.** *Let $R$ be an $F$-finite $F$-pure ring. Let $\mathfrak{a}$, $J$ be two ideals in $R$ such that $\mathfrak{a} \subseteq \sqrt{J}$. Then, $\operatorname{ct}_J(\mathfrak{a}) = c^J(\mathfrak{a})$ if and only if $c^{J_e}(\mathfrak{a}) = c^{J^{[p^e]}}(\mathfrak{a})$ for every $e \in \mathbb{N}$.*

*Proof.* We focus on the first direction, it suffices to show $c^{J^{[p^e]}}(\mathfrak{a}) \leq c^{J_e}(\mathfrak{a})$. As the sequence $\left\{\frac{c^{J_e}(\mathfrak{a})}{p^e}\right\}_{e\geq 0}$ is decreasing and bounded below, it converges to its infimum. By Proposition 3.4.12, $c^J(\mathfrak{a}) \leq \frac{c^{J_e}(\mathfrak{a})}{p^e}$. As a consequence, $c^{J^{[p^e]}}(\mathfrak{a}) = p^e \cdot c^J(\mathfrak{a}) \leq c^{J_e}(\mathfrak{a})$.

We now show the other direction, $\operatorname{ct}_J(\mathfrak{a}) = \lim\limits_{e\to\infty} \frac{c^{J_e}(\mathfrak{a})}{p^e} = \lim\limits_{e\to\infty} \frac{c^{J^{[p^e]}}(\mathfrak{a})}{p^e} = \lim\limits_{e\to\infty} \frac{p^e \cdot c^J(\mathfrak{a})}{p^e} = c^J(\mathfrak{a})$. $\square$

### 3.4.2 Cartier Thresholds in Stanley-Reisner Rings

Throughout this subsection, we denote $S = K[x_1, \dots, x_n]$ with $K$ an $F$-finite field of prime characteristic $p$. Let $I$ be a squarefree monomial ideal of $S$, $R = S/I$, and $\mathfrak{p}_1, \dots, \mathfrak{p}_l$ are the minimal prime ideals of $R$.

**Theorem 3.4.14.** *Suppose $A$ as in Remark 3.1.3 and $B = A/IA$. Let $\mathfrak{a}$, $\mathfrak{q}$ be two ideals in $B$ with $\mathfrak{q}$ a prime monomial ideal, such that $\mathfrak{a} \subseteq \mathfrak{q}$, and $\overline{B} = B/\mathcal{P}(\mathfrak{q})$. Then, the following hold:*

*(1)* $\mathrm{ct}_{\mathfrak{q}}(\mathfrak{a}) = \mathrm{ct}_{\overline{\mathfrak{q}}}(\overline{\mathfrak{a}})$*;*

*(2)* $\mathrm{ct}_{\mathfrak{q}}(\mathfrak{a}) = c^{\overline{\mathfrak{q}}}(\overline{\mathfrak{a}})$*;*

*(3)* $\mathrm{ct}_{\mathfrak{q}}(\mathfrak{a})$ *is a rational number.*

*In particular,* $\mathrm{fpt}(\mathfrak{a})$ *is a rational number.*

*Proof.* We show Part (1). From Proposition 3.3.18 and Lemma 3.4.7, we have

$$\begin{aligned} b_{\mathfrak{a}}^{\mathfrak{q}}(p^e) &= \max\{t \in \mathbb{N} \mid \mathfrak{a}^t \not\subseteq \mathfrak{q}_e\} \\ &= \max\{t \in \mathbb{N} \mid \overline{\mathfrak{a}}^t \not\subseteq (\overline{\mathfrak{q}})_e\} \\ &= b_{\overline{\mathfrak{a}}}^{\overline{\mathfrak{q}}}(p^e). \end{aligned}$$

Therefore, $\mathrm{ct}_{\mathfrak{q}}(\mathfrak{a}) = \mathrm{ct}_{\overline{\mathfrak{q}}}(\overline{\mathfrak{a}})$.

Now, we show Part (2). We claim that $c^{\overline{\mathfrak{q}}}(\overline{\mathfrak{a}}) \le \mathrm{ct}_{\mathfrak{q}}(\mathfrak{a})$. From Proposition 3.3.18, it follows that

$$\begin{aligned} \nu_{\overline{\mathfrak{a}}}^{\overline{\mathfrak{q}}}(p^e) &= \max\{t \in \mathbb{N} \mid \overline{\mathfrak{a}}^t \not\subseteq \overline{\mathfrak{q}}^{[q]}\} \\ &\le \max\{t \in \mathbb{N} \mid \mathfrak{a}^t \not\subseteq \mathfrak{q}_e\} \\ &= b_{\mathfrak{a}}^{\mathfrak{q}}(p^e). \end{aligned}$$

Thus, $c^{\overline{\mathfrak{q}}}(\overline{\mathfrak{a}}) = \lim\limits_{e\to\infty} \frac{\nu_{\overline{\mathfrak{a}}}^{\overline{\mathfrak{q}}}(p^e)}{p^e} \le \lim\limits_{e\to\infty} \frac{b_{\mathfrak{a}}^{\mathfrak{q}}(p^e)}{p^e} = \mathrm{ct}_{\mathfrak{q}}(\mathfrak{a})$.

By Part (1) and Remark 3.4.9, we have $c^{\overline{\mathfrak{q}}}(\overline{\mathfrak{a}}) \le \mathrm{ct}_{\mathfrak{q}}(\mathfrak{a}) = \mathrm{ct}_{\overline{\mathfrak{q}}}(\overline{\mathfrak{a}}) \le c^{\overline{\mathfrak{q}}}(\overline{\mathfrak{a}})$. Therefore, $c^{\overline{\mathfrak{q}}}(\overline{\mathfrak{a}}) = \mathrm{ct}_{\mathfrak{q}}(\mathfrak{a})$.

We show Part (3). Since $\mathfrak{q}$ is a monomial ideal, $\mathcal{P}(\mathfrak{q})$ is also a monomial ideal by Lemma 3.3.14. In addition, $\mathcal{P}(\mathfrak{q})$ is a radical ideal by Remark 3.3.12. Thus, $\mathcal{P}(\mathfrak{q})$ is squarefree monomial ideal. Consequently, $\overline{B}$ is a power series ring modulo a squarefree monomial ideal. Since $\overline{\mathfrak{q}}$ is a monomial ideal in $\overline{B}$, $c^{\overline{\mathfrak{q}}}(\overline{\mathfrak{a}})$ is a rational number by Remark 3.2.2. Therefore, $\mathrm{ct}_{\mathfrak{q}}(\mathfrak{a})$ is a rational number by Part (2).

The last statement follows, since $\mathrm{ct}_{\mathfrak{m}}(\mathfrak{a}) = \mathrm{fpt}(\mathfrak{a})$ and $\mathfrak{m}$ is a monomial prime ideal in $B$. □

Since $\mathrm{ct}_J(\mathfrak{a})$ is preserved under localization and completion, Theorem 3.4.14 allows us to obtain one of the main results of this work.

**Corollary 3.4.15.** *Let $\mathfrak{a}$, $\mathfrak{q}$ be two ideals of $R$, where $\mathfrak{q}$ is a prime ideal and $\mathfrak{a} \subseteq \mathfrak{q}$. Then, $\mathrm{ct}_{\mathfrak{q}}(\mathfrak{a})$ is a rational number.*

*Proof.* We have that $\mathrm{ct}_{\mathfrak{q}}(\mathfrak{a}) = \mathrm{fpt}(\mathfrak{a}\widehat{R_{\mathfrak{q}}})$ by Propositions 3.4.4 and 3.4.6. Therefore, $\mathrm{ct}_{\mathfrak{q}}(\mathfrak{a})$ is a rational number by Theorem 3.4.14. □

**Corollary 3.4.16.** *Let $\mathfrak{a}$, $J$ be two ideals in $R$ with $J$ radical ideal, such that $\mathfrak{a} \subseteq J$. Then, $\mathrm{ct}_J(\mathfrak{a})$ is a rational number. In particular, $\mathrm{fpt}(\mathfrak{a})$ is a rational number.*

*Proof.* Since $J$ is a radical ideal, we have that $J = \bigcap_{i=1}^m \mathfrak{q}_i$ where $\mathfrak{q}_1, \ldots, \mathfrak{q}_m$ are the minimal prime ideals of $J$. From Proposition 3.4.2, $\mathrm{ct}_J(\mathfrak{a}) = \max\{\mathrm{ct}_{\mathfrak{q}_i}(\mathfrak{a})\}$. By Corollary 3.4.15, $\mathrm{ct}_J(\mathfrak{a})$ is a rational number. □

## 3.5 Regularity in Stanley-Reisner Rings

Throughout this section, we denote $S = K[x_1, \ldots, x_n]$ with $K$ an $F$-finite field of prime characteristic $p$. Let $I$ be a squarefree monomial ideal of $S$, $R = S/I$.

**Definition 3.5.1.** Let $\alpha \in \mathbb{N}^n$. The support of $\alpha$ is defined by

$$\mathrm{Supp}(\alpha) = \{i \in \{1, \ldots, n\} \mid \alpha_i \neq 0\}.$$

We also take

$$x^{\mathrm{Supp}(\alpha)} = \prod_{i \in \mathrm{Supp}(\alpha)} x_i$$

**Lemma 3.5.2.** *Given $\alpha \in \mathbb{N}^n$, then*

$$(I : x^\alpha) = (x^{\mathrm{Supp}(\lambda) \setminus \mathrm{Supp}(\alpha)} \mid x^\lambda \text{ minimal generator of } I).$$

*In particular, if $\alpha, \beta \in \mathbb{N}^n$ are such that $\mathrm{Supp}(\alpha) = \mathrm{Supp}(\beta)$, then $(I : x^\alpha) = (I : x^\beta)$.*

*Proof.* Since $I$ is a monomial ideal, it follows that $(I : x^\alpha)$ is a monomial ideal as well. We have $(x^{\mathrm{Supp}(\lambda) \setminus \mathrm{Supp}(\alpha)} \mid x^\lambda \text{ minimal generator of } I) \subseteq (I : x^\alpha)$. Indeed, for every $x^\lambda$ minimal generator of $I$, $x^{\mathrm{Supp}(\lambda) \setminus \mathrm{Supp}(\alpha)} x^\alpha \in I$.

We show that $(I : x^\alpha) \subseteq (x^{\mathrm{Supp}(\lambda) \setminus \mathrm{Supp}(\alpha)} \mid x^\lambda \text{ minimal generator of } I)$. Let $x^\theta$ be a generator of $(I : x^\alpha)$. Thus $x^\theta x^\alpha \in I$. Hence, $x^\lambda | x^\theta x^\alpha$ for some $x^\lambda$ minimal generator of $I$. Then, $\mathrm{Supp}(\lambda) \setminus \mathrm{Supp}(\alpha) \subseteq \mathrm{Supp}(\theta)$, and so, $x^{\mathrm{Supp}(\lambda) \setminus \mathrm{Supp}(\alpha)} | x^\theta$. Therefore, $x^\theta \in (x^{\mathrm{Supp}(\lambda) \setminus \mathrm{Supp}(\alpha)} \mid x^\lambda \text{ minimal generator of } I)$. □

Now, we prove Theorem C.

**Theorem 3.5.3.** *Let $J$ be a homogeneous ideal of $R$. Then,*

$$\lim_{e \to \infty} \frac{\mathrm{reg}(R/J^{[p^e]})}{p^e} = \max_{\substack{1 \le i \le d \\ \alpha \in \mathcal{A}'}} \{a_i(S/(J_\alpha + J)) + |\alpha|\},$$

*where $\mathcal{A}' = \{\alpha \in \mathbb{N}^n \mid 0 \le \alpha_i \le 1 \text{ for } i = 1, \ldots, n\}$, $J_\alpha = (I : x^\alpha)$, and $d = \max\{\dim(S/(J_\alpha + J)) \mid \alpha \in \mathcal{A}'\}$. In particular, this limit is an integer number.*

*Proof.* Without loss of generality, we can take $K$ a perfect field. Let $e$ be a nonnegative integer and $\mathcal{A} = \{\alpha \in \mathbb{N}^n \mid 0 \le \alpha_i \le p^e - 1 \text{ for } i = 1, \dots, n\}$. Then,

$$R^{1/p^e} \cong \bigoplus_{\alpha \in \mathcal{A}} (S/J_\alpha) x^{\alpha/p^e},$$

where $J_\alpha = (I : x^\alpha)$ by Proposition 3.1.2. Applying $- \otimes_R R/J$, we obtain that

$$(R/J^{[p^e]})^{1/p^e} \cong R^{1/p^e}/JR^{1/p^e} \cong \bigoplus_{\alpha \in \mathcal{A}} (S/(J_\alpha + J)) x^{\alpha/p^e},$$

and so

$$H^i_{\mathfrak{m}}((R/J^{[p^e]})^{1/p^e}) \cong \bigoplus_{\alpha \in \mathcal{A}} H^i_{\mathfrak{m}}((S/(J_\alpha + J)) x^{\alpha/p^e}).$$

Hence, we have

$$\begin{aligned}
\frac{a_i(R/J^{[p^e]})}{p^e} &= a_i((R/J^{[p^e]})^{1/p^e}) \\
&= \max_{\alpha \in \mathcal{A}} \{a_i((S/(J_\alpha + J)) x^{\alpha/p^e})\} \\
&= \max_{\alpha \in \mathcal{A}} \left\{ a_i(S/(J_\alpha + J)) + \frac{|\alpha|}{p^e} \right\}.
\end{aligned}$$

For every $\alpha \in \mathcal{A}$ and $\beta \in \mathcal{A}'$, there exist $\gamma \in \mathcal{A}'$ and $\omega \in \mathcal{A}$ such that $\operatorname{Supp}(\alpha) = \operatorname{Supp}(\gamma)$ and $\operatorname{Supp}(\omega) = \operatorname{Supp}(\beta)$. Then, $J_\alpha = J_\gamma$ and $J_\beta = J_\omega$ by Lemma 3.5.2. Hence, we have

$$\frac{a_i(R/J^{[p^e]})}{p^e} = \max_{\alpha \in \mathcal{A}'} \left\{ a_i(S/(J_\alpha + J)) + \frac{|\alpha|(p^e - 1)}{p^e} \right\}.$$

Thus,

$$\begin{aligned}
\lim_{e \to \infty} \frac{\operatorname{reg}(R/J^{[p^e]})}{p^e} &= \lim_{e \to \infty} \max_{i \in \mathbb{Z}} \left\{ \frac{a_i(R/J^{[p^e]})}{p^e} + \frac{i}{p^e} \right\} \\
&= \lim_{e \to \infty} \max_{i \in \mathbb{Z}} \left\{ \max_{\alpha \in \mathcal{A}'} \{a_i(S/(J_\alpha + J)) + \frac{|\alpha|(p^e - 1)}{p^e}\} + \frac{i}{p^e} \right\} \\
&= \lim_{e \to \infty} \max_{\substack{1 \le i \le d \\ \alpha \in \mathcal{A}'}} \left\{ a_i(S/(J_\alpha + J)) + \frac{|\alpha|(p^e - 1)}{p^e} + \frac{i}{p^e} \right\} \\
&= \max_{\substack{1 \le i \le d \\ \alpha \in \mathcal{A}'}} \left\{ \lim_{e \to \infty} a_i(S/(J_\alpha + J)) + \frac{|\alpha|(p^e - 1)}{p^e} + \frac{i}{p^e} \right\} \\
&= \max_{\substack{1 \le i \le d \\ \alpha \in \mathcal{A}'}} \{a_i(S/(J_\alpha + J)) + |\alpha|\}.
\end{aligned}$$

$\square$

# CHAPTER 4

# $F$-Volumes

Motivated by the mixed test ideals associated to a sequence of ideals $I_1, \ldots, I_t$ and their constancy regions, in this chapter we define a numerical invariant called $F$-volume (see Theorem D). This number extends the definition of $F$-threshold of a pair of ideals $I$ and $J$, $c^J(I)$ to a sequence of ideals $J$, $I_1, \ldots, I_t$. We obtain several properties that emulate those of the $F$-threshold. In particular, the $F$-volume detects $F$-pure complete intersections (see Theorem E). In addition, we relate this invariant to the Hilbert-Kunz multiplicity (see Theorem F).

The results presented in this chapter are in joint work with Núñez-Betancourt and Rodríguez-Villalobos [BCNnBRV19].

## 4.1 Existence and Definition

In this section we prove a more general version of Theorem D. In order to do this, we start by introducing a couple of definitions.

**Definition 4.1.1.** A sequence $J_\bullet = \{J_{p^e}\}_{e\in\mathbb{N}}$ of ideals in $R$ whose terms are indexed by the powers of the characteristic is called a $p$-family if $J_{p^e}^{[p]} \subseteq J_{p^{e+1}}$ for all $e \in \mathbb{N}$.

An example of a $p$-family of ideals is the sequence $J_\bullet = \{J^{[p^e]}\}_{e\in\mathbb{N}}$ of Frobenius powers of an ideal $J$.

There are important $p$-families that relate to several limits in prime characteristic that measure singularities [SVdB97, HL02, Yao06, Tuc12, HJ18].

**Definition 4.1.2.** Let $\underline{I} = I_1, \ldots, I_t \subseteq R$ be a sequence of ideals, and $J_\bullet = \{J_{p^e}\}_{e\in\mathbb{N}}$ be a $p$-family of ideals in $R$ such that $I_1, \ldots, I_t \subseteq \sqrt{J_1}$. For each $e \in \mathbb{N}$, we define

$$\mathrm{V}_{\underline{I}}^{J_\bullet}(p^e) = \{(a_1, \ldots, a_t) \in \mathbb{N}^t \mid I_1^{a_1} \cdots I_t^{a_t} \not\subseteq J_{p^e}\}.$$

If $\underline{f} = f_1, \ldots, f_t$ is a sequence of elements of $R$ such that $f_1, \ldots, f_t \in \sqrt{J_1}$, we use $\mathrm{V}_{\underline{f}}^{J_\bullet}(p^e)$ to denote $\mathrm{V}_{\underline{I}}^{J_\bullet}(p^e)$ where $\underline{I} = f_1R, \ldots, f_tR$. In case that the $p$-family is $J_\bullet = \{J^{[p^e]}\}_{e\in\mathbb{N}}$ with $J$ an ideal in $R$, $\mathrm{V}_{\underline{I}}^{J_\bullet}(p^e)$ is denoted by $\mathrm{V}_{\underline{I}}^{J}(p^e)$.

**Remark 4.1.3.** Since $I_1, \ldots, I_t \subseteq \sqrt{J_1}$, for each $i \in \{1, \ldots, t\}$, there exists $\ell_i \in \mathbb{N}$ such that $I_i^{\ell_i} \subseteq J_1$. Additionally, we have that $I_i^{\mu(I_i)p^e} \subseteq I_i^{[p^e]}$ and, as a consequence, $I_i^{\mu(I_i)\ell_i p^e} \subseteq J_1^{[p^e]} \subseteq J_{p^e}$. Hence, if $I_1^{a_1} \cdots I_t^{a_t} \not\subseteq J_{p^e}$, then $a_i < \mu(I_i)\ell_i p^e$ for all $i \in \{1, \ldots, t\}$. Thus, $|\mathrm{V}_{\underline{I}}^{J_\bullet}(p^e)| \leq p^{et} \prod_{i=1}^t \mu(I_i)\ell_i$ for every $e \in \mathbb{N}$. Therefore, the sequence $\left\{\frac{|\mathrm{V}_{\underline{I}}^{J_\bullet}(p^e)|}{p^{et}}\right\}_{e\in\mathbb{N}}$ is bounded.

We now recall a well-known lemma to the experts (see for instance [DSNnBP18, Lemma 3.2]).

**Lemma 4.1.4.** *Let $\mathfrak{a} \subseteq R$ be an ideal. Then, for every $r \geq (\mu(\mathfrak{a}) + s - 1)p^e$, we have that $\mathfrak{a}^r = \mathfrak{a}^{r-sp^e}(\mathfrak{a}^{[p^e]})^s$.*

Towards proving Theorem D, we need to introduce notation to describe different objects.

**Notation 4.1.5.** Let $\underline{I} = I_1, \ldots, I_t \subseteq R$ be a sequence of ideals, and $a = (a_1, \ldots, a_t) \in \mathbb{N}^t$. We denote $(\lceil a_1 \rceil, \ldots, \lceil a_t \rceil)$ by $\lceil a \rceil$. We write $\underline{I}^a$ to denote $I_1^{a_1} \cdots I_t^{a_t}$. Additionally, for each $x = (x_1, \ldots, x_t) \in \mathbb{R}^t$, we write $\widehat{x}^i$ to denote $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_t)$. Let $e_1, e_2 \in \mathbb{N}$. Let $C$ be a subset of $\frac{1}{p^{e_1}}\mathbb{N}^t$. We denote the set

$$\bigcup_{x=(x_1,\ldots,x_t)\in C} \left\{ y = (y_1, \ldots, y_t) \in \frac{1}{p^{e_1+e_2}}\mathbb{N}^t : x_i - \frac{1}{p^{e_1}} < y_i \leq x_i \right\}$$

by $H_{e_1,e_2}(C)$. Finally, we use $\mathbf{1}$ to denote the element of $\mathbb{N}^t$ whose coordinates are all 1.

**Definition 4.1.6.** Let $e_1 \in \mathbb{N}$. Let $C$ be a subset of $\frac{1}{p^{e_1}}\mathbb{N}^t$. We say that $x \in C$ is a border point of $C$ if $x + \frac{1}{p^{e_1}}\mathbf{1} \notin C$. We denote by $\partial C$ the set of all border points in $C$.

**Notation 4.1.7.** Let $\underline{I} = I_1, \ldots, I_t \subseteq R$ be a sequence of ideals, $J_\bullet = \{J_{p^e}\}_{e\in\mathbb{N}}$ be a $p$-family of ideals in $R$ such that $I_1, \ldots, I_t \subseteq \sqrt{J_1}$, and $\mu = \max\{\mu(I_1), \ldots, \mu(I_t)\}$. Consider $e_1, e_2 \in \mathbb{N}$. For each $i \in \{1, ..., t\}$, let $\ell_i = \min\{\ell \mid I_i^\ell \subseteq J_1\}$. Then,

- $\mathcal{B}_n(\underline{I})_{e_1} = \frac{1}{p^{e_1}}\mathbb{N}^{t-1} \cap \left(\prod_{i=1}^{n-1}[0, \mu(I_i)\ell_i] \times \prod_{i=n+1}^{t}[0, \mu(I_i)\ell_i]\right)$ for every $n \in \{1, \ldots, t\}$.
- $\mathcal{B}(\underline{I})_{e_1} = \frac{1}{p^{e_1}}\mathbb{N}^t \cap \left(\bigcup_{j=1}^t \left(\prod_{i=1}^{j-1}[0, \mu(I_i)\ell_i] \times \{0\} \times \prod_{i=j+1}^{t}[0, \mu(I_i)\ell_i]\right)\right)$

- $\mathcal{R}_{e_1,e_2} = H_{e_1,e_2}(\frac{1}{p^{e_1}} \operatorname{V}_{\underline{I}}^{J_\bullet}(p^{e_1}))$, and
- $\mathcal{L}_{e_1,e_2} = H_{e_1,e_2}\left(\partial\left(\frac{1}{p^{e_1}} \operatorname{V}_{\underline{I}}^{J_\bullet}(p^{e_1}) \cup \mathcal{B}(\underline{I})_{e_1}\right) + \frac{1}{p^{e_1}}\{0,\ldots,\mu\}\mathbf{1}\right)$.

Roughly speaking, $\mathcal{R}_{e_1,e_2}$ is the result of filling the set $\frac{1}{p^{e_1}} \operatorname{V}_{\underline{I}}^{J_\bullet}(p^{e_1})$ when considered as a subset of $\frac{1}{p^{e_1+e_2}}\mathbb{N}^t$. Similarly we can think of $\mathcal{L}_{e_1,e_2}$ as the result of filling the subset of $\frac{1}{p^{e_1+e_2}}\mathbb{N}^t$ consisting of points in $\frac{1}{p^{e_1}}\mathbb{N}^t$ that are in the line segments joining $x \in \partial\left(\frac{1}{p^{e_1}} \operatorname{V}_{\underline{I}}^{J_\bullet}(p^{e_1}) \cup \mathcal{B}(\underline{I})_{e_1}\right)$ with $x + \frac{1}{p^{e_1}}\mu\mathbf{1}$.

We now show an example that illustrates the regions previously described.

**Example 4.1.8.** Suppose that $R = K[[x,y]]$ where $K$ is a field of characteristic $p = 2$. Consider $\mathfrak{m} = (x,y)$, the maximal ideal of $R$. Let $\underline{I} = xR, (y^2+x)R$. Then we have that

$$\operatorname{V}_{\underline{I}}^{\mathfrak{m}}(p^{e_1}) = \left(\left([0, 2^{e_1}-1] \times \left[0, \frac{2^{e_1}-2}{2}\right]\right) \cup \left(\left[0, \frac{2^{e_1}-2}{2}\right] \times \left(\frac{2^{e_1}-2}{2}, 2^{e_1}-1\right]\right)\right) \cap \mathbb{N}^2.$$

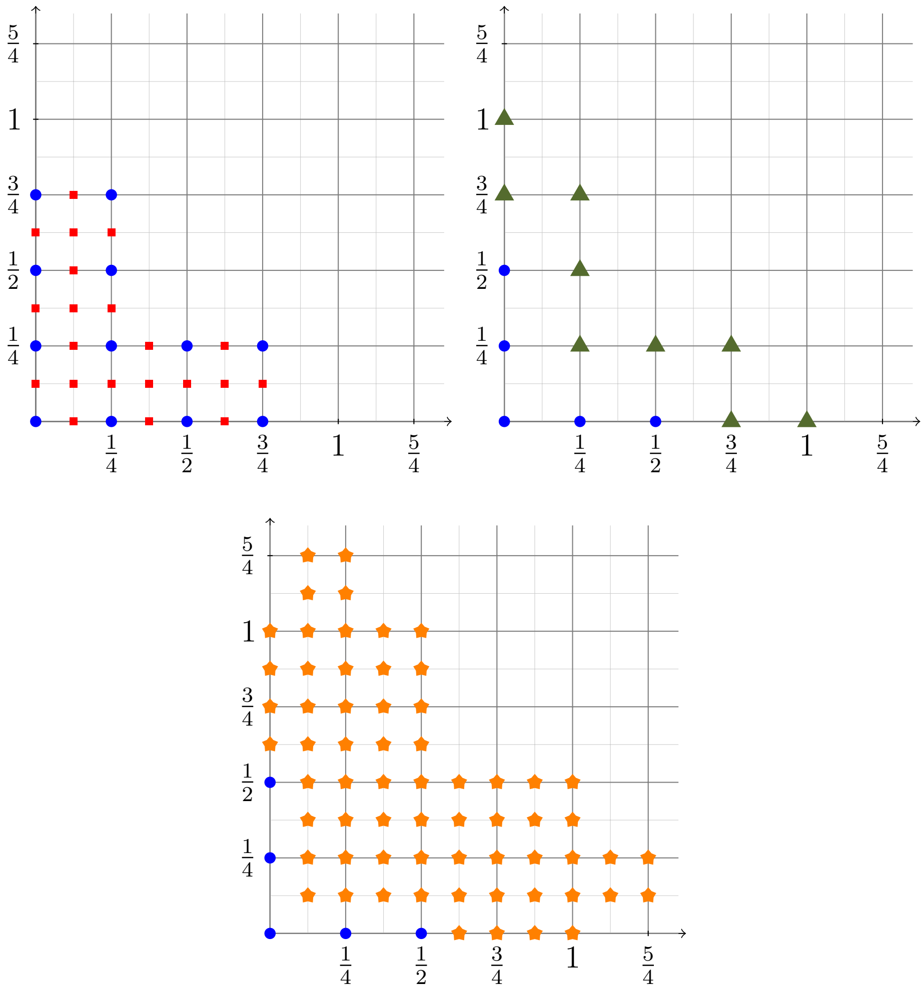
Note that $\mu = 1$ and $\ell_1 = \ell_2 = 1$. It follows that

$$\begin{aligned}
\partial\left(\frac{1}{p^{e_1}} \operatorname{V}_{\underline{I}}^{\mathfrak{m}}(p^{e_1}) \cup \mathcal{B}(\underline{I})_{e_1}\right) = &\frac{1}{2^{e_1}}\Bigg(\Bigg(\left(\{2^{e_1}-1\} \times \left[0, \frac{2^{e_1}-2}{2}\right]\right) \\
&\cup \left(\left[\frac{2^{e_1}-2}{2}, 2^{e_1}-1\right] \times \left\{\frac{2^{e_1}-2}{2}\right\}\right) \\
&\cup \left(\left\{\frac{2^{e_1}-2}{2}\right\} \times \left(\frac{2^{e_1}-2}{2}, 2^{e_1}-1\right]\right) \\
&\cup \left(\left[0, \frac{2^{e_1}-2}{2}\right] \times \{2^{e_1}-1\}\right) \\
&\cup \{(2^{e_1},0),(0,2^{e_1})\}\Bigg) \cap \mathbb{N}^2\Bigg).
\end{aligned}$$

Additionally, we have the following equalities

- $\mathcal{R}_{e_1,e_2} = \left(\bigcup_{x \in \frac{1}{p^{e_1}} \operatorname{V}_{\underline{I}}^{\mathfrak{m}}(p^{e_1})} [0,x_1] \times \cdots \times [0,x_t]\right) \cap \frac{1}{p^{e_1+e_2}}\mathbb{N}^2$,
- $\mathcal{L}_{e_1,e_2} = H_{e_1,e_2}\left(\partial\left(\frac{1}{p^{e_1}} \operatorname{V}_{\underline{I}}^{\mathfrak{m}}(p^{e_1}) \cup \mathcal{B}(\underline{I})_{e_1}\right) + \frac{1}{p^{e_1}}\{0,1\}\mathbf{1}\right)$.

The following figures show the regions of interest in the case $e_1 = 2, e_2 = 1, \mu = 1$. The blue circles represent $\frac{1}{p^{e_1}} \operatorname{V}_{\underline{I}}^{\mathfrak{m}}(p^{e_1})$. The blue circles together with the red squares represent $\mathcal{R}_{e_1,e_2}$. The border points of $\frac{1}{p^{e_1}} \operatorname{V}_{\underline{I}}^{\mathfrak{m}}(p^{e_1}) \cup \mathcal{B}(\underline{I})_{e_1}$ are represented by the green triangles. The orange stars represent the elements of the set $\mathcal{L}_{e_1,e_2}$.




We explore this discussion further in Example 4.1.15.

**Remark 4.1.9.** Let $e_1$ be a positive integer and let $\underline{I} = I_1, \dots, I_t \subseteq R$ be a sequence of ideals. Let $\phi : \partial\left(\frac{1}{p^{e_1}}\operatorname{V}_{\underline{I}}^{J_\bullet}(p^{e_1}) \cup \mathcal{B}(\underline{I})_{e_1}\right) \to \bigcup_{j=1}^{t}\left(\mathcal{B}_j(\underline{I})_{e_1} \times \{j\}\right)$ the map defined by

$$\phi(x_1, \dots, x_t) = (\widehat{y}^s, s)$$

where

$$y = (x_1, \dots, x_t) - \min\{x_i : i \in \{1, \dots, t\}\}\mathbf{1}$$

and

$$s = \min\{i \in \{1, \dots, t\} : x_i = \min\{x_j : j \in \{1, \dots, t\}\}\}.$$

Notice that, if $(x_1, \dots, x_t) \in \frac{1}{p^{e_1}}\operatorname{V}_{\underline{I}}^{J_\bullet}(p^{e_1})$, we have that $y \in \frac{1}{p^{e_1}}\operatorname{V}_{\underline{I}}^{J_\bullet}(p^{e_1})$ and $\widehat{y}^s \in \mathcal{B}_s(\underline{I})_{e_1}$ by Remark 4.1.3. On the other hand, if $x = (x_1, \dots, x_t) \in \mathcal{B}(\underline{I})_{e_1}$, then $\min\{x_i : i \in \{1, \dots, t\}\}) = 0$ and $y = x$. Hence $\widehat{y}^s \in \mathcal{B}_s(\underline{I})_{e_1}$. Thus $\phi$ is well-defined. Now suppose $\phi(x_1, \dots, x_n) = \phi(z_1, \dots, z_n)$. It follows that $(z_1, \dots, z_t) - z_s\mathbf{1} = (x_1, \dots, x_t) - x_s\mathbf{1}$. We can assume without loss of generality that $z_s \geq x_s$. Then, we have that $(z_1, \dots, z_t) =$

$(x_1, \ldots, x_t)+(z_s-x_s)\mathbf{1}$. If $z_s > x_s$, then $z_i \geq x_i + \frac{1}{p^{e_1}}$ and $z_i > 0$ for every $i \in \{1, \ldots, t\}$. Consequently, $(z_1, \ldots, z_t) \in \frac{1}{p^{e_1}} \mathrm{V}_{\underline{I}}^{J_\bullet}(p^{e_1})$ and $(x_1, \ldots, x_t) + \frac{1}{p^{e_1}}\mathbf{1} \in \frac{1}{p^{e_1}} \mathrm{V}_{\underline{I}}^{J_\bullet}(p^{e_1}) \cup \mathcal{B}(\underline{I})_{e_1}$, which contradicts that $(x_1, \ldots, x_t) \in \partial\left(\frac{1}{p^{e_1}} \mathrm{V}_{\underline{I}}^{J_\bullet}(p^{e_1}) \cup \mathcal{B}(\underline{I})_{e_1}\right)$. Hence, $\phi$ is injective. Therefore, we have that

$$\left|\partial\left(\frac{1}{p^{e_1}} \mathrm{V}_{\underline{I}}^{J_\bullet}(p^{e_1}) \cup \mathcal{B}(\underline{I})_{e_1}\right)\right| \leq p^{e_1(t-1)} \sum_{n=1}^{t} \left(\prod_{j=1}^{n-1} (\mu(I_j)\ell_j + 1) \prod_{j=n+1}^{t} (\mu(I_j)\ell_j + 1)\right).$$

**Remark 4.1.10.** Let $e_1, e_2 \in \mathbb{N}$, $C$ be a subset of $\frac{1}{p^{e_1}}\mathbb{N}^t$, and $x$ be an element of $\frac{1}{p^{e_1+e_2}}\mathbb{N}^t$. Suppose that $\frac{1}{p^{e_1}} \lceil p^{e_1}x \rceil \in C$. For every $i \in \{1, \ldots, t\}$ we have that

$$x_i + \frac{1}{p^{e_1}} = \frac{1}{p^{e_1}}\left(p^{e_1}x_i + 1\right) > \frac{1}{p^{e_1}} \lceil p^{e_1}x_i \rceil .$$

Thus,

$$\frac{1}{p^{e_1}} \lceil p^{e_1}x_i \rceil - \frac{1}{p^{e_1}} < x_i \leq \frac{1}{p^{e_1}} \lceil p^{e_1}x_i \rceil .$$

Therefore, $x \in H_{e_1,e_2}(C)$.

We now start a series of lemmas towards proving Theorem D.

**Lemma 4.1.11.** *Let $\underline{I} = I_1, \ldots, I_t \subseteq R$ be a sequence of ideals, and $J_\bullet = \{J_{p^e}\}_{e\in\mathbb{N}}$ be a p-family of ideals in $R$ such that $I_1, \ldots, I_t \subseteq \sqrt{J_1}$. We have that*

$$\frac{1}{p^{e_1+e_2}} \mathrm{V}_{\underline{I}}^{J_\bullet}(p^{e_1+e_2}) \subseteq \mathcal{R}_{e_1,e_2} \cup \mathcal{L}_{e_1,e_2}$$

*Proof.* Let $x = (x_1, \ldots, x_t) \in \frac{1}{p^{e_1+e_2}} \mathrm{V}_{\underline{I}}^{J_\bullet}(p^{e_1+e_2})$ be such that $x \notin \mathcal{R}_{e_1,e_2}$. By Remark 4.1.3, $p^{e_1+e_2}x_i \leq \mu(I_i)\ell_i p^{e_1+e_2}$ for each $i \in \{1, \ldots, t\}$. Hence, $p^{e_1}x_i \leq \mu(I_i)\ell_i p^{e_1}$ and $\lceil p^{e_1}x_i \rceil \leq \mu(I_i)\ell_i p^{e_1}$ for each $i \in \{1, \ldots, t\}$. Thus, if $x_j = \min\{x_1, \ldots, x_t\}$, we have

$$\frac{1}{p^{e_1}}\left(\lceil p^{e_1}x \rceil - \lceil p^{e_1}x_j \rceil \mathbf{1}\right) \in \mathcal{B}(\underline{I})_{e_1}.$$

Hence, $\left\{y \in \frac{1}{p^{e_1}}\mathbb{Z} \mid \frac{1}{p^{e_1}} \lceil p^{e_1}x \rceil - y\mathbf{1} \in \left(\frac{1}{p^{e_1}} \mathrm{V}_{\underline{I}}^{J_\bullet}(p^{e_1}) \cup \mathcal{B}(\underline{I})_{e_1}\right)\right\}$ is not empty.

Since $x \notin \mathcal{R}_{e_1,e_2}$, we have that $\frac{1}{p^{e_1}} \lceil p^{e_1}x \rceil + y\mathbf{1} \notin \frac{1}{p^{e_1}} \mathrm{V}_{\underline{I}}^{J_\bullet}(p^{e_1})$ for every $y \in \frac{1}{p^{e_1}}\mathbb{N}$ by Remark 4.1.10. As a consequence, $\left\{y \in \frac{1}{p^{e_1}}\mathbb{Z} \mid \frac{1}{p^{e_1}} \lceil p^{e_1}x \rceil - y\mathbf{1} \in \left(\frac{1}{p^{e_1}} \mathrm{V}_{\underline{I}}^{J_\bullet}(p^{e_1}) \cup \mathcal{B}(\underline{I})_{e_1}\right)\right\}$ is bounded below by 0.

We take $a = \frac{1}{p^{e_1}} \lceil p^{e_1}x \rceil - r\mathbf{1}$, where

$$r = \min\left\{y \in \frac{1}{p^{e_1}}\mathbb{Z} \mid \frac{1}{p^{e_1}} \lceil p^{e_1}x \rceil - y\mathbf{1} \in \left(\frac{1}{p^{e_1}} \mathrm{V}_{\underline{I}}^{J_\bullet}(p^{e_1}) \cup \mathcal{B}(\underline{I})_{e_1}\right)\right\}.$$

We note that $a \in \partial\left(\frac{1}{p^{e_1}} \mathrm{V}_{\underline{I}}^{J_\bullet}(p^{e_1}) \cup \mathcal{B}(\underline{I})_{e_1}\right)$.

On the other hand, by Lemma 4.1.4 with $s = p^{e_1}a_1, \ldots, p^{e_1}a_t$, we have

$$\begin{aligned}
\underline{I}^{p^{e_2}(p^{e_1}a+\mu\mathbf{1})} &= I_1^{p^{e_2}(p^{e_1}a_1+\mu)} \cdots I_t^{p^{e_2}(p^{e_1}a_t+\mu)} \\
&= I_1^{\mu p^{e_2}}(I_1^{[p^{e_2}]})^{p^{e_1}a_1} \cdots I_t^{\mu p^{e_2}}(I_t^{[p^{e_2}]})^{p^{e_1}a_t} \\
&\subseteq I_1^{[p^{e_2}]}(I_1^{p^{e_1}a_1})^{[p^{e_2}]} \cdots I_t^{[p^{e_2}]}(I_t^{p^{e_1}a_t})^{[p^{e_2}]} \\
&= (I_1^{p^{e_1}a_1+1})^{[p^{e_2}]} \cdots (I_t^{p^{e_1}a_t+1})^{[p^{e_2}]} \\
&= (\underline{I}^{p^{e_1}a+\mathbf{1}})^{[p^{e_2}]}.
\end{aligned}$$

Since $a \in \partial\left(\frac{1}{p^{e_1}} \mathrm{V}_{\underline{I}}^{J_\bullet}(p^{e_1}) \cup \mathcal{B}(\underline{I})_{e_1}\right)$, $p^{e_1}a + \mathbf{1} \notin \mathrm{V}_{\underline{I}}^{J_\bullet}(p^{e_1})$. Then,

$$\begin{aligned}
\underline{I}^{p^{e_2}(p^{e_1}a+\mu\mathbf{1})} &= (\underline{I}^{p^{e_1}a+\mathbf{1}})^{[p^{e_2}]} \\
&\subseteq J_{p^{e_1}}^{[p^{e_2}]} \\
&\subseteq J_{p^{e_1+e_2}}.
\end{aligned}$$

Thus, $p^{e_2}(p^{e_1}a + \mu\mathbf{1}) \notin \mathrm{V}_{\underline{I}}^{J_\bullet}(p^{e_1+e_2})$. Hence, there exists $k \in \{1, \ldots, t\}$ such that $p^{e_1+e_2}x_k \le p^{e_2}(p^{e_1}a_k + \mu)$. This implies, $p^{e_1}x_k \le \lceil p^{e_1}x_k \rceil - p^{e_1}r + \mu$, and so $\lceil p^{e_1}x_k \rceil \le \lceil p^{e_1}x_k \rceil - p^{e_1}r + \mu$. Then, we have that $0 \le r \le \frac{1}{p^{e_1}}\mu$.

Since $\frac{1}{p^{e_1}}\lceil p^{e_1}x \rceil = a + r\mathbf{1} \in \partial\left(\frac{1}{p^{e_1}} \mathrm{V}_{\underline{I}}^{J_\bullet}(p^{e_1}) \cup \mathcal{B}(\underline{I})_{e_1}\right) + \frac{1}{p^{e_1}}\{0, \ldots, \mu\}\mathbf{1}$, it follows that $x \in H_{e_1,e_2}\left(\partial\left(\frac{1}{p^{e_1}} \mathrm{V}_{\underline{I}}^{J_\bullet}(p^{e_1}) \cup \mathcal{B}(\underline{I})_{e_1}\right) + \frac{1}{p^{e_1}}\{0, \ldots, \mu\}\mathbf{1}\right) = \mathcal{L}_{e_1,e_2}$ by Remark 4.1.10. $\square$

**Lemma 4.1.12.** *Let $\underline{I} = I_1, \ldots, I_t \subseteq R$ be a sequence of ideals, and $J_\bullet = \{J_{p^e}\}_{e\in\mathbb{N}}$ be a p-family of ideals in $R$ such that $I_1, \ldots, I_t \subseteq \sqrt{J_1}$. For each $e_1 \in \mathbb{N}$, there exists a subset $A^{e_1}$ of $\frac{1}{p^{e_1}}\mathbb{N}^t$ such that*

*(1)* $\frac{1}{p^{e_1}} \mathrm{V}_{\underline{I}}^{J_\bullet}(p^{e_1}) \subseteq A^{e_1}$,

*(2)* $\frac{1}{p^{e_1+e_2}} \mathrm{V}_{\underline{I}}^{J_\bullet}(p^{e_1+e_2}) \subseteq H_{e_1,e_2}(A^{e_1})$ *for all* $e_2 \in \mathbb{N}$, *and*

*(3)* $\lim_{e_1\to\infty} \frac{|A^{e_1} - \frac{1}{p^{e_1}} \mathrm{V}_{\underline{I}}^{J_\bullet}(p^{e_1})|}{{p^{e_1}}^t} = 0$.

*Proof.* By Lemma 4.1.11, we have that $\frac{1}{p^{e_1+e_2}} \mathrm{V}_{\underline{I}}^{J_\bullet}(p^{e_1+e_2}) \subseteq \mathcal{R}_{e_1,e_2} \cup \mathcal{L}_{e_1,e_2}$. In addition, the set $\mathcal{R}_{e_1,e_2} \cup \mathcal{L}_{e_1,e_2}$ is contained in

$$H_{e_1,e_2}\left(\frac{1}{p^{e_1}} \mathrm{V}_{\underline{I}}^{J_\bullet}(p^{e_1}) \bigcup \left(\partial\left(\frac{1}{p^{e_1}} \mathrm{V}_{\underline{I}}^{J_\bullet}(p^{e_1}) \cup \mathcal{B}(\underline{I})_{e_1}\right) + \frac{1}{p^{e_1}}\{0, \ldots, \mu\}\mathbf{1}\right)\right).$$

Let $A^{e_1} = \frac{1}{p^{e_1}} \mathrm{V}_{\underline{I}}^{J_\bullet}(p^{e_1}) \bigcup \left(\partial\left(\frac{1}{p^{e_1}} \mathrm{V}_{\underline{I}}^{J_\bullet}(p^{e_1}) \cup \mathcal{B}(\underline{I})_{e_1}\right) + \frac{1}{p^{e_1}}\{0, \ldots, \mu\}\mathbf{1}\right)$. Then,

$$\left|A^{e_1} - \frac{1}{p^{e_1}} \mathrm{V}_{\underline{I}}^{J_\bullet}(p^{e_1})\right| \le p^{e_1(t-1)}(\mu+1) \sum_{n=1}^{t} \left(\prod_{j=1}^{n-1}(\mu(I_j)\ell_j + 1) \prod_{j=n+1}^{t}(\mu(I_j)\ell_j + 1)\right)$$

by Remark 4.1.9. Hence, we have that

$$\begin{aligned}
0 &\leq \liminf_{e_1\to\infty} \frac{|A^{e_1} - \frac{1}{p^{e_1}} \operatorname{V}_{\underline{I}}^{J_\bullet}(p^{e_1})|}{p^{e_1 t}} \\
&\leq \limsup_{e_1\to\infty} \frac{|A^{e_1} - \frac{1}{p^{e_1}} \operatorname{V}_{\underline{I}}^{J_\bullet}(p^{e_1})|}{p^{e_1 t}} \\
&\leq \limsup_{e_1\to\infty} \frac{(\mu+1)\sum_{n=1}^{t}\left(\prod_{j=1}^{n-1}(\mu(I_j)\ell_j+1)\prod_{j=n+1}^{t}(\mu(I_j)\ell_j+1)\right)}{p^{e_1}} \\
&= 0.
\end{aligned}$$

It follows that

$$\lim_{e_1\to\infty} \frac{|A^{e_1} - \frac{1}{p^{e_1}} \operatorname{V}_{\underline{I}}^{J_\bullet}(p^{e_1})|}{p^{e_1 t}} = 0.$$

□

We are now ready to prove the main result of this section which appears in the introduction as Theorem D.

**Theorem 4.1.13.** *Let $\underline{I} = I_1, \ldots, I_t \subseteq R$ be a sequence of ideals, and $J_\bullet = \{J_{p^e}\}_{e\in\mathbb{N}}$ be a p-family of ideals in $R$ such that $I_1, \ldots, I_t \subseteq \sqrt{J_1}$. Then, $\lim_{e\to\infty} \frac{|\operatorname{V}_{\underline{I}}^{J_\bullet}(p^e)|}{p^{et}}$ exists.*

*Proof.* For each $e_1 \in \mathbb{N}$, let $A^{e_1}$ be as in Lemma 4.1.12. Then, for each $e_1, e_2 \in \mathbb{N}$, we have

$$\frac{1}{p^{e_1+e_2}} \operatorname{V}_{\underline{I}}^{J_\bullet}(p^{e_1+e_2}) \subseteq H_{e_1,e_2}(A^{e_1}).$$

As a consequence,

$$|\operatorname{V}_{\underline{I}}^{J_\bullet}(p^{e_1+e_2})| \leq |H_{e_1,e_2}(A^{e_1})| \leq p^{e_2 t}|A^{e_1}|.$$

Since $\frac{1}{p^{e_1}} \operatorname{V}_{\underline{I}}^{J_\bullet}(p^{e_1}) \subseteq A^{e_1}$,

$$A^{e_1} = \frac{1}{p^{e_1}} \operatorname{V}_{\underline{I}}^{J_\bullet}(p^{e_1}) \cup \left(A^{e_1} - \frac{1}{p^{e_1}} \operatorname{V}_{\underline{I}}^{J_\bullet}(p^{e_1})\right).$$

Hence,

$$|A^{e_1}| = \left(|\operatorname{V}_{\underline{I}}^{J_\bullet}(p^{e_1})| + \left|A^{e_1} - \frac{1}{p^{e_1}} \operatorname{V}_{\underline{I}}^{J_\bullet}(p^{e_1})\right|\right).$$

It follows that

$$|\operatorname{V}_{\underline{I}}^{J_\bullet}(p^{e_1+e_2})| \leq p^{e_2 t}\left(|\operatorname{V}_{\underline{I}}^{J_\bullet}(p^{e_1})| + \left|A^{e_1} - \frac{1}{p^{e_1}} \operatorname{V}_{\underline{I}}^{J_\bullet}(p^{e_1})\right|\right).$$

Dividing by $p^{e_1 t + e_2 t}$, we obtain

$$\frac{|\operatorname{V}_{\underline{I}}^{J_\bullet}(p^{e_1+e_2})|}{p^{e_1 t+e_2 t}} \leq \frac{|\operatorname{V}_{\underline{I}}^{J_\bullet}(p^{e_1})|}{p^{e_1 t}} + \frac{|A^{e_1} - \frac{1}{p^{e_1 t}} \operatorname{V}_{\underline{I}}^{J_\bullet}(p^{e_1})|}{p^{e_1 t}}.$$

Thus, we have

$$\limsup_{e\to\infty}\frac{|\operatorname{V}_{\underline{I}}^{J_\bullet}(p^e)|}{p^{et}}=\limsup_{e_2\to\infty}\frac{|\operatorname{V}_{\underline{I}}^{J_\bullet}(p^{e_1+e_2})|}{p^{e_1t+e_2t}}\le\frac{|\operatorname{V}_{\underline{I}}^{J_\bullet}(p^{e_1})|}{p^{e_1t}}+\frac{|A^{e_1}-\frac{1}{p^{e_1t}}\operatorname{V}_{\underline{I}}^{J_\bullet}(p^{e_1})|}{p^{e_1t}}.$$

It follows that

$$\limsup_{e\to\infty}\frac{|\operatorname{V}_{\underline{I}}^{J_\bullet}(p^e)|}{p^{et}}\le\liminf_{e_1\to\infty}\frac{|\operatorname{V}_{\underline{I}}^{J_\bullet}(p^{e_1})|}{p^{e_1t}}+\lim_{e_1\to\infty}\frac{|A^{e_1}-\frac{1}{p^{e_1t}}\operatorname{V}_{\underline{I}}^{J_\bullet}(p^{e_1})|}{p^{e_1t}}=\liminf_{e_1\to\infty}\frac{|\operatorname{V}_{\underline{I}}^{J_\bullet}(p^{e_1})|}{p^{e_1t}}.$$

Therefore, the $\lim_{e\to\infty}\frac{|\operatorname{V}_{\underline{I}}^{J_\bullet}(p^e)|}{p^{et}}$ exists. $\square$

Given Theorem 4.1.13, we are able to define the $F$-volume of a sequence of ideals with respect to a $p$-family. We justify the choice of this name in Section 4.3, where we show that this number gives a volume of certain regions for $F$-pure rings.

**Definition 4.1.14.** Let $\underline{I}=I_1,\dots,I_t\subseteq R$ be a sequence of ideals, and $J_\bullet=\{J_{p^e}\}_{e\in\mathbb{N}}$ be a $p$-family of ideals in $R$ such that $I_1,\dots,I_t\subseteq\sqrt{J_1}$. We define the $F$-volume of the sequence $\underline{I}$ with respect to the $p$-family $J_\bullet=\{J_{p^e}\}_{e\in\mathbb{N}}$ by

$$\operatorname{Vol}_F^{J_\bullet}(\underline{I})=\lim_{e\to\infty}\frac{1}{p^{et}}|\operatorname{V}_{\underline{I}}^{J_\bullet}(p^e)|.$$

If $\underline{f}=f_1,\dots,f_t$ is a sequence of elements of $R$ such that $f_1,\dots,f_t\in\sqrt{J_1}$, we use $\operatorname{Vol}_F^{J_\bullet}(\underline{f})$ to denote $\operatorname{Vol}_F^{J_\bullet}(\underline{I})$ where $\underline{I}=f_1R,\dots,f_tR$. In case that the $p$-family is $J_\bullet=\{J^{[p^e]}\}_{e\in\mathbb{N}}$ where $J$ is an ideal in $R$, $\operatorname{Vol}_F^{J_\bullet}(\underline{I})$ is denoted by $\operatorname{Vol}_F^{J}(\underline{I})$, and we call it the $F$-volume of the sequence $\underline{I}$ with respect to $J$.

We end this section providing an example that shows that different generators of an ideal do not necessarily give equal volumes, that is, if we take two ideals $I,J$ such that $I\subseteq\sqrt{J}$, and $I=(f_1,\dots,f_t)=(g_1,\dots,g_s)$ with $\underline{f}\neq\underline{g}$, then it is possible to have $\operatorname{Vol}_F^J(\underline{f})\neq\operatorname{Vol}_F^J(\underline{g})$.

**Example 4.1.15.** We take $R=K[[x,y]]$ with $K$ an $F$-finite field of characteristic $p=2$. Let $I=(x,y^2)=(x,y^2+x)$, $\mathfrak{m}=(x,y)$, $\underline{f}=x,y^2$ and $\underline{g}=x,y^2+x$.

Let us compute $\operatorname{Vol}_F^{\mathfrak{m}}(\underline{f})$. We note that for $a,\ b,\ e\in\mathbb{N}$,

$$\begin{aligned}x^ay^{2b}\notin\mathfrak{m}^{[p^e]}&\Leftrightarrow a\le p^e-1,\ 2b\le p^e-1\\&\Leftrightarrow a\le p^e-1,\ b\le\frac{p^e-1}{2}\\&\Leftrightarrow a\le p^e-1,\ b\le\left\lfloor\frac{p^e-1}{2}\right\rfloor=\frac{p^e-2}{2}.\end{aligned}$$

Hence, $\mathrm{V}^{\mathfrak{m}}_{\underline{f}}(p^e) = [0, p^e-1] \times \left[0, \frac{p^e-2}{2}\right] \cap \mathbb{N}^2$. Thus, $|\,\mathrm{V}^{\mathfrak{m}}_{\underline{f}}(p^e)| = \frac{p^{2e}}{2}$. Therefore,

$$\begin{aligned}\mathrm{Vol}^{\mathfrak{m}}_F(\underline{f}) &= \lim_{e\to\infty} \frac{|\,\mathrm{V}^{\mathfrak{m}}_{\underline{f}}(p^e)|}{p^{2e}}\\ &= \lim_{e\to\infty} \frac{p^{2e}}{2p^{2e}}\\ &= \frac{1}{2}.\end{aligned}$$

Let us compute $\mathrm{Vol}^{\mathfrak{m}}_F(\underline{g})$. We note that $\mathrm{V}^{\mathfrak{m}}_{\underline{f}}(p^e) \subseteq \mathrm{V}^{\mathfrak{m}}_{\underline{g}}(p^e)$. We show that

$$\mathrm{V}^{\mathfrak{m}}_{\underline{g}}(p^e) = \mathrm{V}^{\mathfrak{m}}_{\underline{f}}(p^e) \cup \left(\left[0, \frac{p^e-2}{2}\right] \times \left(\frac{p^e-2}{2}, p^e-1\right] \cap \mathbb{N}^2\right).$$

Since $a,\ b \le p^e - 1$ if $(a,b) \in \mathrm{V}^{\mathfrak{m}}_{\underline{g}}(p^e)$, it is enough to show that $(\frac{p^e-2}{2}, p^e-1) \in \mathrm{V}^{\mathfrak{m}}_{\underline{g}}(p^e)$, and that $(\frac{p^e}{2}, \frac{p^e}{2}) \notin \mathrm{V}^{\mathfrak{m}}_{\underline{g}}(p^e)$.

We have that

$$x^{\frac{p^e-2}{2}}(x+y^2)^{p^e-1} = \sum_{i=0}^{p^e-1} \binom{p^e-1}{i} x^{\frac{p^e-2}{2}+p^e-1-i} y^{2i}.$$

But, $2 \nmid \binom{p^e-1}{\frac{p^e}{2}-1}$, then $\binom{p^e-1}{\frac{p^e}{2}-1} x^{p^e-1} y^{p^e-2} \notin \mathfrak{m}^{[p^e]}$. Therefore, $(\frac{p^e-2}{2}, p^e-1) \in \mathrm{V}^{\mathfrak{m}}_{\underline{g}}(p^e)$.

Moreover, $x^{\frac{p^e}{2}}(x+y^2)^{\frac{p^e}{2}} = x^{\frac{p^e}{2}} y^{p^e} + x^{p^e} \in \mathfrak{m}^{[p^e]}$. Therefore, $(\frac{p^e}{2}, \frac{p^e}{2}) \notin \mathrm{V}^{\mathfrak{m}}_{\underline{g}}(p^e)$.

It follows that

$$\begin{aligned}|\,\mathrm{V}^{\mathfrak{m}}_{\underline{g}}(p^e)| &= |\,\mathrm{V}^{\mathfrak{m}}_{\underline{f}}(p^e)| + \frac{p^e}{2} \cdot \frac{p^e}{2}\\ &= \frac{p^{2e}}{2} + \frac{p^{2e}}{4}\\ &= \frac{3}{4}p^{2e}.\end{aligned}$$

Thus, we have

$$\begin{aligned}\mathrm{Vol}^{\mathfrak{m}}_F(\underline{g}) &= \lim_{e\to\infty} \frac{|\,\mathrm{V}^{\mathfrak{m}}_{\underline{g}}(p^e)|}{p^{2e}}\\ &= \lim_{e\to\infty} \frac{3p^{2e}}{4p^{2e}}\\ &= \frac{3}{4}.\end{aligned}$$

## 4.2 First Properties

In this section we discuss basic properties of the $F$-volumes. In particular, we focus on those properties that resemble the properties of the $F$-thresholds.

**Proposition 4.2.1.** *Let $\underline{I} = I_1, \ldots, I_t \subseteq R$ be a sequence of ideals, and $\mathfrak{a}$, $J \subseteq R$ be two ideals such that $I_1, \ldots, I_t \subseteq \sqrt{J}$. Then, the following statements hold.*

*(1) If $J \subseteq \mathfrak{a}$, then $\operatorname{Vol}_F^{\mathfrak{a}}(\underline{I}) \leq \operatorname{Vol}_F^{J}(\underline{I})$;*

*(2) $\operatorname{Vol}_F^{J^{[p]}}(\underline{I}) = p^t \operatorname{Vol}_F^{J}(\underline{I})$.*

*(3) If $\underline{I}_{t-1} = I_1, \ldots, I_{t-1}$, then $\operatorname{Vol}_F^{J}(\underline{I}) \leq \operatorname{Vol}_F^{J}(\underline{I}_{t-1}) c^J(I_t)$.*

*Proof.*

(1) Since $J \subseteq \mathfrak{a}$, we have $\mathrm{V}_{\underline{I}}^{\mathfrak{a}}(p^e) \subseteq \mathrm{V}_{\underline{I}}^{J}(p^e)$ for every $e \in \mathbb{N}$. Thus, $|\mathrm{V}_{\underline{I}}^{\mathfrak{a}}(p^e)| \leq |\mathrm{V}_{\underline{I}}^{J}(p^e)|$. Therefore, $\operatorname{Vol}_F^{\mathfrak{a}}(\underline{I}) \leq \operatorname{Vol}_F^{J}(\underline{I})$.

(2) We have that $(J^{[p]})^{[p^e]} = J^{[p^{e+1}]}$, then $\mathrm{V}_{\underline{I}}^{J^{[p]}}(p^e) = \mathrm{V}_{\underline{I}}^{J}(p^{e+1})$. Hence $\frac{|\mathrm{V}_{\underline{I}}^{J^{[p]}}(p^e)|}{p^{et}} = \frac{p^t|\mathrm{V}_{\underline{I}}^{J}(p^{e+1})|}{p^{(e+1)t}}$. Therefore, $\operatorname{Vol}_F^{J^{[p]}}(\underline{I}) = p^t \operatorname{Vol}_F^{J}(\underline{I})$.

(3) Let $a \in \mathrm{V}_{\underline{I}}^{J}(p^e)$. Then, $\underline{I}^a \not\subseteq J^{[p^e]}$. We have that $a_t \leq \nu_{I_t}^{J}(p^e)$; otherwise, $\underline{I}^a \subseteq J^{[p^e]}$, which is a contradiction. Then, $\mathrm{V}_{\underline{I}}^{J}(p^e) \subseteq \mathrm{V}_{\underline{I}_{t-1}}^{J}(p^e) \times \{0, \ldots, \nu_{I_t}^{J}(p^e)\}$. Thus,

$$\begin{aligned}
\operatorname{Vol}_F^{J}(\underline{I}) &= \lim_{e\to\infty} \frac{|\mathrm{V}_{\underline{I}}^{J}(p^e)|}{p^{et}} \\
&\leq \lim_{e\to\infty} \frac{|\mathrm{V}_{\underline{I}_{t-1}}^{J}(p^e) \times \{0, \ldots, \nu_{I_t}^{J}(p^e)\}|}{p^{et}} \\
&= \lim_{e\to\infty} \frac{|\mathrm{V}_{\underline{I}_{t-1}}^{J}(p^e)|}{p^{e(t-1)}} \lim_{e\to\infty} \frac{\nu_{I_t}^{J}(p^e)+1}{p^e} \\
&= \operatorname{Vol}_F^{J}(\underline{I}_{t-1}) c^J(I_t).
\end{aligned}$$

□

We now show that $F$-volumes are not affected by integral closure.

**Proposition 4.2.2.** *Let $\underline{I} = I_1, \ldots, I_t \subseteq R$ be a sequence of ideals, and $J_\bullet = \{J_{p^e}\}_{e\in\mathbb{N}}$ be a $p$-family of ideals in $R$ such that $I_1, \ldots, I_t \subseteq \sqrt{J_1}$. Then,*

$$\operatorname{Vol}_F^{J_\bullet}(\underline{I}) = \operatorname{Vol}_F^{J_\bullet}(\overline{I_1}, I_2, \ldots, I_t),$$

*where $\overline{I_1}$ denotes the integral closure of $I_1$. As a consequence,*

$$\operatorname{Vol}_F^{J_\bullet}(\underline{I}) = \operatorname{Vol}_F^{J_\bullet}(\overline{I_1}, \ldots, \overline{I_t}).$$

*Proof.* We have that $I_1 \subseteq \overline{I_1}$ is a reduction by Propositions 2.2.4 and 2.2.5. Then, there exists $\ell \in \mathbb{N}_{>0}$ such that $\overline{I_1}^n \subseteq I_1^{n-\ell}$ for every $n \in \mathbb{N}$ by Definition 2.2.3. We consider the set

$$H = \{\beta \in \mathbb{N}^{t-1} \mid \exists \beta_1 \in \mathbb{N} \text{ such that } (\beta_1, \beta) \in \mathrm{V}^{J_\bullet}_{\overline{I_1}, I_2, \ldots, I_t}(p^e) \setminus \mathrm{V}^{J_\bullet}_{\underline{I}}(p^e)\}.$$

For all $\beta \in H$, we denote $b_\beta$ as the largest nonnegative integer such that

$$(b_\beta, \beta) \in \mathrm{V}^{J_\bullet}_{\overline{I_1}, I_2, \ldots, I_t}(p^e) \setminus \mathrm{V}^{J_\bullet}_{\underline{I}}(p^e).$$

We show that $\mathrm{V}^{J_\bullet}_{\overline{I_1}, I_2, \ldots, I_t}(p^e) \setminus \mathrm{V}^{J_\bullet}_{\underline{I}}(p^e) \subseteq \bigcup_{\beta \in H} (\mathbb{N} \cap (b_\beta - \ell, b_\beta]) \times \{\beta\}$. Let $(a_1, \ldots, a_t)$ be an element of $\mathrm{V}^{J_\bullet}_{\overline{I_1}, I_2, \ldots, I_t}(p^e) \setminus \mathrm{V}^{J_\bullet}_{\underline{I}}(p^e)$. Thus, $a = (a_2, \ldots, a_t) \in H$. We have that $\overline{I_1}^{b_a} I_2^{a_2} \cdots I_t^{a_t} \subseteq I_1^{b_a - \ell} I_2^{a_2} \cdots I_t^{a_t}$. We deduce that $I_1^{b_a - \ell} I_2^{a_2} \cdots I_t^{a_t} \not\subseteq J_{p^e}$. But, $I_1^{a_1} I_2^{a_2} \cdots I_t^{a_t} \subseteq J_{p^e}$. As a consequence, $b_a - \ell < a_1 \leq b_a$. Therefore, $(a_1, a) \in \bigcup_{\beta \in H} (\mathbb{N} \cap (b_\beta - \ell, b_\beta]) \times \{\beta\}$.

We note that $\mathrm{V}^{J_\bullet}_{\underline{I}}(p^e) \subseteq \mathrm{V}^{J_\bullet}_{\overline{I_1}, I_2, \ldots, I_t}(p^e)$. Thus,

$$|\,\mathrm{V}^{J_\bullet}_{\overline{I_1}, I_2, \ldots, I_t}(p^e)| - |\,\mathrm{V}^{J_\bullet}_{\underline{I}}(p^e)| = |\,\mathrm{V}^{J_\bullet}_{\overline{I_1}, I_2, \ldots, I_t}(p^e) \setminus \mathrm{V}^{J_\bullet}_{\underline{I}}(p^e)| \leq \ell |H| \leq \ell \prod_{i=2}^{t} |\,\mathrm{V}^{J_\bullet}_{I_i}(p^e)|,$$

where the last inequality follows because $H \subseteq \mathrm{V}^{J_\bullet}_{I_2, \ldots, I_t}(p^e)$. Since

$$\lim_{e \to \infty} \frac{\prod_{i=2}^{t} |\,\mathrm{V}^{J_\bullet}_{I_i}(p^e)|}{p^{e(t-1)}} = \prod_{i=2}^{t} \mathrm{Vol}^{J_\bullet}_F(I_i),$$

we obtain that

$$0 \leq \lim_{e \to \infty} \frac{|\,\mathrm{V}^{J_\bullet}_{\overline{I_1}, I_2, \ldots, I_t}(p^e)|}{p^{et}} - \frac{|\,\mathrm{V}^{J_\bullet}_{\underline{I}}(p^e)|}{p^{et}} \leq \lim_{e \to \infty} \frac{\ell \prod_{i=2}^{t} |\,\mathrm{V}^{J_\bullet}_{I_i}(p^e)|}{p^{et}} = 0.$$

Therefore,

$$\mathrm{Vol}^{J_\bullet}_F(\underline{I}) = \lim_{e \to \infty} \frac{|\,\mathrm{V}^{J_\bullet}_{\underline{I}}(p^e)|}{p^{et}} = \lim_{e \to \infty} \frac{|\,\mathrm{V}^{J_\bullet}_{\overline{I_1}, I_2, \ldots, I_t}(p^e)|}{p^{et}} = \mathrm{Vol}^{J_\bullet}_F(\overline{I_1}, I_2, \ldots, I_t).$$

□

We now start describing objects that give us an alternative description to $F$-volumes. These descriptions will play an important role in the following sections.

**Definition 4.2.3.** Let $\underline{I} = I_1, \ldots, I_t \subseteq R$ be a sequence of ideals, and $J_\bullet = \{J_{p^e}\}_{e \in \mathbb{N}}$ be a $p$-family of ideals in $R$ such that $I_1, \ldots, I_t \subseteq \sqrt{J_1}$. We take

$$B^{J_\bullet}(\underline{I}; p^e) = \bigcup_{a \in \mathrm{V}^{J_\bullet}_{\underline{I}}(p^e)} [0, a_1/p^e] \times \ldots \times [0, a_t/p^e].$$

If $\underline{f} = f_1, \ldots, f_t$ is a sequence of elements of $R$ such that $f_1, \ldots, f_t \in \sqrt{J_1}$, we use $B^{J_\bullet}(\underline{f}; p^e)$ to denote $B^{J_\bullet}(\underline{I}; p^e)$ where $\underline{I} = f_1R, \ldots, f_tR$. In case that the $p$-family is $J_\bullet = \{J^{[p^e]}\}_{e \in \mathbb{N}}$ where $J$ is an ideal in $R$, $B^{J_\bullet}(\underline{I}; p^e)$ is denoted by $B^J(\underline{I}; p^e)$.

**Remark 4.2.4.** Analogous to Definition 4.1.2, we take

$$\widetilde{\mathrm{V}}_{\underline{I}}^{J_\bullet}(p^e) = \{(a_1, \ldots, a_t) \in \mathbb{N}^t_{>0} \mid I_1^{a_1} \cdots I_t^{a_t} \not\subseteq J_{p^e}\},$$

and

$$\widetilde{B}^{J_\bullet}(\underline{I}; p^e) = \bigcup_{a \in \widetilde{\mathrm{V}}_{\underline{I}}^{J_\bullet}(p^e)} [0, a_1/p^e] \times \ldots \times [0, a_n/p^e].$$

If the $p$-family is $J_\bullet = \{J^{[p^e]}\}_{e\in\mathbb{N}}$ with $J$ an ideal of $R$, we denote $\widetilde{\mathrm{V}}_{\underline{I}}^{J_\bullet}(p^e)$ and $\widetilde{B}^{J_\bullet}(\underline{I}; p^e)$ by $\widetilde{\mathrm{V}}_{\underline{I}}^{J}(p^e)$ and $\widetilde{B}^{J}(\underline{I}; p^e)$ respectively.

We obtain

$$\mathrm{Vol}(\widetilde{B}^{J_\bullet}(\underline{I}; p^e)) = \frac{1}{p^{et}} |\widetilde{\mathrm{V}}_{\underline{I}}^{J_\bullet}(p^e)|,$$

by dividing $\widetilde{B}^{J_\bullet}(\underline{I}; p^e)$ in $t$-cubes of volume $1/p^{et}$ and counting the number of $t$-cubes.

**Proposition 4.2.5.** *Let $\underline{I} = I_1, \ldots, I_t \subseteq R$ be a sequence of ideals, and $J_\bullet = \{J_{p^e}\}_{e\in\mathbb{N}}$ be a $p$-family of ideals in $R$ such that $I_1, \ldots, I_t \subseteq \sqrt{J_1}$. Then,*

$$\mathrm{Vol}_F^{J_\bullet}(\underline{I}) = \lim_{e\to\infty} \frac{|\widetilde{\mathrm{V}}_{\underline{I}}^{J_\bullet}(p^e)|}{p^{et}}.$$

*Proof.* We note that $\widetilde{\mathrm{V}}_{\underline{I}}^{J_\bullet}(p^e) \subseteq \mathrm{V}_{\underline{I}}^{J_\bullet}(p^e)$ and

$$\begin{aligned} \mathrm{V}_{\underline{I}}^{J_\bullet}(p^e) \setminus \widetilde{\mathrm{V}}_{\underline{I}}^{J_\bullet}(p^e) &= \{(a_1, \ldots, a_t) \in \mathbb{N}^t \mid I_1^{a_1} \cdots I_t^{a_t} \not\subseteq J_{p^e} \text{ \& } \exists i \text{ such that } a_i = 0\} \\ &\subseteq \{(a_1, \ldots, a_t) \in \mathbb{N}^t \mid \forall i,\ a_i \leq |\,\mathrm{V}_{I_i}^{J_\bullet}(p^e)| - 1 \text{ \& } \exists i \text{ such that } a_i = 0\}. \end{aligned}$$

Then,

$$|\,\mathrm{V}_{\underline{I}}^{J_\bullet}(p^e)| - |\widetilde{\mathrm{V}}_{\underline{I}}^{J_\bullet}(p^e)| = |\,\mathrm{V}_{\underline{I}}^{J_\bullet}(p^e) \setminus \widetilde{\mathrm{V}}_{\underline{I}}^{J_\bullet}(p^e)| \leq \sum_{i=1}^{t} \left( \prod_{i\neq j} |\,\mathrm{V}_{I_j}^{J_\bullet}(p^e)| \right).$$

We obtain that

$$\begin{aligned} 0 \leq \lim_{e\to\infty} \frac{|\,\mathrm{V}_{\underline{I}}^{J_\bullet}(p^e)|}{p^{et}} - \frac{|\widetilde{\mathrm{V}}_{\underline{I}}^{J_\bullet}(p^e)|}{p^{et}} &\leq \lim_{e\to\infty} \frac{\sum_{i=1}^{t} \left( \prod_{i\neq j} |\,\mathrm{V}_{I_j}^{J_\bullet}(p^e)| \right)}{p^{et}} \\ &= \sum_{i=1}^{t} \lim_{e\to\infty} \frac{\left( \prod_{i\neq j} |\,\mathrm{V}_{I_j}^{J_\bullet}(p^e)| \right)}{p^{et}} = 0, \end{aligned}$$

where the last equality follows from the fact that

$$\lim_{e\to\infty} \frac{\prod_{i\neq j} |\operatorname{V}_{I_j}^{J_\bullet}(p^e)|}{p^{e(t-1)}} = \prod_{i\neq j} \operatorname{Vol}_F^{J_\bullet}(I_j).$$

We conclude that

$$\lim_{e\to\infty} \frac{|\operatorname{V}_{\underline{I}}^{J_\bullet}(p^e)|}{p^{et}} = \lim_{e\to\infty} \frac{|\widetilde{\operatorname{V}}_{\underline{I}}^{J_\bullet}(p^e)|}{p^{et}}.$$

□

We end this section with an upper bound for the $F$-volume in terms of the $F$-threshold.

**Proposition 4.2.6.** *Let $\underline{I} = I_1, \ldots, I_t \subseteq R$ be a sequence of ideals, and $J \subsetneq R$ be an ideal such that $I_1, \ldots, I_t \subseteq \sqrt{J}$. Let $c = c^J(I_1 + \cdots + I_t)$. Then, $\operatorname{Vol}_F^J(\underline{I}) \leq \frac{c^t}{t!}$.*

*Proof.* Let $I = I_1 + \cdots + I_t$. For $\alpha, e \in \mathbb{N}$, we have $I^\alpha \not\subseteq J^{[p^e]}$ if and only if there exists $a = (a_1, \ldots, a_t) \in \mathbb{N}^t$ with $a_1 + \cdots + a_t = \alpha$ such that $\underline{I}^a \not\subseteq J^{[p^e]}$. Hence, $\nu_I^J(p^e) = \max\{|a| \mid a \in \operatorname{V}_{\underline{I}}^J(p^e)\}$. Additionally, for every $a \in \widetilde{B}^J(\underline{I}; p^e)$ there exists $b \in V_{\underline{I}}^J(p^e)$ such that $|p^e a| \leq |b|$. Consequently, we have that $\frac{\nu_I^J(p^e)}{p^e} \geq \max\{|a| \mid a \in \widetilde{B}^J(\underline{I}; p^e)\}$.

Let $\nu(p^e) = \frac{\nu_I^J(p^e)}{p^e}$. We use $H(p^e)$ to denote the set $\{(x_1, \ldots, x_t) \in \mathbb{R}^t_{\geq 0} \mid x_1 + \ldots + x_t \leq \nu(p^e)\}$. Then we have that $\widetilde{B}^J(\underline{I}; p^e) \subseteq H(p^e)$. Thus, $\operatorname{Vol}(\widetilde{B}^J(\underline{I}; p^e)) \leq \operatorname{Vol}(H(p^e)) = \frac{\nu(p^e)^t}{t!}$. As a consequence, we have that

$$\frac{1}{p^{et}} |\widetilde{\operatorname{V}}_{\underline{I}}^J(p^e)| \leq \frac{\nu(p^e)^t}{t!}$$

by Remark 4.2.4. Since $\lim\limits_{e\to\infty} \nu(p^e) = c$, it follows that $\operatorname{Vol}_F^J(\underline{I}) \leq \frac{c^t}{t!}$ by Proposition 4.2.5. □

## 4.3 Properties for $F$-Pure Rings

In this section we focus on $F$-pure rings. In particular, in Proposition 4.3.5 we prove that the $F$-volume is in fact the volume of an object in a real space. We also provide a few properties that hold only in this case.

**Proposition 4.3.1.** *Let $\underline{I} = I_1, \ldots, I_t \subseteq R$ be a sequence of ideals, and $J \subseteq R$ be an ideal such that $I_1, \ldots, I_t \subseteq \sqrt{J}$. If $R$ is $F$-pure, then*

$$B^J(\underline{I}; p^e) \subseteq B^J(\underline{I}; p^{e+1})$$

*Proof.* For every element $a$ in $\mathrm{V}_{\underline{I}}^{J}(p^e)$, we have that $\underline{I}^a \not\subseteq J^{[p^e]}$. Since $R$ is an $F$-pure ring, $(\underline{I}^a)^{[p]} \not\subseteq J^{[p^{e+1}]}$. As a consequence, $\underline{I}^{pa} \not\subseteq J^{[p^{e+1}]}$, thus $pa \in \mathrm{V}_{\underline{I}}^{J}(p^{e+1})$.

In addition, we have that

$$\left[0, \frac{a_1}{p^e}\right] \times \cdots \times \left[0, \frac{a_t}{p^e}\right] \subseteq \left[0, \frac{pa_1}{p^{e+1}}\right] \times \cdots \times \left[0, \frac{pa_t}{p^{e+1}}\right].$$

Therefore,

$$B^J(\underline{I}; p^e) \subseteq B^J(\underline{I}; p^{e+1}).$$

□

**Remark 4.3.2.** Taking the same condition of Proposition 4.3.1, we have that

$$\widetilde{B}^J(\underline{I}; p^e) \subseteq \widetilde{B}^J(\underline{I}; p^{e+1}).$$

**Definition 4.3.3.** Suppose that $R$ is an $F$-pure ring. Let $\underline{I} = I_1, \dots, I_t \subseteq R$ be a sequence of ideals, and $J_\bullet = \{J_{p^e}\}_{e \in \mathbb{N}}$ be a $p$-family of ideals in $R$ such that $I_1, \dots, I_t \subseteq \sqrt{J_1}$. We take

$$B^{J_\bullet}(\underline{I}) = \bigcup_{e \in \mathbb{N}} B^{J_\bullet}(\underline{I}; p^e).$$

If $\underline{f} = f_1, \dots, f_t$ is a sequence of elements of $R$ such that $f_1, \dots, f_t \in \sqrt{J_1}$, we use $B^{J_\bullet}(\underline{f})$ to denote $B^{J_\bullet}(\underline{I})$ where $\underline{I} = f_1R, \dots, f_tR$. In case that the $p$-family is $J_\bullet = \{J^{[p^e]}\}_{e \in \mathbb{N}}$ where $J$ is an ideal in $R$, $B^{J_\bullet}(\underline{I})$ is denoted by $B^J(\underline{I})$.

Suppose that $(R, \mathfrak{m}, K)$ is an $F$-finite regular local ring. Let $\underline{I} = I_1, \dots, I_t \subseteq R$ be a sequence of ideals. The mixed test ideals $\tau(I_1^{a_1} \cdots I_t^{a_t})$ are important objects studied in birational geometry [HY03, BMS08, Pér13]. The set $B^{\mathfrak{m}}(\underline{I})$ is the first constancy region for these ideals [Pér13]. Furthermore, $B^J(\underline{I})$ is the union of the constancy regions whose test ideal is not contained in $J$.

If the ring is not regular, then $B^{\mathfrak{m}}(\underline{I})$ is no longer a constancy region. To see this, it suffices to look at the case where $t = 1$, and take any example where the $F$-threshold $c^{\mathfrak{m}}(\mathfrak{m}) \neq \mathrm{fpt}(\mathfrak{m})$ (e.g. [MOY10]).

**Remark 4.3.4.** Suppose that $R$ is an $F$-pure ring. Let $\underline{I} = I_1, \dots, I_t \subseteq R$ be a sequence of nonzero ideals, and $J \subseteq R$ be an ideal such that $I_1, \dots, I_t \subseteq \sqrt{J}$. We also have

$$B^J(\underline{I}) = \bigcup_{e \in \mathbb{N}} \widetilde{B}^J(\underline{I}; p^e).$$

The following result justifies in part the name of $F$-volume.

**Proposition 4.3.5.** *Suppose that $R$ is an $F$-pure ring. Let $\underline{I} = I_1, \dots, I_t \subseteq R$ be a sequence of ideals, and $J \subseteq R$ be an ideal such that $I_1, \dots, I_t \subseteq \sqrt{J}$. Then, $B^J(\underline{I})$ is a measurable set. Furthermore,*

$$\mathrm{Vol}(B^J(\underline{I})) = \lim_{e \to \infty} \frac{1}{p^{et}} |\widetilde{\mathrm{V}}_{\underline{I}}^{J}(p^e)|.$$

*In particular,*

$$\mathrm{Vol}_F^J(\underline{I}) = \mathrm{Vol}(B^J(\underline{I})).$$

*Proof.* Since $\widetilde{B}^J(\underline{I}; p^e)$ is measurable for every $e \in \mathbb{N}$, we conclude that $B^J(\underline{I})$ is also measurable.

We now focus on the measure of $B^J(\underline{I})$. From Remark 4.2.4, we recall that

$$\mathrm{Vol}(\widetilde{B}^J(\underline{I}; p^e)) = \frac{1}{p^{et}} |\widetilde{\mathrm{V}}_{\underline{I}}^J(p^e))|.$$

Since $\widetilde{B}^J(\underline{I}; p^e) \subseteq \widetilde{B}^J(\underline{I}; p^{e+1})$ for every $e \in \mathbb{N}$, we conclude that

$$\mathrm{Vol}(B^J(\underline{I})) = \lim_{e\to\infty} \frac{1}{p^{et}} |\widetilde{\mathrm{V}}_{\underline{I}}^J(p^e)|.$$

$\square$

**Remark 4.3.6.** Suppose that $R$ is an $F$-pure ring. Let $\underline{I} = I_1, \ldots, I_t \subseteq R$ be a sequence of ideals, and let $J$ be an ideal in $R$ such that $I_1, \ldots, I_t \subseteq \sqrt{J}$. From the Remarks 4.2.4 and 4.3.2, we have that the sequence $\left\{ \frac{|\widetilde{\mathrm{V}}_{\underline{I}}^J(p^e)|}{p^{et}} \right\}_{e\ \in\mathbb{N}}$ is increasing.

**Remark 4.3.7.** Let $\underline{I} = I_1, \ldots, I_t \subseteq R$ be a sequence of ideals, and $J_\bullet = \{J_{p^e}\}_{e\in\mathbb{N}}$ be a $p$-family of ideals in $R$ such that $I_1, \ldots, I_t \subseteq \sqrt{J_1}$. For each $e \in \mathbb{N}$ and $i \in \{1, ..., t\}$, let $\ell_{e,i} = \min\{\ell \mid I_i^\ell \subseteq J_{p^e}\}$ and consider the sets

- $\mathcal{B}(\underline{I})_{e_1}^e = \frac{1}{p^{e_1}}\mathbb{N}^t \cap \left( \bigcup_{j=1}^t \left( \prod_{i=1}^{j-1} [0, \mu(I_i)\ell_{e,i}] \times \{0\} \times \prod_{i=j+1}^t [0, \mu(I_i)\ell_{e,i}] \right) \right)$
- $\mathcal{L}_{e_1,e_2}^e = H_{e_1,e_2} \left( \partial \left( \frac{1}{p^{e_1}} \mathrm{V}_{\underline{I}}^{J_{p^e}}(p^{e_1}) \cup \mathcal{B}(\underline{I})_{e_1}^e \right) + \frac{1}{p^{e_1}} \{0, \ldots, \mu\} \mathbf{1} \right)$

where $\mu = \max\{\mu(I_1), \ldots, \mu(I_t)\}$.

We claim that

$$\frac{1}{p^{e_1+e_2}} \widetilde{\mathrm{V}}_{\underline{I}}^{J_{p^e}}(p^{e_1+e_2}) \subseteq H_{e_1,e_2} \left( \frac{1}{p^{e_1}} \widetilde{\mathrm{V}}_{\underline{I}}^{J_{p^e}}(p^{e_1}) \right) \cup \mathcal{L}_{e_1,e_2}^e.$$

Let $x \in \frac{1}{p^{e_1+e_2}} \widetilde{\mathrm{V}}_{\underline{I}}^{J_\bullet}(p^{e_1+e_2})$. Suppose that $x \notin \mathcal{L}_{e_1,e_2}^e$. From Lemma 4.1.11, we have that

$$\frac{1}{p^{e_1+e_2}} \mathrm{V}_{\underline{I}}^{J_{p^e}}(p^{e_1+e_2}) \subseteq H_{e_1,e_2} \left( \frac{1}{p^{e_1}} \mathrm{V}_{\underline{I}}^{J_{p^e}}(p^{e_1}) \right) \cup \mathcal{L}_{e_1,e_2}^e.$$

We note that $\widetilde{\mathrm{V}}_{\underline{I}}^{J_{p^e}}(p^{e_1+e_2}) \subseteq \mathrm{V}_{\underline{I}}^{J_{p^e}}(p^{e_1+e_2})$. Since $x \notin \mathcal{L}_{e_1,e_2}^e$, $x \in H_{e_1,e_2} \left( \frac{1}{p^{e_1}} \mathrm{V}_{\underline{I}}^{J_{p^e}}(p^{e_1}) \right)$. Then, there exists $y \in \frac{1}{p^{e_1}} \mathrm{V}_{\underline{I}}^{J_{p^e}}(p^{e_1})$ such that $y_i - \frac{1}{p^{e_1}} < x_i \leq y_i$ for every $i$. Since $x_i > 0$

for every $i$, $y_i > 0$ for every $i$. Hence $y \in \frac{1}{p^{e_1}}\widetilde{\mathrm{V}}_{\underline{I}}^{J_{p^e}}(p^{e_1})$ and $x \in H_{e_1,e_2}\left(\frac{1}{p^{e_1}}\widetilde{\mathrm{V}}_{\underline{I}}^{J_{p^e}}(p^{e_1})\right)$. Therefore,

$$\frac{1}{p^{e_1+e_2}}\widetilde{\mathrm{V}}_{\underline{I}}^{J_{p^e}}(p^{e_1+e_2}) \subseteq H_{e_1,e_2}\left(\frac{1}{p^{e_1}}\widetilde{\mathrm{V}}_{\underline{I}}^{J_{p^e}}(p^{e_1})\right) \cup \mathcal{L}^{e}_{e_1,e_2}.$$

Consequently, we have that

$$\frac{|\widetilde{\mathrm{V}}_{\underline{I}}^{J_{p^e}}(p^{e_1+e_2})|}{p^{(e_1+e_2)t}} \leq \frac{|\widetilde{\mathrm{V}}_{\underline{I}}^{J_{p^e}}(p^{e_1})|}{p^{e_1 t}} + \frac{(\mu+1)}{p^{e_1}}\sum_{n=1}^{t}\left(\prod_{j=1}^{n-1}(\mu(I_j)\ell_{e,j}+1)\prod_{j=n+1}^{t}(\mu(I_j)\ell_{e,j}+1)\right).$$

On the other hand, since $I_n^{\mu(I_n)\ell_{0,n}p^e} \subseteq J_1^{[p^e]} \subseteq J_{p^e}$, we have that $\ell_{e,n} \leq \mu(I_n)\ell_{0,n}p^e$. Thus, if $u = (\mu+1)\sum_{n=1}^{t}\left(\prod_{j=1}^{n-1}(\mu(I_j)^2\ell_{0,j}+1)\prod_{j=n+1}^{t}(\mu(I_j)^2\ell_{0,j}+1)\right)$, we obtain

$$\frac{|\widetilde{\mathrm{V}}_{\underline{I}}^{J_{p^e}}(p^{e_1+e_2})|}{p^{(e_1+e_2)t}} \leq \frac{|\widetilde{\mathrm{V}}_{\underline{I}}^{J_{p^e}}(p^{e_1})|}{p^{e_1 t}} + \frac{p^{e(t-1)}u}{p^{e_1}}.$$

We now introduce another basic property for $F$-volumes for $F$-pure rings.

**Proposition 4.3.8.** *Suppose that $R$ is an $F$-pure ring. Let $\underline{I} = I_1,\ldots,I_t \subseteq R$ be a sequence of ideals, and $J_\bullet = \{J_{p^e}\}_{e\in\mathbb{N}}$ be a $p$-family of ideals in $R$ such that $I_1,\ldots,I_t \subseteq \sqrt{J_1}$. Then,*

$$\mathrm{Vol}_F^{J_\bullet}(\underline{I}) = \lim_{e\to\infty}\frac{\mathrm{Vol}_F^{J_{p^e}}(\underline{I})}{p^{et}}.$$

*Proof.* For every $e \in \mathbb{N}$ we have $J_{p^e}^{[p]} \subseteq J_{p^{e+1}}$. Thus, $\mathrm{Vol}_F^{J_{p^{e+1}}}(\underline{I}) \leq \mathrm{Vol}_F^{J_{p^e}^{[p]}}(\underline{I}) = p^t \cdot \mathrm{Vol}_F^{J_{p^e}}(\underline{I})$. Hence,

$$0 \leq \frac{\mathrm{Vol}_F^{J_{p^{e+1}}}(\underline{I})}{p^{(e+1)t}} \leq \frac{\mathrm{Vol}_F^{J_{p^e}}(\underline{I})}{p^{et}},$$

which shows the sequence $\left\{\frac{\mathrm{Vol}_F^{J_{p^e}}(\underline{I})}{p^{et}}\right\}_{e\,\in\mathbb{N}}$ is decreasing, and bounded below by zero. As a consequence, it converges to a limit as $e$ approaches infinity.

Note that, for every nonnegative integer $e$, we have that

$$\begin{aligned}\widetilde{\mathrm{V}}_{\underline{I}}^{J_\bullet}(p^e) &= \{(a_1,\ldots,a_t)\in\mathbb{N}^t_{>0} \mid I_1^{a_1}\cdots I_t^{a_t} \not\subseteq J_{p^e}\}\\ &= \{(a_1,\ldots,a_t)\in\mathbb{N}^t_{>0} \mid I_1^{a_1}\cdots I_t^{a_t} \not\subseteq J_{p^e}^{[p^0]}\}\\ &= \widetilde{\mathrm{V}}_{\underline{I}}^{J_{p^e}}(p^0).\end{aligned}$$

By Remark 4.3.7, there exists $u \in \mathbb{R}_{\geq 0}$ (that does not depend on $e$) such that for every nonnegative integer $s$, we have

$$\frac{|\widetilde{\mathrm{V}}_{\underline{I}}^{J_{p^e}}(p^s)|}{p^{st}} - \frac{|\widetilde{\mathrm{V}}_{\underline{I}}^{J_{p^e}}(p^0)|}{p^{0t}} \leq \frac{p^{e(t-1)}u}{p^0}.$$

Since $R$ is a $F$-pure ring, the sequence $\left\{ \frac{|\widetilde{\mathrm{V}}_{\underline{I}}^{J}(p^s)|}{p^{st}} \right\}_{s\geq 0}$ is increasing by Remark 4.3.6. As a consequence,

$$0 \leq \frac{|\widetilde{\mathrm{V}}_{\underline{I}}^{J_{p^e}}(p^s)|}{p^{st}} - |\widetilde{\mathrm{V}}_{\underline{I}}^{J_{p^e}}(p^0)| \leq p^{e(t-1)}u.$$

Thus,

$$0 \leq \frac{|\widetilde{\mathrm{V}}_{\underline{I}}^{J_{p^e}}(p^s)|}{p^{st}} - |\widetilde{\mathrm{V}}_{\underline{I}}^{J_\bullet}(p^e)| \leq p^{e(t-1)}u.$$

We take limit over $s$ to get

$$0 \leq \mathrm{Vol}_F^{J_{p^e}}(\underline{I}) - |\widetilde{\mathrm{V}}_{\underline{I}}^{J_\bullet}(p^e)| \leq p^{e(t-1)}u,$$

dividing by $p^{et}$ gives

$$0 \leq \frac{\mathrm{Vol}_F^{J_{p^e}}(\underline{I})}{p^{et}} - \frac{|\widetilde{\mathrm{V}}_{\underline{I}}^{J_\bullet}(p^e)|}{p^{et}} \leq \frac{u}{p^e}.$$

Taking limit over $e$ we conclude that

$$\lim_{e\to\infty} \frac{\mathrm{Vol}_F^{J_{p^e}}(\underline{I})}{p^{et}} = \mathrm{Vol}_F^{J_\bullet}(\underline{I}).$$

$\square$

**Proposition 4.3.9.** *Suppose that $R$ is an $F$-finite regular ring. Let $\underline{I} = I_1, \ldots, I_t$ be a sequence of ideals in $R$, $J$ be an ideal of $R$, and $\{J_i\}_i$ be a family of ideals such that $J = \bigcap_i J_i$ and $I_1, \ldots, I_t \subseteq \sqrt{J}$. Then, $B^J(\underline{I}) = \bigcup_i B^{J_i}(\underline{I})$. In particular,* $\mathrm{Vol}(B^J(\underline{I})) = \mathrm{Vol}(\bigcup_i B^{J_i}(\underline{I}))$.

*Proof.* We show that $\bigcup_i \mathrm{V}_{\underline{I}}^{J_i}(p^e) = \mathrm{V}_{\underline{I}}^{J}(p^e)$ for every nonnegative integer $e$. We claim that $\bigcup_i \mathrm{V}_{\underline{I}}^{J_i}(p^e) \subseteq \mathrm{V}_{\underline{I}}^{J}(p^e)$. Indeed, let $a \in \mathrm{V}_{\underline{I}}^{J_i}(p^e)$ for some $i$, then $\underline{I}^a \not\subseteq J_i^{[p^e]}$. Since $J \subseteq J_i$, we have that $\underline{I}^a \not\subseteq J^{[p^e]}$. Thus, $a \in \mathrm{V}_{\underline{I}}^{J}(p^e)$.

We now prove the other inclusion. Let $a \in \mathrm{V}_{\underline{I}}^{J}(p^e)$, then we have that $\underline{I}^a \not\subseteq J^{[p^e]} = (\bigcap_i J_i)^{[p^e]} = \bigcap_i J_i^{[p^e]}$. Consequently, there exists $i$ such that $\underline{I}^a \not\subseteq J_i^{[p^e]}$. Thus, $a \in \mathrm{V}_{\underline{I}}^{J_i}(p^e)$.

It follows that $B^J(\underline{I}; p^e) = \bigcup_i B^{J_i}(\underline{I}; p^e)$, thus $B^J(\underline{I}) = \bigcup_i B^{J_i}(\underline{I})$. Therefore,

$$\mathrm{Vol}(B^J(\underline{I})) = \mathrm{Vol}\left(\bigcup_i B^{J_i}(\underline{I})\right).$$

$\square$

**Remark 4.3.10.** Suppose that $R$ is an $F$-pure ring. Let $\underline{I} = I_1, \ldots, I_t \subseteq R$ be a sequence, and $J \subseteq R$ be an ideal such that $I_1, \ldots, I_t \subseteq \sqrt{J}$. If we take $\alpha \in B^J(\underline{I}; p^e)$, there exists $\beta \in \mathrm{V}^J_{\underline{I}}(p^e)$ such that each $\alpha_i \leq \frac{\beta_i}{p^e}$. Thus, $\lfloor p^e \alpha_i \rfloor \leq \beta_i$. Since $\underline{I}^{\beta} \not\subseteq J^{[p^e]}$, $\underline{I}^{\lfloor p^e \alpha \rfloor} \not\subseteq J^{[p^e]}$. Therefore, $\lfloor p^e \alpha \rfloor \in \mathrm{V}^J_{\underline{I}}(p^e)$.

Suppose that $R$ is an $F$-finite regular ring. Let $\underline{I} = I_1, \ldots, I_t \subseteq R$ be a sequence of ideals, and $I = I_1 + \cdots + I_t$. The mixed test ideals satisfy the following equation

$$\tau(I^{\lambda}) = \sum_{\alpha_1 + \cdots + \alpha_t = \lambda} \tau(I_1^{\alpha_1} \cdots I_t^{\alpha_t}).$$

Motivated by this result, we obtain the following similar properties for $F$-thresholds. This plays an important role to characterize $F$-pure complete intersections in terms of $F$-volume.

**Proposition 4.3.11.** *Suppose that $R$ is an $F$-pure ring. Let $\underline{I} = I_1, \ldots, I_t \subseteq R$ be a sequence, and $J \subseteq R$ be an ideal such that $I_1, \ldots, I_t \subseteq \sqrt{J}$. Then,*

$$c^J(I_1 + \cdots + I_t) = \sup\{|\theta| \mid \theta \in B^J(\underline{I})\}.$$

*Proof.* Let $\lambda = \sup\{|\theta| \mid \theta \in B^J(\underline{I})\}$ and $I = I_1 + \cdots + I_t$.

Since $I^{\nu^J_I(p^e)} \not\subseteq J^{[p^e]}$, there exists $\alpha = (\alpha_1, \ldots, \alpha_t) \in \mathbb{N}^t$ such that $I_1^{\alpha_1} \cdots I_t^{\alpha_t} \not\subseteq J^{[p^e]}$ and $|\alpha| = \nu^J_I(p^e)$. Then, $\frac{1}{p^e}\alpha \in B^J(\underline{I})$. We conclude that $\frac{\nu^J_I(p^e)}{p^e} \leq \lambda$ for every $e$. Then, $c^J(I) \leq \lambda$.

We now show the other inequality. Let $\alpha = (\alpha_1, \ldots, \alpha_t) \in B^J(\underline{I})$. Then, $\alpha = (\alpha_1, \ldots, \alpha_t) \in B^J(\underline{I}; p^e)$ for $e \gg 0$. Then, $(\lfloor p^e \alpha_1 \rfloor, \ldots, \lfloor p^e \alpha_t \rfloor) \in \mathrm{V}^J_{\underline{I}}(p^e)$ for $e \gg 0$ by Remark 4.3.10. We conclude that $I_1^{\lfloor p^e \alpha_1 \rfloor} \cdots I_t^{\lfloor p^e \alpha_t \rfloor} \not\subseteq J^{[p^e]}$ Then, $I^{\lfloor p^e \alpha_1 \rfloor + \cdots + \lfloor p^e \alpha_t \rfloor} \not\subseteq J^{[p^e]}$. Thus, $\lfloor p^e \alpha_1 \rfloor + \cdots + \lfloor p^e \alpha_t \rfloor \leq \nu^J_I(p^e)$ for $e \gg 0$. We have that

$$|\alpha| = \lim_{e \to \infty} \frac{\lfloor p^e \alpha_1 \rfloor + \cdots + \lfloor p^e \alpha_t \rfloor}{p^e} \leq \lim_{e \to \infty} \frac{\nu^J_I(p^e)}{p^e} = c^J(I).$$

We conclude that $\lambda \leq c^J(I)$. □

The following result allows us to obtain the $F$-threshold under special circumstances.

**Proposition 4.3.12.** *Suppose that $R$ is an $F$-pure ring. Let $I, J \subseteq R$ be two ideals such that $I \subseteq J$. Let $\underline{f} = f_1, \ldots, f_t$ be minimal generators of $I$. If $\mathrm{Vol}^J_F(\underline{f}) = 1$, then $c^J(I) = t$.*

*Proof.* We show that $B^J(\underline{f}) = [0,1)^t$. It is enough to prove that $[0,1)^t \subseteq B^J(\underline{f})$. We proceed by contradiction. We suppose that there exists $a \in [0,1)^t$ such that $a \notin B^J(\underline{f})$. Thus, $H \cap B^J(\underline{f}) = \emptyset$, where $H$ denotes the set $[a_1, 1] \times \cdots \times [a_t, 1]$. Hence, $B^J(\underline{f}) \subseteq [0,1]^t - H$. It follows that $\mathrm{Vol}^J_F(\underline{f}) < 1$, and we get a contradiction.

In addition, from Proposition 4.3.11, we have that

$$\begin{aligned} c^J(I) &= \sup\{|\theta| \mid \theta \in B^J(\underline{f})\} \\ &= \sup\{|\theta| \mid \theta \in [0,1)^t\} \\ &= t. \end{aligned}$$

□

We now characterize $F$-pure complete intersections in terms of $F$-volumes. This is along the same lines of how the $F$-pure threshold of a hypersurface characterizes, via Fedder's Criterion [Fed83], when this variety if $F$-pure.

**Theorem 4.3.13.** *Suppose that $(R, \mathfrak{m}, K)$ is a local regular ring. Let $I \subseteq \mathfrak{m}$ be an ideal in $R$, and $\underline{f} = f_1, \ldots, f_t$ be minimal generators of $I$. Then, $\operatorname{Vol}_F^{\mathfrak{m}}(\underline{f}) = 1$ if and only if $I$ is an $F$-pure complete intersection.*

*Proof.* We suppose that $\operatorname{Vol}_F^{\mathfrak{m}}(\underline{f}) = 1$. Then $c^{\mathfrak{m}}(I) = t = \mu(I)$ by Proposition 4.3.12. Thus, $\mu(I) = c^{\mathfrak{m}}(I) \leq \operatorname{ht}(I) \leq \mu(I)$. We conclude that $\operatorname{ht}(I) = \mu(I)$. Hence, $f_1, \ldots, f_t$ is a regular sequence in $R$. Therefore, $I$ is an $F$-pure complete intersection.

For the other direction, we show that $[0,1)^t = B^{\mathfrak{m}}(\underline{f})$. Let $a \in [0,1)^t$. Then, $\max\{a_i\} \leq \frac{p^e-1}{p^e}$ for some $e \in \mathbb{N}$. Since $I$ is an $F$-pure ideal, $R/I$ is an $F$-pure ring. Since $f_1, \ldots, f_t$ is a regular sequence in $R$, we have that $\underline{f}^{p^e-1} \not\in \mathfrak{m}^{[p^e]}$ by Proposition 2.6.7. Then, $(p^e-1, \ldots, p^e-1) \in \operatorname{V}_{\underline{f}}^J(p^e)$. We conclude that $a \in B^{\mathfrak{m}}(\underline{f}; p^e) \subseteq B^{\mathfrak{m}}(\underline{f})$. Therefore, $\operatorname{Vol}_F^{\mathfrak{m}}(\underline{f}) = 1$ by Proposition 4.3.5. □

## 4.4 Relations with Hilbert-Kunz Multiplicities

In this section we relate the $F$-volume with Hilbert-Kunz multiplicities. This is related to previous work done for $F$-thresholds and these multiplicities [NnBS20]. We start proving Theorem F.

**Theorem 4.4.1.** *Suppose that $(R, \mathfrak{m}, K)$ is a local ring. Let $\underline{f} = f_1, \ldots, f_t$ be part of a system of parameters for $R$, $I = (\underline{f})$, and $\overline{R} = R/I$. Then,*

$$\mathrm{e}_{HK}(J; R) \leq \mathrm{e}_{HK}(J\overline{R}; \overline{R}) \operatorname{Vol}_F^J(\underline{f})$$

*for any $\mathfrak{m}$-primary ideal $J$, such that $I \subseteq J$.*

*Proof.* Let $I = (\underline{f})$ and $\mathcal{I}_e = (f_1^{a_1} \cdots f_t^{a_t} \mid a \not\in \operatorname{V}_{\underline{f}}^J(p^e))R$. Then, $R/\mathcal{I}_e$ has a filtration $0 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_m = R/\mathcal{I}_e$ where $N_{t+1}/N_t$ is a homomorphic image of $R/I$ and $m = |\operatorname{V}_{\underline{f}}^J(p^e)|$. Since $J^{[p^e]}$ is $\mathfrak{m}$-primary, we have that

$$\lambda(N_{t+1} \otimes_R R/J^{[p^e]}) \leq \lambda(N_t \otimes_R R/J^{[p^e]}) + \lambda(N_{t+1}/N_t \otimes_R R/J^{[p^e]}).$$

As a consequence,

$$\lambda(R/\mathcal{I}_e \otimes_R R/J^{[p^e]}) \leq |\operatorname{V}^J_{\underline{f}}(p^e)|\lambda(R/I \otimes_R R/J^{[p^e]}).$$

By the definition of $\mathcal{I}_e$, we have that $\mathcal{I}_e \subseteq J^{[p^e]}$. Then,

$$\begin{aligned}\lambda(R/J^{[p^e]}) &= \lambda(R/\mathcal{I}_e + J^{[p^e]})\\ &= \lambda(R/\mathcal{I}_e \otimes_R R/J^{[p^e]})\\ &\leq |\operatorname{V}^J_{\underline{f}}(p^e)|\lambda(R/I \otimes_R R/J^{[p^e]})\\ &\leq |\operatorname{V}^J_{\underline{f}}(p^e)|\lambda(R/(I + J^{[p^e]})).\end{aligned}$$

After dividing by $p^{ed}$, where $d = \dim(R)$, we obtain that

$$\begin{aligned}\frac{\lambda(R/J^{[p^e]})}{p^{ed}} &\leq \frac{|\operatorname{V}^J_{\underline{f}}(p^e)|\lambda(R/(I + J^{[p^e]}))}{p^{ed}}\\ &= \frac{|\operatorname{V}^J_{\underline{f}}(p^e)|}{p^{et}} \cdot \frac{\lambda(R/(I + J^{[p^e]}))}{p^{e(d-t)}}\\ &= \frac{|\operatorname{V}^J_{\underline{f}}(p^e)|}{p^{et}} \cdot \frac{\lambda(\overline{R}/J^{[p^e]})}{p^{e(d-t)}}.\end{aligned}$$

After taking the limit as $e \to \infty$, we obtain the desired inequality. □

We recall a conjecture that relates the Hilbert-Kunz multiplicity and $F$-thresholds.

**Conjecture 4.4.2** ([NnBS20]). Let $(R, \mathfrak{m}, K)$ be a local ring. Let $I \subseteq R$ be an ideal generated by a part of a system of parameters $(f_1, \ldots, f_\ell)$. Let $\overline{R} = R/I$. Let $J$ be an $m$-primary ideal. Then,

$$\operatorname{e}_{HK}(J) \leq \operatorname{e}_{HK}(J\overline{R})\frac{(c^J(I))^\ell}{\ell^\ell}.$$

**Remark 4.4.3.** Related inequalities to Conjecture 4.4.2 that were previously obtained [NnBS20] are the following

$$\operatorname{e}_{HK}(J) \leq \operatorname{e}_{HK}(J\overline{R})\frac{(c^J(I))^\ell}{\ell!},$$

and

$$\operatorname{e}_{HK}(J) \leq \operatorname{e}_{HK}(J\overline{R})c^J(f_1)\cdots c^J(f_\ell).$$

By Proposition 4.2.6,

$$\operatorname{Vol}^J_F(\underline{f}) \leq \frac{(c^J(I))^\ell}{\ell!}.$$

By Proposition 4.2.1(3), we have that

$$\operatorname{Vol}^J_F(\underline{f}) \leq c^J(f_1)\cdots c^J(f_\ell).$$

Therefore, Theorem 4.4.1 is a refinement of the results previously mentioned.

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