CIMAT Centro de Investigación en Matemáticas, A.C.

On F-pure thresholds: computations and relations to others invariants

T H E S I S

To obtain the degree of

Master of Science in Mathematics

by

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Autorización de la versión final

Guanajuato, Gto., 15 de Julio de 2019

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Acknowledgements

I would like to thank my parents, Gladys Velez and Rodrigo Jaramillo, for their unconditional love and support. I also thank my brother, Rodrigo Andres Jaramillo Velez. Without his mentorship, I would never have made it to where I am now. Additionally, I would like to thank my sister-in-law, Yuli Vargas, and my uncle Horacio Velez. They complete a wonderful family.

Without a doubt, Fernanda Lemus represents my best moments in Mexico. I thank Fernanda for filling the void left by my family when I left home, for sharing her thoughts and ideas with me, and for all the joy that she brings to my life.

I am especially grateful to Dr. Luis Núñez Betancourt. I thank him for having the patience to accept me as his student and for showing me the beauty of Commutative Algebra.

I would like to thank Dr. Manuel González Villa for many enlightening discussions, and for guiding me through the process of writing this thesis.

Furthermore, I would like to thank my reviewers, Dr. Daniel Hernández, Dr. Xavier Gomez-Mont, and Dr. Cristhian Garay, for taking the time to read and evaluate my thesis.

A special thanks to Santiago Arenas for our conversations and for being an excellent neighbour.

I thank the Consejo Nacional de Ciencia y Tecnología, CONACYT, for the scholarship granted to me during my studies from September 2017 to July 2019.

Guanajuato, Guanajuato
July 2019

Introduction

The main purpose of this thesis is to present several computations of an algebraic invariant of singularities in positive characteristic: the F-pure threshold of an ideal. This number can be viewed as a prime characteristic analog of the *log canonical threshold*, an invariant of singularities in characteristic 0 [35].

The *log canonical threshold* of a complex polynomial f at $\mathbf{0}$, is defined as

$$\mathbf{lct_0}(f) := \sup\left\{\lambda \in \mathbb{R}_+ \quad \middle| \quad \frac{1}{|f|^{2\lambda}} \quad \text{is integrable around} \quad \mathbf{0}\right\}.$$

It is worthwhile to mention the following algebra-geometric expression of the *log canonical threshold*. Given the resolution of the singularities, in the sense of Hironaka [21], the $\mathbf{lct_0}(f)$ is expressed as the minimum of the set of numbers attached to the exceptional divisor of the resolution.

If f has rational coefficients and $\mathbf{fpt}_{\mathfrak{m}}(f)$ denoted the F-pure threshold of the ideal $I = \langle f \rangle$, then we have the following relation [12, 31]:

$$\lim_{p\to\infty} \mathbf{fpt}_{\mathfrak{m}}(f_p) = \mathbf{lct_0}(f), \tag{1}$$

where f_p is a model over $\mathbb{F}_p$ of f after reducing its coefficients modulo p. Furthermore, the following relation is expected.

Conjecture A. [31, Conjecture 3.6]. *There exist infinitely many primes such that*

$$\mathbf{fpt}_{\mathfrak{m}}(f_p) = \mathbf{lct_0}(f).$$

In Chapter 1 we introduce some of the basic concepts of this work. We define the base p expansion of a number in $(0, 1)$. This definition plays an important role in the computations of the F-pure threshold of a hypersurface [14, 29, 27]. We also introduce the notions of test ideal, F-pure threshold, F-pure rings, and we show some of its basic properties [2]. We focus on the relation of the F-pure threshold with test ideals and F-pure rings [35, 10].

Chapter 2 is devoted to prove the following theorem.

Theorem B. (See Theorem 2.1.6). *Let* $f = g_1 + \cdots + g_l$ *such that* $g_i \in \mathbb{K}[x_{i1}, \ldots, x_{is_i}]$ *and* $\alpha = (\alpha_1, \ldots, \alpha_l) = (\mathbf{fpt}_{\mathfrak{m}_1}(g_1), \ldots, \mathbf{fpt}_{\mathfrak{m}_l}(g_l))$. *If* $|\alpha| := \sum_i \alpha_i \leq 1$, *then*

$$\mathbf{fpt}_{\mathfrak{m}}(f) = \begin{cases} \alpha_1 + \cdots + \alpha_l, & \textit{if} \quad L = \infty, \\ \\ \langle \alpha_1 \rangle_L + \cdots + \langle \alpha_l \rangle_L + \frac{1}{p^L}, & \textit{if} \quad L < \infty, \end{cases}$$

where $\mathfrak{m} = \langle \quad x_{ij} \quad | 1 \leq i \leq l, 1 \leq j \leq \max\{s_1, \ldots, s_l\}\rangle$, $\mathfrak{m}_i = \langle x_{i1}, \ldots, x_{is_i}\rangle$, *and*

$$L = \sup\left\{N \in \mathbb{N} \quad \middle| \quad \alpha_1^{(e)} + \cdots + \alpha_l^{(e)} \leq p - 1 \quad \textit{for all} \quad 0 \leq e \leq N\right\}.$$

Theorem B is a generalization of the formula for the F-pure threshold of diagonal hypersurfaces due to D. Hernández [16]. Beside this, as consequence of Theorem B, we show that the F-pure threshold of $f = z^n + x^a y^b$ is the F-pure threshold of $z^n + y^b$ if $a \leq b$ (see Example 2.1.9). We also discuss the Conjecture A in these cases, and we show a new example where Conjecture A holds (see Example 2.2.3).

In Chapter 3 we compute the F- pure threshold of a determinantal ideal with maximal minors. Since a determinantal ideal is not a principal ideal, the computation of its F-pure threshold requires different methods and concepts than those used in Chapter 2. Some of these concepts are regular sequences, lexicographic orders and Gröbner basis.

Finally, we study the relation between F-pure threshold and the Bernstein-Sato polynomial in Chapter 4. M. Mustaţă, S.Takagi, and K-i. Watanabe [31] have shown how to compute some roots of the Bernstein-Sato polynomial using the F-pure threshold. We use this approach to compare the roots of the Bernstein-Sato polynomial of $f = z^n + x^a y^b$ with the roots of the Bernstein-Sato polynomial of a deformation of f in specific characteristics. In particular, we work with $f = z^4 + x^6 y^5$ and the deformation $g = z^4 + x^6 y^5 + x^5 y^4 z$ [28, Example 3.4].

Chapter 1

Background

The goal of this chapter is describe the F-pure threshold of an ideal and see the relation of this invariant with others concepts such as the test ideals and F-pure rings. See [2] and [19] for more details.

1.1 Expansions in base p

In this section we introduce for a prime number p, the notion of base p expansion of a number in $[0, 1]$, and some of its properties. These properties help us to compute the F-pure threshold of a hypersurface.

Definition 1.1.1. *Let $\alpha \in (0, 1]$, and p be a prime number. A non-terminating base p expansion of α is an expression $\alpha = \sum_{e \geq 1} \frac{\alpha^{(e)}}{p^e}$, with $0 \leq \alpha^{(e)} \leq p - 1$, such that for all $n > 0$, exists $e \geq n$ with $\alpha^{(e)} \neq 0$. The number $\alpha^{(e)}$ is unique and it is called the e^{th} digit of the non-terminating base p expansion of α.*

Example 1.1.2. [19]. *Let $\alpha = \frac{a}{b}$ be a rational number in $(0, 1]$. If p is a prime number such that $p \equiv 1 \mod b$, then $p = bw + 1$ for some $w \geq 1$. Dividing both sides by p, we get $1 = \frac{bw}{p} + \frac{1}{p}$, and multiplying both sides by $\alpha = \frac{a}{b}$ we have*

$$\alpha = \frac{aw}{p} + \frac{1}{p} \cdot \alpha.$$

As $a \leq b$, we have that $aw \leq bw = p - 1$. This shows that the non-terminating base p expansion of α is periodic, and it is given by $\alpha = \sum_{e \geq 1} \frac{aw}{p^e}$.

We now show how adding without carrying works for base 10. Let $a = a_1 10^1 + a_0 10^0$, $b = b_1 10^1 + b_0 10^0$ be two naturals numbers. For the computation of $a + b$, we start with the sum $a_2 + b_2$. If $a_2 + b_2 \geq 10$, then we add a number to the next sum $a_1 + b_1$. If $a_2 + b_2 < 10$, then the first digit of $a + b$ is $a_2 + b_2$ and we say that a_2, b_2 add without carrying. In the next definition we establish the notion of add without carrying for base p.

Definition 1.1.3. *Let $\alpha \in (0, 1]$ and fix a prime number p.*

(1) For $e \geq 1$, we define the e^{th} truncation of the non-terminating base p expansion of α by

$$\langle \alpha \rangle_e = \frac{\alpha^{(1)}}{p} + \cdots + \frac{\alpha^{(e)}}{p^e}.$$

We use the convention $\langle 0 \rangle_e = 0$.

(2) *For* $\overline{\alpha} = (\alpha_1, \ldots, \alpha_n) \in (0,1]^n$, *we set* $\langle \overline{\alpha} \rangle_e = (\langle \alpha_1 \rangle_e, \ldots, \langle \alpha_n \rangle_e)$.

(3) *Let* $(\alpha_1, \ldots, \alpha_n) \in (0,1]^n$. *We say that the non-terminating base* p *expansions of* $\alpha_1, \ldots, \alpha_n$ *add without carrying if*

$$\alpha_1^{(e)} + \cdots + \alpha_n^{(e)} \leq p - 1, \quad \textit{for every} \quad e \geq 1.$$

In such case, we say that $\alpha_1, \ldots, \alpha_n$ *add without carrying in base* p. *g. We say natural numbers* $k_1, \ldots, k_n$ *add without carrying (in base* p*) if the obvious condition holds.*

Lemma 1.1.4. *Let* $\alpha \in (0,1]$. *Then the following hold.*

(1) *If* $\alpha \notin \frac{1}{p^e} \cdot \mathbb{N}$, *then* $\lfloor p^e \alpha \rfloor = p^e \langle \alpha \rangle_e$.

(2) $\lceil p^e \alpha \rceil = p^e \langle \alpha \rangle_e + 1$.

(3) *Let* $(\alpha_1, \ldots, \alpha_n) \in (0,1]^n$. *If* $\alpha_1, \ldots, \alpha_n$ *add without carrying in base* p *and* $\alpha := \sum_{i=1}^n \alpha_i \leq 1$, *then* $\alpha^{(e)} = \alpha_1^{(e)} + \cdots + \alpha_n^{(e)}$.

Proof.

(1) Since $p^e \alpha = p^e \langle \alpha \rangle_e + p^e \sum_{d>e} \frac{\alpha^{(d)}}{p^d}$, we have that

$$\begin{aligned} \lfloor p^e \alpha \rfloor &= \lfloor p^e \langle \alpha \rangle_e + p^e \sum_{d>e} \frac{\alpha^{(d)}}{p^d} \rfloor \\ &= p^e \langle \alpha \rangle_e + \lfloor p^e \sum_{d>e} \frac{\alpha^{(d)}}{p^d} \rfloor \end{aligned}$$

Now, given that $\alpha \notin \frac{1}{p^e} \cdot \mathbb{N}$, we have that $0 < \sum_{d>e} \frac{\alpha^{(d)}}{p^d} < \frac{1}{p^e}$. Therefore,

$$\lfloor p^e \sum_{d>e} \frac{\alpha^{(d)}}{p^d} \rfloor = 0.$$

(2) As $\alpha^{(e)} \leq p - 1$ for all $e \geq 1$, it follows that $0 < \sum_{d>e} \frac{\alpha^{(d)}}{p^d} \leq \frac{1}{p^e}$, and so $\lceil p^e \sum_{d>e} \frac{\alpha^{(d)}}{p^d} \rceil = 1$. Thus,

$$\begin{aligned} \lceil p^e \alpha \rceil &= \lceil p^e \langle \alpha \rangle_e + p^e \sum_{d>e} \frac{\alpha^{(d)}}{p^d} \rceil \\ &= p^e \langle \alpha \rangle_e + \lceil p^e \sum_{d>e} \frac{\alpha^{(d)}}{p^d} \rceil = p^e \langle \alpha \rangle_e + 1. \end{aligned}$$

(3) We have that

$$\begin{aligned} \alpha = \alpha_1 + \cdots + \alpha_n &= \sum_{e \geq 1} \frac{\alpha_1^{(e)}}{p^e} + \cdots + \sum_{e \geq 1} \frac{\alpha_n^{(e)}}{p^e} \\ &= \sum_{e \geq 1} \frac{\alpha_1^{(e)} + \cdots + \alpha_n^{(e)}}{p^e}. \end{aligned}$$

As $\alpha_1, \ldots, \alpha_n$ add without carrying in base p, then $\sum_{i=1}^n \alpha_i^{(e)} \leq p - 1$ for all $e \geq 1$. Thus, by the uniqueness in the expansion in base p we conclude that $\alpha^{(e)} = \alpha_1^{(e)} + \cdots + \alpha_n^{(e)}$.

□

The following theorems have an important role in this thesis.

Lemma 1.1.5. [27, 29]. *Let* $\mathbf{k} = (k_1, \dots, k_n) \in \mathbb{N}^n$ *and set* $N = |\mathbf{k}| = \sum_{i=1}^{n} k_i$. *Then*

$$\begin{pmatrix} N \\ \mathbf{k} \end{pmatrix} := \frac{N!}{k_1! \cdots k_n!} \not\equiv 0 \mod p,$$

if and only if $k_1, \dots, k_n$ *add without carrying in base* p.

Theorem 1.1.6 (Dirichlet). *Given* n, m *non-zero natural numbers with* $\gcd(n, m) = 1$, *there exist infinitely many prime numbers* p *such that* $p \equiv n \mod m$.

Theorem 1.1.7 (Dirichlet). *For any collection* $\alpha_1, \dots, \alpha_n$ *of rational numbers, there exist infinitely many prime numbers* p *such that* $(p-1) \cdot \alpha_i \in \mathbb{N}$ *for* $1 \le i \le n$.

1.2 Test ideals

In this section we present the notion of test ideal, and we show some of its basic properties. These ideals were first introduced by M. Hochster and C. Huneke [22], and they are used in the study of singularities in algebraic geometry in positive characteristic.

Let R any commutative ring of prime characteristic p. The Frobenius map is the p^{th} power map

$$F : R \to R, \qquad r \mapsto r^p.$$

Since $(r+s)^p = r^p + s^p$ in characteristic p, the Frobenius map is a ring homomorphism.

Our goal is to use the Frobenius map to understand the singularities of the ring R.

Definition 1.2.1. *A ring* R *of positive characteristic* p *is said to be* F*-finite if* R *is a finitely generated module over its ring* $F(R) := R^p$.

Definition 1.2.2. *For every ideal* $I \subseteq R$, *let* $I^{[p^e]}$ *denotes the ideal generated by the set* $\left\{g^{p^e} | g \in I\right\}$ *which we call the* e^{th} *Frobenius power of* I.

Throughout this section, R denotes a regular F-finite ring of positive characteristic p.

Remark 1.2.3. *Since* R *is a regular ring, then the Frobenius map* $F : R \to R$ *is flat* [26]. *Moreover its* e*-iteration for a positive integer* $e \ge 1$,

$$F^e : R \to R, \qquad r \mapsto r^{p^e}.$$

is a flat morphism. Therefore,

$$(I^{[p^e]} :_R J^{[p^e]}) = (F^e(I) :_R F^e(J)) = F^e((I :_R J)) = (I :_R J)^{[p^e]}.$$

Let L be a fixed algebraic closure of the fraction field of R. We define the ring of p^e-roots by

$$R^{1/p^e} := \left\{ r \in L \quad | \quad r^{p^e} \in R \right\}.$$

The ring inclusion $R^{p^e} \subseteq R$ is equivalent to the ring inclusion $R \subseteq R^{1/p^e}$ given by

$$\begin{array}{ccc} R & \hookrightarrow & R^{1/p^e} \\ r & \mapsto & (r^{p^e})^{1/p^e}. \end{array}$$

Example 1.2.4. *Let* $R = \mathbb{F}_p[x]$. *Then* $R^{1/p} = \mathbb{F}_p[x^{1/p}]$.

Remark 1.2.5. *Let* $R = \mathbb{K}\left[x_1, \ldots, x_n\right]$ *the polynomial ring over a perfect field of positive characteristic* p, *and* $\mathfrak{m} = \langle x_1, \ldots, x_n \rangle$. *The ring* R *is finitely generated and free over* $R^{p^e} = \mathbb{K}\left[x_1^{p^e}, \ldots, x_n^{p^e}\right]$, *with basis* $\left\{\mu | \mu \notin \mathfrak{m}^{[p^e]}\right\}$.

Now, in order to define the test ideal we introduce the ideal $I^{[1/p^e]}$ for an ideal $I \subseteq R$, in the case that R is free over R^{p^e}.

Definition 1.2.6. [2, Proposition 2.5] Suppose that R is free over R^{p^e}. For an ideal $I = \langle f_1, \ldots, f_r \rangle \subseteq R$, let $I^{[1/p^e]}$ denotes the ideal

$$\langle a_{i,j} | i \leq r, j \leq s \rangle,$$

where $f_i = \sum_{j=1}^{s} a_{i,j}^{p^e} e_j$ and $e_1, \ldots, e_s$ is a base of R over R^{p^e}.

Remark 1.2.7. *Observe that if* S *is a multiplicative system in* R, *and if* $e_1, \ldots, e_s$ *generate* R *over* R^{p^e}, *then* $\frac{e_1}{1}, \ldots, \frac{e_s}{1}$ *generate* $S^{-1}R$ *over* $(S^{-1}R)^{p^e}$.

Lemma 1.2.8. *Suppose that* R *is free over* R^{p^e}. *Let* $I = \langle f_1, \ldots, f_r \rangle$ *and* $J = \langle g_1, \ldots, g_t \rangle$ *be ideals in* R.

(1) If $I \subseteq J$, *then* $I^{[1/p^e]} \subseteq J^{[1/p^e]}$.

(2) $\left(I^{\left[p^{e'}\right]}\right)^{[1/p^e]} = I^{\left[p^{e'}/p^e\right]} \subseteq \left(I^{[1/p^e]}\right)^{\left[p^{e'}\right]}$

(3) $J \subseteq I^{[p^e]}$ *if and only if* $J^{[1/p^e]} \subseteq I$.

Proof.

(1) Let $I^{[1/p^e]} = \langle a_{i,j} | i \leq r, j \leq s \rangle$ and $J^{[1/p^e]} = \langle b_{k,j} | k \leq t, j \leq s \rangle$. By definition we know that $J \subseteq \langle b_{k,j} \rangle^{[p^e]}$, then

$$f_i = \sum_{n=1}^{N} h_{i,n} b_{k_n,j_n}^{p^e}.$$

We express $h_{i,n}$ as a combination of $e_1, \ldots, e_s$,

$$h_{i,n} = \sum_{j=1}^{s} c_{i_j,n_j}^{p^e} e_j.$$

Replacing $h_{i,n}$, we have

$$f_i = \sum_{n=1}^{N} \left(\sum_{j=1}^{s} c_{i_j,n_j}^{p^e} e_j \right) b_{k_n,j_n}^{p^e} = \sum_{j=1}^{s} \left(\sum_{n=1}^{N} c_{i_j,n_j}^{p^e} b_{k_n,j_n}^{p^e} \right) e_j.$$

This shows that $a_{ij}^{p^e} = \sum_{n=1}^{N} c_{i_j,n_j}^{p^e} b_{k_n,j_n}^{p^e}$. Therefore $a_{ij} = \sum_{n=1}^{N} c_{i_j,n_j} b_{k_n,j_n} \in J^{[1/p^e]}$.

(2) This statement follows from the definition when $e' \geq e$, so we assume the $e \leq e'$. Let $e_1, \dots, e_s$ and $v_1, \dots, v_t$ basis for R over R^{p^e} and for R over $R^{p^{e-e'}}$ respectively. For each generator of I we have the following expressions

$$f_i^{p^{e'}} = \sum_{j=1}^{s} b_{i,j}^{p^e} e_j, \quad \text{for all} \quad i \in \{1, \dots, r\}, \tag{1.1}$$

$$f_i = \sum_{k=1}^{t} c_{i,k}^{p^{e-e'}} v_k, \quad \text{for all} \quad i \in \{1, \dots, r\}, \tag{1.2}$$

$$f_i^{p^{e'}} = \sum_{k=1}^{t} c_{i,k}^{p^e} v_k^{p^{e'}}, \quad \text{for all} \quad i \in \{1, \dots, r\}. \tag{1.3}$$

We also have that

$$v_k^{p^{e'}} = \sum_{j=1}^{s} d_{k,j}^{p^e} e_j \quad \text{for all} \quad k \in \{1, \dots, t\}.$$

Then, replacing $v_k^{p^{e'}}$ in (1.3) we obtain

$$\sum_{j=1}^{s} b_{i,j}^{p^e} e_j = f_i^{p^{e'}} = \sum_{k=1}^{t} c_{i,k}^{p^e} v_k^{p^{e'}} = \sum_{j=1}^{s} \left(\sum_{k=1}^{t} c_{i,k}^{p^e} d_{k,j}^{p^e} \right) e_j.$$

This implies, $b_{i,j} = \sum_{k=1}^{t} c_{i,j} d_{k,j}$. Therefore we conclude that

$$\left(I^{\left[p^{e'}\right]} \right)^{[1/p^e]} \subseteq I^{\left[p^{e'}/p^e\right]}.$$

In order to prove the equality, notice that

$$I \subseteq \left(I^{[1/p^e]} \right)^{[p^e]} \subseteq \left(\left(I^{[1/p^e]} \right)^{[p^{e'}]} \right)^{[p^{e-e'}]}.$$

So, we get

$$I^{[p^{e'}/p^e]} = I^{[p^{e-e'}]} \subseteq \left(\left(\left(I^{[1/p^e]} \right)^{[p^{e'}]} \right)^{[p^{e-e'}]} \right)^{[1/p^{e-e'}]} \subseteq \left(I^{[1/p^e]} \right)^{[p^{e'}]}.$$

For the last inclusion. If $f_i = \sum_{j=1}^{s} a_{i,j}^{p^e} e_j$ and $e_j = \sum_{k=1}^{t} g_{j,k}^{p^{e-e'}} v_k$, then

$$\begin{aligned} \sum_{k=1}^{t} c_{i,k}^{p^{e-e'}} v_k = f_i &= \sum_{j=1}^{s} a_{i,j}^{p^e} e_j \\ &= \sum_{k=1}^{t} \left(\sum_{j=1}^{s} \left(a_{i,j}^{p^{e'}} g_{j,k} \right)^{p^{e-e'}} \right) v_k. \end{aligned}$$

Thus $c_{i,k} = \sum_{j=1}^{s} a_{i,j}^{p^{e'}} g_{j,k}$ and

$$I^{\left[p^{e'}/p^e\right]} \subseteq \left(I^{[1/p^e]} \right)^{\left[p^{e'}\right]}.$$

(3) First assume that $J \subseteq I^{[p^e]}$. If we apply (1) and (2), then we get

$$J^{[1/p^e]} \subseteq \left(I^{[p^e]}\right)^{[1/p^e]} = I.$$

On other hand, if $J^{[1/p^e]} \subseteq I$ then

$$J \subseteq \left(J^{[1/p^e]}\right)^{[p^e]} \subseteq I^{[p^e]}.$$

□

The following lemma shows that the formation of the ideal $I^{[1/p^e]}$ commutes with localization.

Lemma 1.2.9. *Suppose that R is free over R^{p^e}. Let I be an ideal of R and S a multiplicative system in R. Then $(S^{-1}I)^{[1/p^e]} = S^{-1}\left(I^{[1/p^e]}\right)$.*

Proof. Since $I \subseteq \left(I^{[1/p^e]}\right)^{[p^e]}$, we deduce

$$S^{-1}I \subseteq S^{-1}\left(\left(I^{[1/p^e]}\right)^{[p^e]}\right) = \left(S^{-1}I^{[1/p^e]}\right)^{[p^e]}.$$

Therefore, by Lemma 1.2.8, $(S^{-1}I)^{[1/p^e]} \subseteq S^{-1}\left(I^{[1/p^e]}\right)$.

For the reverse inclusion, write $(S^{-1}I)^{[1/p^e]} = S^{-1}J$, for some ideal J such that $(J : s) = J$ for every $s \in S$. The Remark 1.2.3 implies that

$$\left(J^{[p^e]} : s\right) = \left(J^{[p^e]} : s^{p^e}\right) = J^{[p^e]}.$$

Now, from Lemma 1.2.8 (2) we obtain that

$$S^{-1}I = \left(S^{-1}I\right)^{[p^e/p^e]} \subseteq \left(\left(S^{-1}I\right)^{[1/p^e]}\right)^{[p^e]} \subseteq \left(S^{-1}J\right)^{[p^e]} = S^{-1}\left(J^{[p^e]}\right).$$

Then, it follows that $I \subseteq J^{[p^e]}$ and again Lemma 1.2.8 gives $I^{[1/p^e]} \subseteq J$. Therefore,

$$S^{-1}\left(I^{[1/p^e]}\right) \subseteq S^{-1}J = \left(S^{-1}I\right)^{[1/p^e]}.$$

□

Remark 1.2.10. *If $(R, \mathfrak{m})$ is a regular local ring that is F-finite , then its completion $\hat{R}$ is also regular and F-finite. In fact, If $e_1, \ldots, e_s$ is a basis of R over R^{p^e}, then these elements also is a basis of $\hat{R}$ over $(\hat{R})^{p^e}$. Therefore, Definition 1.2.6 implies that*

$$\left(I\hat{R}\right)^{[1/p^e]} = \left(I^{[1/p^e]}\right)\hat{R}, \quad \textit{for an ideal } I \textit{ in } R.$$

The definition of test ideal was introduced by N. Hara and K-I. Yoshida [13], as a positive characteristic analog of multiplier ideals. Furthermore, The test ideal is an ideal reflecting the Frobenius properties of a prime characteristic ring. For example, the test ideal defines the closed locus of Spec(R) consisting of points $\mathfrak{p}$ at which $R_{\mathfrak{p}}$ is not a F-regular. In this notes, we give the definition of M. Blickle, M. Mustaţă, and K. Smith [2].

Definition 1.2.11. *Suppose that R is free over R^{p^e}. Let I be an ideal of R, c a positive real number, and e a positive integer. Then, the test ideal of I with exponent c is*

$$\tau\left(I^c\right) = \bigcup_{e \geq 1} \left(I^{\lceil cp^e \rceil}\right)^{[1/p^e]}.$$

Lemma 1.2.12. [2, Lemma 2.8] Suppose that R is free over R^{p^e}. For an ideal $I \subseteq R$, r, r' positive real numbers and $e' \geq e$ positive integers such that $\frac{r}{p^e} \geq \frac{r'}{p^{e'}}$, we have that

$$\left(I^r\right)^{[1/p^e]} \subseteq \left(I^{r'}\right)^{\left[1/p^{e'}\right]}.$$

Proof. From the hypothesis $\frac{r}{p^e} \geq \frac{r'}{p^{e'}}$, we have that $rp^{e'-e} \leq r'$. Hence $I^{rp^{e'-e}} \subseteq I^{r'}$. Lemma 1.2.8 (1) and (2) imply

$$\left(I^r\right)^{[1/p^e]} \subseteq \left(I^{rp^{e'-e}}\right)^{\left[1/p^{e'}\right]} \subseteq \left(I^{r'}\right)^{\left[1/p^{e'}\right]}.$$

□

Remark 1.2.13. *Let c be a positive real number. Then, we obtain*

$$\frac{\lceil cp^e \rceil}{p^e} \geq \frac{\lceil cp^{e+1} \rceil}{p^{e+1}}, \quad \textit{for every positive integer } e.$$

This implies that

$$\left(I^{\lceil cp^e \rceil}\right)^{[1/p^e]} \subseteq \left(I^{\lceil cp^{e+1} \rceil}\right)^{\left[1/p^{e+1}\right]}.$$

Since R is a Noetherian ring, we deduce that the union in the definition of test ideal stabilizes after finitely many steps. This means that the test ideal $\tau(I^c)$ is equal to $\left(I^{\lceil cp^e \rceil}\right)^{[1/p^e]}$ for all sufficiently large e.

Remark 1.2.14. *Suppose that R is free over R^{p^e}, and R is a local ring with maximal ideal $\mathfrak{m}$. If $d = \dim(R)$, and λ is a positive real number such that $\lambda > d$, then $\lceil p^e \lambda \rceil > p^e d$ for all $e \in \mathbb{N}$. Now, if $e >> 0$ the Pigeonhole principle implies*

$$\begin{aligned}\tau\left(\mathfrak{m}^{\lambda}\right) = \left(\mathfrak{m}^{\lceil p^e \lambda \rceil}\right)^{[1/p^e]} &\subseteq \left(\mathfrak{m}^{p^e d}\right)^{[1/p^e]} \\ &\subseteq \left(\mathfrak{m}^{[p^e]}\right)^{[1/p^e]} \\ &\subseteq \mathfrak{m}.\end{aligned}$$

Therefore, we conclude that $\tau\left(\mathfrak{m}^{\lambda}\right) \neq S$ for all $\lambda > d$.

For an ideal I, we present an example of how to compute the ideal $\left(I^{\lceil p^e c \rceil}\right)^{[1/p^e]}$ and its test ideal for a positive real number c.

Example 1.2.15. *Let $R = \mathbb{F}_2[x, y, z]$ and $f = x^2 + y^5 + z^5$. We compute the test ideal of $I = \langle f \rangle$ with exponent $c = \frac{1}{2}$. In this case we denote $\tau(I^{1/2})$ by $\tau(f^{1/2})$. Since R is free over R^2, we write f as combination of the elements of the basis $\left\{\mu | \mu \notin \langle x, y, z \rangle^{[2]}\right\}$. This is,*

$$f = (x)^2(1) + (y^2)^2(y) + (z^2)^2(z).$$

Then, by Definition 1.2.6, we conclude that $I^{[1/2]} = (f)^{[1/2]} = \langle x, y^2, z^2\rangle$. *For* e *sufficiently large the test ideal of* I *with exponent* $1/2$ *is equal to*

$$\tau(f^{1/2}) = (f^{\lceil 2^e(1/2)\rceil})^{[1/2^e]} = (f^{2^{e-1}})^{[1/2^e]}$$

Now, by Lemma 1.2.8, we have that $\tau(I^{1/2}) = \tau(f^{1/2}) = (f^{2^{e-1}})^{[1/2^e]} = (f)^{[1/2]} = \langle x, y^2, z^2\rangle$.

Proposition 1.2.16. *Suppose that* R *is free over* R^{p^e}, *and let* I *and* J *be ideals of* R.

(1) *If* $r_1 \leq r_2$, *then* $\tau(I^{r_2}) \subseteq \tau(I^{r_1})$.

(2) *If* $I \subseteq J$, *then* $\tau(I^c) \subseteq \tau(J^c)$.

Proof.

(1) Since $r_1 \leq r_2$, we deduce $\lceil r_1 p^e\rceil \leq \lceil r_2 p^e\rceil$ and $I^{\lceil r_2 p^e\rceil} \subseteq I^{\lceil r_1 p^e\rceil}$. Lemma 1.2.8 (1) implies that

$$\left(I^{\lceil r_2 p^e\rceil}\right)^{[1/p^e]} \subseteq \left(I^{\lceil r_1 p^e\rceil}\right)^{[1/p^e]}$$

By taking the limit $e \to \infty$, we get the assertion.

(2) If $I \subseteq J$, then $I^{\lceil cp^e\rceil} \subseteq J^{\lceil cp^e\rceil}$. Lemma 1.2.8 gives that

$$\left(I^{\lceil cp^e\rceil}\right)^{[1/p^e]} \subseteq \left(J^{\lceil cp^e\rceil}\right)^{[1/p^e]}$$

By taking the limit $e \to \infty$ we conclude

$$\tau(I^c) \subseteq \tau(J^c).$$

□

A direct application of Lemma 1.2.9 shows that the formation of test ideals commutes with localization.

Proposition 1.2.17. *Let* I *be an ideal of* R, *and* S *a multiplicative system in* R, *then* $\tau((S^{-1}I)^c) = S^{-1}\tau(I^c)$.

Proof. For e sufficiently large,

$$\tau((S^{-1}I)^c) = \left((S^{-1}I)^{\lceil cp^e\rceil}\right)^{[1/p^e]} = \left(S^{-1}\left(I^{\lceil cp^e\rceil}\right)\right)^{[1/p^e]} = S^{-1}\left(I^{\lceil cp^e\rceil}\right)^{[1/p^e]} = S^{-1}\left(\tau(I^c)\right).$$

□

Remark 1.2.18. *Suppose that* $(R, \mathfrak{m})$ *is a local ring and let* $\hat{R}$ *be its completion. Remark 1.2.10 implies that*

$$\tau\left(\left(I\hat{R}\right)^c\right) = \tau\left(I^c\right)\hat{R}, \quad \textit{for an ideal } I \textit{ in } R.$$

1.3 F-pure threshold

This section introduces the F-pure threshold of an ideal. We show a few properties of the F-pure threshold and we discuss its relation with test ideals [31, 35, 10].

Throughout this section, R denotes the polynomial ring $\mathbb{K}[x_1,\dots,x_n]$ over a perfect field $\mathbb{K}$ of positive characteristic p, and $\mathfrak{m}$ denotes the maximal ideal $\langle x_1,\dots,x_n\rangle$.

Definition 1.3.1. *Let I be a fixed ideal, such that $(0)\neq I\subseteq\mathfrak{m}$. For I and every positive integer e we define*

$$\nu_I^{\mathfrak{m}}(p^e) := \max\left\{l\in\mathbb{N}|I^l\not\subseteq\mathfrak{m}^{[p^e]}\right\}. \tag{1.4}$$

Remark 1.3.2. *If I is an ideal of R, then $f^{p^e}\in I^{[p^e]}$ if and only if $f\in I$. This follows because R is finitely generated and free over R^{p^e}.*

Remark 1.3.3. *For a non-zero proper ideal $I\subseteq\mathfrak{m}\subseteq R$ generated by r elements, and a positive integer e we have*

$$I^{r(p^e-1)+1}\subseteq I^{[p^e]}\subseteq\mathfrak{m}^{[p^e]}$$

Therefore,

$$0\leq\nu_f^{\mathfrak{m}}(p^e)<r(p^e-1)+1. \tag{1.5}$$

Lemma 1.3.4. *For every proper ideal $I\in\mathfrak{m}\subset R$ and every non-negative integer e, we have*

$$p\nu_I^{\mathfrak{m}}(p^e)\leq\nu_I^{\mathfrak{m}}(p^{e+1}). \tag{1.6}$$

Proof. The Definition 1.3.1 tells us that $I^{\nu_I^{\mathfrak{m}}(p^e)}\not\subseteq\mathfrak{m}^{[p^e]}$, then there exists an element $f\in I^{\nu_I^{\mathfrak{m}}(p^e)}$ such that $f\notin\mathfrak{m}^{[p^e]}$. Now, since $f^p\in I^{p\nu_I^{\mathfrak{m}}(p^e)}$ and remark 1.3.2 implies that $f^p\notin\left(\mathfrak{m}^{[p^e]}\right)^{[p]}=\mathfrak{m}^{[p^{e+1}]}$ we conclude that

$$I^{p\nu_I^{\mathfrak{m}}(p^e)}\not\subseteq\left(\mathfrak{m}^{[p^e]}\right)^{[p]}=\mathfrak{m}^{[p^{e+1}]}$$

and this concludes the proof.

□

From the Lemma 1.3.4 and Remark 1.3.3 follows that $\left\{\frac{\nu_I^{\mathfrak{m}}(p^e)}{p^e}\right\}_{e\geq0}$ is a non-decreasing and bounded sequence, therefore its limit exists.

Definition 1.3.5. *The F-pure threshold of a non-zero proper ideal $I\subset\mathfrak{m}$ is*

$$\mathbf{fpt}(I):=\lim_{e\to\infty}\frac{\nu_I^{\mathfrak{m}}(p^e)}{p^e}=\sup_e\frac{\nu_I^{\mathfrak{m}}(p^e)}{p^e}. \tag{1.7}$$

It is known that $\mathbf{fpt}(I)$ is a rational number [2].

In the case of a hypersurface $f\in\mathfrak{m}\subset R$, we denote the F-pure threshold of $I=\langle f\rangle$ by $\mathbf{fpt}_{\mathfrak{m}}(f)$ and $\nu_I^{\mathfrak{m}}(p^e)$ by $\nu_f^{\mathfrak{m}}(p^e)$. This means that

$$\mathbf{fpt}_{\mathfrak{m}}(f)=\mathbf{fpt}(I).$$

Example 1.3.6. *If $f = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, where $\frac{1}{\alpha_i} \notin \frac{1}{p^e} \cdot \mathbb{N}$ for $i \in \{1, \ldots, n\}$, then*

$$f^l = x_1^{l \cdot \alpha_1} \cdots x_n^{l \cdot \alpha_n} \notin \mathfrak{m}^{[p^e]} \Leftrightarrow l \leq p^e \cdot a, \quad \text{where} \quad a = \min\left\{\frac{1}{\alpha_1}, \ldots, \frac{1}{\alpha_n}\right\}.$$

Thus, $\nu_f^{\mathfrak{m}}(p^e) = \lfloor p^e a \rfloor$. Finally, Lemma 1.1.4 implies that

$$\mathbf{fpt}_{\mathfrak{m}}(f) = \lim_{e \to \infty} \frac{\nu_f^{\mathfrak{m}}(p^e)}{p^e} = \lim_{e \to \infty} \frac{\lfloor p^e a \rfloor}{p^e} = \lim_{e \to \infty} \frac{p^e \langle a \rangle_e}{p^e} = a.$$

It is known that $\mathbf{lct}_0(x_1^{\alpha_1} \cdots x_n^{\alpha_n}) = \min\left\{\frac{1}{\alpha_1}, \ldots, \frac{1}{\alpha_n}\right\}$ [1].

The next lemma shows how the F-pure threshold relates to test ideal. This connection is a reflection of the relation between the log canonical and the multiplier ideal in zero characteristic.

Lemma 1.3.7. *For a non-zero proper ideal $I \subseteq \mathfrak{m}$, we have*

$$\mathbf{fpt}(I) = \sup\{c \in \mathbb{R}_{\geq 0} | \tau(I^c) = R\}$$

Proof. Let $a = \mathbf{fpt}(I)$ be the F-pure threshold of I. Notice that for every $\epsilon > 0$, we have that $a - \epsilon \leq \frac{\nu_I^{\mathfrak{m}}(p^e)}{p^e}$ for e sufficiently large. Then

$$\lceil p^e(a - \epsilon) \rceil \leq \nu_I^{\mathfrak{m}}(p^e)$$

and $I^{\lceil p^e(a-\epsilon) \rceil} \nsubseteq \mathfrak{m}^{[p^e]}$. Lemma 1.2.8 (3) implies that $\left(I^{\lceil p^e(a-\epsilon) \rceil}\right)^{[1/p^e]} \nsubseteq \mathfrak{m}$. Hence $\tau(I^{a-\epsilon}) = R$. Now, if $\lambda \in \{c \in \mathbb{R}_{\geq 0} | \tau(I^c) = R\}$, then $\tau(I^\lambda)_{\mathfrak{m}} = R_{\mathfrak{m}}$. This implies that $\left(I^{\lceil p^e \lambda \rceil}\right)^{[1/p^e]} \nsubseteq \mathfrak{m}$; therefore, we get

$$p^e \lambda - 1 \leq \lceil p^e \lambda \rceil \leq \nu_I^{\mathfrak{m}}(p^e)$$

by Lemma 1.2.8 (3). Dividing by p^e and taking the limit $e \to \infty$, it follows that $\lambda \leq \mathbf{fpt}(I)$. This implies the assertion. □

Theorem 1.3.8. *If $I \subseteq \mathfrak{m}$ is a non-zero ideal, then*

$$\mathbf{fpt}(I) \leq \mathrm{ht(I)}$$

Proof. Let λ be a positive real number such that $\lambda \geq \mathrm{ht(I)}$. For a prime ideal Q containing I with $\lambda \geq \mathrm{ht(Q)}$. We have $\tau\left((QR_Q)^\lambda\right) \neq R_Q$ by Remark 1.2.14. Note that $IR_Q \subseteq QR_Q$. Then, Proposition 1.2.16 implies that

$$\tau\left(I^\lambda\right) \subseteq \tau\left((IR_Q)^\lambda\right) \cap R \subseteq \tau\left((QR_Q)^\lambda\right) \cap R \neq R$$

for all $\lambda \geq \mathrm{ht(I)}$. Finally from Lemma 1.3.7 we get the conclusion. □

Lemma 1.3.9. *If $I \subseteq J \subseteq \mathfrak{m}$ are non-zero ideals, then*

$$\mathbf{fpt}(I) \leq \mathbf{fpt}(J)$$

Proof. Since $I \subseteq J$, we obtain that

$$I^{\nu_J^{\mathfrak{m}}(p^e)+1} \subseteq J^{\nu_J^{\mathfrak{m}}(p^e)+1} \subseteq \mathfrak{m}^{[p^e]}.$$

Thus $\nu_I^{\mathfrak{m}}(p^e) < \nu_J^{\mathfrak{m}}(p^e) + 1$. Finally, we conclude that

$$\mathbf{fpt}(I) = \lim_{e\to\infty} \frac{\nu_I^{\mathfrak{m}}(p^e)}{p^e} \leq \mathbf{fpt}(I) = \lim_{e\to\infty} \frac{\nu_J^{\mathfrak{m}}(p^e)}{p^e} + \lim_{e\to\infty} \frac{1}{p^e} = \mathbf{fpt}(J).$$

□

Lemma 1.3.10. *For every $f \in \mathfrak{m} \subset R$ and every non-negative integer e, we have*

$$p\nu_f^{\mathfrak{m}}(p^{e-1}) + \nu_f^{\mathfrak{m}}(p) \leq \nu_f^{\mathfrak{m}}(p^e) \tag{1.8}$$

Proof. In this proof we use that the Frobenius morphism is flat (Remark 1.2.3) and by simplicity we denote $\nu_f^{\mathfrak{m}}(p^e)$ by ν_e. Then, it follows from the next implications

$$\begin{aligned} f^{\nu_{e-1}} \notin \mathfrak{m}^{[p^{e-1}]} &\Leftrightarrow \left(\mathfrak{m}^{[p^{e-1}]} : f^{\nu_{e-1}}\right) \subseteq \mathfrak{m} \\ &\Rightarrow \left(\mathfrak{m}^{[p^{e-1}]} : f^{\nu_{e-1}}\right)^{[p]} \subseteq \mathfrak{m}^{[p]} \\ &\Rightarrow \left(\mathfrak{m}^{[p^e]} : f^{p\nu_{e-1}}\right) \subseteq \mathfrak{m}^{[p]} \\ &\Rightarrow \left(\left(\mathfrak{m}^{[p^e]} : f^{p\nu_{e-1}}\right) : f^{\nu_1}\right) \subseteq \left(\mathfrak{m}^{[p]} : f^{\nu_1}\right) \subseteq \mathfrak{m} \\ &\Rightarrow \left(\mathfrak{m}^{[p^e]} : f^{p\nu_{e-1}+\nu_1}\right) \subseteq \mathfrak{m} \\ &\Rightarrow f^{p\nu_{e-1}+\nu_1} \notin \mathfrak{m}^{[p^e]} \end{aligned}$$

The last implication gives that $p\nu_{e-1} + \nu_1 \leq \nu_e$ for all $e \geq 1$. □

Remark 1.3.11. *Using the Lemma 1.3.10 and induction, we obtain that*

$$\nu_f^{\mathfrak{m}}(p) = p - 1 \Leftrightarrow \nu_f^{\mathfrak{m}}(p^e) = p^e - 1, \quad \forall e \geq 1.$$

Proposition 1.3.12. [31] Let $f \in \mathfrak{m} \subset R$ be non-zero. For every integer $e \geq 1$ we have

$$\mathbf{fpt}_{\mathfrak{m}}(f) \leq \frac{\nu_f^{\mathfrak{m}}(p^e) + 1}{p^e}.$$

Proof. As $f^{\nu_f^{\mathfrak{m}}(p^e)+1} \in \mathfrak{m}^{[p^e]}$, we obtain $f^{p(\nu_f^{\mathfrak{m}}(p^e)+1)} \in \mathfrak{m}^{[p^{e+1}]}$, and so,

$$\begin{aligned} &\nu_f^{\mathfrak{m}}(p^{e+1}) \leq p(\nu_f^{\mathfrak{m}}(p^e) + 1) \\ &\frac{\nu_f^{\mathfrak{m}}(p^{e+1})}{p^{e+1}} \leq \frac{\nu_f^{\mathfrak{m}}(p^e) + 1}{p^e}, \quad \text{for every integer} \quad e \geq 1. \end{aligned}$$

Thus, this implies that $\mathbf{fpt}_{\mathfrak{m}}(f) \leq \frac{\nu_f^{\mathfrak{m}}(p^e)+1}{p^e}$.

□

Lemma 1.3.13. [19] Let $f \in \mathfrak{m} \subset R$ be a non-zero polynomial. For every positive integer e, we have

$$\langle \mathbf{fpt}_{\mathfrak{m}}(f) \rangle_e = \frac{\nu_f^{\mathfrak{m}}(p^e)}{p^e}$$

Proof. From the above proposition and the definition of *F-pure threshold*, we get

$$\frac{\nu_f^{\mathfrak{m}}(p^e)}{p^e} \leq \mathbf{fpt}_{\mathfrak{m}}(f) \leq \frac{\nu_f^{\mathfrak{m}}(p^e)+1}{p^e}$$
$$\nu_f^{\mathfrak{m}}(p^e) \leq p^e \mathbf{fpt}_{\mathfrak{m}}(f) \leq \nu_f^{\mathfrak{m}}(p^e)+1$$

Thus, $\lceil p^e \mathbf{fpt}_{\mathfrak{m}}(f) \rceil = \nu_f^{\mathfrak{m}}(p^e)+1$. Now, by Lemma 1.1.4, we have $\lceil p^e \mathbf{fpt}_{\mathfrak{m}}(f) \rceil = p^e \langle \mathbf{fpt}_{\mathfrak{m}}(f) \rangle_e + 1$, and this implies that

$$p^e \langle \mathbf{fpt}_{\mathfrak{m}}(f) \rangle_e = \nu_f^{\mathfrak{m}}(p^e).$$

□

Lemma 1.3.14. [15] Let $\mathfrak{m} = \langle x_1, \dots, x_n \rangle$ and $\mathfrak{n} == \langle y_1, \dots, y_n \rangle$ the homogeneous maximal ideals in $\mathbb{K}[x_1, \dots, x_n]$ and in $\mathbb{K}[y_1, \dots, y_l]$ respectively. If $f \in \mathfrak{m}$ and $g \in \mathfrak{n}$, then

$$\mathbf{fpt}_{\mathfrak{m}+\mathfrak{n}}(fg) = \min \{\mathbf{fpt}_{\mathfrak{m}}(f), \mathbf{fpt}_{\mathfrak{n}}(g)\}.$$

Proof. Suppose that $\mathbf{fpt}_{\mathfrak{m}}(f) = \beta_1 \leq \beta_2 = \mathbf{fpt}_{\mathfrak{n}}(g)$. Then $\langle \beta_1 \rangle_e \leq \langle \beta_2 \rangle_e$ for all $e \geq 1$. Hence, by Lemma 1.3.13 we obtain $f^{p^e \langle \beta_1 \rangle_e} \notin \mathfrak{m}^{[p^e]}$ and $g^{p^e \langle \beta_1 \rangle_e} \notin \mathfrak{n}^{[p^e]}$. Since f and g are in different sets of variables, $(fg)^{p^e \langle \beta_1 \rangle_e} \notin (\mathfrak{m}+\mathfrak{n})^{[p^e]}$. Lemma 1.3.13 also implies that $f^{p^e \langle \beta_1 \rangle_e + 1} \in \mathfrak{m}^{[p^e]}$, thus we get $(fg)^{p^e \langle \beta_1 \rangle_e + 1} \in (\mathfrak{m}+\mathfrak{n})^{[p^e]}$. Finally, we conclude that

$$p^e \langle \beta_1 \rangle_e = \nu_{fg}^{\mathfrak{m}+\mathfrak{n}}(p^e) = p^e \left\langle \mathbf{fpt}_{\mathfrak{m}+\mathfrak{n}}(fg) \right\rangle_e.$$

The claim follows by letting $e \to \infty$.

□

1.4 *F*-pure rings

The notion of F-pure ring are defined by M. Hochster and J. Roberts by using the Frobenius map in positive characteristic $p > 0$ [23]. This concept have some similarity to the approach of rational singularities defined for singularities of characteristic zero.

Definition 1.4.1. *Let $T \to S$ be any homomorphism of rings. Considering S as a T-module via restriction of scalars. We say that $T \to S$ splits if there is a T-module map $S \to T$ such that the composition*

$$T \to S \to T$$

is the identity map on T. Equivalently, $T \to S$ splits if there exists $\phi \in \operatorname{Hom}_T(S, T)$ such that $\phi(1) = 1$.

Definition 1.4.2. *Let R be a regular F-finite ring of positive characteristic p. R is said to be F-pure (F-split) if the inclusion $R \subseteq R^{1/p^e}$ splits as a map of R-modules for some $e \geq 1$.*

For $f \in R$ and m positive integer we have that the inclusion $R \cdot f^{m/p^e} \subset R^{1/p^e}$ splits (as map of R-modules) if there exists an map $\phi \in \mathrm{Hom_R(R^{1/p^e}, R)}$ with $\phi(f^{m/p^e}) = 1$

Theorem 1.4.3 (Fedder's criterion)**.** *Suppose that $(R, \mathfrak{m})$ is a F-finite regular local ring, or else a polynomial ring over a perfect field and its homogeneous maximal ideal. Then R/I is F-pure if and only if*

$$\left(I^{[p^e]} : I\right) \not\subseteq \mathfrak{m}^{[p^e]}$$

Example 1.4.4. *Let R be the polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$ over a perfect field $\mathbb{K}$ of positive characteristic p, and $\mathfrak{m} = \langle x_1, \ldots, x_n \rangle$. If $I \subseteq R$ is a square-free monomial ideal, then*

$$\left(I^{[p]} : I\right) = \left\{f \in R | fI \subseteq I^{[p]}\right\} \subseteq \left\langle x_1^{p-1}, \ldots, x_n^{p-1}\right\rangle \not\subseteq \mathfrak{m}^{[p]}.$$

Therefore, Fedder's criterion implies that R/I is a F-pure ring. The ring R/I is called a Stanley–Reisner ring

Theorem 1.4.5. *Let R be the polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$ over a perfect field $\mathbb{K}$ of positive characteristic p, and $\mathfrak{m} = \langle x_1, \ldots, x_n \rangle$. For a hypersurface $f \in \mathfrak{m} \subseteq R$. The ring $R/\langle f \rangle$ is F-pure if and only if $\mathbf{fpt}_{\mathfrak{m}}(f) = 1$*

Proof. The Fedder's Criterion tells us that $R/\langle f \rangle$ is F-pure if and only if $f^{p-1} \in \mathfrak{m}^{[p]}$. By definition, $f^{p-1} \in \mathfrak{m}^{[p]}$ if and only if $\nu_f^{\mathfrak{m}}(p) = p - 1$. Then by Remark 1.3.11, we have that $R/\langle f \rangle$ is F-pure if and only if $\nu_f^{\mathfrak{m}}(p^e) = p^e - 1$. Finally, Definition 1.3.5 and Lemma 1.3.13 give us that $R/\langle f \rangle$ is F-pure if and only if $\mathbf{fpt}_{\mathfrak{m}}(f) = 1$. □

Example 1.4.6. *Let $R = \mathbb{K}[x, y, z]$ be the polynomial ring in three variables over a perfect field, and $\mathfrak{m} = \langle x, y, z \rangle$ its maximal ideal. If $p = 2$ and $f \in \mathfrak{m}$, then*

$$\begin{aligned} f \notin \left\langle x^2, y^2, z^2 \right\rangle &\Leftrightarrow v_f^m(p) = p - 1 \\ &\Leftrightarrow v_f^m(p^e) = p - 1, \quad \textit{By Remark 1.3.11} \\ &\Leftrightarrow \mathbf{fpt}_{\mathfrak{m}}(f) = 1, \quad \textit{By Definition of} \quad \mathbf{fpt}_{\mathfrak{m}}(f) \\ &\Leftrightarrow R/\langle f \rangle \quad \textit{is} \quad F - \textit{pure}. \end{aligned}$$

So, for example $f = x^2 - y^2 - z$ yields a F-pure ring.

Chapter 2

Computations on F-pure thresholds

The F-pure threshold of a polynomial f is a numerical invariant in positive characteristic, measuring the singularity of the hypersurface $I = \langle f \rangle$ at the origin. For instance, Theorem 1.4.5 tells us a criterion to decide if f is smooth at the origin. There are different questions around this invariant, some of them are: Is there a formula for the F-pure threshold of f as a function of the characteristic $p > 0$? Does Conjecture A holds for the polynomial f ? We are interested in obtaining an answer for these two questions, when $f = g_1 + \cdots + g_l$ is a polynomial, where $g_1, \ldots, g_l$ are polynomials in different sets of variables.

2.1 Statement of the Main Theorem

For concrete hypersurfaces there are formulas for their F-pure thresholds in terms of the characteristic. For example, elliptic curves [33], binomial hypersurfaces and diagonal hypersurfaces [15, 16]. Our mainly motivation to compute the F-pure threshold of $f = g_1 + \cdots + g_l$ is the formula for the F-pure threshold of a diagonal hypersurface due to D. Hernández.

Throughout this chapter, $\mathbb{K}$ is a perfect field of positive characteristic p, R denotes the polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$, and $\mathfrak{m}$ denotes the maximal ideal $\langle x_1, \ldots, x_n \rangle$.

Definition 2.1.1. *Let $\delta = (\delta_1 \ldots, \delta_n) \in \mathbb{N}^n$. For $f = \sum_{\delta \in \mathbb{N}^n} a_\delta \cdot x_1^{\delta_1} \cdots x_n^{\delta_n} \in R$ we define the support of f as*

$$\mathbf{supp}(f) := \left\{ x_1^{\delta_1} \cdots x_n^{\delta_n} \quad | \quad a_\delta \neq 0 \right\}$$

Lemma 2.1.2. *For $i \in \{1, 2\}$, let $\mathfrak{m}_i = \langle x_{i1}, \ldots, x_{in} \rangle$ and $\mathfrak{m} = \left\langle \underline{x_{i1}}, \underline{x_{i2}} \right\rangle$ the maximal ideals of the polynomial rings $R_i = \mathbb{K}[x_{i1}, \ldots, x_{in}]$ and $R = \mathbb{K}\left[\underline{x_{i1}}, \underline{x_{i2}}\right]$, respectively. If $f \in R_1$ and $g \in R_2$, then*

$$fg \notin \mathfrak{m}^{[p^e]} \text{ if and only if } f \notin \mathfrak{m}_1^{[p^e]} \text{ and } g \notin \mathfrak{m}_2^{[p^e]}.$$

Proof. Since $\mathfrak{m}_i \subseteq \mathfrak{m}$ we have that if $f \in \mathfrak{m}_1^{[p^e]}$ or $g \in \mathfrak{m}_2^{[p^e]}$, then

$$fg \in \mathfrak{m}_i^{[p^e]} \subseteq \mathfrak{m}^{[p^e]}.$$

Thus, if $fg \notin \mathfrak{m}^{[p^e]}$ then $f \notin \mathfrak{m}_1^{[p^e]}$ and $g \notin \mathfrak{m}_2^{[p^e]}$.

We assume that $f \notin \mathfrak{m}_1^{[p^e]}$ and $g \notin \mathfrak{m}_2^{[p^e]}$ and let $u_1 \in \mathbf{supp}(f)$ and $u_2 \in \mathbf{supp}(g)$ such that $u_i \notin \mathfrak{m}_i^{[p^e]}$. As u_1 and u_2 are in different sets of variables, we have $u_1 u_2 \in \mathbf{supp}(fg)$ and $u_1 u_2 \notin \mathfrak{m}^{[p^e]}$. Therefore,

$$fg \notin \mathfrak{m}^{[p^e]}.$$

□

Now, we present the formula for the F-pure threshold of a diagonal hypersurface. This formula gives us a systematic method to compute the F-pure threshold in this case.

Theorem 2.1.3. [16]. Consider the diagonal hypersurface given by $f = x_1^{d_1} + \cdots + x_n^{d_n} \in R$ with $(d_1, \ldots, d_n) \in \mathbb{N}^n$. Let

$$L = \sup\left\{ N \in \mathbb{N} \quad \middle| \quad \left(\frac{1}{d_1}\right)^{(e)} + \cdots + \left(\frac{1}{d_n}\right)^{(e)} \leq p-1 \quad \text{for all} \quad 0 \leq e \leq N \right\}.$$

Then, we have

$$\mathbf{fpt}_{\mathfrak{m}}(f) = \begin{cases} \frac{1}{d_1} + \cdots + \frac{1}{d_n}, & \text{if} \quad L = \infty, \\ \left\langle \frac{1}{d_1} \right\rangle_L + \cdots + \left\langle \frac{1}{d_n} \right\rangle_L + \frac{1}{p^L}, & \text{if} \quad L < \infty. \end{cases}$$

In next the example we use Theorem 2.1.3 to compute the F-pure threshold of a specific polynomial in different characteristics.

Example 2.1.4. *Let $f = x^2 + y^7 \in \mathbb{K}[x, y]$ and $\mathfrak{m} = \langle x, y \rangle$. We use the above theorem to compute $\mathbf{fpt}_{\mathfrak{m}}(f)$ for some p.*

If $p = 2$, then

$$\left(\frac{1}{2}\right)^{(e)} = \begin{cases} 0 & \textit{if} \quad e = 1 \\ 1 & \textit{if} \quad e > 1 \end{cases}, \qquad \left(\frac{1}{7}\right)^{(e)} = \begin{cases} 1 & \textit{if} \quad e = 3k, k \geq 1 \\ 0 & \textit{otherwise.} \end{cases}$$

Thus $L = 2 < \infty$, and

$$\mathbf{fpt}_{\mathfrak{m}}(f) = \frac{1}{2^2} + 0 + \frac{1}{2^2} = \frac{1}{2}.$$

If $p = 7$, then

$$\left(\frac{1}{2}\right)^{(e)} = 3 \quad \textit{for all} \quad e \geq 1, \qquad \left(\frac{1}{7}\right)^{(e)} = \begin{cases} 0 & \textit{if} \quad e = 1 \\ 6 & \textit{if} \quad e > 1. \end{cases}$$

Thus $L = 1 < \infty$, and

$$\mathbf{fpt}_{\mathfrak{m}}(f) = \frac{3}{7} + 0 + \frac{1}{7} = \frac{4}{7}.$$

If $p = 14w + 1$ for some $w \in \mathbb{N} \setminus \{0\}$, then

$$\left(\frac{1}{2}\right)^{(e)} = 7w \quad \textit{for all} \quad e \geq 1, \qquad \left(\frac{1}{7}\right)^{(e)} = 2w \quad \textit{for all} \quad e \geq 1.$$

This implies that $L = \infty$, and

$$\mathbf{fpt}_{\mathfrak{m}}(f) = \frac{1}{2} + \frac{1}{7} = \frac{9}{14}.$$

If $p = 14w + 5$ for some $w \in \mathbb{N} \setminus \{0\}$, then

$$\left(\frac{1}{2}\right)^{(e)} = 7w + 2 \quad \textit{for all} \quad e \geq 1, \qquad \frac{1}{7} = \frac{2w}{p} + \frac{10w+3}{p^2} + \frac{8w+2}{p^3} + \cdots.$$

This implies that $L = 1 < \infty$, *and*

$$\mathbf{fpt}_{\mathfrak{m}}(f) = \frac{7w+2}{p} + \frac{2w}{p} + \frac{1}{p} = \frac{9w+3}{p}.$$

Since $\frac{9w+3}{p} + \frac{3}{14p} = \frac{9}{14}$, *we obtain*

$$\mathbf{fpt}_{\mathfrak{m}}(f) = \frac{9}{14} - \frac{3}{14p}.$$

Finally, we have the following formula

$$\mathbf{fpt}_{\mathfrak{m}}(x^2+y^7) = \begin{cases} \frac{1}{2}, & \textit{if} \quad p=2, \\ \frac{4}{7}, & \textit{if} \quad p=7, \\ \frac{9}{14}, & \textit{if} \quad p \equiv 1 \mod 14, \\ \frac{9}{14} - \frac{3}{14p}, & \textit{if} \quad p \equiv 5 \mod 14. \end{cases} \tag{2.1}$$

Remark 2.1.5. *By Formulas* (1) *and* (2.1) *we have that* $\mathbf{lct}_0(x^2+y^7) = \frac{9}{14}$. *The Dirichlet's Theorem on arithmetic progressions* 1.1.6 *guarantees that there exists infinite prime numbers such that*

$$\mathbf{fpt}_{\mathfrak{m}}(x^2+y^7) = \frac{9}{14} = \mathbf{lct}_0(x^2+y^7).$$

Therefore, Conjecture A holds for $f = x^2 + y^7$.

The following theorem is a generalization of Theorem 2.1.3 and its proof uses methods from [16, 19].

Theorem 2.1.6 (Main Theorem)**.** *Let* $f = g_1 + \cdots + g_l$ *such that* $g_i \in \mathbb{K}[x_{i1}, \ldots, x_{is_i}]$ *and* $\alpha = (\alpha_1, \ldots, \alpha_l) = (\mathbf{fpt}_{\mathfrak{m}_1}(g_1), \ldots, \mathbf{fpt}_{\mathfrak{m}_l}(g_l))$. *If* $|\alpha| := \sum_i \alpha_i \leq 1$, *then*

$$\mathbf{fpt}_{\mathfrak{m}}(f) = \begin{cases} \alpha_1 + \cdots + \alpha_l, & \textit{if} \quad L = \infty, \\ \langle \alpha_1 \rangle_L + \cdots + \langle \alpha_l \rangle_L + \frac{1}{p^L}, & \textit{if} \quad L < \infty, \end{cases}$$

where $\mathfrak{m} = \langle \ x_{ij} \ \mid \ 1 \leq i \leq l, 1 \leq j \leq \max\{s_1, \ldots, s_l\}\rangle$, $m_i = \langle x_{i1}, \ldots, x_{is_i}\rangle$ *and*

$$L = \sup\left\{N \in \mathbb{N} \ \mid \ \alpha_1^{(e)} + \cdots + \alpha_l^{(e)} \leq p-1 \quad \textit{for all} \quad 0 \leq e \leq N\right\}.$$

Before discussing the proof of Theorem 2.1.6, let us state and prove the following preliminary result .

Lemma 2.1.7. *With the same hypothesis from the above theorem we have*

$$\mathbf{fpt}_{\mathfrak{m}}(f) \geq \langle \alpha_1 \rangle_L + \cdots + \langle \alpha_l \rangle_L + \frac{1}{p^L},$$

when $L < \infty$.

Proof. First note that by definition of L, $\alpha_1^{(L+1)} + \cdots + \alpha_l^{(L+1)} \geq p$. Moreover there exists $\delta = (\delta_1, \ldots, \delta_l) \in \mathbb{N}^l$ such that $\delta_1 + \cdots + \delta_l = p - 1$ with $0 \leq \delta_i \leq \alpha_i^{(L+1)}$ and $\delta_j < \alpha_j^{(L+1)}$ for some $j \in \{1, \ldots, n\}$.

We can assume that $\delta_1 < \alpha_1^{(L+1)}$. If we set

$$\eta(e) = (\eta_1(e), \ldots, \eta_l(e)) = \langle \alpha \rangle_L + \left(\frac{\delta_1}{p^{L+1}} + \frac{p-1}{p^{L+2}} + \cdots + \frac{p-1}{p^e}, \frac{\delta_2}{p^{L+1}}, \ldots, \frac{\delta_l}{p^{L+1}} \right),$$

then

$$\eta(e) = \langle \alpha \rangle_L + \left(\frac{1}{p^{L+1}} \cdot \left(\delta_1 + \frac{p^{e-(L+1)} - 1}{p^{e-(L+1)}} \right), \frac{\delta_2}{p^{L+1}}, \ldots, \frac{\delta_l}{p^{L+1}} \right).$$

Thus,

$$\begin{aligned} \eta_i(e) &\leq \langle \alpha_i \rangle_{L+1}, \quad \text{for} \quad 2 \leq i \leq l, \\ \eta_1(e) &< \langle \alpha_1 \rangle_{L+1} \end{aligned}$$

.

If $\prec$ denotes component-wise strict inequality, then $\mathbf{0} \prec \langle \alpha \rangle_e - \eta(e) \in \frac{1}{p^e} \cdot \mathbb{N}^l$ for $e \geq L+2$, where $\mathbf{0} = (0, \ldots, 0)$.

Now, since $\delta_1 + \cdots + \delta_l = p - 1$, and

$$\begin{aligned} p^e \eta_1(e) &= p^{e-(L+1)} \delta_1 + p^{e-(L+2)}(p-1) + \cdots + p - 1, \\ p^e \eta_2(e) &= p^{e-(L+1)} \delta_2, \\ &\vdots \\ p^e \eta_l(e) &= p^{e-(L+1)} \delta_l, \end{aligned}$$

we conclude that the elements $p^e \eta_1(e), \ldots, p^e \eta_l(e)$ add without carrying in base p. Therefore, by Lemma 1.1.5, we get

$$\binom{p^e |\eta(e)|}{p^e \eta(e)} \not\equiv 0 \mod p. \tag{2.2}$$

Since $\eta_i(e) \leq \langle \alpha_i \rangle_e$ for $e \geq L+2$ and all $i < l$, we obtain

$$p^e \eta_i(e) \leq p^e \langle \alpha_i \rangle_e = \nu_{g_i}^{m_i}(p^e).$$

Therefore,

$$g_i^{p^e \eta_i(e)} \notin \mathfrak{m_i}^{[p^e]}.$$

By Lemma 2.1.2, we have

$$g_1^{p^e \eta_1(e)} \cdots g_l^{p^e \eta_l(e)} \notin \mathfrak{m}^{[p^e]}. \tag{2.3}$$

Now from (2.2) and (2.3) we conclude that

$$f^{p^e |\eta(e)|} \notin \mathfrak{m}^{[p^e]},$$

and it follows that

$$\begin{aligned}
p^e \left\langle \mathbf{fpt}_{\mathfrak{m}}(f) \right\rangle_e = \nu_f^{\mathfrak{m}}(p^e) &\geq p^e |\eta(e)| \\
\left\langle \mathbf{fpt}_{\mathfrak{m}}(f) \right\rangle_e &\geq |\eta(e)| \\
&= |\left\langle \alpha \right\rangle_L| + \frac{1}{p^{L+1}}\left(\sum_{l=1}^{l} \delta_i\right) + \frac{p-1}{p^{L+2}} + \cdots + \frac{p-1}{p^e} \\
&= |\left\langle \alpha \right\rangle_L| + \frac{p-1}{p^{L+1}} + \frac{p-1}{p^{L+2}} + \cdots + \frac{p-1}{p^e}.
\end{aligned}$$

Finally, taking the limit when $e \to \infty$, we get

$$\begin{aligned}
\mathbf{fpt}_{\mathfrak{m}}(f) &\geq |\left\langle \alpha \right\rangle_L| + \frac{p-1}{p^{L+1}}\left(1 + \frac{1}{p} + \frac{1}{p^2} + \dots\right) \\
&= |\left\langle \alpha \right\rangle_L| + \frac{p-1}{p^{L+1}}\left(\frac{1}{1-1/p}\right) \\
&= |\left\langle \alpha \right\rangle_L| + \frac{p-1}{p^{L+1}}\left(\frac{p}{p-1}\right) \\
&= |\left\langle \alpha \right\rangle_L| + \frac{1}{p^L}.
\end{aligned}$$

□

We now show Theorem 2.1.6.

Proof of the Main Theorem.

Case 1: $L = \infty$.
Since $f^{\nu_f^{\mathfrak{m}}(p^e)} \notin \mathfrak{m}^{[p^e]}$, we conclude that there exists $\mathbf{k} = (k_1, \dots, k_l) \in \mathbb{N}^l$ such that $|\mathbf{k}| = \nu_f^{\mathfrak{m}}(p^e)$ and $g_1^{k_1} \cdots g_l^{k_l} \notin \mathfrak{m}^{[p^e]}$. Since the polynomials $g_1, \dots, g_l$ are in different sets of variables, by Lemma 2.1.2, we have

$$g_i^{k_i} \notin \mathfrak{m_i}^{[p^e]},$$

and it follows that $k_i \leq \nu_{g_i}^{\mathfrak{m}_i}(p^e)$ for all $i \in \{1, \dots, n\}$. Since

$$\begin{aligned}
\nu_f^{\mathfrak{m}}(p^e) = |\mathbf{k}| &\leq \sum_{i=1}^{l} \nu_{g_i}^{\mathfrak{m}_i}(p^e) \\
\frac{\nu_f^{\mathfrak{m}}(p^e)}{p^e} &\leq \sum_{i=1}^{l} \frac{\nu_{g_i}^{\mathfrak{m}_i}(p^e)}{p^e},
\end{aligned}$$

taking the limit when $e \to \infty$, we have

$$\begin{aligned}\mathbf{fpt}_{\mathfrak{m}}(f) = \lim_{e\to\infty} \frac{\nu_f^{\mathfrak{m}}(p^e)}{p^e} &\le \lim_{e\to\infty} \sum_{i=1}^{l} \frac{\nu_{g_i}^{\mathfrak{m}_i}(p^e)}{p^e} \\ &= \sum_{i=1}^{l} \lim_{e\to\infty} \frac{\nu_{g_i}^{\mathfrak{m}_i}(p^e)}{p^e} \\ &= \sum_{i=1}^{l} \alpha_i = |\alpha|.\end{aligned}$$

Now we check that $|\alpha| \le \mathbf{fpt}_{\mathfrak{m}}(f)$. Since $L = \infty$, the entries of α add without carrying in base p. Note that the coefficients of the expansion of $p^e \langle \alpha_i \rangle_e$ appears in the expansion of α_i in base p, hence we conclude that the entries of $p^e \langle \alpha \rangle_e$ add without carrying. By Lemma 1.1.5

$$\binom{p^e |\langle \alpha \rangle_e|}{p^e \langle \alpha \rangle_e} \not\equiv 0 \mod p. \tag{2.4}$$

Since

$$\left(\nu_{g_1}^{\mathfrak{m}_1}(p^e), \ldots, \nu_{g_l}^{\mathfrak{m}_l}(p^e)\right) = p^e \langle \alpha \rangle_e ,$$

we have

$$g_i^{p^e \langle \alpha_i \rangle_e} \notin \mathfrak{m_i}^{[p^e]} \quad \text{for all} \quad i.$$

Lemma 2.1.2 shows that

$$g_1^{p^e \langle \alpha_1 \rangle_e} \cdots g_l^{p^e \langle \alpha_l \rangle_e} \notin \mathfrak{m}^{[p^e]}. \tag{2.5}$$

Therefore, from (2.5) and (2.4), we get

$$f^{p^e |\langle \alpha \rangle_e|} \notin \mathfrak{m}^{[p^e]},$$

and, this implies that

$$p^e |\langle \alpha \rangle_e| \le \nu_f^{\mathfrak{m}}(p^e) \qquad \forall e >> 1$$

and, dividing by p^e

$$|\langle \alpha \rangle_e| \le \frac{\nu_f^{\mathfrak{m}}(p^e)}{p^e} \qquad \forall e >> 1.$$

Finally, taking the limit when $e \to \infty$, we get $|\alpha| \le \mathbf{fpt}_{\mathfrak{m}}(f)$.

Therefore, we deduce that

$$\mathbf{fpt}_{\mathfrak{m}}(f) = |\alpha|, \quad \text{if} \quad L = \infty.$$

Case 2: $L < \infty$.

By Lemma 2.1.7 we have that

$$\mathbf{fpt}_{\mathfrak{m}}(f) \ge \langle \alpha_1 \rangle_L + \cdots + \langle \alpha_l \rangle_L + \frac{1}{p^L}. \tag{2.6}$$

Reasoning by contradiction, we assume that the inequality in (2.6) is strict, then

$$\frac{\nu_f^{\mathfrak{m}}(p^e)}{p^e} > |\left\langle \alpha \right\rangle_L| + \frac{1}{p^L}, \quad e >> 1,$$

and multiplying by p^e,

$$\nu_f^{\mathfrak{m}}(p^e) > p^e|\left\langle \alpha \right\rangle_L| + p^{e-L} = p^{e-L}\left(p^L|\left\langle \alpha \right\rangle_L| + 1\right), \quad e >> 1.$$

Therefore,

$$f^{p^{e-L}\left(p^L|\langle\alpha\rangle_L|+1\right)} = \left(f^{p^L|\langle\alpha\rangle_L|+1}\right)^{p^{e-L}} \notin \mathfrak{m}^{[p^e]} = \left(\mathfrak{m}^{[p^L]}\right)^{[p^{e-L}]}.$$

By Remark 1.3.1, it follows that

$$f^{p^L|\langle\alpha\rangle_L|+1} \notin \mathfrak{m}^{[p^L]}. \tag{2.7}$$

This implies that there exists $\mathbf{k} = (k_1, \ldots, k_l) \in \mathbb{N}^l$ such that $|\mathbf{k}| = p^L|\left\langle \alpha \right\rangle_L| + 1$ and $g_1^{k_1} \cdots g_l^{k_l} \notin \mathfrak{m}^{[p^L]}$. The Lemma 2.1.2 implies that

$$g_i^{k_i} \notin \mathfrak{m}_i^{[p^L]}, \quad \forall i \in \{1 \ldots, l\} \tag{2.8}$$

from the previous formula we conclude that

$$k_i \leq \nu_{g_i}^{\mathfrak{m}_i}(p^L) = p^L \left\langle \alpha_i \right\rangle_L, \quad \forall i \in \{1 \ldots, l\}.$$

Thus,

$$\frac{1}{p^L} \cdot \mathbf{k} \preceq \left\langle \alpha \right\rangle_L;$$

since $|\mathbf{k}| = p^L|\left\langle \alpha \right\rangle_L| + 1$, we get

$$|\left\langle \alpha \right\rangle_L| + \frac{1}{p^L} = \frac{1}{p^L}|\mathbf{k}| \leq |\left\langle \alpha \right\rangle_L|,$$

which is a contradiction. Therefore, we obtain that the equality

$$\mathbf{fpt}_{\mathfrak{m}}(f) = |\left\langle \alpha \right\rangle_L| + \frac{1}{p^L} \tag{2.9}$$

holds.

□

Below we present some consequences of Theorem 2.1.6. For instance, we compute the F-pure threshold for another family polynomials.

Corollary 2.1.8. *For $i \in \{1, \ldots, n\}$, let $f_i \in \mathbb{K}[y_{i1}, \ldots, y_{il_i}]$ be polynomials in different sets of variables. Let $(d_1, \ldots, d_n) \in \mathbb{N}^n$, and let*

$$F = f_1 x_1^{d_1} + \cdots + f_n x_n^{d_n}.$$

If $\mathbf{fpt}_{\mathfrak{n}_i}(f_i) \geq \frac{1}{d_i}$ for all i, then

$$\mathbf{fpt}_{\mathfrak{n}}(F) = \mathbf{fpt}_{\mathfrak{m}}(x_1^{d_1} + \cdots + x_n^{d_n}),$$

where $\mathfrak{n}_i = \langle y_{i1}, \ldots, y_{il_i} \rangle$, $\mathfrak{m} = \langle x_1, \ldots, x_n \rangle$, and $\mathfrak{n} = \langle y_{ij}, x_i \rangle$.

Proof. It follows from Example 1.3.6 and Lemma 1.3.14 that $\mathbf{fpt}_{\mathfrak{n}_i+\langle x_i\rangle}(f_i x_i^{d_i}) = \frac{1}{d_i}$. If $g_i = f_i x_i^{d_i}$, then by Theorem 2.1.6 we have that

$$L = \sup\left\{N \in \mathbb{N} \quad \Big| \quad \left(\frac{1}{d_1}\right)^{(e)} + \cdots + \left(\frac{1}{d_n}\right)^{(e)} \leq p-1 \quad \text{for all} \quad 0 \leq e \leq N\right\},$$

and,

$$\mathbf{fpt}_{\mathfrak{n}}(F) = \begin{cases} \frac{1}{d_1} + \cdots + \frac{1}{d_n}, & \text{if} \quad L = \infty, \\ \left\langle \frac{1}{d_1} \right\rangle_L + \cdots + \left\langle \frac{1}{d_n} \right\rangle_L + \frac{1}{p^L}, & \text{if} \quad L < \infty. \end{cases}$$

Therefore,

$$\mathbf{fpt}_{\mathfrak{n}}(F) = \mathbf{fpt}_{\mathfrak{m}}(x_1^{d_1} + \cdots + x_n^{d_n}).$$

□

Example 2.1.9. *Let $f = z^n + x^a y^b \in \mathbb{K}[x, y, z]$ such that $b \geq a$, and $\mathfrak{m} = \langle x, y, z\rangle$. Let $\beta = \langle z\rangle$ and $\gamma = \langle x, y\rangle$ the maximal ideals in $\mathbb{K}[z]$ and $\mathbb{K}[x, y]$ respectively. Since $\mathbf{fpt}_\beta(z^n) = \frac{1}{n}$ and $\mathbf{fpt}_\gamma(x^a y^b) = \frac{1}{b}$, by Theorem 2.1.6 we obtain*

$$\mathbf{fpt}_{\mathfrak{m}}(f) = \begin{cases} \frac{1}{n} + \frac{1}{b}, & \textit{if} \quad L = \infty, \\ \left\langle \frac{1}{n} \right\rangle_L + \left\langle \frac{1}{b} \right\rangle_L + \frac{1}{p^L}, & \textit{if} \quad L < \infty. \end{cases}$$
$$= \mathbf{fpt}_{\mathfrak{m}}(z^n + y^b)$$

where $L = \sup\left\{N \in \mathbb{N} \quad | \quad \left(\frac{1}{n}\right)^{(e)} + \left(\frac{1}{b}\right)^{(e)} \leq p-1 \quad \text{for all} \quad 0 \leq e \leq N\right\}$. The log canonical threshold of $f = z^n + x^a y^b$ is $\frac{b+n}{nb}$ [4]. Note that formula (1) gives us the same result.

In the particular case of $n = 2$, $b = 2$ and $a = 1$, we have the Whitney umbrella.

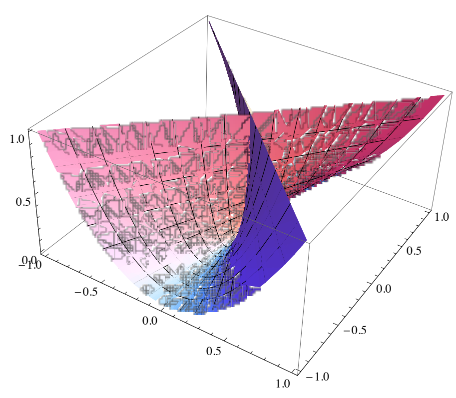

Figure 2.1: Whitney umbrella [24].

If $p = 2$, then

$$\mathbf{fpt}_{\mathfrak{m}}(z^2 + xy^2) = \mathbf{fpt}_{\mathfrak{m}}(z^2 + y^2) = \frac{3}{4}.$$

If $p \equiv 3 \mod 4$, then

$$\mathbf{fpt}_{\mathfrak{m}}(z^2 + xy^2) = \mathbf{fpt}_{\mathfrak{m}}(z^2 + y^2) = \frac{1}{2} + \frac{1}{2} = 1.$$

We also have that

$$\mathbf{lct_0}(z^2 + xy^2) = \mathbf{lct_0}(z^2 + y^2) = \frac{1}{2} + \frac{1}{2} = 1.$$

Since, there is infinitely many primes p such that $p \equiv 3 \mod 4$, we obtain that Conjecture A holds for $f = z^2 + xy^2$.

Remark 2.1.10. *Theorem 2.1.6 generalizes Theorem 2.1.3. For $i \in \{1, \dots, n\}$, let $\mathfrak{m}_i = \langle x_i \rangle$ the homogeneous maximal ideal of $\mathbb{K}[x_i]$. By Example 1.3.6 we know that $\mathbf{fpt}_{\mathfrak{m}_i}(x_i^{d_i}) = \frac{1}{d_i}$. Hence, by Theorem 2.1.6, we have*

$$L = \sup \left\{ N \in \mathbb{N} \quad | \quad \left(\frac{1}{d_1}\right)^{(e)} + \cdots + \left(\frac{1}{d_n}\right)^{(e)} \leq p - 1 \quad \textit{for all} \quad 0 \leq e \leq N \right\},$$

and,

$$\mathbf{fpt}_{\mathfrak{m}}(x_1^{d_1} + \cdots + x_n^{d_n}) = \begin{cases} \mathbf{fpt}_{\mathfrak{m}_1}(x_i^{d_1}) + \cdots + \mathbf{fpt}_{\mathfrak{m}_n}(x_i^{d_n}), & \textit{if} \quad L = \infty, \\ \left\langle \mathbf{fpt}_{\mathfrak{m}_1}(x_1^{d_1}) \right\rangle_L + \cdots + \left\langle \mathbf{fpt}_{\mathfrak{m}_n}(x_n^{d_n}) \right\rangle_L + \frac{1}{p^L} & \textit{if} \quad L < \infty. \end{cases}$$

$$= \begin{cases} \frac{1}{d_1} + \cdots + \frac{1}{d_n}, & \textit{if} \quad L = \infty, \\ \left\langle \frac{1}{d_1} \right\rangle_L + \cdots + \left\langle \frac{1}{d_n} \right\rangle_L + \frac{1}{p^L} & \textit{if} \quad L < \infty. \end{cases}.$$

2.2 Relation with log canonical threshold.

We want to investigate if the Conjecture A holds for the polynomial f in Theorem 2.1.6. Specifically, we are interested in studying when there exists a positive integer N with the following property: for every prime number p with $p \equiv 1 \mod N$ we have $\mathbf{fpt}_{\mathfrak{m}}(f) = \mathbf{lct_0}(f)$.

Lemma 2.2.1. *If f is the polynomial in Theorem 2.1.6, then there exist infinitely many prime numbers such that*

$$\mathbf{fpt}_{\mathfrak{m}}(f) = \sum_{i=1}^{l} \mathbf{fpt}_{\mathfrak{m}_i}(g_i)$$

Proof. In Definition 1.3.5 we mentioned that the F-pure threshold is a rational number. So, under the conditions from Theorem 2.1.6. For $\alpha_i = \frac{a_i}{b_i}$ where $a_i, b_i \in \mathbb{N}$, Theorem 1.1.7 implies that there is an infinite set B of prime numbers such that if $p \in B$, then

$$\begin{aligned} p &\equiv 1 \mod b_1 \\ &\vdots \\ p &\equiv 1 \mod b_l. \end{aligned}$$

From $\alpha_1 + \cdots + \alpha_l \leq 1$ and Example 1.1.2, we conclude that $\alpha_1, \dots, \alpha_l$ add without carrying. Thus, $\mathbf{fpt}_{\mathfrak{m}}(f) = \sum_{i=1}^{l} \alpha_i = \sum_{i=1}^{l} \mathbf{fpt}_{\mathfrak{m}_i}(g_i)$. □

Theorem 2.2.2. *Let f be a polynomial as in Theorem 2.1.6, and B the set in Lemma 2.2.1. Suppose that there is an infinite subset $A \subseteq B$, and $(c_1, \ldots, c_l) \in \mathbb{N}^l$ such that if $p \in A$, then*

$$p \equiv 1 \mod c_i \quad \textit{implies} \quad \mathbf{fpt}_{\mathfrak{m}_i}(g_i) = \mathbf{lct}_0(g_i) \quad \textit{for all} \quad i.$$

Then, there is a positive integer N such that for every prime number p with $p \equiv 1 \mod N$, we have $\mathbf{fpt}_{\mathfrak{m}}(f) = \mathbf{lct}_{\mathbf{0}}(f)$.

Proof. Let $N = \operatorname{lcm}(b_1, \ldots b_l, c_1, \ldots, c_l)$. If $p \equiv 1 \mod N$, then $p \in A$. Therefore, by Lemma 2.2.1, we have

$$\mathbf{fpt}_{\mathfrak{m}}(f) = \sum_{i=1}^{l} \mathbf{fpt}_{\mathfrak{m}_i}(g_i) = \sum_{i=1}^{l} \mathbf{lct}_0(g_i) = \mathbf{lct}_0(f).$$

□

The next example illustrates the situation in Theorem 2.2.2.

Example 2.2.3. *Let $\mathfrak{m} = \langle t, u, v, x, y, w, z \rangle$, $\mathfrak{m}_1 = \langle x, y \rangle$, $\mathfrak{m}_2 = \langle w, z \rangle$ and $\mathfrak{m}_3 = \langle t, u, v \rangle$ the homogeneous maximal ideals of $\mathbb{K}[t, u, v, x, y, w, z]$, $\mathbb{K}[x, y]$, $\mathbb{K}[w, z]$ and $\mathbb{K}[t, u, v]$ respectively.*

- *If $p \equiv 1 \mod 16$, Theorem 2.1.3 implies $\mathbf{fpt}_{\mathfrak{m}_1}(x^4 + y^4) = \frac{8}{16}$. Moreover the relation (1) implies that $\mathbf{lct}_0(x^4 + y^4) = \frac{8}{16}$.*
- *In* [15] it is proved that $\mathbf{fpt}_{\mathfrak{m}_2}(z^7 w^2 + z^5 w^6) = \frac{3}{16}$ if $p \equiv 1 \mod 32$ and using *Macaulay2* we get $\mathbf{lct}_0(z^7 w^2 + z^5 w^6) = \frac{3}{16}$.
- By Example 1.3.12 we have that

$$\mathbf{fpt}_{\mathfrak{m}_3}(v^2 u^3 t^8) = \frac{2}{16} = \mathbf{lct}_0(v^2 u^3 t^8),$$

  for all prime p.

In this way Theorem 2.2.2 and Dirichlet's Theorem 1.1.6 imply that there exists infinitely many primes p such that $p \equiv 1 \mod 32$ implies

$$\begin{aligned}
\mathbf{fpt}_{\mathfrak{m}}(x^4 + y^4 + z^7 w^2 + z^5 w^6 + v^2 u^3 t^8) &= \mathbf{fpt}_{\mathfrak{m}_1}(x^4 + y^4) + \mathbf{fpt}_{\mathfrak{m}_2}(z^7 w^2 + z^5 w^6) + \mathbf{fpt}_{\mathfrak{m}_3}(v^2 u^3 t^8) \\
&= \frac{8}{16} + \frac{3}{16} + \frac{2}{16} = \frac{13}{16} \\
&= \mathbf{lct}_0(x^4 + y^4) + \mathbf{lct}_0(z^7 w^2 + z^5 w^6) + \mathbf{lct}_0(v^2 u^3 t^8) \\
&= \mathbf{lct}_0(x^4 + y^4 + z^7 w^2 + z^5 w^6 + v^2 u^3 t^8).
\end{aligned}$$

Thus, Conjecture A holds for the polynomial $f = x^4 + y^4 + z^7 w^2 + z^5 w^6 + v^2 u^3 t^8$.

Chapter 3

F-pure threshold of determinantal ideals

In this chapter we compute the F- pure threshold of determinantal ideals generated by minors with maximal size. In the previous computations we presented a few techniques to calculate the F-pure threshold of a hypersurface. These techniques are not very effective in the case of non-principal ideals, for this reason we need others methods and concepts such that regular sequences and Gröbner basis

3.1 Determinantal ideals

We consider the polynomial ring $R = \mathbb{K}[X]$, where $\mathbb{K}$ is a perfect field of prime characteristic p, and

$$X = \begin{pmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \dots & x_{mn} \end{pmatrix}$$

is a matrix of indeterminates with $m \leq n$.

We also use the lexicographical term order on R with

$$x_{11} > x_{12} > \cdots > x_{1n} > x_{21} > x_{22} > \cdots > x_{m1} > \cdots > x_{mn}$$

Definition 3.1.1. *An ideal $I_t \subseteq R$ is called a determinantal ideal of degree $t \in \mathbb{N}$ if I_t is generated by the size t minors of X.*

Notation 3.1.2. *In our case we consider minors of maximal size m. We set the next notation for the first size m minors of X,*

$$\Delta_1 = \det \begin{pmatrix} x_{11} & x_{12} & \dots & x_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \dots & x_{mm} \end{pmatrix},$$

$$\Delta_2 = \det \begin{pmatrix} x_{12} & x_{13} & \dots & x_{1m+1} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m2} & x_{m3} & \dots & x_{mm+1} \end{pmatrix},$$

$$\vdots$$

$$\Delta_{n-m+1} = \det \begin{pmatrix} x_{1n-m+1} & \dots & x_{1n-1} & x_{1n} \\ \vdots & \ddots & \vdots & \vdots \\ x_{mn-m+1} & \dots & x_{mn-1} & x_{mn} \end{pmatrix}.$$

And we denote the leading term of a polynomial $f \in R$ by $\operatorname{Lt}(f)$.

It is important to note that leading terms respect to lexicographical order of these minors are the product of the elements of their diagonals

$$\operatorname{Lt}(\Delta_1) = x_{11}x_{22}\cdots x_{mm}$$

$$\operatorname{Lt}(\Delta_2) = x_{12}x_{23}\cdots x_{mm+1}$$

$$\vdots$$

$$\operatorname{Lt}(\Delta_{n-m+1}) = x_{1n-m+1}x_{2n-m+2}\cdots x_{mn}.$$

The next lemma presents sufficient conditions to decide when a sequence of polynomials is a regular sequence.

Lemma 3.1.3. [25]. Let $f_1, \ldots, f_n \in R$ be polynomials such that with respect to lexicographical order on R, $\gcd(\operatorname{Lt}(f_i), \operatorname{Lt}(f_j)) = 1$ for $i \neq j$. Then, $f_1, \ldots, f_n$ is a regular sequence in R.

Proof. The polynomial f_1 is a regular element in R, since R is a domain and $f_1 \neq 0$. By induction we assume that for $k \leq n-1$, $\{f_1, f_2, \ldots, f_k\}$ forms a regular sequence in R. If

$$J = \langle f_1, f_2, \ldots, f_k \rangle,$$

then we can say that the set $\{f_1, f_2, \ldots, f_k\}$ is a Gröbner basis for J, since $\gcd(\operatorname{Lt}(f_i), \operatorname{Lt}(f_j)) = 1$ for every $i \neq j$. Let $gf_{k+1} \in J$. Then, $\operatorname{Lt}(g)\operatorname{Lt}(f_{k+1})$ must be divisible by $\operatorname{Lt}(f_i)$ for some $1 \leq i \leq k$. But, $\gcd(\operatorname{Lt}(f_i), \operatorname{Lt}(f_{k+1})) = 1$, and hence $\operatorname{Lt}(f_i)$ divides $\operatorname{Lt}(g)$. Let $r = g - \frac{\operatorname{Lt}(g)}{\operatorname{Lt}(f_i)} f_i$. If $r = 0$, then $g \in J$. If $r \neq 0$, then $\operatorname{Lt}(r) < \operatorname{Lt}(g)$ and $rf_{k+1} \in J$. Now, we repeat the same argument with rf_{k+1}. □

Remark 3.1.4. *Note that for the minors $\Delta_1, \Delta_2, \ldots, \Delta_{n-m+1}$ we have $\gcd(\operatorname{Lt}(\Delta_i), \operatorname{Lt}(\Delta_j)) = 1$ for $i \neq j$. Because they do not share any variable. Hence, by Lemma 3.1.3 we conclude that $\Delta_1, \Delta_2, \ldots, \Delta_{n-m+1}$ is a regular sequence in R.*

In order to find a formula for the F-pure threshold of the determinantal ideal I_m, we prove the following lemma.

Lemma 3.1.5. *For the ideal $J = \langle \Delta_1, \Delta_2, \ldots, \Delta_{n-m+1} \rangle \subseteq I_m$ we have that R/J is a F-pure ring.*

Proof. In Remark 3.1.4 we established that the sequence $\Delta_1, \Delta_2, \ldots, \Delta_{n-m+1}$ is a regular sequence. Then we find a maximal regular sequences, such that, it has as first terms $\Delta_1, \Delta_2, \ldots, \Delta_{n-m+1}$. This is, there exists a regular sequences

$$\Delta_1, \Delta_2, \ldots, \Delta_{n-m+1}, \Delta_{n-m+2}, \Delta_2, \ldots, \Delta_d$$

where d is the Krull dimension of R. Since R is Cohen-Macaulay ring, Hironaka's criterion [32, Theorem 25.16] implies that there exists a free and flat morphism

$$\phi : \mathbb{K}[y_1, y_2, \ldots, y_d] \to R$$
$$y_i \mapsto \Delta_i$$

Note that $((y_1,\dots,y_l)^{[p]}:(y_1,\dots,y_l)) = (y_1,\dots,y_l)^{[p]} + (y_1\cdots y_l)^{p-1}$ for $0\le l\le d$. Since ϕ is flat, we deduce

$$\begin{aligned}(J^{[p]}:J) &= \phi(((y_1,\dots,y_{n-m+1})^{[p]}:(y_1,\dots,y_{n-m+1})))\\ &= \phi((y_1,\dots,y_{n-m+1})^{[p]} + (y_1\cdots y_{n-m+1})^{p-1})\\ &= J^{[p]} + (\Delta_1\Delta_2\cdots\Delta_{n-m+1})^{p-1}\end{aligned}$$

The polynomial $(\Delta_1\Delta_2\cdots\Delta_{n-m+1})^{p-1}\notin \mathfrak{m}^{[\mathfrak{p}]}$, therefore $(J^{[p]}:J)\not\subseteq \mathfrak{m}^{[\mathfrak{p}]}$. Finally by Fedder Criterion 1.4.3 we conclude that R/J is a F-pure. □

L. Miller, A. Singh and M. Varbaro have computed that F-pure thresholds of a determinantal ideal I_t [30]. In the particular case of determinantal ideals with maximal size, I_m. We show that F- pure threshold of I_m does not depend of the characteristic p.

Theorem 3.1.6. *The F-pure threshold of a determinantal ideal of degree m, $I_m\subseteq R$ is*

$$\mathbf{fpt}(I_m) = n-m+1$$

Proof. First we consider the height of the ideal J in Lemma 3.1.5 and the height of I_m. Since J is generated by a regular sequence, we conclude that its height is equal to the number of generators, $n-m+1$. The Eagon-Northcott formula [8, 7] gives that $\mathrm{ht}(I_m) = n-m+1$. Therefore $\mathrm{ht}(J) = \mathrm{ht}(I_m)$. Now we pay attention to the F-pure threshold of J. In Lemma 3.1.5 we proved that R/J is F-pure ring and $(J^{[p]}:J) = J^{[p]} + (\Delta_1\Delta_2\cdots\Delta_{n-m+1})^{p-1}$. Moreover, by Frobenius iteration we conclude

$$(J^{[p^e]}:J) = J^{[p^e]} + (\Delta_1\Delta_2\cdots\Delta_{n-m+1})^{p^e-1}$$

The Fedder Criterion 1.4.3 implies that $(\Delta_1\Delta_2\cdots\Delta_{n-m+1})^{p^e-1}\notin\mathfrak{m}^{[p^e]}$. Then by pigeonhole principle we conclude

$$\nu_J^{\mathfrak{m}}(p^e) = (n-m+1)(p^e-1).$$

Taking the limit $e\to\infty$, we deduce that $\mathbf{fpt}(J) = n-m+1$. Finally by Lemma 1.3.9 and Theorem 1.3.8 we get

$$n-m+1 = \mathrm{ht}(J) = \mathbf{fpt}(J) \le \mathbf{fpt}(I_m) \le \mathrm{ht}(I_m) = n-m+1$$

□

Example 3.1.7. *Consider that matrix*

$$X = \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14}\\ x_{21} & x_{22} & x_{23} & x_{24}\end{pmatrix}$$

Then the ideal I_m is generated by

$$\begin{aligned}\Delta_1 &= x_{11}x_{22}-x_{21}x_{12} & \Delta_4 &= x_{12}x_{24}-x_{22}x_{14}\\ \Delta_2 &= x_{12}x_{23}-x_{22}x_{13} & \Delta_5 &= x_{11}x_{23}-x_{21}x_{13}\\ \Delta_3 &= x_{13}x_{24}-x_{14}x_{23} & \Delta_6 &= x_{11}x_{24}-x_{21}x_{14}\end{aligned}$$

and the ideal J is generated by

$$\Delta_1 = x_{11}x_{22}-x_{21}x_{12}$$
$$\Delta_2 = x_{12}x_{23}-x_{22}x_{13}$$
$$\Delta_3 = x_{13}x_{24}-x_{14}x_{23}$$

The Theorem 3.1.6 implies that

$$\mathbf{fpt}(I_2) = 4-2+1 = 3$$

Chapter 4

Bernstein-Sato Polynomial and F-pure threshold

The Bernstein-Sato polynomial is an important and subtle invariant associated to a polynomial. The roots of Bernstein-Sato polynomial are related to others algebraic and topological invariants like the eigenvalues of monodromy, the jumping numbers, and the log canonical thresholds [3]. However, the Bernstein-Sato polynomial is in general very difficult to compute. M. Mustaţă, S. Takagi, and K-i. Watanabe have proposed a method to find some roots of this polynomial from the F-pure threshold [31]. We apply this method to understand the differences between the Bernstein-Sato polynomials of $f = z^n + x^a y^b$ and a deformation of f.

4.1 Bernstein-Sato Polynomial

Definition 4.1.1. *Let $f \in R = \mathbb{C}[x_1, \dots, x_n]$ be a polynomial with rational coefficients. The Bernstein-Sato polynomial of f is the non-zero monic polynomial $b_f(s)$ of minimal degree among those $b \in \mathbb{Q}[s]$ such that*

$$b(s) f^s = P f^{s+1}, \tag{4.1}$$

for some operator $P \in \mathbb{C}[x, \partial/\partial_x, s]$, with $x = (x_1, \dots, x_n)$, and $\partial/\partial_x = (\partial/\partial_{x_1}, \dots, \partial/\partial_{x_n})$

Remark 4.1.2. *The existence of non-zero Bernstein-Sato polynomials is a deep fact whose proof by I.N. Bernstein started the theory of D-modules. The existence of this polynomial was proved independently by M. Sato using different techniques.*

Remark 4.1.3. *In general it is difficult and computationally expensive to compute Bernstein-Sato polynomials. Nowadays, computer programs like Macaulay2 and Singular have packages allowing the computation of Bernstein-Sato polynomial in simple cases. Below we show some examples.*

Example 4.1.4. *If $f = x^2 + y^2 \in \mathbb{C}[x, y]$, we have*

$$\begin{aligned}
\partial_x^2 \left(x^2+y^2\right)^{s+1} &= (s+1)\left(4sx^2\left(x^2+y^2\right)^{s-1} + 2\left(x^2+y^2\right)^{s}\right)\\
\partial_y^2 \left(x^2+y^2\right)^{s+1} &= (s+1)\left(4sy^2\left(x^2+y^2\right)^{s-1} + 2\left(x^2+y^2\right)^{s}\right).
\end{aligned}$$

And the following equality holds

$$\begin{aligned}
\left(\partial_x^2+\partial_y^2\right)\left(x^2+y^2\right)^{s+1} &= (s+1)\left(4sx^2\left(x^2+y^2\right)^{s-1} + 2\left(x^2+y^2\right)^{s} + 4sy^2\left(x^2+y^2\right)^{s-1} + 2\left(x^2+y^2\right)^{s}\right)\\
&= 4(s+1)^2\left(x^2+y^2\right)^{s}.
\end{aligned}$$

Therefore, the Bernstein-Sato polynomial of f is $b_f(s) = 4(s+1)^2$ with differential operator $P = \partial_x^2 + \partial_y^2$.

Example 4.1.5. *Using Macaulay2 we have that the Bernstein-Sato polynomial of $f = x^2 + y^7 \in \mathbb{C}[x, y]$ is*

$$b_f(s) = (1/7529536)\,(s+1)(14s+9)(14s+11)(14s+13)(14s+15)(14s+17)(14s+19)$$

Remark 4.1.6. *Let A be the localization of $\mathbb{Z}$ at some nonzero integer m, and $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$. The roots of $b_f(s)$ are rational, so $b_f(s) \in \mathbb{Q}[s]$. Therefore for every integer $m \in \mathbb{Z}$, b_f has a well-defined class in $\mathbb{F}_p$*

M. Mustaţă, S. Takagi, and K-i. Watanabe have proposed a method to find some roots of the Bernstein-Sato polynomial from the F-pure threshold. The following result is a particular case of [28, Proposition 3.11].

Theorem 4.1.7. *Let $f \in \langle x_1, \ldots, x_n \rangle\, A[x_1, \ldots, x_n]$ a non-zero polynomial, then for every prime $p >> 0$, we have*

$$b_f(\nu_f^{\mathfrak{m}}(p^e)) = 0 \quad in \quad \mathbb{F}_p \tag{4.2}$$

for all e.

Remark 4.1.8. *Note that the value of $\nu_f^{\mathfrak{m}}(p^e)$ depends on the prime characteristic p and on the positive integer e. Then, different prime numbers could give the same root, and if we fix a prime number, different positive integers e can give different roots. Moreover, in a general case $\nu_f^{\mathfrak{m}}(p^e)$ can be computed respect to others ideals, it means that we calculate $\nu_f^J(p^e)$ for others ideals $J \subset \mathfrak{m}$. So, the general case of Theorem 4.1.7 establishes that the numbers $\nu_f^J(p^e)$ also give roots of the Bernstein-Sato polynomial* [28, Proposition 3.11].

Remark 4.1.9. *Let $b_f(s) = s^m + b_1 s^{m-1} + \cdots + b_{m-1}s + b_m$ be the Bernstein-Sato polynomial of f. Suppose that for a positive integer e, there is a polynomial*

$$P_f(x) = a_r x^r + \cdots + a_1 x + a_0 \quad in \quad \mathbb{Q}[x],$$

and natural numbers N, k such that $\nu_f^{\mathfrak{m}}(p^e) = P_f(p)$ for every $p >> 0$ with $p \equiv k \mod N$. Theorem 4.1.7 gives us that $b_f(\nu_f^{\mathfrak{m}}(p^e))$ is divisible by p. This means that p divide the expression

$$(a_r p^r + \cdots + a_1 p + a_0)^m + b_1(a_r p^r + \cdots + a_1 p + a_0)^{m-1} + \cdots + b_{m-1}(a_r p^r + \cdots + a_1 p + a_0) + b_m.$$

Therefore, Dirichlet Theorem on arithmetic progressions implies that there are infinite prime numbers p such that, $a_0^m + b_1 a_0^{m-1} + \cdots + b_{m-2}a_0^2 + b_{m-1}a_0 + b_m$ is divisible by p. So, we conclude

$$b_f(P_f(0)) = 0 \quad in \quad \mathbb{Q}[x].$$

Remark 4.1.10. *For a polynomial f, Theorem 4.1.7 tells us how to find roots of its Bernstein-Sato polynomial. But, can we find all the roots of the Bernstein-Sato polynomial of f using Theorem 4.1.7? The answer of this question is negative* [31, Example 4.1]. *However, if f is a monomial, Theorem 4.1.7 gives us all the roots of the Bernstein-Sato polynomial* [5].

Example 4.1.11. *In the Example 2.1.4 we computed the F-pure threshold of $f = x^2 + y^7$ for a $p \equiv 5 \mod 14$. By the Lemma 1.3.13 we have that*

$$\begin{aligned}\nu_f^{\mathfrak{m}}(p) &= \lceil p\,\mathbf{fpt}_{\mathfrak{m}}(f)\rceil - 1\\ &= \lceil p\left(\frac{9}{14} - \frac{3}{14p}\right)\rceil - 1\\ &= \frac{9p}{14} - \frac{3}{14} - 1 = \frac{9p}{14} - \frac{17}{14}.\end{aligned}$$

In this case we have $\nu_f^{\mathfrak{m}}(p)$ is equal to the polynomial $P_f(p) = \frac{9p}{14} - \frac{17}{14}$. Then, Remark 4.1.9 shows that $P_f(0) = -\frac{17}{14}$ is a root of the Bernstein-Sato polynomial of f.

4.2 Bernstein-Sato polynomial of $f = z^4 + x^6y^5$ and a deformation of f

Bernstein-sato polynomials are a tool to investigate strata of deformations of polynomials; such as, analytic deformations with fixed topological type or Milnor number [20, 6]. D.Hernández and E. Witt are systematically studying the roots of Bernstein-Sato polynomials of $x^a + y^b$ and their deformations, via prime characteristic methods [36]. The next natural case to study are polynomials of the form $z^n + x^ay^b$.

Our goal is investigate the differences between the Bernstein-Sato polynomials of $f = z^n + x^ay^b$ and some deformations of f by means of Theorem 4.1.7. In particular, we study to the next example due to V. Levandovskyy and J. Martín-Morales [28].

Example 4.2.1. [28, Example 3.4]. *Let $f = z^4 + x^6y^5$ and $g = f + x^5y^4z$. The Bernstein-Sato polynomial of f has 27 roots with multiplicity one except -1 which has multiplicity two.*

$$-1, \frac{-5}{6}, \frac{-9}{10}, \frac{-4}{3}, \frac{-13}{10}, \frac{-2}{3}, \frac{-3}{4}, \frac{-19}{20}, \frac{-5}{12}, \frac{-11}{10}, \frac{-17}{12}, \frac{-17}{20}, \frac{-11}{12}, \frac{-7}{10}, \frac{-19}{12},$$
$$\frac{-13}{20}, \frac{-27}{20}, \frac{-7}{6}, \frac{-21}{20}, \frac{-9}{20}, \frac{-13}{12}, \frac{-5}{4}, \frac{-3}{2}, \frac{-7}{12}, \frac{-31}{20}, \frac{-7}{4}, \frac{-23}{20}.$$

For g we have that the roots of its Bernstein-Sato polynomial are

$$-1, \frac{-5}{6}, \frac{-9}{10}, \frac{-4}{3}, \frac{-13}{10}, \frac{-2}{3}, \frac{-3}{4}, \frac{-19}{20}, \frac{-5}{12}, \frac{-11}{10}, \frac{-17}{12}, \frac{-17}{20}, \frac{-11}{12}, \frac{-7}{10}, -,$$
$$\frac{-13}{20}, \frac{-27}{20}, \frac{-7}{6}, \frac{-21}{20}, \frac{-9}{20}, \frac{-13}{12}, \frac{-5}{4}, \frac{\mathbf{-1}}{\mathbf{2}}, \frac{-7}{12}, \frac{\mathbf{-11}}{\mathbf{20}}, -, \frac{-23}{20}.$$

In this case we do not have information about the multiplicities.

Note that in the Example 4.2.1 the roots $\frac{-19}{12}$ and $\frac{-7}{4}$ do not appear as roots of the Bernstein-Sato polynomial of $g = z^4 + x^6y^5 + x^5y^4z$. In order to give a justification of this fact, we check that $\frac{-19}{12}$ is a root of the Bernstein-Sato polynomial of $f = z^4 + x^6y^5$ using $p \equiv 11 \mod 12$ by means of Theorem 4.1.7. Secondly, we show partial evidence that $\frac{-19}{12}$ is not a root of $b_g(s)$.

Proposition 4.2.2. *If p is a prime number, such that, $p \equiv 11 \mod 12$ and $f = z^4 + x^6y^5$, then*

$$\nu_f^{\mathfrak{m}}(p) = \frac{5p}{12} - \frac{19}{12}$$

Proof. Example 2.1.9 tells us how to calculate the F-pure threshold of $f = z^4 + x^6y^5$. If $p \equiv 11 \mod 12$, then exists a $w \in \mathbb{N}$ such that $p = 12w + 11$. The expansions of $\frac{1}{4}$ and $\frac{1}{6}$ in base p are

$$\frac{1}{4} = \frac{3w+2}{p} + \frac{9w+8}{p^2} + \dots,$$
$$\frac{1}{6} = \frac{2w+1}{p} + \frac{10w+9}{p^2} + \dots,$$

therefore $\frac{1}{4}$ and $\frac{1}{6}$ not add without carry and the F-pure threshold of $f = z^4 + x^6y^5$ is

$$\mathbf{fpt}_{\mathfrak{m}}(f) = \frac{3w+2}{p} + \frac{2w+1}{p} + \frac{1}{p} = \frac{5w+4}{p} = \frac{5}{12} - \frac{7}{12p}. \tag{4.3}$$

By Lemma 1.3.13 we have

$$\begin{aligned} \nu_f^{\mathfrak{m}}(p^e) &= \lceil p^e \mathbf{fpt}_{\mathfrak{m}}(f) \rceil - 1 \\ &= \lceil p^e \left(\frac{5}{12} - \frac{7}{12p} \right) \rceil - 1 \\ &= \begin{cases} \frac{5p}{12} - \frac{7}{12} - 1 = \frac{5p}{12} - \frac{19}{12} & \text{if} \quad e = 1, \\ -1 & \text{if} \quad e \geq 2. \end{cases} \end{aligned}$$

□

Remark 4.2.3. *With $p \equiv 11 \mod 12$, Remark 4.1.9 gives that $\nu_f^{\mathfrak{m}}(p)$ is equal to the polynomial $P_f(p) = \frac{5p}{12} - \frac{19}{12}$. Therefore, $P_f(0) = -\frac{19}{12}$ is a root of the Bernstein-Sato polynomial of $f = z^4 + x^6y^5$.*

For the polynomial $g = z^4 + x^6y^5 + x^5y^4z$, next proposition shows that $p \equiv 11 \mod 12$ gives a root of the Bernstein-Sato polynomial of g different from $\frac{-19}{12}$.

Proposition 4.2.4. *Let p be a prime number such that $p \equiv 11 \mod 12$. For $f = z^4 + x^6y^5$ and $g = z^4 + x^6y^5 + x^5y^4z$ we have,*

$$\nu_f^{\mathfrak{m}}(p^e) \leq \nu_g^{\mathfrak{m}}(p^e)$$

for all e. In particular $\nu_f^{\mathfrak{m}}(p) < \nu_g^{\mathfrak{m}}(p)$.

Proof. We have that $g = f + x^5y^4z$, thus

$$g^{\nu_f^{\mathfrak{m}}(p^e)} = f^{\nu_f^{\mathfrak{m}}(p^e)} + \nu_f^{\mathfrak{m}}(p^e) f^{\nu_f^{\mathfrak{m}}(p^e)-1}(x^5y^4z) + \dots$$

In this expression the term $f^{\nu_f^{\mathfrak{m}}(p^e)}$ does not belong to $\mathfrak{m}^{[p^e]}$. Therefore, $g^{\nu_f^{\mathfrak{m}}(p^e)} \notin \mathfrak{m}^{[p^e]}$. This implies that

$$\nu_f^{\mathfrak{m}}(p^e) \leq \nu_g^{\mathfrak{m}}(p^e),$$

for all e.
Now, for $e = 1$ it guarantees that $\nu_f^{\mathfrak{m}}(p) < \nu_g^{\mathfrak{m}}(p)$. In fact, rewriting $\nu_f^{\mathfrak{m}}(p)$ as

$$\begin{aligned}\nu_f^{\mathfrak{m}}(p) &= \frac{5p}{12} - \frac{19}{12}\\ &= \frac{5(12w+11)}{12} - \frac{19}{12}\\ &= \frac{60w+36}{12} = 5w+3,\end{aligned}$$

and considering expansion

$$g^{\nu_f^{\mathfrak{m}}(p)+1} = f^{\nu_f^{\mathfrak{m}}(p)+1} + (\nu_f^{\mathfrak{m}}(p)+1)f^{\nu_f^{\mathfrak{m}}(p)}(x^5y^4z) + \frac{(\nu_f^{\mathfrak{m}}(p)+1)\nu_f^{\mathfrak{m}}(p)}{2}f^{\nu_f^{\mathfrak{m}}(p)-1}(x^5y^4z)^2 + \dots,$$

it can be shown $\frac{(\nu_f^{\mathfrak{m}}(p)+1)\nu_f^{\mathfrak{m}}(p)}{2}f^{\nu_f^{\mathfrak{m}}(p)-1}(x^5y^4z)^2 \notin \mathfrak{m}^{[p]}$. Since,

$$f^{\nu_f^{\mathfrak{m}}(p)-1} \notin \mathfrak{m}^{[p]},$$

then there exists $0 \le l \le \nu_f^{\mathfrak{m}}(p)$ such that

$$(z^4)^{\nu_f^{\mathfrak{m}}(p)-1-l}(x^6y^5)^l = (z^4)^{5w+2-l}(x^6y^5)^l \notin \mathfrak{m}^{[p]} \quad \text{and} \quad \binom{\nu_f^{\mathfrak{m}}(p)-1}{l} \not\equiv 0 \mod p.$$

Looking at the exponent of the variables, we have

$$20w+8-4l < 12w+11 \quad \text{and} \quad 6l < 12w+11,$$

equivalently

$$\frac{8w-3}{4} < l < \frac{12w+11}{6}$$

Therefore, the only possibles values for l are $2w$ and $2w+1$. Let us discuss each case separately.

- <u>$l = 2w+1$</u>:

$$\begin{aligned}(z^4)^{\nu_f^{\mathfrak{m}}(p)-1-l}(x^6y^5)^l &= (z^4)^{3w+1}(x^6y^5)^{2w+1}\\ &= z^{12w+4}x^{12w+6}y^{10w+5}.\end{aligned}$$

  Multiplying by the monomial $(x^5y^4z)^2$ we get

$$\begin{aligned}(z^4)^{\nu_f^{\mathfrak{m}}(p)-1-l}(x^6y^5)^l(x^5y^4z)^2 &= z^{12w+4}x^{12w+6}y^{10w+5}(x^{10}y^8z^2)\\ &= z^{12w+6}x^{12w+14}y^{10w+13}.\end{aligned}$$

  Since $x^{12w+14} \in \mathfrak{m}^{[p]}$, we conclude $(z^4)^{\nu_f^{\mathfrak{m}}(p)-1-l}(x^6y^5)^l(x^5y^4z)^2 \in \mathfrak{m}^{[p]}$.

- <u>$l = 2w$</u>:

$$\begin{aligned}(z^4)^{\nu_f^{\mathfrak{m}}(p)-1-l}(x^6y^5)^l &= (z^4)^{3w+2}(x^6y^5)^{2w}\\ &= z^{12w+8}x^{12w}y^{10w}\end{aligned}$$

Multiplying by the monomial $(x^5y^4z)^2$ we get

$$\begin{aligned}(z^4)^{\nu_f^{\mathfrak{m}}(p)-1-l}(x^6y^5)^l(x^5y^4z)^2 &= z^{12w+8}x^{12w}y^{10w}(x^{10}y^8z^2)\\ &= z^{12w+10}x^{12w+10}y^{10w+8}.\end{aligned}$$

We conclude $(z^4)^{\nu_f^{\mathfrak{m}}(p)-1-l}(x^6y^5)^l(x^5y^4z)^2 \notin \mathfrak{m}^{[p]}$. Moreover, since $\nu_f^{\mathfrak{m}}(p)-1-2w < p$, $2w < p$ and $\nu_f^{\mathfrak{m}}(p)-1+2w < p-1$, we deduce that $\nu_f^{\mathfrak{m}}(p)-1-2w$ and $2w$ add without carrying and by Theorem 1.1.5,

$$\binom{\nu_f^{\mathfrak{m}}(p)-1}{2w} \not\equiv 0 \mod p.$$

Since the term $\binom{\nu_f^{\mathfrak{m}}(p)-1}{2w}(z^4)^{\nu_f^{\mathfrak{m}}(p)-1-2w}(x^6y^5)^{2w}(x^5y^4z)^2$ is in the expansion of $f^{\nu_f^{\mathfrak{m}}(p)-1}(x^5y^4z)^2$, we have that

$$f^{\nu_f^{\mathfrak{m}}(p)-1}(x^5y^4z)^2 \notin \mathfrak{m}^{[p]}$$

Note, that $\nu_f^{\mathfrak{m}}(p)-1 < p$ and $\nu_f^{\mathfrak{m}}(p)+1 < p-1$, then $\nu_f^{\mathfrak{m}}(p)-1$ and 2 add without carrying. Theorem 1.1.5 gives us

$$\binom{\nu_f^{\mathfrak{m}}(p)+1}{2} \not\equiv 0 \mod p.$$

Therefore,

$$g^{\nu_f^{\mathfrak{m}}(p)+1} \notin \mathfrak{m}^{[p]}.$$

Finally, we have shown

$$\nu_f^{\mathfrak{m}}(p) < \nu_f^{\mathfrak{m}}(p)+1 \leq \nu_g^{\mathfrak{m}}(p).$$

□

Remark 4.2.5. *In the same condition of the above proposition, Prof. Daniel Hernández has proved* $\nu_f^{\mathfrak{m}}(p)+1 \geq \nu_g^{\mathfrak{m}}(p)$, *therefore by Proposition 4.2.4 we have* $\nu_f^{\mathfrak{m}}(p)+1 = \nu_g^{\mathfrak{m}}(p)$. *Let me show the Daniel Hernández's prove of* $\nu_f^{\mathfrak{m}}(p)+1 \geq \nu_g^{\mathfrak{m}}(p)$. *Since we have that* $\nu_f^{\mathfrak{m}}(p) = \frac{5p}{12} - \frac{19}{12}$, *it sufficient to show that* $\nu_g^{\mathfrak{m}}(p) \leq \frac{5p}{12} - \frac{7}{12}$. *By simplicity, set* $\nu = \nu_g^{\mathfrak{m}}(p)$, *so* $g^\nu \notin \langle x^p, y^p, z^p\rangle$. *If you expand out* g^ν *using the multinomial theorem, the fact that* $\langle x^p, y^p, z^p\rangle$ *is a monomial ideal implies that there exist* $i, j, k \in \mathbb{N}$ *with* $i+j+k = \nu$ *such that*

$$z^{4i}(x^6y^5)^j(x^5y^4z)^k \notin \langle x^p, y^p, z^p\rangle.$$

The exponent of x *above is* $6j+5k$, *the exponent of* z *is* $4i+k$, *and each of these must be less than* p. *In others words,*

$$\begin{aligned}6j+5k &\leq p-1\\ 4i+k &\leq p-1\end{aligned}$$

Multiply the first equation by 4, the second by 6, and then add the two resulting equations to obtain $24i+24j+26k \leq 10p-10$, *and since* $k \geq 0$, *this implies that* $24(i+j+k) \leq 10p-10$, *or equivalently,*

$$\nu = i+j+k \leq \frac{10p-10}{24} = \frac{5p-5}{12} = \frac{5p-7}{12} + \frac{2}{12}$$

Finally, because $p \equiv 11 \mod 12$*, we have that* $\frac{5p-7}{12} \in \mathbb{N}$*, and so taking the floor function of the above inequality shows that*

$$\nu \leq \frac{5p-7}{12}$$

as claimed.

Remark 4.2.6. *Let* $P_g \in \mathbb{Q}[x]$ *be a polynomial such that* $\nu_g^{\mathfrak{m}}(p) = P_g(p)$ *with* $p \equiv 11 \mod 12$*. From Proposition 4.2.4 we conclude that*

$$P_f(0) < P_f(0) + 1 \leq P_g(0).$$

Therefore, Remark 4.1.9 implies that $\nu_g^{\mathfrak{m}}(p)$ *provides a different root to* $\frac{-19}{12}$ *for infinite many primes numbers* p*.*

Remark 4.2.7. *From the computation in Example 4.2.1 due to V. Levandovskyy and J. Martín-Morales we know that* $\frac{-19}{12}$ *is not a root of* $b_g(s)$*. Our computations above give partial evidence that* $b_g(-19/12) \neq 0$ *in the following sense. Using* $p \equiv 11 \mod 12$ *and Theorem 4.1.7 we were able to recover that* $b_f(-19/12) = 0$*. Moreover, again using Theorem 4.1.7 we show that* $p \equiv 11 \mod 12$ *render different roots (*$\neq -19/12$*) of the Bernstein-Sato polynomial of* g*. Ideally one should check with all* p *and all ideals* J *4.1.8. However, it is known that Theorem 4.1.7 do not get in general all roots of the Bernstein-Sato polynomial 4.1.10.*

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