CIMAT

Centro de Investigación en Matemáticas, A. C.

Subordinators in Banach Spaces

Tesis

que para obtener el Grado de

Doctor en Ciencias
con orientación en
Probabilidad y Estadística

Presenta

Alfonso Rocha Arteaga

Director de Tesis

Dr. Víctor Pérez-Abreu

Guanajuato, México, Junio del 2003

To my wife Eldegar

and my daughters Ana Karen and Daniela

Acknowledgements

I wish to express my deep gratitude to my advisor, Víctor Pérez-Abreu, for his valuable direction of this thesis and for many long discussions during the last three years.

I am grateful to my external examiner, Ken-iti Sato, who carefully revised this work.

My thanks to the examination panel, Luis Gorostiza Ortega, Maite Fernández Unzueta, María Emilia Caballero Acosta and Constantin Tudor for their helpful suggestions.

Special thanks to Maite Fernández Unzueta for her kind readiness to discuss Analysis all the time I spent at CIMAT and to Stephanie Dunbar for her patience and generous interest in teaching me English.

I thank CIMAT for the facilities to carry out this research as well as the Universidad Autónoma de Sinaloa and the grant PROMEP UASIN-236 for their economic support.

Finally, I extend my gratitude to my family.

Contents

Preface

Additive processes are stochastically continuous processes with independent increments starting at zero, whose paths are right-continuous with left-limits almost surely. An additive process is called a Lévy process if the increments are stationary. Since the pioneering work of Paul Lévy in the 1930's, Lévy processes have been largely researched and have revealed their importance as a fundamental class of stochastic processes with increasing contemporary applications. The books of Jean Bertoin (1996) and Ken-iti Sato (1999) provide a comprehensive modern knowledge of Lévy processes in finite dimensions. The programs and proceedings of the recently celebrated Conferences on Lévy Processes are sources of ideas and references for applied and theoretical developments in the subject; see Barndorff-Nielsen, Mikosch and Resnik (2001) and Barndorff-Nielsen (2002).

Lévy processes possess a rich structure. They are characterized by the so-called generating triplet (A, ν, γ) of their Lévy-Khintchine representation for the characteristic function (Fourier transform) at a fixed time. Lévy (1934) derived this characterization by analyzing the jumps structure of the sample functions, splitting a Lévy process into a sum of two independent processes: a continuous process and a process which is a compensated sum of independent jumps. Later on, Itô (1942) formulated and proved this characterization rigorously; in what is now referred to as the Lévy-Itô decomposition.

The generating triplet (A, ν, γ) provides useful information about the Lévy process and it is intimately connected with its distributional properties. The nonnegative term A is related to the Gaussian part, ν is the so-called Lévy measure which is related to the jumps of the process and γ is an element in the state space.

Subordinators are increasing Lévy processes and they constitute a very important class. One dimensional subordinators are characterized in terms of their generating triplet as follows: a one dimensional Lévy process $\{\sigma_t : t \geq 0\}$ is a subordinator if and only if its generating triplet satisfies

$$A = 0, \quad \nu((-\infty, 0)) = 0, \quad \gamma^0 = \gamma - \int_{(0,1]} x\nu(dx) \geq 0 \quad \text{and} \tag{1}$$

$$\int_{(0,1]} x\nu(dx) < \infty, \tag{2}$$

where γ^0 is called the *drift;* see Bertoin (1999, Th. 1.2) or Sato (1999, Th. 21.5). This means that a one dimensional subordinator has no Gaussian part, the Lévy measure is concentrated in the cone $K = [0, \infty)$ and the drift belongs to K. Since $\sigma_0 = 0$, a one dimensional subordinator is a process with values in K.

Furthermore, the law of the subordinator is specified by its Laplace transform

$$Ee^{-u\sigma_t} = \exp\{-t\Phi(u)\} \quad u \geq 0, \tag{3}$$

where Φ is the *Laplace exponent*

$$\Phi(u) = \int_{(0,\infty)} \left(1 - e^{-ux}\right) \nu(dx) + \gamma^0 u, \quad u \geq 0. \tag{4}$$

The expressions (1)-(4) give the *special form of the Lévy-Khintchine representation* of the Fourier and Laplace transforms of a subordinator. Furthermore, at time one, (3)-(4) gives the characterization of the Laplace transform of a nonnegative infinitely divisible scalar random variable as presented in Feller (1971, Th. 13.7.2).

One dimensional subordinators have important sample path properties and they are useful in building other Lévy processes via the so-called method of Bochner's subordination. On one hand, for example, sample paths of one dimensional subordinators have bounded variation and the rate of growth is such that $\sigma_t/t \to \gamma^0$ when $t \to 0$, almost surely. Further-

more, under the assumption of regularly varying of Φ, subordinators obey a law of iterated logarithm whose rate depends on the Laplace exponent (Bertoin (1996, Th. 3.11)).

On the other hand, subordination of a finite dimensional Lévy process $\{X_t : t \geq 0\}$ by an independent one dimensional subordinator $\{\sigma_t : t \geq 0\}$ is a largely studied area, see for example Bochner (1955), Sato (1999, 2001) and references therein. The random time change process $\{Y_t = X_{\sigma_t} : t \geq 0\}$ is still a finite dimensional Lévy process, whose generating triplet is explicitly described from those of X and σ (Sato (1999, Th. 30.1)).

Cone-parameter Lévy processes and cone-valued additive processes in the case of cones of $\mathbb{R}^d$ are already discussed in the classical books by Bochner (1955, pp. 106-108) and Skorohod (1991, Th. 3.21), respectively. Recently, there has been some interest in the structure of cone-valued subordinators, cone-parameter additive processes and their subordination in the case of finite dimensional cones; see for example Barndorff-Nielsen, Pedersen and Sato (2001), Pedersen and Sato (2003a, 2003b) and Pedersen (2003), among others.

From the mathematical point of view, it is of natural interest to extend the study of additive and Lévy processes to infinite dimensional Banach spaces and investigate whether there is an intrinsic relation between probability and functional analytic aspects.

Relevant references in this setting are the pioneering works by Gihman and Skorohod (1975), Kuelbs (1979) and Dettweiler (1982) who study the Lévy-Khintchine representation, rates of growth of sample paths and the Lévy-Itô decomposition, respectively. A recent reference is by Albeverio and Rudiger (2002), who deal with the Lévy-Itô decomposition and stochastic integrals.

This thesis deals with subordinators taking values in cones of infinite dimensional Banach spaces. It contributes to the theory of additive and Lévy processes in Banach spaces in three specific topics.

The first subject is the characterization of the subordinated Banach space valued process $\{Y_t = X_{\sigma_t} : t \geq 0\}$, in the case when X is a Banach space valued Lévy process and σ is an independent one dimensional subordinator. In order to explicitly obtain the generating triplet of the Lévy process Y in terms of the generating triplets of X and σ, it is first required to derive some appropriate distributional properties for Lévy processes in Banach spaces. More specifically, we first extend to Banach spaces results on the relation between g-moments of Lévy processes and g-moments of Lévy measures due to Kruglov (1970, 1972) for the finite dimensional and Hilbert space cases respectively. In addition, we prove useful tail and conditional moment estimates for Lévy processes in Banach spaces, extending and generalizing analogous ones for the Euclidean case.

The second problem deals with the structure of additive processes and subordinators taking values in a cone K of an infinite dimensional Banach space and with the question of whether there exists a special form of the Lévy-Khintchine representation for K-valued additive processes similar to the one dimensional case (1)-(2). Gihman and Skorohod (1975) deal with a restricted class of Banach spaces where the answer is affirmative. In the present work we conduct a systematic and detailed study on the structure of additive processes and subordinators in general Banach spaces and on the type of convergence of their non-compensated jumps. In particular, we find that the existence of a special Lévy-Khintchine representation for additive processes analogous to (1)-(2) is related to the geometry of the Banach space state, a fact that is not unusual to find in the study of convergence of independent random variables in Banach spaces, as found for symmetric random variables, among others, by Kwapień (1974) and Hoffmann-Jorgensen (1974); see also the survey by Woyczyński (1978). Specifically, we show that the special Lévy-Khintchine representation for cone-valued additive processes in Banach spaces is valid for proper cones containing no copy of c_0^+ (the cone of nonnegative scalar sequences converging to zero) and whose dual cone is generating. We propose to call a cone with these three properties an *LK-cone*. Several examples of cone-additive processes and subordinators are constructed, including the new class of α-tempered subordinators, $0 < \alpha < 2$, which extends the corresponding results for matrices introduced in Barndorff-Nielsen and Pérez-Abreu (2002) to Banach spaces. The case $\alpha = 1$ may be thought as the Banach space version of the important one dimensional inverse Gaussian subordinator (Barndorff-Nielsen (1998)).

Finally, the third topic deals with properties of subordinators taking values in a special

class of cones in Banach spaces where there exists a continuous linear functional that agrees with the norm in the cone. These spaces are called of Birkhoff-Kakutani type in this work, after Birkhoff (1938) and Kakutani (1941) who studied Banach lattices and the so-called *AL-spaces* where the norm is additive in the positive elements. The latter lattices form a subclass of the more general Birkhoff-Kakutani spaces. Specific examples are the Euclidean space $\mathbb{R}^d$, the space of symmetric real $d \times d$-matrices $\mathbb{M}^d$, the space $L_1(H)$ of trace class operators in a Hilbert space H and duals of C^*-algebras. In addition, it is found that for a type of countably Hilbertian nuclear spaces studied in Kallianpur and Xiong (1995) -of which the Schwartz space $S(\mathbb{R}^d)$ of rapidly decreasing functions is an example- it is always possible to find a sequence of norms of Birkhoff-Kakutani type generating the original nuclear topology. In connection with Lévy processes, Birkhoff-Kakutani-valued subordinators inherit some interesting properties of the one dimensional subordinator associated to their norm-processes. As an application of this inheritance, sample path properties on the asymptotic behavior of one dimensional subordinators are transferred to this Banach space setting, such as the law of large numbers and the law of iterated logarithm in Bertoin (1996, Prop. 3.8 and Th. 3.11). These asymptotic properties give new results even in finite dimensional spaces of dimensions higher than one, specifically, for subordinators taking values in the octant cone $\mathbb{R}_+^d$, $d \geq 2$, and in the cone $\mathbb{M}_+^d$ of nonnegative definite symmetric real $d \times d$-matrices.

This thesis is organized as follows. Chapter 1 presents material on cones in Banach spaces. The first part contains notation and preliminaries, stressing the convergence of sequences and series in cones. A collection of cones relevant for this work is also provided. The second part introduces Birkhoff-Kakutani spaces including the above mentioned examples.

Chapter 2 is devoted to Bochner's subordination of a Banach space valued Lévy process by an independent one dimensional subordinator. The first part presents well known results on additive and Lévy processes in Banach spaces, with emphasis in the Lévy-Khintchine representation, the description of Lévy measures, the construction of such processes and the self-decomposability property. The second part finds conditions for the existence of g-moments and derives several distributional properties for Lévy processes in Banach spaces.

They are applied in the last part of the chapter to explicitly identify the generating triplet of the Banach space valued process obtained by Bochner's transformation.

Chapter 3 deals with additive processes and subordinators taking values in cones of Banach spaces. A systematic and detailed analysis of cone-valued additive processes is done, with special accent on the convergence of the uncompensated jumps and the nature of conditions that give rise to the special Lévy-Khintchine representation. As important implications, regular subordinators in cones of Banach spaces are introduced, their corresponding Laplace transforms are obtained and self-decomposability properties are pointed out. The last part of this chapter describes three useful methods for constructing subordinators in Banach spaces, providing several examples, including a construction of subordinators in cones with bases.

Finally, Chapter 4 studies additive processes and subordinators with values in cones of Birkhoff-Kakutani spaces. It first presents their one dimensional similarities and establishes a connection with the norm processes. It includes an application of this intimate relation to the study of rates of growth of some sample path behavior of Birkhoff-Kakutani valued subordinators, including laws of large numbers and laws of iterated logarithm at small and large times.

Chapter 1

Cones in Banach Spaces

The study of infinite dimensional subordinators leads naturally to consider convergence of elements in infinite dimensional cones. The first part of this chapter includes preliminaries on cones in Banach spaces, establishing notation and recalling two results on convergence in cones as well as basic facts about cones with bases. We refer to the books by Kamthan and Gupta (1985) and Schaefer (1999, Ch. 5) for a systematic treatment of cones in topological vector spaces.

For the sake of completeness of presentation we also include a collection of interesting examples of the so-called LK-cones. We then introduce Birkhoff-Kakutani spaces and present several important Banach spaces of this type.

1.1 Preliminaries

1.1.1 Definitions and convergence in cones

We recall that a nonempty closed convex set K of B is said to be a *cone* if $\lambda \geq 0$ and $x \in K$ imply $\lambda x \in K$. It is noted that a cone is closed under finite sums and contains the zero element. A cone K is said to be *generating* if $B = K - K$, that is, every $x \in B$ can be written as $x = x_1 - x_2$ for $x_1 \in K$ and $x_2 \in K$. A cone K is called a *proper cone* if $x = 0$

whenever x and $-x$ are in K. The *dual cone* K^* of K is defined as

$$K^* = \{f \in B^* : f(s) \geq 0 \text{ for every } s \in K\} \tag{1.1}$$

the set of continuous linear functionals on B which are nonnegative on the cone K. It is easy to see that K^* is a cone of B^*. A linear functional $f \in B^*$ is called *positive linear functional* (with respect to K) if $f \in K^*$.

We recall that a vector space V over $\mathbb{R}$ endowed with an order $\leq$ is called an *ordered vector space* if $x \leq y$ implies $x + z \leq y + z$ for all $x, y, z \in V$ and if $x \leq y$ implies $\lambda x \leq \lambda y$ for all $x, y \in V$ and $\lambda \geq 0$. A proper cone K of a Banach space B introduces a partial order on B (and therefore B becomes an ordered vector space) by defining $x_1 \leq_K x_2$ whenever $x_2 - x_1 \in K$ for $x_1 \in B$ and $x_2 \in B$. This allows us to define the notions of increasingness and decreasingness in B. Namely, given a sequence $(x_n) = (x_n)_{n=1}^{\infty}$ in B, if $x_n \leq_K x_{n+1}$ for each $n \geq 1$, the sequence is called K-*increasing*. If $x_{n+1} \leq_K x_n$, for each $n \geq 1$, the sequence is called K-*decreasing*. Likewise, a function $f : [0, \infty) \to B$ is called K-*increasing* if $f(t_1) \leq_K f(t_2)$ for $t_1 \leq t_2$ and it is called K-*decreasing* if $f(t_2) \leq_K f(t_1)$ for $t_1 \leq t_2$.

A sequence (x_n) in K is said to be K-*majorized* if there exists $x \in K$ with $x_n \leq_K x$, for $n \geq 1$. A cone K is said to be *regular* if every K-increasing and K-majorized sequence in K is norm convergent. A cone K is called *normal* if $0 \leq_K x \leq_K y$ where $y \in K$, implies $\|x\| \leq \lambda \|y\|$, where $\lambda > 0$ is a constant.

Let c_0 denote the Banach space of real sequences $a = (a_n)$ converging to zero with norm $\|a\| = \sup_{n \geq 1} |a_n|$ and let c_0^+ denote the cone of c_0 consisting of all sequences with nonnegative terms. We denote by (e_n) the canonical sequence $e_n = (0, ..., 0, 1, 0, ...)$ where 1 is in the n^{th}-term, $n \geq 1$. It is called the unit vector basis of c_0 and it is noted that it belongs to c_0^+.

The following is a well known result of Bessaga-Pelczynski type on the convergence of series in Banach spaces. We have formulated Proposition 1.1.2 below in a way that is convenient for this work. We have included its proof which follows step by step the proof for Banach spaces elements (Diestel (1984, Th. 5.8)), but using only the fact that the

elements in the series belong to a cone.

Recall that a series $\sum_{k=1}^{\infty} x_k$ in B is *weakly unconditionally Cauchy (w.u.C.)* if for all $f \in B^*$, $\sum_{k=1}^{\infty} |f(x_k)|$ is a real convergent series. Note that $\sum_{k=1}^{\infty} e_k$ is w.u.C. where (e_n) is the unit vector basis of c_0 since for every $f \in c_0^*$ we have

$$\sum_{k=1}^{n} |f(e_k)| = \sum_{k=1}^{n} sign(f(e_k))f(e_k) = f\left(\sum_{k=1}^{n} sign(f(e_k))e_k\right) \leq \|f\|',$$

for each $n \geq 1$, where $\|f\|' = \sup_{\|x\|=1} |f(x)|$. It is clear that $\sum_{k=1}^{\infty} e_k$ does not converge strongly because it is not a Cauchy sequence. We shall use systematically the following terminology.

Definition 1.1.1 Let K_1 and K_2 be two cones of the Banach spaces B_1 and B_2. It is said that K_1 is *isomorphic to* K_2 if there is an isomorphism φ between $\overline{Span}(K_1)$ and $\overline{Span}(K_2)$ such that $\varphi(K_1) = K_2$.

Proposition 1.1.2 *Let K be a cone of a Banach space B. In order that any w.u.C. series $\sum_{k=1}^{\infty} x_k$, with $x_n \in K$, $n \geq 1$, be (norm) unconditionally convergent, it is necessary and sufficient that K contain no subcone isomorphic to c_0^+.*

Proof. Necessity. Suppose that K contains a subcone K_2 isomorphic to c_0^+. Then there is an isomorphism φ from c_0 onto $\overline{Span}(K_2)$ with $\varphi(c_0^+) = K_2$. Note that $\varphi(e_n) \in K_2$ since for the unit vector basis $e_n \in c_0^+$. Since the series $\sum_{k=1}^{\infty} e_k$ is w.u.C. but not unconditionally convergent then $\sum_{k=1}^{\infty} \varphi(e_k)$, where $\varphi(e_n) \in K$, is w.u.C. but not unconditionally convergent. This is a contradiction.

Sufficiency. Suppose that there exists a sequence (x_n) in K where $\sum_{k=1}^{\infty} x_k$ is w.u.C. but not unconditionally convergent. Then exist sequences (p_n) and (q_n) of positive integers with $p_1 < q_1 < p_2 < q_2 ...$ such that $\inf_n \left\|\sum_{k=p_n}^{q_n} x_k\right\| > 0$. Let $y_n = \sum_{k=p_n}^{q_n} x_k$ and note that $y_n \in K$ for all n. By the Bessaga-Pelczynski principle (Diestel (1984, p. 42)) we select a basic subsequence (y_{n_k}) of (y_n) since $f(y_n) \to 0$ for all $f \in B^*$ and $\inf_n \|y_n\| > 0$. Note that the basic subsequence (y_{n_k}) also satisfies $f(y_{n_k}) \to 0$ for all $f \in B^*$ and $\inf_k \|y_{n_k}\| > 0$ and hence it is equivalent to the unit vector basis (e_k) of c_0 (Diestel (1984, Cor. 5.7)). In consequence there is an isomorphism φ from c_0 onto $\overline{Span}(y_{n_k})$ which carries each e_k to y_{n_k}. Moreover $\varphi(\sum_{k=1}^{\infty} a_k e_k) = \sum_{k=1}^{\infty} a_k y_{n_k}$. Considering only $a_n \geq 0$ we get $\varphi(c_0^+) \subset K$ by closedness of K. We have shown that K has a subcone isomorphic to c_0^+. This is a contradiction. ∎

Remark 1.1.3 a) A Banach space that contains no copy of c_0 does not have cones isomorphic to c_0^+. Indeed, if the space contains a subcone isomorphic to c_0^+ then by Definition 1.1.1 there is an isomorphism between c_0 and the closedness of the linear span of such a cone. This implies that the whole space contains a copy of c_0 which is a contradiction.

b) Every Banach space has a cone with no subcones isomorphic to c_0^+. Indeed, let B be a Banach space. If B contains no copy of c_0 the assertion follows from (a). Suppose B contains a copy of c_0. Take the cone of B which is isomorphic to the cone c_0^{++} of c_0 defined in the next example. Now the assertion follows from Proposition 1.1.2.

Example 1.1.4 Let $c_0^{++} = \{(a_n) \in c_0 : a_n \geq a_{n+1}, n \geq 1\}$. Clearly c_0^{++} is a cone of c_0. Note that the continuous linear functional f on c_0 defined by $f(a) = a_1$ where $a = (a_n)$ coincides with the norm in c_0^{++}. Thus every weakly unconditionally Cauchy series in c_0^{++} is unconditionally convergent.

A major result in the convergence of cone-valued sequences is given by the proposition below. It was proved by Kamthan and Gupta (1985, Th. 12.4.5) in the case of Fréchet spaces. We formulate it in a way that is useful for our purposes, providing a simpler proof of the sufficiency part.

Proposition 1.1.5 *Let K be a cone of B whose dual cone K^* is generating for B^*. Then K is regular if and only if K contains no subcones isomorphic to c_0^+.*

Proof. *Necessity.* Suppose that K is regular. If it contains a subcone isomorphic to c_0^+, by definition, B contains a subspace isomorphic to c_0. Since K^* is a generating cone of B^* (i.e. K is normal), from Theorem 12.4.5 in Kamthan and Gupta (1985) the cone K is not regular. This is a contradiction and therefore K contains no subcones isomorphic to c_0^+.

Sufficiency. Assume that K contains no subcones isomorphic to c_0^+. We shall prove that any sequence (x_n) in K which is K-increasing and K-majorized by some $x \in K$ is norm convergent. Let $s_n = \sum_{k=1}^{n}(x_{k+1} - x_k)$ for $n \geq 1$. Note that $s_n = x_{n+1} - x_1 \in K$. In view of Proposition 1.1.2 is enough to prove that $\sum_{k=1}^{\infty} |f(x_{k+1} - x_k)| < \infty$ for every $f \in B^*$. Decompose f into $f = f^+ - f^-$ where $f^+ \in K^*$ and $f^- \in K^*$ and notice that $\{f^+(s_n)\}$ and $\{f^-(s_n)\}$ are nonnegative increasing sequences which are bounded by $f^+(x - x_1)$ and $f^-(x - x_1)$ respectively. Hence they are real convergent sequences and since $|f(x)| = f^+(x) + f^-(x)$, we obtain $\sum_{k=1}^{\infty} |f(x_{k+1} - x_k)| < \infty$. ∎

We introduce the following often used terminology throughout this thesis.

Definition 1.1.6 (LK-Cone) A proper cone K is said to be an *LK-cone* if it contains no copy of c_0^+ and its dual cone K^* is generating for B^*.

Examples of LK-cones are presented in Section 1.2. On the other hand, for a cone containing a copy of c_0^+ -and therefore an example of a cone that is not of LK-type- we mention the positive compact self-adjoint operators $K^+(H)$ in a Hilbert space H with the operator norm (Brown (1995)). We observe that the sufficiency part of Proposition 1.1.2 gives a method to construct examples of cones K containing a copy of c_0^+ whenever there exists a w.u.C. series with elements in K which is not unconditionally convergent. As for an example of a cone whose dual is not generating, we mention the subcone of c_0 defined by $K = \{(a_n) \in c_0 : a_n + a_{n+1} \geq 0\}$; see Example 12.3.5 in Kamthan and Gupta (1985).

1.1.2 Cones with bases

Given a basis of a Banach space, there is a natural cone associated to it. In this section we recall basic definitions and a characterization theorem for cones generated by bases of type l_+. We refer to Singer (1970) for an extensive discussion on cones generated by bases.

Recall that a sequence (x_n) of a Banach space B is called a *basis* of B if for every $x \in B$

there exists a unique sequence of real numbers $\{\alpha_n\}$ such that

$$x = \sum_{k=1}^{\infty} \alpha_k x_k,$$

i.e. $lim_{n\to\infty} \|x - \sum_{k=1}^{n} \alpha_k x_k\| = 0$. A basis (x_n) of a Banach space is said to be a *bounded basis* if

$$0 < \inf_{n\geq 1} \|x_n\| \leq \sup_{n\geq 1} \|x_n\| < \infty.$$

A basis (x_n) of a Banach space B gives rise the natural cone

$$K_{(x_n)} = \left\{ \sum_{k=1}^{\infty} \alpha_k x_k \in B : \alpha_k \geq 0 \text{ for all } k \right\},$$

called the *cone associated to the basis* (x_n) (or the cone generated by (x_n)). In fact, $K_{(x_n)}$ is a proper cone of B and it coincides with the *cone generated by* (x_n), that is, the smallest cone containing the basis (x_n).

Definition 1.1.7 A basis (x_n) of a Banach space B is said to be *of type* l_+ if there exists a constant $\eta > 0$ such that

$$\left\| \sum_{k=1}^{n} \alpha_k x_k \right\| \geq \eta \sum_{k=1}^{n} \alpha_k \tag{1.2}$$

for any finite sequence of nonnegative numbers $\alpha_1, \alpha_2, ..., \alpha_n$.

For instance, the natural basis (e_n) of l_1, the Banach space of of scalar sequences $x = (x_n)$ with norm $\|x\| = \sum_{k=1}^{\infty} |x_k| < \infty$, is of type l_+. Proposition 1.3.4 below provides additional examples.

The next result gives an easy characterization of cones generated by a basis of type l_+. It is presented in Theorem 2.10.2 in Singer (1970).

Theorem 1.1.8 *Let $K_{(x_n)}$ be the associated cone to the bounded basis (x_n) of a real Banach space B. Then the following are equivalent:*

a) (x_n) is of type l_+

b) We have that

$$K_{(x_n)} = \left\{ \sum_{k=1}^{\infty} \alpha_k x_k \in B : \alpha_k \geq 0 \text{ for all } k, \ \sum_{k=1}^{\infty} \alpha_k < \infty \right\},$$

that is, $\sum_{k=1}^{\infty} \alpha_k x_k \in K_{(x_n)}$ if and only if $\alpha_k \geq 0$ for all k and $\sum_{k=1}^{\infty} \alpha_k < \infty$.

The cone $K_{(x_n)}$ in the former characterization is called of *type* l_+. We end this section by proving that $K^*_{(x_n)}$ is generating for B^*.

Lemma 1.1.9 *Let $K_{(x_n)}$ be the associated cone to the bounded basis (x_n) of a real Banach space B. Then the dual cone $K^*_{(x_n)}$ is generating for B^*.*

Proof. From Kamthan and Gupta (1985, Th. 1.5.4 and Prop. 1.5.7) we have the following equivalence: $K^*_{(x_n)}$ is generating cone if and only if K is normal. It is then enough to prove that $0 \leq_{K_{(x_n)}} x \leq_{K_{(x_n)}} y$ with $y \in K$ implies $\|x\| \leq \lambda \|y\|$, where $\lambda > 0$ is a constant. Indeed, we first notice from (1.2) that for every $\sum_{k=1}^{\infty} \alpha_k x_k \in B$ with $\alpha_n \geq 0$,

$$\left\| \sum_{k=1}^{\infty} \alpha_k x_k \right\| \geq \eta \sum_{k=1}^{\infty} \alpha_k,$$

where $\eta > 0$ is the constant in (1.2). Now, assume that $x = \sum_{k=1}^{\infty} \alpha_k x_k$ and $y = \sum_{k=1}^{\infty} \beta_k x_k$ belong to K. Note that $0 \leq \alpha_n \leq \beta_n$. Then, using the former inequality and the fact that (x_n) is a bounded basis, we have

$$\begin{aligned} \left\| \sum_{k=1}^{\infty} \alpha_k x_k \right\| &\leq \left(\sup_{n\geq 1} \|x_n\| \right) \sum_{k=1}^{\infty} \alpha_k \leq \left(\sup_{n\geq 1} \|x_n\| \right) \sum_{k=1}^{\infty} \beta_k \\ &\leq \left(\frac{1}{\eta} \sup_{n\geq 1} \|x_n\| \right) \left\| \sum_{k=1}^{\infty} \beta_k x_k \right\|. \end{aligned}$$

■

1.2 Examples of LK-cones

For the sake of completeness and presentation, this section collects several examples of LK-cones in Banach spaces. We recall from Remark 1.1.3 (a), that a cone in a Banach space containing no copy of c_0 cannot contain any subcone isomorphic to c_0^+.

The class of reflexive Banach spaces provides spaces containing no copy of c_0 (Guerre-Delabrière (1992, p. 36)).We first point out concrete examples of LK-cones in three reflexive spaces. Let $1 < p, q < \infty$ such that $1/p + 1/q = 1$.

Example 1.2.1 (l_p spaces, $1 < p < \infty$) Let l_p be *the separable Banach space of scalars sequences* $x = (x_n)$ with norm $\|x\|_p = \{\sum_{k=1}^{\infty} |x_k|^p\}^{1/p}$. The dual of l_p is l_q via the isometric isomorphism $l_q \ni y \longmapsto \varphi_y \in l_p^*$ given by $\varphi_y(x) = \sum_{k=1}^{\infty} x_k y_k$. Consider the proper cone $l_p^+ = \{(x_n) \in l_p : x_n \geq 0\}$. Then l_q^+ is the dual cone of l_p^+ and it is a generating cone for the dual l_q. In fact, any $(a_n) \in l_q$ is decomposed into $(a_n) = (a_n^+) - (a_n^-)$ where $a_n^+ \geq 0$ and $a_n^- \geq 0$. Clearly the sequences (a_n^+) and (a_n^-) are in l_q^+ because $a_n^+ \leq |a_n|$ and $a_n^- \leq |a_n|$. Hence l_p^+ is an LK-cone in l_p.

Example 1.2.2 (L_p-spaces, $1 < p < \infty$) Let $(X, \mathcal{A}, \mu)$ be a measure space and $L_p(\mu)$ *the Banach space of equivalence classes of measurable functions* defined on X which are p-integrable with respect to μ. Here $\|x\|_p = \left\{\int_X |x(s)|^p \mu(ds)\right\}^{1/p}$ is the norm of the element represented by the function x. The dual of $L_p(\mu)$ is isometrically isomorphic to $L_q(\mu)$. In fact, for each $\varphi \in L_p^*(\mu)$ there is a $y \in L_q(\mu)$ such that $\varphi(x) = \int_X xyd\mu$. The cone $L_p^+(\mu) = \{x \in L_p(\mu) : x \geq 0\}$ is proper with dual cone $L_q^+(\mu)$. The latter is a generating cone for $L_q(\mu)$, since for $y \in L_q(\mu)$, $y = y^+ - y^-$, the positive and negative function parts of y which are in $L_q^+(\mu)$. Then for any $1 < p < \infty$, $L_p(\mu)$ is an LK-cone in $L_p(\mu)$.

Example 1.2.3 (p-trace class spaces, $1 < p < \infty$) Let $(H, \langle \cdot, \cdot \rangle)$ be a separable Hilbert space and let $L_1(H)$ denote *the separable (not reflexive) Banach space of selfadjoint trace class operators of* H so that $tr(|S|) = \sum_{k=1}^{\infty} s_k < \infty$ where s_n, $n \geq 1$, are the singular values

of S, i.e., the eigenvalues of the positive operator $|S| := \{S^*S\}^{1/2}$. The trace-norm is defined by $\|S\|_1 = tr(|S|)$ and the trace of S by $tr(S) = \sum_{k=1}^{\infty} \langle S\phi_k, \phi_k \rangle$ where (ϕ_n) is a complete orthonormal set of H. Let $L_p(H)$ be the set of selfadjoint compact operators S of H such that $|S|^p \in L_1(H)$ and let $\|S\|_p = \{tr(|S|^p)\}^{1/p} = \{\sum_{k=1}^{\infty} |s_k|^p\}^{1/p}$. Then $L_p(H)$ is a Banach space with norm $\|\cdot\|_p$. The dual space $L_p^*(H)$ is isometrically isomorphic to $L_q(H)$ where for each $\varphi \in L_p^*(H)$ there is a $T \in L_q(H)$ such that $\varphi(S) = tr(TS)$ (Reed and Simon (1975, Prop. 7)). Consider the proper cone $L_p^+(H) = \{S \in L_p(H) : S \text{ is a positive operator}\}$. Then $L_q^+(H)$ is the dual cone of $L_p^+(H)$ and it is a generating cone for $L_q(H)$. Indeed, for $S \in L_q(H)$ its spectral representation $\sum_{k=1}^{\infty} s_k \langle \cdot, \phi_k \rangle \phi_k$ is decomposed it into the difference of positive compact operators $S^+ = \sum_{k=1}^{\infty} s_k^+ \langle \cdot, \phi_k \rangle \phi_k$ and $S^- = \sum_{k=1}^{\infty} s_k^- \langle \cdot, \phi_k \rangle \phi_k$, where $s_n = s_n^+ - s_n^-$. It is clear that $S^+, S^- \in L_q^+(H)$. Then for any $1 < p < \infty$, $L_p^+(H)$ is an LK-cone.

Example 1.2.4 (Weakly sequentially complete spaces) A Banach space B is called *weakly sequentially complete*, if every weakly Cauchy sequence is weakly convergent. These spaces contain no subcones isomorphic to c_0^+ (Guerre-Delabrière (1992, p. 36)). Let K be a proper cone of B with generating dual cone K^*. Then K is a LK-cone. In particular, since $L_1(\mu)$ is weakly sequentially complete space (not reflexive), then $L_1^+(\mu) = \{x \in L_1(\mu) : x \geq 0\}$ is an LK-cone.

Example 1.2.5 (Schur spaces) A Banach space has the *Schur property* or it is a *Schur space*, if every sequence which converges weakly to zero, converges strongly to zero. Schur spaces contain no copy of c_0, since every weakly Cauchy sequence is norm convergent (Wnuk (1993)). Then, any proper cone K with generating dual is an LK-cone. In particular, since the Banach space l_1 is a Schur space, its proper cone $l_1^+ = \{(x_n) \in l_1 : x_n \geq 0\}$ has generating dual cone and therefore is an LK-cone. An interesting class of new examples of Schur spaces can be found in Ülger (1997), who characterized the closed subspaces of the compact operators $K(H)$ of a Hilbert space H, whose duals have the Schur property.

We recall that a *vector lattice* is an ordered vector space $(V, \leq)$ such that $\sup\{x, y\}$ and $\inf\{x, y\}$ exist for all x, $y \in V$. This implies in particular that the subset $V_+ = \{x \in V :$

$x \geq 0\}$ is a generating proper cone for V. In a vector lattice $(V, \leq)$ every $x \in V$ has a unique representation $x = x^+ - x^-$ as a difference of two elements $x^+ \in V_+$ and $x^- \in V_+$ satisfying $\inf\{x^+, x^-\} = 0$. The absolute value of $x \in V$ is defined as $|x| = x^+ + x^-$. A *Banach lattice* is a vector lattice $(V, \leq)$ with a complete norm $\|\cdot\|$ satisfying $\|x\| \leq \|y\|$ whenever $|x| \leq |y|$ for all $x, y \in V$. We refer to the book by Schaefer (1974) for a detailed treatment on Banach lattices.

Example 1.2.6 (p-Schur spaces) It is said that a Banach lattice has the *positive Schur property* if every weakly null sequence with positive terms is norm null. These Banach lattices contain no copy of c_0 (Wnuk (1993, p.18)). Examples of these spaces are the AL-spaces (see Example 1.3.10 below). We refer to Wnuk (1993) for a survey on this particular subject.

1.3 Birkhoff-Kakutani Spaces

1.3.1 Basic properties

Birkhoff (1938) and Kakutani (1941) studied the so-called AL-spaces. In this section we introduce a more general class of important Banach spaces which includes the formers.

Definition 1.3.1 Let $(B, \|\cdot\|)$ be a Banach space and let K be a proper cone of B. The triplet $(B, \|\cdot\|, K)$ is called a *Birkhoff-Kakutani space* if there exists a continuous linear functional $f_0 \in B^*$ such that $f_0(x) = \|x\|$ for every $x \in K$. In particular, the norm is additive in K. In this case, it is said that K is a cone *of Birkhoff-Kakutani type*.

Birkhoff-Kakutani spaces have important properties which are collected in the proposition below.

Proposition 1.3.2 *Let $(B, \|\cdot\|, K)$ be a Birkhoff-Kakutani space. Then, K is an LK-cone. In particular,*

a) The dual cone K^* is a generating cone of B^*.

b) Every weakly unconditionally Cauchy series in K is norm convergent.

c) K contains no subcones isomorphic to c_0^+.

d) K is a regular cone.

Proof. a) Assume that f_0 is a continuous linear functional on B such that $f_0(x) = \|x\|$ for all $x \in K$. Let f be any nonzero continuous linear functional on B. Define $f_1(x) = f(x) + \|f\|' f_0(x)$ and $f_2(x) = \|f\|' f_0(x)$ for all $x \in B$. Clearly f_1 and f_2 are continuous linear functionals on B which are nonnegative on K and $f_1(x) - f_2(x) = f(x)$.

b) Let (x_n) be a sequence in K and let $s_n = \sum_{k=1}^{n} x_k$. For every $f \in B^*$, $\sum_{k=1}^{\infty} |f(x_k)|$ is finite. In particular for f_0, if $n > m$, $s_n - s_m \in K$ and therefore $f_0(s_n - s_m) = \|s_n - s_m\|$. Then (s_n) is norm convergent.

Trivially, (c) follows from Proposition 1.1.2. Now, that K is an LK-cone follows from (a). Finally, (e) follows from (c). ■

It is possible to construct a Birkhoff-Kakutani space starting from a norm which is already additive in a generating cone, as it is shown in the following example.

Example 1.3.3 Let $(B, \|\cdot\|)$ be a Banach space with generating cone K. That is, for every $x \in B$, $x = x^+ - x^-$ where $x^+, x^- \in K$. Assume that the norm is additive in K. Then the function $f : B \to \mathbb{R}$ defined by $f(x) = \|x^+\| - \|x^-\|$ is a well defined continuous linear functional on B. Thus $(B, \|\cdot\|, K)$ is a Birkhoff-Kakutani space.

Indeed, notice that if $x = x_1 - x_2$ where $x_1 \in K$ and $x_2 \in K$ is another representation of x, then $x_1 + x^- = x^+ + x_2$ and hence $\|x_1\| + \|x^-\| = \|x^+\| + \|x_2\|$. Therefore the function $f : B \to \mathbb{R}$ is well defined. We shall prove that f is linear. Let $x = x^+ - x^-$ and $y = y^+ - y^-$ where x^+, x^-, y^+ and y^- are in K. Let $a \in \mathbb{R}$. If $a \geq 0$ we have

$$f(ax+y) = \|ax^+ + y^+\| - \|ax^- + y^-\| = a\|x^+\| + \|y^+\| - a\|x^-\| - \|y^-\| = af(x) + f(y).$$

If $a < 0$ we write $ax = -ax^- - (-a)x^+$. Then

$$f(ax+y) = -a(\|x^-\| - \|x^+\|) + (\|y^+\| - \|y^-\|) = af(x) + f(y).$$

Finally, it is clear that f is continuous.

Under a simply and not an unusual assumption, cones of Birkhoff-Kakutani type contains subcones of type l_+.

Proposition 1.3.4 *Let $(B, \|\cdot\|, K)$ be a Birkhoff-Kakutani space with a bounded basis (x_n) in K. Then the cone $K_{(x_n)}$ is of type l_+.*

Proof. Notice that for any finite sequence of nonnegative numbers $\alpha_1, \alpha_2, ..., \alpha_n$, using the additivity of the norm in K, we have

$$\left\| \sum_{k=1}^{n} \alpha_k x_k \right\| = \sum_{k=1}^{n} \alpha_k \|x_k\| \geq \eta \sum_{k=1}^{n} \alpha_k,$$

where $\eta = \inf_{n\geq 1} \|x_n\| > 0$. Now the assertion follows from Theorem 1.1.8. ■

1.3.2 Examples

Several important and interesting Banach spaces are examples of Birkhoff-Kakutani type spaces, which provide examples of their associated LK-cones.

Example 1.3.5 The space $(c_0, \|\cdot\|, c_0^{++})$ introduced in Example 1.1.4 is of Birkhoff-Kakutani type.

Example 1.3.6 Consider $\mathbb{R}^n$ with the norm $\|x\| = |x_1| + |x_2| + ... + |x_n|$ where $x = (x_1, x_2, ..., x_n)$. Let $\mathbb{R}^n_+$ be the set of x in $\mathbb{R}^n$ with nonnegative coordinates. Then $\mathbb{R}^n_+$ is a proper cone. The linear functional $x \longmapsto x_1 + x_2 + ... + x_n$ coincides with the norm of x in $\mathbb{R}^n_+$. Thus $(\mathbb{R}^n, \|\cdot\|, \mathbb{R}^n_+)$ is a Birkhoff-Kakutani space.

Example 1.3.7 (Matrices) Let $M_{n\times n} = M_{n\times n}(\mathbb{R})$ be the finite dimensional Banach space of $n \times n$ real symmetric matrices with norm $\|A\| = tr(|A|)$ where $tr(A)$ denote the trace of A and $|A|$ is the square root matrix of AA^T. Let $M^+_{n\times n}$ denote the set of nonnegative definite matrices in $M_{n\times n}$. Then $M^+_{n\times n}$ is a proper cone and $(M_{n\times n}, tr(|\bullet|), M^+_{n\times n})$ is a Birkhoff-Kakutani space.

The Banach space of trace class operators in a separable Hilbert space provides an important example of a Birkhoff-Kakutani space, which is not a Banach lattice. Namely,

Example 1.3.8 (Trace class operators) Let $(L_1(H), \|\cdot\|)$ be the Banach space of selfadjoint trace class operators of a separable Hilbert space H with norm $\|S\| = \sum_{n=1}^{\infty} s_n$ (where the s_n are the eigenvalues of $|S|$). The set $L_1^+(H)$ of positive trace class (covariance) operators in $L_1(H)$ is a proper cone.

Let $V \longmapsto tr(V)$, for all $V \in L_1(H)$, be the trace functional that provides a continuous linear functional, which restricted to the cone $L_1^+(H)$ satisfies $tr(S) = \|S\|$, since the trace of a positive operator coincides with the sum of its nonnegative eigenvalues. Therefore $(L_1(H), tr(|\bullet|), L_1^+(H))$ is a Birkhoff-Kakutani space.

More generally, we have that the Banach dual of any C^*-algebra is a Birkhoff-Kakutani space.

Example 1.3.9 (Dual of a C^*-algebra) We recall that a Banach algebra A over $\mathbb{C}$ with involution $x \longrightarrow x^*$ satisfying $\|xx^*\| = \|x\|^2$ is called a *C^*-algebra*. Let A^* denote the Banach dual of A and let $A_{sa} = \{x \in A : x = x^*\}$ be the selfadjoint elements of A, i.e., $\phi \in A^*$ is selfadjoint if $\phi(x) \in \mathbb{R}$ for all $x \in A_{sa}$. Then $A_+ = \{xx^* : x \in A\}$ is a proper cone of A_{sa}.

A positive linear functional $\phi \in A^*$ is defined with respect to the cone A^+, i.e., $\phi(A_+) \subset \mathbb{R}_+$. Let A^*_+ denote the proper cone of positive linear functionals of A^*. The cone A^*_+ is generating for A^* (Schaefer (1999, p. 270)). Moroever, the dual norm denoted by $\|\cdot\|'$ is additive in A^*_+, that is, $\|\phi + \psi\|' = \|\phi\|' + \|\psi\|'$ for every $\phi, \psi \in A^*_+$ (Schaefer (1999, Cor. 6.4.2)). Then, using the construction of Example 1.3.3, a continuous linear functional f on

A can be defined satisfying $f(\phi) = \|\phi\|$ for every $\phi \in A_+^*$. Therefore $(A^*, \|\cdot\|', A_+^*)$ is a Birkhoff-Kakutani space. Conditions for separability of duals of C^*-algebras are given in Tomiyama (1963).

Example 1.3.10 (Abstract Lebesgue-spaces) Abstract L-spaces were introduced by Birkhoff (1938) and studied by Kakutani (1941). Recall from Example 1.2.6 that a Banach lattice is a vector lattice $(B, \leq)$ with a complete norm $\|\cdot\|$ satisfying $\|x\| \leq \|y\|$ whenever $|x| \leq |y|$ for all $x, y \in B$.

A Banach lattice $(B, \leq)$ whose norm satisfies $\|x + y\| = \|x\| + \|y\|$ for all $x, y \in B_+$ is called *Abstract L-space or AL-space.* By Example 1.3.3, AL-spaces are Birkhoff-Kakutani spaces for $(B, \|\cdot\|, B_+)$. In particular we have the following examples of AL-spaces:

a) Let V be any vector lattice and let f be a strictly positive linear functional on V. Then V is a normed lattice with the norm $x \longmapsto \|x\| := f(|x|)$ for all $x \in V$, which is additive in V_+. The completion of $(V, \|\cdot\|)$ is an AL-space. (Schaefer (1974, Ex. 2.8.2)).

b) Let X be a locally compact space and let $C(X)$ denote the *vector lattice of continuous functions* from X into $\mathbb{R}$ with compact support. Let $\mu \in C(X)^*$ be a strictly positive Radon measure. Then the completion of $(C(X), \mu)$ (Example a)) is an AL-space. This AL-space is isomorphic to $L_1(\mu)$, the Banach lattice of μ-integrable functions. The $L_1(\mu)$ spaces are the most general AL-spaces up to isomorphism (Schaefer (1974, Th. 2.8.5))

We finally mention that Schaefer (1974, Cor. Th. 2.8.5) gives conditions for separability of AL-spaces.

1.3.3 Countably Birkhoff-Kakutani type nuclear spaces

This section introduces an uncountable family of Birkhoff-Kakutani spaces whose norms generate the topology of a countably Hilbertian nuclear (CHN) space. Besides providing a broad class of Banach spaces of this type, we believe these examples will be useful in studying subordinators in CHN spaces.

We first recall that a family of norms in a vector space is said to be *compatible,* if for any two norms and any Cauchy sequence (with respect to both norms) that converges to zero

with respect to one norm, it converges to zero with respect to the other norm. We also recall that a separable Fréchet space Λ is called *countably Hilbertian* if its topology is given by an increasing sequence $\{\|\cdot\|_n\}_{n\geq 0}$, of compatible Hilbertian norms. A countably Hilbertian space Λ is said to be *nuclear* if for each $n \geq 0$, there exists $m > n$ such that the canonical injection from Λ_m into Λ_n is Hilbert-Schmidt where Λ_n is the completion of Λ with respect to $\|\cdot\|_n$.

We consider the following class of countably Hilbertian nuclear spaces presented in Kallianpur and Xiong (1995). Let H be a real separable Hilbert space with inner product $\langle\cdot,\cdot\rangle_H$ and let $-L$ be a closed densely defined selfadjoint operator of H such that $\langle -L\phi, \phi\rangle_H \leq 0$ for ϕ in the domain of L. Assume that there exist $r_1 > 0$ such that $(I + L)^{-r_1}$ is Hilbert-Schmidt with I the identity operator. Then there exists a complete orthonormal set $\{\phi_j\}$ in H and a sequence $0 \leq \lambda_1 \leq \lambda_2 \leq \ldots$ such that

$$L\phi_j = \lambda_j \phi_j \qquad j \geq 1$$

and

$$\theta^2 := \sum_{j=1}^{\infty} (1 + \lambda_j)^{-2r_1} < \infty. \tag{1.3}$$

Let

$$\Lambda = \left\{ \phi \in H : \sum_{j=1}^{\infty} (1 + \lambda_j)^{2r} \langle \phi, \phi_j \rangle_H^2 < \infty \text{ for every } r \in \mathbb{R} \right\}.$$

Define the inner product $\langle\cdot,\cdot\rangle_r$ on Λ by

$$\langle \phi, \psi \rangle_r = \sum_{j=1}^{\infty} (1 + \lambda_j)^{2r} \langle \phi, \phi_j \rangle_H \langle \psi, \phi_j \rangle_H$$

and $\|\phi\|_r^2 = \langle \phi, \phi \rangle_r$. Let Λ_r be the $\|\cdot\|_r$-completion of Λ.

Note that $\Lambda_s \subset \Lambda_r$ for $r \leq s$ since $\|\cdot\|_r \leq \|\cdot\|_s$, where $\Lambda_0 = H$. On the oter hand, identifying Λ_0^* with Λ_0 by Riesz representation theorem, denote the dual Λ_r^* by Λ_{-r} with norm $\|\cdot\|_{-r}$ for $r \geq 0$ and observe that $\Lambda_{-r} \subset \Lambda_{-s}$ for $0 \leq r \leq s$. Moroever

$$\Lambda = \bigcap_{r=0}^{\infty} \Lambda_r \quad \text{and} \quad \Lambda^* = \bigcup_{r=0}^{\infty} \Lambda_{-r}$$

Condition (1.3) implies that the canonical injection from Λ_p into Λ_q for $p \geq q + r_1$ is Hilbert-Schmidt. Hence Λ is a CHN space. The *Schwartz space* $\mathcal{S}(\mathbb{R})$ of rapidly decreasing functions is an example of a CHN space generated in this way; see Remark 1.3.5 in Kallianpur and Xiong (1995).

We now show that the nuclear topology of Λ is also generated by a sequence of Birkhoff-Kakutani type norms equivalent to the sequence of norms $\{\|\cdot\|_r\}_{r\geq 0}$. For each $r \in \mathbb{R}$ and for $\phi \in \Lambda$, let

$$|\phi|_r = \sum_{j=1}^{\infty} (1+\lambda_j)^r \left|\langle \phi, \phi_j \rangle\right|.$$

We observe that $|\phi|_r$ is finite for $\phi \in \Lambda$. Indeed,

$$\begin{aligned} |\phi|_r &= \sum_{j=1}^{\infty} (1+\lambda_j)^{r+r_1-r_1} \left|\langle \phi, \phi_j \rangle\right| \\ &\leq \left(\sum_{j=1}^{\infty} (1+\lambda_j)^{-2r_1}\right)^{1/2} \left(\sum_{j=1}^{\infty} (1+\lambda_j)^{2(r+r_1)} \left|\langle \phi, \phi_j \rangle\right|^2\right)^{1/2} \leq \theta \, \|\phi\|_{r+r_1} < \infty. \end{aligned}$$

Clearly $|\cdot|_r$ defines a norm in Λ. Since trivially we have that $\|\phi\|_r \leq |\phi|_r$, then

$$\|\phi\|_r \leq |\phi|_r \leq \theta \, \|\phi\|_{r+r_1} \qquad \phi \in \Lambda.$$

We observe that $|\cdot|_r \leq |\cdot|_s$ whenever $r \leq s$. Thus the nuclear topology of Λ is also generated by the compatible sequence of norms $\{|\cdot|_n\}_{n\geq 0}$. Moroever, we have that

$$\Lambda = \left\{ \phi \in H : \sum_{j=1}^{\infty} (1+\lambda_j)^r \left|\langle \phi, \phi_j \rangle\right|_H < \infty \text{ for every } r \in \mathbb{R} \right\}.$$

Next define a cone of Λ by

$$K = \left\{ \phi \in \Lambda : \langle \phi, \phi_j \rangle_H \geq 0 \text{ for } j \geq 1 \right\},$$

and observe that $\Lambda = K - K$. Let $\tilde{\Lambda}_r$ be the $|\cdot|_r$-completion of Λ and let K_r be the $|\cdot|_r$-closure of K. It is clear that $\tilde{\Lambda}_r \subset \Lambda_r$. We observe that for each $r \in \mathbb{R}$, the norm $|\cdot|_r$ is additive in K. Moreover

$$\Lambda = \cap_r \tilde{\Lambda}_r \quad \text{and} \quad \Lambda^* = \cup_r \tilde{\Lambda}_r \tag{1.4}$$

Then we have the following result

Proposition 1.3.11 $(\tilde{\Lambda}_r, |\cdot|_r, K_r)$ *is a Birkhoff-Kakutani space, for each* $r \in \mathbb{R}$.

Proof. According to Example 1.3.3, it is enough to prove that: (i) the norm $|\cdot|_r$ is additive in the cone K_r and that (ii) $\tilde{\Lambda}_r = K_r - K_r$. To prove (i) note that $K \subset K_r$ and obviously the statement is valid for $\phi \in K$. If $\phi^1, \phi^2 \in K_r$ take sequences (ϕ_j^1) and (ϕ_j^2) in K such that $\phi^1 = |\cdot|_r\text{-}lim_{j\to\infty}\, \phi_j^1$ and $\phi^2 = |\cdot|_r\text{-}lim_{j\to\infty}\, \phi_j^2$. Then $|\phi^1 + \phi^2|_r = |\phi^1|_r + |\phi^2|_r$. To prove (ii) recall that the decomposition $\Lambda = K - K$ is true. By a limiting argument we obtain $\tilde{\Lambda}_r = K_r - K_r$. ■

We finally observe that Λ and Λ^* contain no copy of c_0^+. This follows from (1.4) and Propositions 1.3.11 and 1.3.2 (c).

Chapter 2

Lévy Processes in Banach Spaces

The first part of this chapter recalls basic known results and establishes notation on additive and Lévy processes as well as self-decomposable laws in Banach spaces, which are used throughout this thesis.

As the main contribution of this chapter, a complete description of Bochner's subordination of a Banach space valued Lévy process by an independent one dimensional subordinator is given in the second part. Conditions for the existence of g-moments and appropriate estimates for tails and conditional moments of Banach space valued Lévy processes are first derived. They are then applied to obtain in an explicit form the generating triplet of the Banach space valued subordinated Lévy process obtained via this transformation.

2.1 Preliminaries

Additive and Lévy processes in Banach spaces have been studied by several authors. In general, the structure of these processes is similar as for the finite dimensional case. The corresponding Lévy-Khintchine representation for additive processes is given by Gihman and Skorohod (1975) whereas Dettweiler (1982) derives their Lévy-Itô decomposition and Kuelbs (1979) study rates of growth of the sample paths. Recent results on Lévy processes in Banach spaces are by Albeverio and Rudiger (2002) and references therein.

It is known that Lévy processes are in one to one correspondence to infinitely divisible laws. In this direction we refer to the books of Araujo and Giné (1980) and Linde (1986) for a systematic study of infinitely divisible laws in Banach spaces.

2.1.1 Additive processes

Definition 2.1.1 A *B-valued additive process* $\{X_t : t \geq 0\}$ is a stochastic process with values in B defined on a probability space $(\Omega, \mathcal{F}, P)$ satisfying the following

i) For any $0 \leq t_1 < t_2 < ... < t_n$, the random variables $X_{t_2} - X_{t_1}, ..., X_{t_n} - X_{t_{n-1}}$ are independent (*independent increments*).

ii) Almost surely, $X_0 = 0$.

iii) It is *stochastically continuous*, i.e., for every $\varepsilon > 0$, $P(\|X_t - X_s\| > \varepsilon) \to 0$ as $s \to t$.

iv) Almost surely, it is right-continuous in $t \geq 0$ and has left-limits in $t > 0$ (*càdlàg*).

As we already mentioned, the Lévy-Khintchine representation of an additive process in a Banach space is given by Gihman and Skorohod (1975). To obtain such a representation they consider a stochastic process with independent increments on a separable Banach space and proceed in a similar manner as in the one dimensional case to decompose the process into a sum of nonrandom, discrete and stochastically continuous parts. Next, they concentrate in the study of a stochastically continuous process with independent increments on a separable Banach space.

Let $\{X_t : t \geq 0\}$ be a stochastically continuous process with independent increments on a separable Banach space B. Let $\mathcal{B}_0$ denote the ring of Borel sets of B with positive distance from 0. Denote $X_{s-} := \lim_{s \uparrow t} X_s$. Define for any $t > 0$ and C in $\mathcal{B}_0$,

$$N_t(C) = \sum_{s<t} 1_C(X_s - X_{s-}) \quad \text{and} \tag{2.1}$$

$$X_t^C = \sum_{s<t} (X_s - X_{s-}) 1_C(X_s - X_{s-}), \tag{2.2}$$

the *number of jumps* and the *sum of jumps* of the process which have occurred up to time t and took place in C, respectively. Both processes are well defined for each C in $\mathcal{B}_0$.

For any continuous linear functional f on B, the one dimensional process $\{f(X_t)\}$ is stochastically continuous and has independent increments. With the aid of this fact, that supplies a useful information about $\{X_t\}$, one can proceed essentially as in the one dimensional case to arrive at the Lévy-Khintchine representation presented in the following theorem for the case of B-additive processes.

Theorem 2.1.2 (Gihman and Skorohod 1975) *Let $\{X_t : t \geq 0\}$ be an additive process in a separable Banach space B. Then, its characteristic functional $Ee^{if(X_t)}$ has the form*

$$\exp\left\{-\frac{1}{2}(A_t f, f) + if(\gamma_t) + \int \left[e^{if(x)} - 1 - if(x)1_{\{\|x\|\leq 1\}}(x)\right] \nu_t(dx)\right\} \quad f \in B^*, \tag{2.3}$$

where for each $t \geq 0$, γ_t is in B, $(A_t f, f)$ is a nonnegative quadratic functional in f and non-decreasing in t, the Lévy measure ν_t on $B\backslash\{0\}$ is such that for any $f \in B^*$

$$\int_{0<\|x\|\leq 1} f^2(x)\, \nu_t(dx) < \infty. \tag{2.4}$$

Remark 2.1.3 a) For fixed $C \in \mathcal{B}_0$, the processes (2.1) and (2.2) are additive processes and moreover $\{N_t(C)\}$ is a *Poisson process* on $B\backslash\{0\}$ with parameter $EN_t(C) = \nu_t(C)$. The function $\nu_t(C)$ is continuous and non-decreasing in t.

For fixed $t > 0$, $\{N_t(\cdot)\}$ is a *Poisson random measure* on $\mathcal{B}_0$ with intensity measure $EN_t(C) = \nu_t(C)$. Notice that $\nu_t(C) < \infty$ whenever $C \in \mathcal{B}_0$.

b) Let $\triangle_\varepsilon = \{\|x\| > \varepsilon\}$. The process $\left\{X_t - X_t^{\triangle_\varepsilon} : t \geq 0\right\}$ has moments of any order. The centered process $\left\{\left(X_t - X_t^{\triangle_\varepsilon}\right) - E\left(X_t - X_t^{\triangle_\varepsilon}\right) : t \geq 0\right\}$ converges (uniformly on each closed interval $[0, t]$) almost surely as $\varepsilon \downarrow 0$ to a continuous process with independent increments $\{G_t\}$. The process $\{G_t\}$ is *Gaussian*, that is, $\{f(G_t)\}$ is a one dimensional Gaussian process for any continuous linear functional f.

c) $(A_t f, f)$ is a quadratic functional, i.e., A_t is a selfadjoint nonnegative operator from B^* into B and $(A_t f, f)$ denotes the real number $f(A_t f)$.

d) For every $t > 0$, ν_t has a unique σ-finite extension to the σ-algebra $\mathcal{B}(B\backslash\{0\})$ of sets of the form $C \cap B\backslash\{0\}$ with C belonging to the σ-algebra of Borel sets of B. Here $\mathcal{B}(B\backslash\{0\})$ coincides with the σ-algebra generated by the ring $\mathcal{B}_0$.

e) For each $t \geq 0$, (2.3) is the characteristic functional of the *infinitely divisible random variable* X_t taking values in the Banach space B. We shall often used the notation $\hat{\mu}_t$ for the characteristic functional of the random variable X_t.

Remark 2.1.4 a) It is important to recall that for infinite dimensional Banach spaces, Lévy measures are not characterized by the integrability condition $\int_{0<\|x\|\leq 1} \|x\|^2 \nu_t(dx) < \infty$; see Araujo and Giné (1980) and Linde (1986). They are rather identified by the fact that the mapping

$$f \longmapsto \exp\{\int \left[e^{if(x)} - 1 - if(x)1_{\{\|x\|\leq 1\}}(x)\right] \nu_t(dx)\} \quad f \in B^* \tag{2.5}$$

is a characteristic functional of some probability measure on B.

b) (2.5) is the characteristic functional of the process in (2.3) without its continuous part $(A_t, 0, \gamma_t)$ which corresponds to the jump nature of the original process (Lévy part).

c) The system of parameters (A_t, ν_t, γ_t) in Theorem 2.1.2 is called the *system of generating triplets of the process* $\{X_t\}$. The system (A_t, ν_t, γ_t) is unique in the sense that, if the characteristic function of $\{X_t\}$ has the form (2.3) with parameters $(A_t', \nu_t', \gamma_t')$ where $A_t' : B^* \to B$ is a selfadjoint nonnegative operator, $\gamma_t' \in B$ and $\nu_t' : \mathcal{B}(B\backslash\{0\}) \to [0, \infty]$ is a Lévy measure, that is, it satisfies $\nu_t'(\{0\}) = 0$ and (2.5) is a characteristic functional then $A_t = A_t'$, $\nu_t = \nu_t'$ and $\gamma_t = \gamma_t'$ for all $t \geq 0$.

The *Lévy-Itô decomposition* of an additive process $\{X_t : t \geq 0\}$ in B is done by Dettweiler (1982) (see also Albeverio and Rudiger (2002)). Here the Lévy part process is expressed as a compensated sum of independent jumps. In fact, the process is decomposed into

$$X_t = \gamma_t' + G_t + \int_{0<\|x\|\leq 1} x\left(N_t(dx) - \nu_t(dx)\right) + \int_{\|x\|\geq 1} x N_t(dx), \tag{2.6}$$

where $\{G_t : t \geq 0\}$ is a centered Gaussian process, γ_t' is in B and the integral

$\int_{0<\|x\|\leq 1} x\left(N_t(dx) - \nu_t(dx)\right)$ is a $L_p(B)$ limit, for $p \geq 1$, of the *compensated sum* of independent jumps

$$\sum_{k=1}^{\infty} X_t^{\triangle_{k,k+1}} - E\sum_{k=1}^{\infty} X_t^{\triangle_{k,k+1}}, \tag{2.7}$$

where $\triangle_{k,k+1} = \{x : \frac{1}{k+1} < \|x\| \leq \frac{1}{k}\}$, $k \geq 1$. Here $N_t(dx) - \nu_t(dx)$ is a signed random measure where N_t and ν_t are the Poisson random measure and the Lévy measure of the process X, respectively. The compensated sum (2.7) also converges almost surely.

The last integral in (2.6) is a compound Poisson part in the following sense. Notice that for each $C \in \mathcal{B}_0$ the process in (2.2) satisfy

$$X_t^C = \int_C x N_t(dx)$$

and its characteristic functional is given by

$$Ee^{if(X_t^C)} = \exp\left\{\int_C \left(e^{if(x)} - 1\right) \nu_t(dx)\right\} \quad f \in B^*.$$

Then the additive process $\{X_t^C : t \geq 0\}$ is called *compound Poisson*. We will often use the following important result, which is obtained from Dettweiler (1982).

Lemma 2.1.5 *Let $\{X_t : t \geq 0\}$ be a B-valued additive process with system of generating triplets (A_t, ν_t, γ_t). Then there exist an additive process $\{X_t^0 : t \geq 0\}$ with generating triplets $(A_t, \nu_{0,t}, \gamma_t')$ and an independent compound Poisson process $\{X_t^1 : t \geq 0\}$ with generating triplets $(0, \nu_{1,t}, 0)$ such that*

$$\{X_t\} \stackrel{d}{=} \{X_t^0 + X_t^1\}, \tag{2.8}$$

where $\gamma_t' \in B$, $\nu_{0,t} = [\nu_t]_{\|x\|\leq 1}$ and $\nu_{1,t} = [\nu_t]_{\|x\|>1}$ are the restrictions of ν_t to the sets $\{\|x\| \leq 1\}$ and $\{\|x\| > 1\}$, respectively.

Proof. We shall use the fact that the processes $\{X_t - X_t^C : t \geq 0\}$ and $\{X_t^C : t \geq 0\}$ are independent for each $C \in \mathcal{B}_0$, being this analogous as in the one-dimensional case presented in Gihman and Skorohod (1975, Th 4.3). For each $t \geq 0$ define

$$X_t^0 = \gamma_t' + G_t + \int_{0<\|x\|\leq 1} x\,(N_t(dx) - \nu_t(dx)) \text{ and} \tag{2.9}$$

$$X_t^1 = \int_{\|x\|\geq 1} x N_t(dx). \tag{2.10}$$

Then from the Lévy-Itô decomposition (2.6) we have that $\{X_t^0\}$ and $\{X_t^1\}$ are independent. The processes (2.9) and (2.10) have the required system of generating triplets. ■

Gihman and Skorohod (1975) also proved the proposition below on the bounded variation of B-valued additive process. We first recall.

Definition 2.1.6 A stochastic process $\{Z_t : t \geq 0\}$ in B is said to be *of bounded variation* in the interval $[c, d]$ if $var_{c\leq s\leq d} Z(s) := \sup_{t_0<t_1<...<t_n} \sum_{k=1}^{n} \|Z_{t_k} - Z_{t_{k-1}}\| < \infty$, where the supremum is taken over all the partitions $c = t_0 < t_1 < ... < t_n = d$ of $[c, d]$.

Proposition 2.1.7 *Let $\{Z_t : t \geq 0\}$ be a B-valued additive process. Then, $\{Z_t\}$ has bounded variation on each interval $[0, t]$, with probability 1, if and only if, it has characteristic functional given by*

$$Ee^{if(Z_t)} = exp\left\{\int_B \left(e^{if(x)} - 1\right)\nu_t(dx) + if(\gamma_t')\right\} \qquad f \in B^*,$$

where γ_t' is a B-valued continuous function of bounded variation on each interval $[0, t]$ and the Lévy measure ν_t satisfies

$$\int_{0<\|x\|\leq 1} \|x\|\,\nu_t\,(dx) < \infty. \tag{2.11}$$

Moreover, $V(t) = var_{0\leq s\leq t} Z(s)$ is an additive process in $\mathbb{R}$ and

$$Ee^{izV(t)} = exp\left\{\int_B \left(e^{iz\|x\|} - 1\right)\nu_t(dx) + izu_t\right\} \qquad z \in \mathbb{R}$$

where $u_t = var_{0\leq s\leq t}\,\gamma_s'$.

We now deal with the existence of additive processes in Banach spaces. This problem has been addressed by Dettweiler (1982) for Banach spaces of type $1 \leq p \leq 2$ (see Araujo and Giné (1980)). We present below a general result that follows on the lines of Dettweiler's proof.

Proposition 2.1.8 *Let $(\nu_t)_{t\geq 0}$ be a family of Lévy measures on a separable Banach space B. Assume that the function $\nu_t(C)$ is continuous and increasing in t for each $C \in \mathcal{B}_0$. Then there exists a stochastically continuous process $\{X_t : t \geq 0\}$ with independent increments and càdlàg such that ν_t is the Lévy measure of the distribution of X_t for each $t \geq 0$.*

Proof. The proof follows as in the one dimensional case and in Dettweiler (1982, Prop. 2.3). For each $t \geq 0$ there exists a unique infinitely probability measure μ_t on B with characteristic functional (2.5). Let $0 \leq s \leq t$ and denote by $\mu_{s,t}$ the probability measure whose Lévy measure is $\nu_t - \nu_s$. The family $(\mu_{s,t})_{s\leq t}$ satisfies $\mu_{s,t} \to \delta_0$ as $s \uparrow t$ or $s \downarrow t$ because of $(\nu_t - \nu_0)(C) \downarrow 0$ as $t \downarrow 0$ for each $C \in \mathcal{B}_0$. Hence, using the standard construction from Kolmogorov' s extension theorem (Parthasarathy (1967, Th. 5.5.1)), there exists a stochastic process $\{X_t : t \geq 0\}$ with independent increments where $\mu_{s,t}$ is the distribution of the increment $X_t - X_s$. The process is stochastically continuous since $P(\|X_t - X_s\| > \varepsilon) = \mu_{s,t}(\|x\| > \varepsilon) \to 0$ for each $\varepsilon > 0$ as $s \uparrow t$ or $s \downarrow t$. Now the result follows from the fact that every stochastically continuous process with independet increments in a Banach space has a version whose paths are càdlàg, see Dynkin (1961). ■

2.1.2 Lévy processes

Definition 2.1.9 A *Lévy process* $\{X_t : t \geq 0\}$ in a separable Banach space B is a B-valued additive process with *stationary increments*, that is, the random variables $X_{t+s} - X_s$ and X_t have the same probability law for any choice of $s \geq 0$ and $t \geq 0$.

For a Lévy process, the Lévy-Khintchine representation (2.3) is

$$Ee^{if(X_t)} = \exp\left\{t\left(-\frac{1}{2}(Af, f) + if(\gamma) + \int \left[e^{if(x)} - 1 - if(x)1_{\{\|x\|\leq 1\}}(x)\right]\nu(dx)\right)\right\} \tag{2.12}$$

for every $f \in B^*$, where $\gamma_t = t\gamma$, $\nu_t = t\nu$ and $A_t = tA$. The number of jumps $N_t(C)$ of the process which fall into the set C up to time t, defined in (2.1), satisfies that, for a fixed $t > 0$, $\{N_t(C)\}$ is a Poisson random measure in C on $\mathcal{B}_0$ with intensity measure being the product of Lebesgue measure on $[0, \infty)$ with ν, i.e., $EN_t(C) = t\nu(C)$.

Definition 2.1.10 The triplet of parameters (A, ν, γ) in (2.12) is called *generating triplet*. The measure ν is called the associated *Lévy measure*.

As in the finite dimensional case, there is a one-to-one relation between Lévy processes and infinitely divisible laws in Banach spaces.

Proposition 2.1.11 *Let μ be an infinitely divisible probability measure with generating triplet (A, ν, γ). Then there exists a Lévy process $\{X_t : t \geq 0\}$ with generating triplet (A, ν, γ) such that X_1 has the law μ and viceversa.*

Proof. The existence of the process with the required properties follows from Proposition 2.1.8 and (2.12). ■

Linear transformations of Lévy processes are still Lévy processes and it is possible to describe their generating triplets, as it is pointed out below.

Proposition 2.1.12 *Let $\{X_t : t \geq 0\}$ be a B-valued Lévy process with generating triplet (A, ν, γ). Let B_1 be a separable Banach space and let the map $u \longmapsto f_u$ be an isomorphism of B_1 onto B^*. Let $T : B \to B$ and let $T' : B_1 \to B_1$ be continuous linear transformations with the property*

$$f_u(Ts) = f_{T'u}(s), \tag{2.13}$$

for every $u \in B_1$ and $s \in B$. Then $\{T(X_t) : t \geq 0\}$ is a B-valued Lévy process with generating

triplet (A_T, ν_T, γ_T) given by

$$\begin{aligned} A_T &= TAT', \\ \nu_T &= (\nu T^{-1})\,|_{B\backslash\{0\}}\,, \\ \gamma_T &= T\gamma + \textstyle\int Tx[1_{\{\|Tx\|\leq 1\}}(Tx) - 1_{\{\|x\|\leq 1\}}(x)]\nu(dx). \end{aligned} \tag{2.14}$$

Here $\nu T^{-1}(C) = \nu(\{x : Tx \in C\})$ where $(\nu T^{-1})\,|_{B\backslash\{0\}}$ denotes the restriction of the measure νT^{-1} to $B\backslash\{0\}$ and the last integral is a Bochner integral.

We refer to the book by Yosida (1980, Ch. 5) for Bochner integral with respect to non finite measures.

Proof. From the continuity of T and the fact that $\{X_t\}$ is a B-valued Lévy process it is clear that $\{T(X_t)\}$ is also a B-valued Lévy process. We now obtain its generating triplet. From (2.13)

$$Ee^{if_u(TX_1)} = Ee^{if_{T'u}(X_1)} = \exp\left\{-\frac{1}{2}(Af_{T'u}\,,\, f_{T'u}) + if_{T'u}(\gamma) + \int \left(e^{if_{T'u}(x)} - 1 - if_{T'u}(x)1_{\{\|x\|\leq 1\}}(x)\right)\nu(dx)\right\},$$

where $(Af_{T'u}\,,\, f_{T'u}) = f_{T'u}\,(Af_{T'u}) = f_u\,(TAf_{T'u})$ (Remark 2.1.3 (c)). Using (2.13) the above expression becomes

$$\begin{aligned} \exp\Bigg\{ &-\frac{1}{2}f_u\,(TAf_{T'u}) + if_u\,(T\gamma) \\ &+ i\int f_u(Tx)\left[1_{\{\|Tx\|\leq 1\}}(Tx) - 1_{\{\|x\|\leq 1\}}(x)\right]\nu(dx) \\ &+ \int \left[e^{if_u(y)} - 1 - if_u(y)1_{\{\|y\|\leq 1\}}(y)\right]\nu T^{-1}(dy)\Bigg\}. \end{aligned} \tag{2.15}$$

Notice that, for any $f_u \in B^*$,

$$\int \left(f_u^2(y) \wedge 1\right) \nu T^{-1}(dy) = \int \left(f_{T'u}^2(x) \wedge 1\right) \nu(dx) < \infty$$

and therefore the last integral in (2.15) is well defined. Since the infinitely divisible random variable TX_1 has the characteristic functional (2.15) in which the last integral is a characteristic functional, then by Remark 2.1.4 (b) we get (2.14). ∎

2.1.3 Self-decomposability

Definition 2.1.13 A random variable X in B (or its distribution) is called *self-decomposable* if for each $0 < c < 1$ there exists an independent random variable X_c of X such that

$$X \overset{law}{=} cX + X_c. \tag{2.16}$$

As in the finite dimensional setting the following fact holds, see Kumar and Schreiber (1975).

Theorem 2.1.14 *If a random variable X in B is self-decomposable then X and its component X_c, is (2.16), are infinitely divisible, for each $0 < c < 1$.*

Self-decomposable laws in Banach spaces are characterized in terms of their Lévy measures, without imposing restrictions on A nor γ (Jurek (1983a)).

Theorem 2.1.15 *Let X be an infinitely divisible random variable in B with Lévy measure ν. Then X is self-decomposable if and only if*

$$\nu(C) = \int_{\partial U} \int_{(0,\infty)} 1_C(ry) \frac{k(y,r)}{r} dr \lambda(dy) \qquad C \in \mathcal{B}(B \backslash \{0\}), \tag{2.17}$$

where λ is a probability measure on ∂U and the function $k(y,r) : \partial U \times (0,\infty) \to (0,\infty)$ is non-increasing and left-continuous in r for each $y \in \partial U$ and measurable in y for each $r \in (0,\infty)$.

As important subclass of self-decomposable distributions is the class of stable laws in Banach spaces, as proved in Kumar and Schreiber (1975, Prop. 1.9).

Definition 2.1.16 A random variable X in B (or its distribution) is said to be *stable* if for each $a > 0$ and $b > 0$ there exist $c > 0$ and $x \in B$ such that

$$aX_a + bX_b \overset{law}{=} cX + x$$

where X_a, X_b and X are independent and have same distribution. It is said to be that X is *strictly stable* if $x = 0$.

It is known that there exists a uniquely determined $\alpha \in (0,2]$ such that $c = (a^\alpha + b^\alpha)^{1/\alpha}$. We call a random variable *α-stable* (respect. *strictly α-stable*) if it is stable (respect. strictly stable) with index $\alpha \in (0,2]$. As usual, $\alpha = 2$ corresponds to the Gaussian case. For $0 < \alpha < 2$ we have.

Example 2.1.17 Let $0 < \alpha < 2$. A B-valued random variable X is α-stable (see Linde (1986, Prop. 6.3.1)) if and only if its characteristic functional has the form

$$Ee^{if(X)} = \exp\left\{c_\alpha^{-1} \int_{(0,\infty)} \int_{\partial U} \left(e^{irf(y)} - 1 - irf(y)1_U(ry)\right) \frac{\lambda(dy)}{r^{1+\alpha}} dr + f(\gamma)\right\} \quad f \in B^*.$$

where $\gamma \in B$, $c_1 = \pi/2$, $c_\alpha = -\cos(\pi\alpha/2)\Gamma(-\alpha)$ for $\alpha \neq 1$ and λ is a finite measure on the unit sphere ∂U of the unit closed ball U of B. The measure λ is called the *spectral measure* of X.

Here $c_\alpha > 0$ for all $\alpha \in (0,2)$ since the Gamma function Γ is negative in $(-1,0)$ and positive in $(-2,-1)$.

Using polar coordinates in Banach spaces (Jurek (1983b)), we observe that the measure ν defined by $\nu(C) = c_\alpha^{-1} \int_{(0,\infty)} \int_{\partial U} 1_C(ry)\lambda(dy)\frac{dr}{r^{1+\alpha}}$ for each $C \in \mathcal{B}(B\backslash\{0\})$ and $\nu(\{0\}) = 0$, is a Lévy measure on B.

From Proposition 2.1.11 we have that there exists an α-stable Lévy process $\{X_t : t \geq 0\}$ such that X_1 has the law of X.

Remark 2.1.18 The corresponding function $k(y,r)$ in Theorem 2.1.15 for the α-stable case, $0<\alpha<2$, is $k(y,r)=(const)\frac{1}{r^\alpha}$.

2.2 Distributional properties

There are many distributional properties of the finite dimensional Lévy processes. They can be divided in two sorts of properties: time dependent and time independent; see Sato (1999, Ch. 5). In this section we derive only two kinds of properties for Banach space valued Lévy processes, both of interest in their own and useful in dealing with Bochner's subordination. The g-moment properties are time independent while the tail and conditional moment estimates depend -of course- on the time.

2.2.1 Existence of g-moments

For certain class of functions g defined on a separable Banach space, g-moments of Lévy processes are intimately connected with g-moments of their Lévy measures. Kruglov ((1970), (1972)) proved that these are equivalent for Lévy processes on $\mathbb{R}$ and even on Hilbert spaces. For an infinitely divisible probability measure on a Banach space with characteristic functional (2.5), the equivalence is due to de Acosta (1980). In Proposition 2.2.1 below we prove this equivalence for a general Lévy process on a separable Banach space.

Let $g: B \to \mathbb{R}_+$ be a function which is bounded on the closed unit ball. We say that g is a *submultiplicative function* if there exists a constant $a>0$ such that $g(x+y)\le ag(x)g(y)$ for every $x\in B, y\in B$.

Proposition 2.2.1 *Let $\{X_t : t\ge 0\}$ be Lévy process in a separable Banach space B with generating triplet (A, ν, γ). In order that $Eg(X_t)<\infty$ for some t (or equivalently for every t), it is necessary and sufficient that*

$$\int_{\|x\|>1} g(x)\nu(dx)<\infty.$$

Proof. The proof of the "if " part is done on the same lines of the finite dimensional case of Sato (1999, Th. 25.3). Let $b>1$ such that $sup_{\|x\|\le 1}|g(x)|\le b$ and let us choose n such that $n-1<\|x\|\le n$, then

$$g(x)=g(\frac{x}{n}+\dots+\frac{x}{n})\le a^{n-1}g(\frac{x}{n})^n\le a^{n-1}b^n\le be^{c\|x\|}, \tag{2.18}$$

where $c=\log(ab)>0$ choosing $a>1$.

Let $\nu_0=[\nu]_{\|x\|\le 1}$ and $\nu_1=[\nu]_{\|x\|>1}$ denote the restrictions of ν to the sets $\{\|x\|\le 1\}$ and $\{\|x\|>1\}$, respectively. By Lemma 2.1.5, let $\{X_t^0\}$ and $\{X_t^1\}$ be independent Lévy processes in B such that $\{X_t\}\overset{d}{=}\{X_t^0+X_t^1\}$ where $\{X_t^1\}$ is a compound Poisson process with Lévy measure ν_1. Denote by μ^t, μ_0^t and μ_1^t the distributions of X_t, X_t^0 and X_t^1 respectively. Assume that $Eg(X_t)<\infty$ for some $t>0$. Then, since $X_t\overset{d}{=}X_t^0+X_t^1$,

$$Eg(X_t)=\int_B g(x)\mu^t(dx)=\int_B\int_B g(x+y)\mu_0^t(dx))\mu_1^t(dy)<\infty,$$

which implies, for some $x\in B$, that $\int_B g(x+y)\mu_1^t(dy)<\infty$ and hence

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}\int_B g(x+y)\nu_1^{n*}(dy)<\infty,$$

where $\nu_1^{n*}=\nu_1*\nu_1*\dots*\nu_1$ (n times) is the n-th convolution of ν_1 with itself. In view of (2.18), $g(y)\le ag(-x)g(x+y)\le abe^{c\|x\|}g(x+y)$ and

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}\int_B g(y)\nu_1^{n*}(dy)<\infty. \tag{2.19}$$

Then, for $n=1$, $\int_B g(y)\nu_1(dy)<\infty$.

For the "only if" part we assume that $\int_B g(y)\nu_1(dy)<\infty$. By submultiplicativity, we have for every n,

$$\int_B g(y)\nu_1^{n*}(dy)\le a^{n-1}\left\{\int_B g(y)\nu_1(dy)\right\}^n<\infty.$$

This implies (2.19) and therefore $Eg(X_t^1) < \infty$ for every $t > 0$. It remains to prove that $Eg(X_t^0) < \infty$ for every $t > 0$ since $Eg(X_t) \leq aE[g(X_t^0)]\,E[g(X_t^1)]$. Let $\{G_t^0\},\{P_t^0\}$ and $\{D_t^0\}$ be homogeneous and independent random processes in B, where $\{G_t^0\}$ is Gaussian, $\{P_t^0\}$ is generalized Poisson (that is P_1^0 has the characteristic functional (2.5)) with Lévy measure ν_0 and $\{D_t^0\}$ is a non random function, such that $X_t^0 \overset{d}{=} G_t^0 + P_t^0 + D_t^0$. We only need to prove that $E[g(G_t^0)]\,E[g(P_t^0)]\,E[g(D_t^0)] < \infty$. For simplicity we take $t = 1$. That $E[g(P_1^0)]$ is finite follows from de Acosta (1980, Cor. 3.3) since ν_0 is concentrated in the closed unit ball of B. Now, since G_1^0 is a centered Gaussian random variable then $G_1^0 \overset{d}{=} \frac{1}{\sqrt{2}}G_1 + \frac{1}{\sqrt{2}}G_2$ where G_1 and G_2 are independent copies of G_1^0 (Araujo and Giné (1980, Prop. 3.6.4)). Then

$$\begin{aligned}
Eg(G_1^0) &= Eg(\frac{1}{\sqrt{2}}G_1 + \frac{1}{\sqrt{2}}G_2) \\
&\leq aE\left[g\left(\frac{1}{\sqrt{2}}G_1^0\right)\right]^2 = a\left[Eg\left(\frac{1}{(\sqrt{2})^2}G_1 + \frac{1}{(\sqrt{2})^2}G_2\right)\right]^2 \\
&\leq aa^2\left[\left\{Eg\left(\frac{1}{(\sqrt{2})^2}G_1^0\right)\right\}^2\right]^2 \leq a^7\left[Eg\left(\frac{1}{(\sqrt{2})^3}G_1^0\right)\right]^8 \\
\cdots &\leq a^{2^n-1}\left[Eg\left(\frac{1}{(\sqrt{2})^n}G_1^0\right)\right]^{2^n} \qquad \text{for } n \geq 1.
\end{aligned}$$

We can choose $\beta > 0$ such that $Ee^{\beta\|G_1^0\|} < \infty$ which is a consequence of Fernique's theorem, see Araujo and Giné (1980, Th. 3.6.5). Choosing n sufficiently large such that $c\frac{1}{(\sqrt{2})^n} < \beta$, then $Eg(G_1^0) < \infty$, which concludes the proof. ■

For exponential and regular moments of Lévy processes we obtain the following conditions for their existence, which are independent of t.

Corollary 2.2.2 *a) For every $f \in B^*$, $Ee^{f(X_t)} < \infty$ for all $t > 0$ if and only if*

$$\int_{\|x\|>1} e^{f(x)}\nu(dx) < \infty.$$

b) Let $q > 0$. Then $Ee^{q\|X_t\|} < \infty$ for all $t > 0$ if and only if $\int_{\|x\|>1} e^{q\|x\|}\nu(dx) < \infty$.

c) Let $p > 0$. Then $E\|X_t\|^p < \infty$ for all $t > 0$ if and only if $\int_{\|x\|>1}\|x\|^p\,\nu(dx) < \infty$.

Proof. (a) and (b) follows since $g(x) = e^{f(x)}$ and $g(x) = e^{q\|x\|}$ are submultiplicative functions. Let us prove (c). Assume that $E\|X_t\|^p < \infty$ for every $t > 0$, which is equivalent to the finiteness of $E[1 \vee \|X_t\|^p]$ for every $t > 0$, which in turn is equivalent to $\int_{\|x\|>1}(1 \vee \|x\|^p)\,\nu(dx) < \infty$ since $g(x) = 1 \vee \|x\|^p$ is a submultiplicative function. This is equivalent to $\int_{\|x\|>1}\|x\|^p\,\nu(dx) < \infty$, which concludes the proof. ■

As an application of the above corollary, we point out that for the α-stable Banach space valued Lévy process $\{X_t : t \geq 0\}$ given by Proposition 2.1.17, $E\|X_t\|^p < \infty$ if and only if $p < \alpha$, since $\int_{\|x\|>1}\|x\|^p\,\nu(dx) = \int_{\partial U}\int_{(1,\infty)} r^{p-\alpha-1}dr\lambda(dy) = \lambda(\partial U)\int_{(1,\infty)} r^{p-\alpha-1}dr$.

2.2.2 Tail and moment estimates

We now derive useful tail and conditional moment estimates for Lévy processes with values in a separable Banach space. Let $\{X_t^0 : t \geq 0\}$ be the Lévy process in the decomposition of $\{X_t : t \geq 0\}$ in Lemma 2.1.5. Observe by Corollary 2.2.2 that $E\|X_t^0\|^2 < \infty$ for all $t \geq 0$. In the following lemma we shall assume that X_t^0 is such that $E\|X_t^0\|^2 \leq \lambda t$ for some positive constant λ.

Lemma 2.2.3 *Let $\{X_t : t \geq 0\}$ be a Lévy process in a separable Banach space B with generating triplet (A, ν, γ) and assume that $\{X_t\}$ satisfies the above assumption. Then*
a) There exist positive constants $C(\varepsilon)$, C_1 and C_2 such that, for every $t > 0$,

$$P(\|X_t\| > \varepsilon) \leq C(\varepsilon)t \qquad \textit{for } \varepsilon > 0, \tag{2.20}$$

$$E\left[\|X_t\|^2 ; \|X_t\| \leq 1\right] \leq C_1 t, \tag{2.21}$$

$$E\left[\|X_t\| ; \|X_t\| \leq 1\right] \leq C_2 t^{1/2}. \tag{2.22}$$

b) For any $f \in B^*$ there exist positive constants $C_1(f)$ and $C_2(f)$ such that, for every $t > 0$,

$$E\left[|f(X_t)|^2 ; \|X_t\| \le 1\right] \le C_1(f)t, \tag{2.23}$$

$$|E\left[f(X_t); \|X_t\| \le 1\right]| \le C_2(f)t. \tag{2.24}$$

c) There exists a positive constant C_3 such that, for every $t > 0$,

$$\|E\left[X_t; \|X_t\| \le 1\right]\| \le C_3 t. \tag{2.25}$$

Proof. a) Let $\nu_0 = [\nu]_{\|x\|\le 1}$ and $\nu_1 = [\nu]_{\|x\|>1}$. By Lemma 2.1.5, let $\{X_t^0\}$ and $\{X_t^1\}$ be independent Lévy processes in B such that $\{X_t\} \stackrel{d}{=} \{X_t^0 + X_t^1\}$ where $\{X_t^1\}$ is a compound Poisson process with Lévy measure ν_1. For $t \le 1$ we have

$$P(\|X_t\| > \varepsilon) = P(\|X_t^0 + X_t^1\| > \varepsilon) \le P(X_t^1 \ne 0) + P(X_t^1 = 0,\ \|X_t^0\| > \varepsilon)$$
$$\le 1 - e^{-t\nu_1(\{\|x\|>1\})} + \frac{E\|X_t^0\|^2}{\varepsilon^2} \le \left\{\nu_1(\{\|x\| > 1\}) + \frac{\lambda}{\varepsilon^2}\right\} t,$$

where we have applied Chevyshev's inequality and the assumption on $\{X_t^0\}$. This proves (2.20). Next, note that $\{\|X_t^0 + X_t^1\| \le 1\} = \{\|X_t^0\| \le 1, X_t^1 = 0\} \cup \{\|X_t^0 + X_t^1\| \le 1, X_t^1 \ne 0\}$ then

$$E\left[\|X_t\|^2 ; \|X_t\| \le 1\right] \le E\left[\|X_t^0\|^2 1_{\{\|X_t^0\|\le 1, X_t^1 = 0\}}\right] + E\left[\|X_t\|^2 1_{\{\|X_t^0+X_t^1\|\le 1, X_t^1 \ne 0\}}\right]$$
$$\le E\left[\|X_t^0\|^2 1_{\{\|X_t^0\|\le 1\}}\right] + P\left(X_t^1 \ne 0\right) \le E\left\|X_t^0\right\|^2 + P\left(X_t^1 \ne 0\right)$$

since $\|X_t\|^2 1_{\{\|X_t\|\le 1\}\cap\{X_t^1 \ne 0\}} \le 1_{\{X_t^1 \ne 0\}}$. This reduces to the above case and hence (2.21) holds. To obtain (2.22) apply Hölder's inequality to get
$E\left[\|X_t\| ; \|X_t\| \le 1\right] \le \left(E\left[\|X_t\|^2 ; \|X_t\| \le 1\right]\right)^{1/2} \le C_1^{1/2} t^{1/2}$ where C_1 is given by (2.21).

b) Let f be a continuous linear functional on B.
Since $E\left[|f(X_t)|^2 ; \|X_t\| \le 1\right] \le \left(\|f\|'\right)^2 E\left[\|X_t\|^2 ; \|X_t\| \le 1\right]$, (2.23) follows from (2.21). To

prove (2.24) we use the three elementary estimates $|e^{it} - 1 - it| \le t^2/2$, $t \in \mathbb{R}$, $|e^z - 1| \le e^{|z|} - 1$, $z \in \mathbb{C}$, and $|e^z - 1| \le \frac{7}{4}|z|$, $|z| \le 1$. Note that

$$|E\left[f(X_t); \|X_t\| \le 1\right]|$$
$$= \left|E\left[e^{if(X_t)} - 1\right] - E\left[e^{if(X_t)} - 1; \|X_t\| > 1\right] - E\left[e^{if(X_t)} - 1 - if(X_t); \|X_t\| \le 1\right]\right|$$
$$\le \left|E\left[e^{if(X_t)} - 1\right]\right| + 2P\left(\|X_t\| > 1\right) + \frac{1}{2}E\left[f^2(X_t); \|X_t\| \le 1\right].$$

The last two terms of the sum are bounded by constant multiples of t by (2.20) and (2.23) respectively. Next, $E\left|e^{if(X_t)} - 1\right| = \left|e^{t\Psi_X(f)} - 1\right|$ where
$\Psi_X(f) = if(\gamma) - \frac{1}{2}(Af, f) + \int \left[e^{if(x)} - 1 - if(x)1_{\{\|x\|\le 1\}}(x)\right] \nu(dx)$ is given by (2.12). Let $z = \Psi_X(f)$. When $t \ge 1$, $E\left|e^{if(X_t)} - 1\right| \le 2t$. Now consider the case $0 \le t < 1$. If $0 \le t \le \frac{1}{|z|}$ then $|e^{tz} - 1| \le \frac{7}{4}|z|t$. Finally, if $\frac{1}{|z|} \le t \le 1$, $|e^{tz} - 1| \le e^{t|z|} - 1 \le e^{|z|t}|z|t \le e^{|z|}|z|t$.

c) By (2.22) $E\left[\|X_t\| ; \|X_t\| \le 1\right]$ is finite for each t and hence $X_t 1_{\{\|X_t\|\le 1\}}$ is Bochner integrable (Araujo and Giné (1980, Prop. 3.2.2)). Let $V = X_t 1_{\{\|X_t\|\le 1\}}$. Since V is also Pettis integrable (Araujo and Giné (1980, p. 101)), we have

$$f(EV) = Ef(V) \qquad \text{for every } f \in B^*.$$

On the other hand, by the Hahn-Banach Theorem there exists a continuous linear functional $\tilde{f} \in B^*$ satisfying $\|EV\| = \tilde{f}(EV)$ and then $\|EV\| = E\tilde{f}(V)$. This means $\|E\left[X_t; \|X_t\| \le 1\right]\|$ $= E\left[\tilde{f}(X_t); \|X_t\| \le 1\right]$. The assertion follows from (2.24) applied to $\tilde{f}$. ■

2.3 Subordination

Recall that a one dimensional subordinator is an increasing Lévy process $\{\sigma_t : t \ge 0\}$ in $\mathbb{R}$ which is characterized in terms of its generating triplet by (1). Let $\gamma_\sigma^0 \ge 0$ be its drift and let ν_σ be its Lévy measure that satisfies

$$\int_{(0,\infty)} (1 \wedge s)\, \nu_\sigma (ds) < \infty. \tag{2.26}$$

Furthermore, $Ee^{w\sigma_t} = e^{t\Psi_\sigma(w)}$ where

$$\Psi_\sigma (w) = \int_{(0,\infty)} (e^{ws} - 1)\, \nu_\sigma (ds) + \gamma_\sigma^0 w, \tag{2.27}$$

for any $w \in \mathbb{C}$ whose $\mathrm{Re}(w) \leq 0$ (see Sato (1999, Th. 24.11)).

The theorem below holds for general spaces other than Banach spaces and it is referred to as *Bochner's subordination*.

Theorem 2.3.1 *Let $\{X_t : t \geq 0\}$ be a Lévy process in B and let $\{\sigma_t : t \geq 0\}$ be a one dimensional subordinator. Assume that $\{X_t\}$ and $\{\sigma_t\}$ are independent and define $Y_t = X_{\sigma_t}$, $t \geq 0$. Then the subordinated process $\{Y_t : t \geq 0\}$ is a Lévy process.*

The proof of the above theorem is omitted since it is essentially the same as for the one dimensional case in Sato (1999, Prop. 1.16) and for the multivariate case in Barndorff-Nielsen, Pedersen and Sato (2003a). One should use the fact that if X and Y are independent random variables in B and if $f : B \times B \to \mathbb{R}$ is a bounded measurable function, then $g(y) = Ef(X, y)$ is also bounded measurable and $Ef(X, Y) = Eg(Y)$.

It is possible to find the explicit form of the generating triplet of Y from the generating triplets of X and σ. The following is the main result of this chapter.

Theorem 2.3.2 *Let $\{X_t : t \geq 0\}$ be a Lévy process in B with generating triplet (A_X, ν_X, γ_X) and let $\{\sigma_t : t \geq 0\}$ be a one dimensional subordinator with Lévy measure ν_σ and drift $\gamma_\sigma^0 \geq 0$. Assume that $\{X_t\}$ and $\{\sigma_t\}$ are independent. Let $\{Y_t : t \geq 0\}$ be a B-valued Lévy process subordinate to $\{X_t\}$ by $\{\sigma_t\}$, that is $Y_t = X_{\sigma_t}$, $t \geq 0$. Then*

a) Let $\mu = \mathcal{L}(X_1)$ and denote by $\hat{\mu}$ its characteristic functional. Then the characteristic functional of $\{Y_t\}$ is given by $Ee^{if(Y_t)} = e^{t\Psi_\sigma(\log \hat{\mu}(f))}$ for all real-valued $f \in B^*$.

b) Let $\mu_s = \mathcal{L}(X_s)$ and assume that $\{X_t\}$ is as in Lemma 2.2.3. Then the generating triplet (A_Y, ν_Y, γ_Y) of $\{Y_t\}$ is given by

$$A_Y = \gamma_\sigma^0 A_X, \tag{2.28}$$

$$\nu_Y(C) = \int_{(0,\infty)} \mu_s(C)\nu_\sigma(ds) + \gamma_\sigma^0 \nu_X(C) \qquad C \in \mathcal{B}(B\backslash\{0\}), \tag{2.29}$$

$$\gamma_Y = \int_{(0,\infty)} \int_{\|x\|\leq 1} x\mu_s(dx)\nu_\sigma(ds) + \gamma_\sigma^0 \gamma_X, \tag{2.30}$$

where the last integral is in the sense of Bochner.

c) If $\gamma_\sigma^0 = 0$ and $\int_{(0,1]} s^{1/2}\nu_\sigma(ds) < \infty$, then $A_Y = 0$, $\int_{\|x\|\leq 1} \|x\|\, \nu_Y(dx) < \infty$ and $\gamma_Y - \int_{\|x\|\leq 1} x\nu_Y(dx) = 0$. Moreover, the Banach-valued Lévy process $\{Y_t\}$ has bounded variation on each interval $[0, t]$.

Proof. a) Let f be a real-valued continuous linear functional on B. We observe that $Ee^{if(X(s))} = e^{s \log \hat{\mu}(f)}$ for each $s \geq 0$. Since $\mathrm{Re}\,(\log \hat{\mu}(f)) \leq 0$ we use (2.27) to get

$$Ee^{if(Y_t)} = E\left[\left(Ee^{if(X(s))}\right)_{s=\sigma_t}\right] = E\left[\left(e^{s\log\hat{\mu}(f)}\right)_{s=\sigma_t}\right] = e^{t\Psi_\sigma(\log\hat{\mu}(f))}.$$

b) We use Lemma 2.2.3 in order to find the generating triplet of $\{Y_t\}$. From (2.27) and (a) we have

$$\begin{aligned} Ee^{if(Y_t)} = e^{t\Psi_\sigma(\log\hat{\mu}(f))} &= \exp\left\{t\left(\int_{(0,\infty)} (e^{s\log\hat{\mu}(f)} - 1)\nu_\sigma(ds) + \gamma_\sigma^0 \log\hat{\mu}(f)\right)\right\} \\ &= \exp\left\{t\left(\int_{(0,\infty)} (\hat{\mu}_s(f) - 1)\nu_\sigma(ds) + \gamma_\sigma^0 \log\hat{\mu}(f)\right)\right\}. \end{aligned} \tag{2.31}$$

Let $g(f, x) = e^{f(x)} - 1 - if(x)1_{\{\|x\|\leq 1\}}(x)$. Note that

$$\gamma_\sigma^0 \log\hat{\mu}(f) = -\frac{1}{2}\gamma_\sigma^0 (A_X f, f) + i\gamma_\sigma^0 f(\gamma_X) + \int_B g(f, x)\, \gamma_\sigma^0 \nu_X(dx). \tag{2.32}$$

On the other hand, by Lemma 2.2.3 $\int_{\|x\|\leq 1} x\mu_s(dx)$ is a well defined Bochner integral. From (2.25) and (2.26) we have that $\int_{(0,\infty)} \left\|\int_{\|x\|\leq 1} x\mu_s(dx)\right\| \nu_\sigma(ds)$ is finite and hence

$\int_{(0,\infty)} \int_{\|x\|\le 1} x\mu_s(dx)\nu_\sigma(ds)$ is a well defined Bochner integral. Therefore

$$\int_{(0,\infty)} \int_{\|x\|\le 1} f(x)\mu_s(dx)\nu_\sigma(ds) = f\left(\int_{(0,\infty)} \int_{\|x\|\le 1} x\mu_s(dx)\nu_\sigma(ds)\right).$$

Then we have

$$\int_{(0,\infty)} (\hat{\mu}_s(f) - 1)\nu_\sigma(ds) = \int_{(0,\infty)} \int_B (e^{if(x)} - 1)\mu_s(dx)\nu_\sigma(ds)$$

$$= \int_{(0,\infty)} \int_B g(f,x)\mu_s(dx)\nu_\sigma(ds) + if\left(\int_{(0,\infty)} \int_{\|x\|\le 1} x\mu_s(dx)\nu_\sigma(ds)\right). \qquad (2.33)$$

From (2.31), (2.32) and (2.33) we get

$$Ee^{if(Y_t)} = \exp\left\{t\left[-\frac{1}{2}\gamma_\sigma^0 A_X(f,f) + if\left(\int_{(0,\infty)} \int_{\|x\|\le 1} x\mu_s(dx)\nu_\sigma(ds) + \gamma_\sigma^0\gamma_X\right)\right.\right.$$
$$\left.\left. + \int_B g(f,x)\left(\int_{(0,\infty)} \mu_s(\cdot)\nu_\sigma(ds) + \gamma_\sigma^0 \nu_X(\cdot)\right)(dx)\right]\right\}. \qquad (2.34)$$

Next, define $\nu_1(C) = \int_{(0,\infty)} \mu_s(C)\nu_\sigma(ds)$ for $C \in \mathcal{B}(B\backslash\{0\})$. It remains to prove that ν_1 is a Lévy measure, i.e., the function $f \longmapsto \exp \int_B g(f,x)\nu_1(dx)$ for $f \in B^*$ is a characteristic functional of some B-valued random variable (see (2.5)) which satisfies (2.4). Indeed,

$$\exp \int_B g(f,x)\,\nu_1(dx)$$
$$= \exp\left\{\int_{(0,\infty)} (Ee^{if(X(s))} - 1)\nu_\sigma(ds) - i\int_{(0,\infty)} \int_{\|x\|\le 1} f(x)\mu_s(dx)\nu_\sigma(ds)\right\}$$
$$= \exp\left\{\int_{(0,\infty)} (e^{s\log\hat{\mu}(f)} - 1)\nu_\sigma(ds)\right\} \exp\left\{-if\left(\int_{(0,\infty)} \int_{\|x\|\le 1} x\mu_s(dx)\nu_\sigma(ds)\right)\right\}.$$

The last two factors are characteristic functionals, since the first one corresponds to the characteristic functional of a subordinated process at time 1 obtained by subordination of X_s by $\sigma_t - t\gamma_\sigma^0$ and the second one corresponds to the characteristic functional of a degenerated distribution. The integrability condition (2.4) for ν_1 follows from (2.23) and (2.26). Hence the measure $\int_{(0,\infty)} \mu_s(\cdot)\nu_\sigma(ds) + \gamma_\sigma^0\nu_X(\cdot)$ in (2.34) is a Lévy measure.

In view of (2.34) and the uniqueness of the generating triplet (Remark 2.1.4 (c)), we get (2.28), (2.29) and (2.30).

c) Assume that $\gamma_\sigma^0 = 0$ and $\int_{(0,1]} s^{1/2}\nu_\sigma(ds) < \infty$. Then $A_Y = 0$ by (2.28) and $\nu_Y(dx) = \int_{(0,1]} \mu_s(dx)\nu_\sigma(ds)$ by (2.29). We have

$$\int_{\|x\|\le 1} \|x\|\,\nu_Y(dx) = \int_{(0,\infty)} \int_{\|x\|\le 1} \|x\|\,\mu_s(dx)\nu_\sigma(ds) < \infty$$

by (2.26) and (2.22) in Lemma 2.2.3. Now, it follows from (2.30) and (2.34) that $\gamma_Y - \int_{\|x\|\le 1} x\nu_Y(dx) = 0$ and moreover

$$Ee^{if(Y_t)} = \exp\left\{t\int_B (e^{if(x)} - 1)\nu_Y(dx)\right\}.$$

By Proposition 2.1.7 we conclude that $\{Y_t\}$ has bounded variation on each interval $[0,t]$. ■

Remark 2.3.3 The finite dimensional case of (a) and (b) of the above theorem appears in Sato (1999, Th. 30.1) and is due to several authors. A partial analogue of (c) is in Rocha-Arteaga and Sato (2003, Th. 4.3.7). There arises the natural question whether it is possible to do Bochner's subordination and identify the generating triplet of the subordinated process Y, when the subordinator σ takes values in a higher dimensional cone, compatible in some sense with a cone-parameter Lévy process X. Until now, this has been shown to be true only for cones generated by bases in some finite and infinite dimensional settings. The case of cones in $\mathbb{R}^d$ is considered in Barndorff-Nielsen et. al (2001) and Pedersen and Sato (2003a), while the case of matrices and trace class operators is done in Pérez-Abreu and Rocha-Arteaga (2003). In all these cases it was first needed to derive appropriate tail and conditional moment estimates analogous to those presented in this chapter.

Chapter 3

Cone-Additive Processes and Subordinators

As a main contribution of this thesis, in the first part of this chapter we conduct a systematic and detailed analysis on the structure of additive processes taking values in proper cones of infinite dimensional Banach spaces. Special emphasis is put on the convergence of the non-compensated jumps and the validity or not of a special Lévy-Khintchine representation. It is proved that for an LK-cone K, a Banach space-valued additive process has the special Lévy-Khintchine representation (with respect to K) if and only if it lies in K. In particular, it is shown that the type of cones considered by Gihman and Skorohod (1975) form a subclass of LK-cones. In the second part of this chapter subordinators in Banach spaces are introduced and their corresponding Laplace transforms are obtained. As a consequence, infinitely divisible laws in cones of Banach spaces are described, recovering a result by Dettweiler (1976). Finally, many concrete examples of subordinators in Banach spaces are provided, via three different methods that use results of previous chapters.

3.1 The Lévy-Khintchine representation

3.1.1 Cone-valued additive processes

We begin this section by giving sufficient conditions on the system of generating triplets (A_t, ν_t, γ_t) of an additive process $\{Z_t : t \geq 0\}$ in order that Z is *concentrated* on a proper cone K. By this we mean that $Z_t(\omega) \in K$, ω-almost surely. We emphasize that throughout this subsection the only assumption on the cone K is to be proper.

We use the notation $\int_{0<\|x\|\leq 1} x\nu_t(dx)$ for the Pettis integral of $x 1_{\{x:0<\|x\|\leq 1\}}(x)$ with respect to the σ-finite measure ν_t on $\mathcal{B}(B\backslash\{0\})$. Namely, as in the finite measure case the function $h(x) = x1_{\{x:0<\|x\|\leq 1\}}(x)$ defined on B is *Pettis integrable* if (3.1) is satisfied and there exists and element β in B such that $f(\beta) = \int_{0<\|x\|\leq 1} f(x)\nu_t(dx)$ for every $f \in B^*$. The element $\beta \in B$ is called the Pettis integral.

Proposition 3.1.1 *Let K be a proper cone of a separable Banach space B. Let $\{Z_t : t \geq 0\}$ be a B-valued additive process in B with generating triplets (A_t, ν_t, γ_t). Assume that*

a) $A_t = 0$,

b) $\nu_t(B\backslash K) = 0$, i.e., ν_t is concentrated on K,

c)

$$\int_{0<\|x\|\leq 1} |f(x)|\, \nu_t(dx) < \infty, \qquad \textit{for every } f \in B^* \tag{3.1}$$

and d)

$$\gamma_t^0 := \gamma_t - \int_{0<\|x\|\leq 1} x\nu_t(dx) \in K, \tag{3.2}$$

is continuous in t where the last integral is in the Pettis sense.

Then the process Z is concentrated on K and its characteristic functional has the form

$$Ee^{if(Z_t)} = exp\left\{\int_K \left(e^{if(x)} - 1\right)\nu_t(dx) + if(\gamma_t^0)\right\}. \tag{3.3}$$

Here the non-random function $\gamma_t^0, t \geq 0$, is called the drift of the additive process Z.

Proof of Proposition 3.1.1. Since K is convex and closed there exists a sequence of

continuous linear functionals g_k such that $K = \cap_{k=1}^{\infty}\{x : g_k(x) \geq 0\}$, this is a consequence of the second separation theorem in Schaefer (1999, Cor. 9.2.1). Clearly $f(\gamma_t^0) = f(\gamma_t) - \int_{0<\|x\|\leq 1} f(x)\nu_t(dx)$ for all $f \in B^*$ and this gives the formula (3.3). Here we have used assumptions (a)-(d). Define, for each $\varepsilon > 0$,

$$Z_t^{\triangle_\varepsilon} = \sum_{s<t}(Z_s - Z_{s-})\, 1_{\triangle_\varepsilon}(Z_s - Z_{s-})$$

where $\triangle_\varepsilon = \{x : \|x\| > \varepsilon\}$. We first prove that $Z_t^{\triangle_\varepsilon} \in K$ almost surely. Indeed, suppose that there exists $C \in \mathcal{B}_0$ contained in $B\backslash K$ such that $Z_t^C \neq 0$ with positive probability, then $0 < P\left(Z_t^C \neq 0\right) \leq 1 - e^{-\nu_t(C)}$ which is not possible since $\nu_t(C) = 0$. Therefore $Z_t^{\triangle_\varepsilon} \in K$ a.s. for each $\varepsilon > 0$.

Since $\gamma_t^0 \in K$ we show that $Z_t - \gamma_t^0 \in K$ almost surely. Let $J_t = Z_t - \gamma_t^0$ and notice from the continuity of γ_t^0 that J_t and Z_t have the same jumps and therefore $J_t^{\triangle_\varepsilon} = Z_t^{\triangle_\varepsilon}$ a.s. Hence $\left\{J_t^{\triangle_\varepsilon}\right\}$ is a K-valued additive process and for $u \geq 0$

$$Ee^{-ug_k(J_t^{\triangle_\varepsilon})} = exp\left\{\int_{K\cap\{x:\|x\|\leq\varepsilon\}}\left(e^{-ug_k(x)} - 1\right)\nu_t(dx)\right\} \qquad \text{for } k \geq 1.$$

Letting $\varepsilon \downarrow 0$ we get

$$Ee^{-u\lim_{\varepsilon\downarrow 0} g_k(J_t^{\triangle_\varepsilon})} = exp\left\{\int_K \left(e^{-ug_k(x)} - 1\right)\nu_t(dx)\right\}.$$

The right hand is finite from (3.1) and tends to 1 as u decreases to zero. Therefore $\lim_{\varepsilon\downarrow 0} g_k(J_t^{\triangle_\varepsilon})$ exists and it is nonnegative for each k. From (3.3) and the fact that $Z_t^{\triangle_\varepsilon}$ and $Z_t - Z_t^{\triangle_\varepsilon}$ are independent

$$Ee^{ig_k(J_t - J_t^{\triangle_\varepsilon})} = exp\left\{\int_{K\cap\{x:\|x\|\leq\varepsilon\}}\left(e^{ig_k(x)} - 1\right)\nu_t(dx)\right\}$$

where the right hand side tends to 1 as $\varepsilon \downarrow 0$ and therefore $g_k(J_t) = \lim_{\varepsilon\downarrow 0} g_k(J_t^{\triangle_\varepsilon})$ almost surely. Since $g_k\left(J_t^{\triangle_\varepsilon}\right)$ is nonnegative and increasing as $\varepsilon \downarrow 0$ for all k, then $g_k(J_t - J_t^{\triangle_\varepsilon}) \geq 0$.

Hence $J_t - J_t^{\triangle_\varepsilon}$ and $J_t^{\triangle_\varepsilon}$ are in K and therefore $J_t \in K$. ■

The following fact is a trivial but useful consequence.

Corollary 3.1.2 *The zero drift process $\{Z_t - \gamma_t^0 : t \geq 0\}$ is an additive process concentrated on K with characteristic functional*

$$Ee^{if(Z_t-\gamma_t^0)} = exp\left\{\int_K \left(e^{if(x)} - 1\right) \nu_t(dx)\right\}. \tag{3.4}$$

There are cone valued additive processes whose Lévy measures satisfy the stronger integrability condition (3.5) below (see Examples 3.3.1 and 3.3.2). This implies the existence of the Bochner integral $\int_{0<\|x\|\leq 1} x\nu_t(dx)$. Chapter 4 gives examples of Banach spaces where this is always the case.

Corollary 3.1.3 *Let K be a proper cone of separable Banach space B. Let $\{Z_t : t \geq 0\}$ be a B-valued additive process with generating triplets (A_t, ν_t, γ_t) satisfying (a), (b), (d) in Proposition 3.1.1 as well as the additional condition*

$$\int_{0<\|x\|\leq 1} \|x\| \, \nu_t(dx) < \infty, \tag{3.5}$$

Then $\int_{0<\|x\|\leq 1} x\nu_t(dx)$ is a Bochner integral and the process Z is concentrated on K.

3.1.2 Convergence of jumps

Whether the converse of Proposition 3.1.1 is true or not, relies strongly in the analysis of the jumps of the process falling into the cone. This section is focused in analyzing sums of non-compensated jumps of cone-valued additive processes. While in the finite dimensional case these sums are always convergent (Skorohod (1991, Th. 3.21)), the situation for general infinite dimensional Banach spaces seems to be different. However, in the two theorems below, we are able to give a "*weak type*" result for general Banach spaces and a "*strong type*" result under an additional condition, respectively.

We first derive a technical lemma, where we also introduce some notation.

Lemma 3.1.4 *Let K be a proper cone of a separable Banach space B and let $\{Z_t : t \geq 0\}$ be a K-valued additive process. Fix $t > 0$ and let for each $\varepsilon > 0$,*

$$Z_t^{\triangle_\varepsilon} = \sum_{s<t} (Z_s - Z_{s-}) 1_{\triangle_\varepsilon} (Z_s - Z_{s-}) \tag{3.6}$$

where $\triangle_\varepsilon = \{x : \|x\| > \varepsilon\}$. Then for any $f \in K^*$, the one dimensional family $f(Z_t^{\triangle_\varepsilon})$ is nonnegative, increasing as $\varepsilon \downarrow 0$ and bounded by $f(Z_t)$, almost surely.

Proof. The one dimensional process $\{f(Z_t) : t \geq 0\}$ is a nonnegative additive process since $\{Z_t\}$ is a K-valued Lévy process and $f \in K^*$. Its corresponding sum of non-compensated jumps $[f(Z_t)]^{\triangle_\varepsilon^f} := \sum_{s<t} (f(Z_s) - f(Z_{s-})) 1_{\{f(x):|f(x)|>\varepsilon\}} (f(Z_s) - f(Z_{s-}))$ is nonnegative, increasing as $\varepsilon \downarrow 0$ and bounded by $f(Z_t)$ (see Gihman and Skorohod (1975, Th. 4.1.7)). Hence $lim_{\varepsilon\downarrow 0} [f(Z_t)]^{\triangle_\varepsilon^f}$ exists.

Let ε_n be any decreasing sequence to 0 and let $\varepsilon_n(f) = \varepsilon_n \|f\|'$. Assume, without loss of generality, that $\|f\|' > 0$. If $f(Z_s - Z_{s-}) > \varepsilon_n(f)$ then $\|Z_s - Z_{s-}\| > \varepsilon_n$. Therefore $[f(Z_t)]^{\triangle_{\varepsilon_n(f)}^f} \leq f(Z_t^{\triangle_{\varepsilon_n}})$ almost surely for every $n \geq 1$ and hence $f(Z_t^{\triangle_{\varepsilon_n}})$ is nonnegative almost surely. This proves the first assertion. Next, when $\varepsilon_2 < \varepsilon_1$ we have that $\triangle_{\varepsilon_1} \subset \triangle_{\varepsilon_2}$ and hence $f(Z_t^{\triangle_{\varepsilon_2}} - Z_t^{\triangle_{\varepsilon_1}}) = \sum_{s<t} f(Z_s - Z_{s-}) 1_{\{x:\varepsilon_2<\|x\|\leq\varepsilon_1\}} (Z_s - Z_{s-}) \geq 0$. The second assertion is proved.

Now let $\varepsilon > 0$ and note that $f(Z_t^{\triangle_\varepsilon}) = \sum_{s<t} f(Z_s - Z_{s-}) 1_{\{x:\|x\|>\varepsilon\}} (Z_s - Z_{s-})$ is bounded by $f(Z_t)$ almost surely since it represents a finite number of jumps of the process $\{f(X_t)\}$. This proves the third assertion. ■

We point out the following observation.

Lemma 3.1.5 *Under the notation of Lemma 3.1.4, the jumps sum $Z_t^{\triangle_\varepsilon}$ is K-increasing as ε decreases to 0 and it is K-majorized by Z_t. Namely, $Z_t^{\triangle_{\varepsilon_2}} - Z_t^{\triangle_{\varepsilon_1}} \in K$ as $\varepsilon_2 < \varepsilon_1$ and $Z_t - Z_t^{\triangle_\varepsilon} \in K$ for $\varepsilon > 0$, respectively.*

Proof. Let g_k, $k \geq 1$, be a sequence of continuous linear functionals g_k such that $K = \cap_{k=1}^{\infty} \{x : g_k(x) \geq 0\}$ since K is convex and closed. The one dimensional additive process $\{g_k(Z_t)\}$ is nonnegative for each k and hence its jumps are also nonnegative almost surely. Observe that for a jump $Z_s - Z_{s-} \neq 0$ either $g_k(Z_s - Z_{s-}) = 0$ or $g_k(Z_s - Z_{s-}) \neq 0$. Therefore $g_k(Z_s - Z_{s-}) \geq 0$ a.s. for every k. Then $Z_s - Z_{s-} \in K$ and hence the jumps sum $Z_t^{\triangle_\varepsilon} \in K$ almost surely.

If $\varepsilon_1 < \varepsilon_2$ then $Z_t^{\triangle_{\varepsilon_2}} - Z_t^{\triangle_{\varepsilon_1}} = \sum_{s<t} (Z_s - Z_{s-}) 1_{\{x : \varepsilon_2 < \|x\| \leq \varepsilon_1\}} (Z_s - Z_{s-}) \in K$. This proves the first assertion. Next, by Lemma 3.1.4 we have that $g_k\left(Z_t - Z_t^{\triangle_\varepsilon}\right) \geq 0$ a.s. for every k and therefore $Z_t - Z_t^{\triangle_\varepsilon} \in K$ a.s. for $\varepsilon > 0$. Thus $Z_t^{\triangle_\varepsilon}$ is K-majorized by Z_t. ■

The following result provides useful information on the type of convergence of the jumps of a K-valued additive process in a general separable Banach space B. Fix $t > 0$ and let $Z_t^{\triangle_\varepsilon}$ be the sum of jumps in (3.6) where $\triangle_\varepsilon = \{x : \|x\| > \varepsilon\}$. For any sequence $\varepsilon_n \downarrow 0$, we have the alternative representation of the jump process (3.6) as sums of independent random elements

$$Z_t^{\triangle_{\varepsilon_n}} = \sum_{k=1}^{n} \xi_k \tag{3.7}$$

where $\xi_1 = Z_t^{\triangle_{\varepsilon_1}}$ and for $k \geq 2$

$$\xi_k = \sum_{s<t} (Z_s - Z_{s-}) 1_{\triangle_{\varepsilon_k, \varepsilon_{k-1}}} (Z_s - Z_{s-})$$

with $\triangle_{\varepsilon_k, \varepsilon_{k-1}} = \{x : \varepsilon_k < \|x\| \leq \varepsilon_{k-1}\}$.

Remark 3.1.6 It is emphasized that in all the previous results in this and the former subsection, the only assumption on the cone K is to be proper. In order to get more insight into the structure of infinite dimensional Banach space-valued additive processes and of their special Lévy-Khintchine representations, additional assumptions on the cone are required.

Theorem 3.1.7 *Let K be a proper cone of a separable Banach space B such that K^* is a generating cone of B^*. Let $\{Z_t : t \geq 0\}$ be a K-valued additive process. Then the jumps sum $Z_t^{\triangle_{\varepsilon_n}}$ is w.u.C. almost surely for any sequence $\varepsilon_n \downarrow 0$.*

Proof. By assumption, any continuous linear functional f on B can be decomposed into $f = f^+ - f^-$ where $f^+ \in K^*$ and $f^- \in K^*$. It is enough to prove the assertion for any positive (with respect to K) linear functional since $|f(x)| = f^+(x) + f^-(x)$. Let $f \in K^*$. Then the one dimensional additive process $\{f(Z_t)\}$ is nonnegative and hence possesses only nonnegative jumps. Notice that $\sum_{k=1}^{n} f(\xi_k) = \sum_{s<t} f(Z_s - Z_{s-}) 1_{\triangle_{\varepsilon_n}} (Z_s - Z_{s-}) = f(Z_t^{\triangle_{\varepsilon_n}})$. From Lemma 3.1.4 $f(Z_t^{\triangle_{\varepsilon_n}})$ is increasing as function of ε_n as $n \to \infty$ and bounded. Therefore $\sum_{k=1}^{n} f(\xi_k)$ converges a.s. ■

The theorem below shows that under the additional condition on the norm convergence of the sum of the non-compensated jumps, the converse of Proposition 3.1.1 is true for cones with generating duals.

Theorem 3.1.8 *Let K be a proper cone of a separable Banach space B such that K^* is a generating cone of B^*. Let $\{Z_t : t \geq 0\}$ be a K-valued additive process with generating triplets (A_t, ν_t, γ_t) and for fix $t > 0$ define $Z_t^{\triangle_\varepsilon}$ as in (3.6).*

If the jumps sum $Z_t^{\triangle_\varepsilon}$ converges a.s. in norm as $\varepsilon \downarrow 0$, then the characteristic functional of the process is given by

$$Ee^{if(Z_t)} = exp\left\{\int_K \left(e^{if(x)} - 1\right) \nu_t(dx) + if(\gamma_t^0)\right\}, \qquad f \in B^*, \tag{3.8}$$

where $\gamma_t^0 \in K$, the Lévy measure ν_t is concentrated on K and satisfies

$$\int_{0<\|x\|\leq 1} |f(x)| \, \nu_t(dx) < \infty, \qquad f \in B^*. \tag{3.9}$$

Moreover, $\int_{0<\|x\|\leq 1} x\nu_t(dx)$ is a well defined Pettis integral such that $\gamma_t^0 = \gamma_t - \int_{0<\|x\|\leq 1} x\nu_t(dx)$ is continuous in t.

Proof. Let $K = \cap_{k=1}^{\infty} \{x : g_k(x) \geq 0\}$ for some sequence of continuous linear functionals g_k. Then each one dimensional additive process $\{g_k(Z_t)\}$ has only nonnegative jumps since it is nonnegative. So if C is contained in $\cup_{k=1}^{\infty} \{x : g_k(x) < 0\}$ then $\nu_t(C) = 0$. Thus ν_t is concentrated on K. Let $\triangle_\varepsilon = \{x : \|x\| > \varepsilon\} \cap K$. By assumption $Z_t^{\triangle_\varepsilon}$ converges strongly to some Z_t^0 as $\varepsilon \downarrow 0$ almost surely. Therefore the process $\{Z_t - Z_t^0\}$ is continuous almost surely and from Remark 2.1.4

$$Ee^{if(Z_t - Z_t^0)} = exp\left\{-\frac{1}{2}(A_t f, f) + if(\gamma_t^0)\right\}. \tag{3.10}$$

Since $Z_t - Z_t^{\triangle_\varepsilon} \in K$ for all $\varepsilon > 0$ (Lemma 3.1.5) then $Z_t - Z_t^0 \in K$ by closedness of K. Hence for every $f^+ \in K^*$ the process $\{f^+(Z_t - Z_t^0)\}$ is nonnegative, continuous and Gaussian, therefore $var\,(f^+(Z_t - Z_t^0)) = A_t(f^+, f^+) = 0$. Now, for $f \in B^*$ write $f = f^+ - f^-$ where $f^+ \in K^*$ and $f^- \in K^*$, then

$$(A_t f, f) = var\left(f(Z_t - Z_t^0)\right) = var\left[f^+(Z_t - Z_t^0) - f^-(Z_t - Z_t^0)\right] = 0,$$

since $\{f^+(Z_t - Z_t^0)\}$ and $\{f^-(Z_t - Z_t^0)\}$ are constants almost surely. This shows that the covariance operator $A_t = 0$. Since $f(Z_t - Z_t^0) \geq 0$ for every $f \in K^*$ then from (3.10) we get $\gamma_t^0 \in K$ which clearly is continuous in t.

Now from the fact that $A_t = 0$, (3.10) and

$$\begin{aligned} Ee^{if(Z_t^0)} = \lim_{\varepsilon\downarrow 0} Ee^{if(Z_t^{\triangle_\varepsilon})} &= \lim_{\varepsilon\downarrow 0} exp\left\{\int_{\triangle_\varepsilon} \left(e^{if(x)} - 1\right)\nu_t(dx)\right\} \\ &= exp\left\{\int_K \left(e^{if(x)} - 1\right)\nu_t(dx)\right\} \end{aligned} \tag{3.11}$$

we obtain the formula (3.8).

Let $f^+ \in K^*$. Then from (2.3), (3.11) and

$$\begin{aligned} Ee^{if^+(Z_t^0)} = \lim_{\varepsilon\downarrow 0} exp&\left\{\int_{\{x\in K:\varepsilon<\|x\|\leq 1\}} \left[e^{if^+(x)} - 1 - if^+(x)\right]\nu_t(dx)\right. \\ &+ \left.\int_{\{x\in K:\|x\|>1\}} \left(e^{if^+(x)} - 1\right)\nu_t(dx) + i\int_{\{x\in K:\varepsilon<\|x\|\leq 1\}} f^+(x)\nu_t(dx)\right\} \end{aligned}$$

we have that $exp\left\{i\int_{\varepsilon<\|x\|\leq 1} f^+(x)\nu_t(dx)\right\}$ converges as $\varepsilon \downarrow 0$. This last convergence is equivalent to the convergence of the degenerate distribution at point $\int_{\varepsilon<\|x\|\leq 1} f^+(x)\nu_t(dx)$ and

consequently $\int_{\varepsilon<\|x\|\leq 1} f^+(x)\nu_t(dx) \to \int_{0<\|x\|\leq 1} f^+(x)\nu_t(dx)$ as $\varepsilon \downarrow 0$. Since ν_t is concentrated on K and K^* is a generating cone of B^* (3.9) holds for every $f \in B^*$.

Finally, $\int_{0<\|x\|\leq 1} x\nu_t(dx) := \gamma_t - \gamma_t^0$ is a well defined element in B since (3.9) holds and $f\left(\int_{0<\|x\|\leq 1} x\nu_t(dx)\right) = f(\gamma_t - \gamma_t^0)$. The last part now follows from the uniqueness of the generating triplet of the process Z. ■

It is possible to construct examples of additive processes where the assumption of the norm convergence of the jumps sum (3.6) always holds for any general separable Banach space, as shown as follows.

Example 3.1.9 Let K be a proper cone of a separable Banach space B. Let $\{N(t) : t \geq 0\}$ be a one dimensional Poisson process with parameter 1 and let $\{\tau_j\}_{j=1}^\infty$ be its sequence of arrival times. Let $\{\xi_j\}_{j=1}^\infty$ be a sequence of independent and identically distributed random variables independent of $N(t)$. Let $G : (0, \infty) \times \mathbb{R} \to K$ be a measurable map. Then, from Rosinski (1990, Lem. 2.1), the K-valued process given by

$$Z_t = \sum_{j=1}^{N(t)} G(\tau_j, \xi_j) \quad t \geq 0$$

is a K- valued additive process. The jumps sum of Z is norm convergent. Indeed, for any $\varepsilon > 0$ and $t > 0$ we have that

$$Z_t^{\triangle_\varepsilon} = \sum_{s<t} (Z_s - Z_{s-})1_{\triangle_\varepsilon}(Z_s - Z_{s-}) = \sum_{j=1}^{N(t)} G(\tau_j, \xi_j)1_{\triangle_\varepsilon}(G(\tau_j, \xi_j)),$$

since $Z_s - Z_{s-} = G(\tau_j, \xi_j)1_{\{\tau_j = s\}}$ for all $s \geq 0$. Then, since the number of jumps up to time t is finite we have norm-$\lim_{\varepsilon\downarrow 0} Z_t^{\triangle_\varepsilon} = \sum_{j=1}^{N(t)} G(\tau_j, \xi_j)$ exists almost surely. If in addition K has a generating dual cone, then (3.8) is the characteristic functional of Z and the integrability condition (3.9) is satisfied.

Definition 3.1.10 Let K be a proper cone of a separable Banach space. We shall say that a K-valued additive process has the *special Lévy-Khintchine representation* (with respect to

K) if its characteristic functional is given by (3.8), where $\gamma_t^0 \in K$ and the Lévy measure ν_t is concentrated on K and it satisfies the integrability condition (3.9).

We finally end this section with an important conclusion.

Theorem 3.1.11 *Let K be a proper regular cone of a separable Banach space B and let Z be a any K-valued additive process. Then*

a) The jump process $Z_t^{\triangle_\epsilon}$ in (3.6) is always norm convergent a.s.

b) If in addition the dual cone of K is generating, then Z has the special Lévy-Khintchine representation.

Proof. (a) follows from Lemma 3.1.5 and recalling that for a regular cone K all K-increasing and K-majorized sequences are norm convergent. (b) follows from (a) and Theorem 3.1.8. ■

3.1.3 Additive processes in LK-cones

A consequence of Theorem 3.1.11 and Propositions 1.1.5 and 3.1.1 is that additive processes in LK-cones are characterized by the special Lévy-Khintchine representation.

Theorem 3.1.12 *Let K be an LK-cone of a separable Banach space B. A B-valued additive process possesses the special Lévy-Khintchine representation (3.8) if and only if it is a K-valued additive process.*

Proof. Assume the special Lévy-Khintchine representation, then by Definition 3.1.10 and Proposition 3.1.1 the process Z is K-valued. Conversely, assume Z is a K-valued additive process. Recall that an LK-cone is proper, has generating dual cone and it contains no copy of c_0^+. Then the assertion follows from Proposition 1.1.5 and Theorem 3.1.11 (b). ■

We recover, as a special case of Theorem 3.1.12, a result formulated in Gihman and Skorohod (1975, Cor. p. 278), who considered a restricted class of cones. Here we give a rigorous proof of this result by proving that their cones are LK-cones.

Proposition 3.1.13 *Let B be a separable Banach space with a proper cone K such that there is a continuous linear functional f_0 satisfying*

$$k_0 = \inf_{x \in K, \|x\|=1} f_0(x) > 0. \tag{3.12}$$

Then, a B-valued additive process has the special Lévy-Khintchine representation, if and only if it is a K-valued process.

Proof. We first observe that for $0 \neq x \in K$, $0 < k_0 \leq f_0(x/\|x\|)$ and therefore

$$\|x\| \leq k_0^{-1} f_0(x), \qquad x \in K. \tag{3.13}$$

For any nonzero continuous linear functional f on B, define $f_1(x) = f(x) + \|f\|' k_0^{-1} f_0(x)$ and $f_2(x) = \|f\|' k_0^{-1} f_0(x)$ for all $x \in B$. Then f_1 and $f_2 \in K^*$ and $f_1(x) - f_2(x) = f(x)$. Indeed, observe that f_2 is nonnegative on K since $f_0(x) > 0$ for $x \in K$ and using (3.13) f_1 is also nonnegative on K. Then K^* is a generating cone for B^*.

Next, let $\sum_{k=1}^{\infty} x_k$ be any w.u.C. series of elements in K and let $s_n = \sum_{k=1}^{n} x_k$. Then $\sum_{k=1}^{\infty} |f_0(x_k)|$ is finite and for $n > m$, $s_n - s_m \in K$ and therefore $\|s_n - s_m\| \leq k_0^{-1} f_0(s_n - s_m)$. Then (s_n) is norm convergent and hence from Proposition 1.1.2, K does not contain a copy of c_0^+. Then, K is an LK-cone and the result then follows from Theorem 3.1.12. ■

There are examples of LK-cones for which condition (3.12) is not satisfied. Namely,

Example 3.1.14 Consider the Hilbert space l_2 of scalars sequences $x = (x_n)$ such that $\sum_{k=1}^{\infty} |x_k|^2 < \infty$ with inner product $\langle x, y\rangle = \sum_{k=1}^{\infty} x_k y_k$ and norm $\|x\|_2 = \{\sum_{k=1}^{\infty} |x_k|^2\}^{1/2}$. The proper cone $l_2^+ = \{(x_n) \in l_2 : x_n \geq 0\}$ is an LK-cone in l_2 (see Example 1.2.1).

We shall prove that l_2^+ does not satisfy condition (3.12). Indeed, let $f \in l_2^*$. By the Riesz representation theorem there exists $y \in l_2$ such that $f(x) = \langle x, y\rangle$ for every $x \in l_2$. Then we write

$$\langle x, y\rangle = \sum_{k=1}^{\infty} x_k y_k, \tag{3.14}$$

where $x = (x_n)$ and $y = (y_n)$.

If $y \notin l_2^+$ there is at least one $y_i < 0$ and we can choose $x = e_i \in l_2^+$ such that $\|x\| = 1$ where e_i is the i-th unit vector of the canonical basis of l_2. Observe from (3.14) that $f(x) < 0$ and hence condition (3.12) is not verified for $f \in l_2^*$. Now assume that $y \in l_2^+$ and let us choose the sequence $(x^n)_{n\geq 1}$ in l_2^+ defined by $x^n = e_n$ where clearly $\|x^n\| = 1$. Then from (3.14) we have that $f(x^n) = \langle x^n, y \rangle = y_n \to 0$ as $n \to \infty$. Therefore $\inf_{x \in l_2^+, \|x\|_2 = 1} f(x) \leq 0$. We have proved that condition (3.12) is not satisfied for any $f \in l_2^*$.

More generally, for any separable Hilbert space its LK-cone generated by a complete orthonormal set does not satisfy condition (3.12). Moroever, in a similar way as in the Example 3.1.14, we can prove that the natural LK- cones of the space of scalar sequences l_p and the space of p-trace class compact operators $L_p(H)$ on a separable Hilbert space H, for $p > 1$, do not satisfy condition (3.12).

Remark 3.1.15 a) A result of Bessaga-Pelczynski already mentioned in Section 1.1.1 establishes that in order that each w.u.C. series $\sum_k x_k$ in a Banach space B be unconditionally convergent, it is both necessary and sufficient that B contain no copy of c_0 (Diestel (1984, Th. 5. 8)). This gives a characterization of Banach spaces where weakly unconditionally Cauchy series are unconditionally (norm) convergent. This characterization has been used in the probabilistic literature by Hoffmann-Jorgensen (1974) and Kwapien (1974). Specifically, they characterize the class of Banach spaces B for which the boundedness of the partial sums of independent symmetric B-valued random variables implies the convergence of the series, as those containing no copy of c_0. In this direction we point out that in our analysis on the convergence of the sums of non compensated jumps, we are dealing with non symmetric (in law) terms, since our process lies in the cone K.

b) As mentioned in Section 2.1.1, in the case of Banach-valued additive processes, the convergence of the compensated jumps in (2.7) was proved by Dettweiler (1982). He uses purely probabilistic tools such as the Itô-Nisio Theorem, Kolmogorov 's inequality and a result on L_p-convergence of series of independent random variables due to Hoffmann-Jorgensen

(1977, Th. 5.5). In contrast, a complete analysis of the convergence of the uncompensated jumps (3.6) of cone-vaued additive processes, requires probabilistic tools as well as to take into consideration the underlying geometry of the cone.

c) It remains an open question whether LK-cones are the only type of cones for which every cone-additive process has the special Lévy-Khintchine representation. We have some ideas about this problem which lead to an affirmative answer, but we have not been able to give a proof of this conjecture.

3.2 Subordinators

In this section we concentrate on the important case of cone valued additive processes with stationary increments.

3.2.1 Cone-increasing Lévy processes

We begin by pointing out a simple fact for general cone-increasing processes.

Lemma 3.2.1 *Let K be a proper cone of a separable Banach space B. Then every K-increasing process $\{Z_t : t \geq 0\}$ such that $Z_0 = 0$ a.s. is a K-valued process.*

Proof. Since $Z_0 = 0$ a.s. then, by K-increasingness, $Z_t = Z_t - Z_0 \in K$ a. s. ■

In the case of cone-increasing Lévy processes, the converse holds. That is, a cone-valued Lévy process is necessarily cone-increasing and viceversa. The proof is as in the finite dimensional case in Rocha-Arteaga and Sato (2003, Th. 4.1.9).

Proposition 3.2.2 *Let $\{Z_t : t \geq 0\}$ be a Lévy process. Let K be a proper cone of a separable Banach space B. Then the following are equivalent:*

a) For any fixed $t \geq 0$, Z_t is concentrated on K almost surely.

b) Almost surely, $Z_t(\omega)$ is K-increasing in t.

Proof. That (b) implies (a) follows from the above lemma. On the other hand, if (a) holds, using the stationarity of the increments, for all $s \le t$, $P(Z_t - Z_s \in K) = P(Z_{t-s} \in K) = 1$. Then

$$P(Z_t - Z_s \in K \text{ for all } s,t \text{ in } \mathbb{Q}^+ \text{ such that } s \le t) = 1.$$

From the right continuity of $\{Z_t\}$ and the closedness of K, we have that $Z_{t_n} - Z_{s_n} \to Z_t - Z_s \in K$ as $n \to \infty$, where s_n and t_n are decreasing sequences in $\mathbb{Q}^+$ to s and t respectively. Thus, almost surely, for any choice of s,t with $s \le t$, we have $Z_s \le_K Z_t$. ∎

In view of the above proposition and in analogy with the one dimensional case we introduce the following natural definition.

Definition 3.2.3 Let K be a proper cone of a separable Banach space B. A K-increasing Lévy process in B will be called a *K-valued subordinator, K-subordinator* or simply *subordinator* if the underlying cone is well understood.

As a consequence of Proposition 3.2.2, we write the special form of Proposition 3.1.1 for subordinators.

Proposition 3.2.4 *Let K be a proper cone of a separable Banach space B. Let $\{Z_t : t \ge 0\}$ be a B-valued Lévy process with generating triplet (A, ν, γ). Assume that*

a) $A = 0$,

b) $\nu(B \backslash K) = 0$, i.e., ν is concentrated on K,

c)

$$\int_{0<\|x\|\le 1} |f(x)|\,\nu(dx) < \infty, \qquad f \in B^* \tag{3.15}$$

and d)

$$\gamma^0 := \gamma - \int_{0<\|x\|\le 1} x\nu(dx) \in K, \tag{3.16}$$

where the last integral is in the sense of Pettis.

Then the process Z is a K-subordinator with the special Lévy-Khintchine representation

$$Ee^{if(Z_t)} = exp\left\{ t\left(\int_K \left(e^{if(x)} - 1\right)\nu(dx) + if(\gamma^0) \right) \right\}. \tag{3.17}$$

For K-subordinators with special Lévy-Khintchine respresentation (3.17) we introduce the following terminology.

Definition 3.2.5 Let K be a proper cone of separable Banach space B. A K-subordinator $\{Z_t : t \ge 0\}$ is called a *regular subordinator* in K or a *K-valued regular subordinator* if its has the special Lévy-Khintchine representation (3.17) where (3.15) is satisfied.

As in the K-valued additive processes case, in order that all K-subordinators be K-regular subordinators it is required that the converse of Proposition 3.2.4 holds; for which additional assumptions on the cone K are needed.

Proposition 3.2.6 *Let K be an LK-cone of a separable Banach space B. A B-valued Lévy process is a K-subordinator if and only if its characteristic functional has the special Lévy-Khintchine representation (3.17). That is, $A = 0$, ν is concentrated on K satisfying (3.15) and $\gamma^0 \in K$.*

3.2.2 The Laplace transform

The Laplace transform of a regular subordinator in a Banach space is sometimes more useful than the Fourier transform. It is obtained, by the usual analytic continuation argument, from the Fourier transform.

Proposition 3.2.7 *Let K be a proper cone of a separable Banach space and let $\{Z_t : t \ge 0\}$ be a K-valued regular subordinator. Let S be a K-valued infinitely divisible random variable with the same law as Z_1. Then*

a) The Laplace transform of S is of the form

$$Ee^{-f(S)} = exp\{-\Phi(f)\} \qquad f \in K^* \tag{3.18}$$

where

$$\Phi(f) = \int_K \left(1 - e^{-f(x)}\right)\nu(dx) + f(\gamma^0) \tag{3.19}$$

and ν, the Lévy measure of S, is concentrated on K and satisfies (3.15) and γ^0 is in K.

b) The Laplace transform of Z_t is given by

$$Ee^{-f(Z_t)} = exp\{-t\Phi(f)\} \qquad f \in K^*.$$

Here the exponent Φ is called the *Laplace exponent of the subordinator.*

Proof. a) Let $f \in K^*$. We claim that

$$Ee^{zf(S)} = exp\left\{\int_K \left(e^{zf(x)} - 1\right)\nu(dx) + \omega f(\gamma_0)\right\} \tag{3.20}$$

for any $z \in \mathbb{C}$ with $Re(z) \leq 0$. Define

$$\begin{aligned}\varphi_1(z) &= Ee^{zf(S)},\\ \varphi_2(z) &= exp\left\{\int_K \left(e^{zf(x)} - 1\right)\nu(dx) + zf(\gamma_0)\right\}\end{aligned}$$

for any $z \in \mathbb{C}$ with $Re(z) \leq 0$.

First, φ_1 and φ_2 are finite and continuous in $\{Re(z) \leq 0\}$. We only prove the statements for φ_2. Choose $\delta(z, f) = \min(\frac{1}{|z|\|f\|'}, 1) > 0$, where without loss of generality we have taken $z \neq 0$ and $f \neq 0$, then

$$\int_{0<\|x\|<\delta(z,f)} \left|\left(e^{zf(x)} - 1\right)\right|\nu(dx) \leq \frac{7}{4}|z| \int_{0<\|x\|<\delta(z,f)} |f(x)|\,\nu(dx).$$

This proves that φ_2 is well defined. To prove the continuity observe that for any $z_0 \in \{Re(z) \leq 0\}$,

$$|\varphi_2(z) - \varphi_2(z_0)| \to 0, \qquad \text{as } |z - z_0| \to 0,$$

is equivalent to show that

$$\left|\int_K \left(e^{zf(x)} - 1\right)\nu(dx) - \int_K \left(e^{z_0f(x)} - 1\right)\nu(dx)\right| \leq \int_K \left|e^{(z-z_0)f(x)} - 1\right|\nu(dx) \longrightarrow 0,$$

as $|z - z_0| \to 0$. Now for z sufficiently close to z_0,

$$\begin{aligned}\int_K \left|e^{(z-z_0)f(x)} - 1\right|\nu(dx) &= \int_{\{\|x\|<1\}\cap K} \left|e^{(z-z_0)f(x)} - 1\right|\nu(dx) + \int_{\{\|x\|\geq1\}\cap K} \left|e^{(z-z_0)f(x)} - 1\right|\nu(dx)\\ &\leq \frac{7}{4}|z - z_0| \int_{\{\|x\|<1\}\cap K} |f(x)|\,\nu(dx) + \left(e^{|z-z_0|\|f\|'} - 1\right)\nu(\|x\| \geq 1)\end{aligned}$$

which converges to 0 as $|z - z_0| \to 0$. Here the last integral is well defined by (3.15). This proves that Φ_2 is continuous in $\{Re(z) \leq 0\}$.

Second, both φ_1 and φ_2 are analytic in $A = \{Re(z) < 0\}$. That φ_1 is analytic on A is obvious. In order to prove the statement for φ_2, define for $n = 1, 2, ...$,

$$I_n(z) = exp\left\{\int_{K_n} \left(e^{zf(x)} - 1\right)\nu(dx) + zf(\gamma_0)\right\} \qquad z \in A,$$

where $K_n = \{x \in K : \|x\| \leq n\}$. As in the first part, we choose $\delta(z, f) = \min(\frac{1}{|z|\|f\|'}, 1) > 0$, then, in the set $\{0 < \|x\| < \delta(z, f)\}$ the following are valid,

i) the former integral is finite for each $z \in A$,

ii) for each x in K_n, the integrand is differentiable in $z \in A$,

iii) and the integrand is uniformly bounded in z and x.

These imply that $\int_{0<\|x\|<\delta(z,f)} \left(e^{zf(x)} - 1\right)\nu(dx)$ is analytic in $z \in A$ and therefore

$$\int_{K_n} \left(e^{zf(x)} - 1\right)\nu(dx) \tag{3.21}$$

is analytic in A. Finally, the integral in (3.20) is analytic in A since it appears as an uniform limit, in z, of the analytic functions in (3.21) because

$$\left|\int_{K\setminus K_n} \left(e^{zf(x)} - 1\right)\nu(dx)\right| \leq \int_{K\setminus K_n} \left|\left(e^{zf(x)} - 1\right)\right|\nu(dx) \leq 2\nu(\{x \in K : \|x\| > n\}) \to 0$$

as n tends to ∞. Here the last inequality follows from the fact that $\left|e^{zf(x)}\right| = e^{\mathrm{Re}(z)f(x)} \leq 1$ for every z such that $\mathrm{Re}(z) \leq 0$ and $f \in K^*$. This proves that φ_2 is analytic in A.

In summary, $\varphi_1 - \varphi_2$ is continuous in $\{Re(z) \leq 0\}$ and analytic in $\{Re(z) < 0\}$. From (3.8) $\varphi_1(z) - \varphi_2(z) = 0$ when $Re(z) = 0$. Then, by the H.A. Schwartz's principle of reflection we can extend the range of analyticity of $\varphi_1 - \varphi_2$ to $\mathbb{C}$. We get $\varphi_1(z) - \varphi_2(z) = 0$ for $z \in \mathbb{C}$. In particular we get (3.18) for $z = -1$ which proves (a). Finally from (3.17) and (a), we obtain (b). ■

As a consequence of the preceding proposition and Proposition 2.1.11, infinitely divisible random variables in LK-cones are characterized as follows.

Corollary 3.2.8 *Let K be an LK-cone of a separable Banach space B. In order that a K-valued random variable S have an infinitely divisible law it is necessary and sufficient that S have the Laplace transform (3.18) where $\gamma^0 \in K$ and the Lévy measure be concentrated on K satisfying (3.15).*

Proof. Suppose S is an infinitely divisible random variable in K and let $\{Z_t : t \geq 0\}$ be the associated K-subordinator such that S and Z_1 have the same law, see Proposition 2.1.11. From Proposition 3.2.6 the process Z is a K-regular subordinator. Then from Proposition 3.2.7 (b) S has the Laplace transform (3.18). The converse is immediate. ■

The last result may be thought of the Banach space analogue of the characterization (see (3)-(4) in the Preface) of the Laplace transform of a real nonnegative infinitely divisible random variable in Theorem 13.7.2 in Feller (1971). It was essentially proved (by a different method) in Dettweiler (1976) for normal and regular cones of general ordered vector spaces. Since in a Banach space a cone is normal if and only if its dual cone is generating (Schaefer (1999, Cor. 5.3)), then from Proposition 1.1.5 (which is based in a result of Kamthan and Gupta (1985)), we have a more natural statement of the above corollary than Dettweiler's formulation for Banach space case. Closed results for other infinite dimensional topological spaces are given by Mase (1979) and Giné and Hahn (1985) for the case of cones of compact

convex sets of $\mathbb{R}^n$ and by Jonasson (1998) for locally compact cones, including those of finite measures on a compact Polish space and of increasing nonnegative upper semicontinuous functions.

3.2.3 Self-decomposability

This section ends with an application of Proposition 3.2.7 to self-decomposability. We first see that self-decomposable regular subordinators are easily described from their Lévy measures. A Lévy process $\{X_t : t \geq 0\}$ is called *self-decomposable* if the law of X_1 is self-decomposable.

Proposition 3.2.9 *Let K be a proper cone of separable Banach space B. A K-regular subordinator is self-decomposable if and only if its Lévy measure is given by*

$$\nu(C) = \int_{S_K} \int_{(0,\infty)} 1_C(ry) \frac{k(y,r)}{r} dr \lambda(dy) \qquad C \in \mathcal{B}(B \backslash \{0\}), \tag{3.22}$$

where S_K is unit sphere of K, λ is a finite measure on S_K and the function $k(y,r) : S_K \times (0,\infty) \to (0,\infty)$ is non-increasing and left-continuous in r for each $y \in S_K$ and measurable in y for each $r \in (0,\infty)$. Moreover,

$$\int_{S_K} \int_{(0,1)} |f(y)|\, k(y,r) dr \lambda(dy) < \infty \qquad f \in B^*. \tag{3.23}$$

Proof. It follows from (3.17), (3.15) and Theorem 2.1.15. ■

The following proposition shows that the infinitely divisible random element S_c, in the definition (2.16) of self-decomposability, is also a K-valued element if S is K-valued.

Proposition 3.2.10 *Let K be a proper cone of separable Banach space B and consider a K-valued regular self-decomposable subordinator. Let S be a random variable in K with the same law that of the subordinator at time 1. Then the component S_c in (2.16), for each $0 < c < 1$, is a K-valued infinitely divisible random variable.*

Proof. Theorem 2.1.14 implies the infinite divisibility of the component S_c. It remains to prove that it takes values in K. Without loss of generality assume that the drift γ^0 of S is zero. From the decomposition in law $S = cS + S_c$, (3.18) and Proposition 3.2.9

$$\begin{aligned} \log Ee^{-f(S_c)} &= \log Ee^{-f(S)} - \log Ee^{-f(cS)} \\ &= \int_K \left(e^{-f(x)} - 1\right) \nu(dx) - \int_K \left(e^{-f(cx)} - 1\right) \nu(dx) \\ &= \int_{\mathcal{S}_K} \int_{(0,\infty)} \left(e^{-f(x)} - 1\right) \left[k(y,r) - k(c^{-1}y,r)\right] \frac{dr}{r} \lambda(dy). \end{aligned}$$

Notice that $k_c(y,r) := k(y,r) - k(c^{-1}y,r)$ is nonnegative in r since $k(y,r)$ is non-increasing in r. Let us define the nonnegative measure ν_c on K as

$$\nu_c(C) = \int_{\mathcal{S}_K} \int_{(0,\infty)} 1_C(ry) k_c(y,r) \frac{dr}{r} \lambda(dy).$$

Since $k(y,r)$ and $k(c^{-1}y,r)$ satisfy (3.23) we get

$$\int_{0<\|x\|\leq 1} |f(x)|\, \nu_c(dx) = \int_{\mathcal{S}_K} \int_{(0,1)} |f(y)|\, k_c(y,r) dr \lambda(dy) < \infty \qquad f \in B^*.$$

We have shown that $Ee^{if(S_c)} = \exp\left\{\int_K \left(e^{if(x)} - 1\right) \nu_c(dx)\right\}$ with ν_c concentrated on K satisfying $\int_{0<\|x\|\leq 1} |f(x)|\, \nu_c(dx) < \infty$ for every $f \in B^*$. Next, by Proposition 2.1.11 there exists a Lévy process $\{Z_t : t \geq 0\}$ in B whose law at time 1 agrees with the law of S_c. It is a K-valued regular subordinator by Proposition 3.2.4. Now the assertion follows from Proposition 3.2.2. ■

3.3 Examples

Various examples of cone additive processes and subordinators are constructed in this section, by using three general different methods.

3.3.1 Stable and tempered subordinators

Proposition 3.1.1 and 3.2.4 give a general method for constructing cone-valued additive processes and subordinators from the original Lévy-Khintchine representation (2.3). We use this and known results for α-stable laws in Banach spaces (Example 2.1.17) to construct examples of α-stable cone-valued additive processes and regular subordinators. Then, using an "exponential tilting" argument for these α-stable cone-valued processes, we introduce a new class of α-tempered cone-additive processes and subordinators. When $\alpha = 1$ they may be regarded as the cone-valued version of the one dimensional inverse Gaussian random variable.

Throughout this section K will be an arbitrary proper cone of a real separable Banach space B. Let $\mathcal{S}_K = \{x \in K : \|x\| = 1\}$.

Example 3.3.1 (α-Stable additive processes) Let $t \geq 0$. A probability measure μ_t on a Banach space B has α-stable distribution, $0 < \alpha < 2$, (see Example 2.1.17) if and only if its characteristic functional has the form

$$\hat{\mu}_t(f) = \exp\left\{c_\alpha^{-1} \int_{(0,\infty)} \int_{\partial U} \left(e^{irf(y)} - 1 - irf(y) 1_U(ry)\right) \frac{\lambda_t(dy)}{r^{1+\alpha}} dr + f(\gamma_t)\right\} \qquad f \in B^*.$$

where $\gamma_t \in B$, $c_\alpha > 0$ is a constant and λ_t is the spectral measure of μ_t concentrated on the unit sphere ∂U of the unit closed ball U of B. Assume that for each $D \in \mathcal{B}_0(\partial U \backslash \{0\})$ the function $\lambda_t(D)$ is continuous and increasing in t. Then, from Proposition 2.1.8 there exists an additive process $\{Z_t : t \geq 0\}$ such that Z_t has the α-stable distribution μ_t.

Consider $0 < \alpha < 1$ and assume that λ_t is concentrated on $\mathcal{S}_K$. Using polar coordinates as in Example 2.1.17, let $\nu_t(C) = c_\alpha^{-1} \int_{(0,\infty)} \int_{\mathcal{S}_K} 1_C(ry) \lambda_t(dy) \frac{dr}{r^{1+\alpha}}$ for $C \in \mathcal{B}(K \backslash \{0\})$ which is concentrated on K for each t. Since $\int_{0<\|x\|\leq 1} \|x\|\, \nu_t(dx) = c_\alpha^{-1} \int_{(0,1)} \int_{\mathcal{S}_K} \lambda_t(dy) \frac{dr}{r^\alpha} = c_\alpha^{-1} \frac{1}{1-\alpha} \lambda_t(\mathcal{S}_K) < \infty$, then condition (3.5) is satisfied and we have that $\int_{0<\|x\|\leq 1} x \nu_t(dx)$ and $\int_{\mathcal{S}_K} y \lambda_t(dy)$ are well defined Bochner integrals and in particular $\int_{0<\|x\|\leq 1} |f(x)|\, \nu_t(dx) < \infty$ for each $f \in B^*$ and every t. Next, choose γ_t^0 such that $\gamma_t^0 = \gamma_t - c_\alpha^{-1} \frac{1}{1-\alpha} \int_{\mathcal{S}_K} y \lambda_t(dy)$ belongs to K. Then assumptions (a)-(d) in Proposition 3.1.1 are satisfied. Therefore the additive process Z is concentrated on K and has characteristic functional

$$Ee^{if(Z_t)} = \exp\left\{\int_K \left(e^{if(x)} - 1\right)\nu_t(dx) + if(\gamma_t^0)\right\}. \qquad (3.24)$$

Furthermore, using the fact that $\int_{(0,\infty)} \left(e^{-rf(y)} - 1\right)\frac{dr}{r^{1+\alpha}} = \{f(y)\}^\alpha \Gamma(-\alpha)$ for every $f \in K^*$, the Laplace transform of Z given by Proposition 3.2.7 becomes

$$Ee^{-f(Z_t)} = \exp\left\{c_\alpha^{-1}\Gamma(-\alpha)\int_{S_K}\{f(y)\}^\alpha \lambda_t(dy) - f(\gamma_t^0)\right\} \qquad f \in K^*. \qquad (3.25)$$

The K-valued additive process $\{Z_t : t \geq 0\}$ with system of generating triplets $(0, \nu_t, \gamma_t^0)$ is called *α-stable additive process*.

The family of tempered stable distributions is obtained by exponential tilting of positive stable distributions (also called Esscher transformations, see Barndorff-Nielsen, Mikosh and Resnik (2001, pp. 24-27)). The one dimensional case has been recently studied by Barndorff-Nielsen and Levendorskiĭ (2001) and Barndorff-Nielsen and Shepard (2001). The matrix case is studied in Barndorff-Nielsen and Pérez-Abreu (2002). For Lévy processes in $\mathbb{R}^d$ this transformation is studied in Sato (1999, Ex. 33.14 and 33.15).

Example 3.3.2 (α-Tempered additive processes) Let $0 < \alpha < 2$. Fix $t \geq 0$ and consider an $\alpha/2$-stable probability measure μ_t concentrated on K given by Example 3.3.1 with Laplace transform (3.25). Fix $p \in K^*$ and define a distribution on K by the exponential tilting

$$\mu_{p,t}(dy) = \frac{e^{-p(y)}}{L_{\mu_t}(p)}\mu_t(dy)$$

where $L_{\mu_t}(p)$ denotes the Laplace transform of μ_t. Then the Laplace transform of $\mu_{p,t}$ is

given by $L_{\mu_{p,t}}(f) = L_{\mu_t}(f+p)/L_{\mu_t}(p)$ for every $f \in K^*$. Hence

$$\begin{aligned}
\log L_{\mu_{p,t}}(f) &= \log L_{\mu_t}(f+p) - \log L_{\mu_t}(p)\\
&= c_{\alpha/2}^{-1}\Gamma(-\alpha/2)\left\{\int_{S_K}\{(f+p)(y)\}^\alpha \lambda_t(dy) - \int_{S_K}\{p(y)\}^\alpha \lambda_t(dy) - f(\gamma_t^0)\right\}\\
&= c_{\alpha/2}^{-1}\left\{\int_{S_K}\int_{(0,\infty)}\left(e^{-r(f+p)(y)} - 1\right)\frac{dr}{r^{1+\alpha/2}}\lambda_t(dy)\right.\\
&\qquad \left. - \int_{S_K}\int_{(0,\infty)}\left(e^{-rp(y)} - 1\right)\frac{dr}{r^{1+\alpha/2}}\lambda_t(dy) - f(\gamma_t^0)\right\}\\
&= c_{\alpha/2}^{-1}\left\{\int_{S_K}\int_{(0,\infty)}\left(e^{-rf(y)} - 1\right)e^{-rp(y)}\frac{dr}{r^{1+\alpha/2}}\lambda_t(dy) - f(\gamma_t^0)\right\}\\
&:= \int_K \left(e^{-f(x)} - 1\right)\nu_t(dx) - f(\gamma_t^0)
\end{aligned}$$

where $\nu_t(C) = c_{\alpha/2}^{-1}\int_{(0,\infty)}\int_{S_K} 1_C(ry)\frac{dr}{r^{1+\alpha/2}}\lambda_t(dy)$ for $C \in \mathcal{B}(K\backslash\{0\})$. As in Example 3.3.1, we have that $\int_{0<\|x\|\leq 1}\|x\|\,\nu_t(dx) < \infty$ and hence the Lévy measure ν_t satisfies $\int_{0<\|x\|\leq 1}|f(x)|\,\nu_t(dx) < \infty$ for $f \in B^*$. Assume that $\lambda_t(D)$ is continuous and increasing in t for each $D \in \mathcal{B}_0(\mathcal{S}_K\backslash\{0\})$. Then by Proposition 2.1.8 there exists an additive process $\{Z_t : t \geq 0\}$ in B such that $\mu_{p,t}$ is the law of Z_t. Now by Proposition 3.1.1 Z is concentrated on K. The K-valued additive process $\{Z_t : t \geq 0\}$ with system of generating triplets $(0, \nu_t, \gamma_t^0)$ is called *α-tempered additive process*.

Example 3.3.3 (α-Stable subordinators) Let $0 < \alpha < 1$ and let S be an α-stable random variable with values in K. Let $\{Z_t : t \geq 0\}$ be he K-valued Lévy process such that Z_1 has the law of S, see Proposition 2.1.11. We have that $\{Z_t\}$ is a regular subordinator by Definition 3.2.5 and (3.24). It is called *α-stable subordinator*.

Since its Lévy measure is $\nu(C) = c_\alpha^{-1}\int_{(0,\infty)}\int_{S_K} 1_C(ry)\frac{dr}{r^{1+\alpha}}\lambda(dy)$ for $C \in \mathcal{B}(K\backslash\{0\})$ where λ is the spectral measure of Z_1, from Remark 2.1.18 and Proposition 3.2.9 we have that $\{Z_t\}$ is self-decomposable. Furthermore, the Laplace transform of Z_t (see 3.25) has the form

$-tf(\gamma^0)$

$$Ee^{-f(Z_t)} = \exp\left\{tc_\alpha^{-1}\Gamma(-\alpha)\int_{S_K}\{f(y)\}^\alpha \lambda(dy)\right\} \qquad f \in K^*. \qquad (3.26)$$

The following example, in the case $\alpha = 1$, extends to the infinite dimensional case, the inverse Gaussian matrix introduced in Barndorff-Nielsen and Pérez-Abreu (2002). The corresponding subordinator is the Banach-valued version of the one dimensional inverse Gaussian Lévy process; see Barndorff-Nielsen (1998) for important applications of the latter.

Example 3.3.4 (α-Tempered subordinators) Let $0 < \alpha < 2$ and let T be a tempered K-valued random variable obtained by exponential tilting of a K-valued $\alpha/2$-stable random variable with spectral measure λ concentrated on $\mathcal{S}_K$ and a fixed $p \in K^*$. From Example 3.3.2 we have

$$Ee^{if(T)} = \exp\left\{\int_K \left(e^{if(x)} - 1\right)\nu(dx)\right\} \qquad f \in B^*$$

where $\int_{0<\|x\|\leq 1} |f(x)|\,\nu(dx) < \infty$ and

$$\nu(dx) = c_{\alpha/2}^{-1}\int_{(0,\infty)}\int_{\mathcal{S}_K} 1_C(ry)e^{-rp(y)}\frac{dr}{r^{1+\alpha/2}}\lambda(dy) \qquad C \in \mathcal{B}(K\backslash\{0\}). \tag{3.27}$$

A K-valued Lévy process $\{Y_t : t \geq 0\}$ such that Y_1 has the law of T is a regular subordinator (see Proposition 2.1.11). Such a process is called *α-tempered subordinator*.

Tempered subordinators are self-decomposable. Indeed, $k(y,r) = c_{\alpha/2}^{-1}\frac{e^{-rp(y)}}{r^{\alpha/2}}$ is the required function in (3.22), i.e., it is nonnegative, non-increasing and left-continuous for each $y \in \mathcal{S}_K$.

3.3.2 Iterated subordination

We now describe a second method to construct cone-valued subordinators by iterated subordination of a cone-valued regular subordinator by a one dimensional subordinator; similar to the subordination of Theorem 2.3.1.

Theorem 3.3.5 *Let $\{Z_t : t \geq 0\}$ be a K-valued regular subordinator and let $\{\sigma_t : t \geq 0\}$ be a one dimensional subordinator. Assume Z and σ are independent and let $Y_t = Z_{\sigma_t}$, for $t \geq 0$. Then the subordinated process $\{Y_t : t \geq 0\}$ is a K-valued subordinator and its Laplace*

transform is given by

$$Ee^{-f(Y_t)} = e^{-t\Phi_\sigma(\Phi_Z(f))} \qquad f \in K^*, \tag{3.28}$$

where $\Phi_Z(f)$ with $f \in K^*$ and $\Phi_\sigma(u)$ with $u \geq 0$ are the Laplace exponents of $\{Z_t\}$ and $\{\sigma_t\}$ respectively, see (3.19) and (4) in the Preface.

Proof. From Theorem 2.3.1 we have that Y is a Lévy process which is K-valued since Z is K-valued. The proof of (3.28) follows by using the usual conditional argument

$$\begin{aligned} Ee^{-f(Y_t)} &= E\left[\left(Ee^{-f(X(s))}\right)_{s=\sigma_t}\right] = E\left[\left(e^{-s\Phi_Z(f)}\right)_{s=\sigma_t}\right] \\ &= \exp\left[-t\Phi_\sigma\left(\Phi_Z(f)\right)\right] \end{aligned}$$

for every $f \in K^*$. ■

An application of the former theorem allows us to construct a concrete example of a K-regular subordinators with an explicit expression for its Laplace exponent.

Example 3.3.6 (Mittag-Leffler subordinator) Let $0 < \alpha < 1$ and let $\{Z_t : t \geq 0\}$ be a K-valued α-stable subordinator with Laplace exponent Φ_Z given in (3.26). Let $\{\sigma_t : t \geq 0\}$ be a one dimensional Γ-subordinator, that is, its law is the Gamma distribution denoted by $\Gamma(p,q)$ whose probability density function is $q^p\Gamma(p)x^{p-1}e^{-qx}1_{(0,\infty)}(x)$. Its corresponding Laplace exponent is $\Phi_\sigma(u) = \log(1+\frac{u}{q})^p$ (see Sato (1999, Ex. 8.10)).

Let $\{Y_t : t \geq 0\}$ be the K-subordinator process obtained from subordination of $\{Z_t\}$ by $\{\sigma_t\}$. Then, its corresponding Laplace exponent Φ has the form (see 3.28)

$$\Phi(f) = p\log\left(1 + \frac{\int_{\mathcal{S}_K}\{f(y)\}^\alpha\,\tilde{\lambda}(dy)}{q}\right) \qquad f \in K^*$$

where $\tilde{\lambda}(dy) = -c_\alpha^{-1}\Gamma(-\alpha)\lambda(dy)$ and λ is the spectral measure of Z_1.

If, in particular, $\{\sigma_t\}$ has law $\Gamma(1,1)$ and $\tilde{\lambda}$ is concentrated at the point $y_0 \in \mathcal{S}_K$,

$$\Phi(f) = \log\left(1 + \{f(y_0)\}^\alpha\,\tilde{\lambda}(\{y_0\})\right) \qquad f \in K^*.$$

Hence the subordinator Y may be thought as the cone valued analogue of the one dimensional Mittag-Leffler subordinator (see (3.36) in Example 3.3.14 below).

3.3.3 Subordinators in cones with bases

A class of subordinators on cones generated by bases of Banach spaces can be constructed via a third method described as follows. Suppose the separable Banach space B has a bounded basis (x_n) of type l_+. Then the associated cone $K_{(x_n)}$ of B is of type l_+, that is,

$$K_{(x_n)} = \left\{ \sum_{k=1}^{\infty} \alpha_k x_k \in B : \alpha_k \geq 0 \text{ for all } k, \ \sum_{k=1}^{\infty} \alpha_k < \infty \right\}. \tag{3.29}$$

Theorem 3.3.7 *Let B be a real separable Banach space with cone $K_{(x_n)}$ of type l_+ generated by the bounded basis (x_n). Let $\{\sigma_k(t) : t \geq 0\}$, for $k = 1, 2, ...$, be independent subordinators in $\mathbb{R}$ with Φ_k the Laplace exponent for σ_k, i.e., $Ee^{-u\sigma_k(t)} = e^{-t\Phi_k(u)}$ $u \geq 0$. Let*

$$S = \sum_{k=1}^{\infty} \eta_k \sigma_k(1) x_k, \tag{3.30}$$

where (η_k) is a sequence of positive real numbers. Assume that

$$\sum_{k=1}^{\infty} \Phi_k(\eta_k) < \infty. \tag{3.31}$$

Then S is an infinitely divisible $K_{(x_n)}$-valued random variable and

$$Z_t = \sum_{k=1}^{\infty} \eta_k \sigma_k(t) x_k, \qquad t \geq 0 \tag{3.32}$$

is a $K_{(x_n)}$-valued subordinator such that Z_1 has the law of S.

Proof. From the dominated convergence theorem we get, for each n,

$$Ee^{-\sum_{k=1}^{\infty} \eta_k \sigma_k(1)} = e^{-\sum_{k=1}^{\infty} \Phi_k(\eta_k)} > 0$$

which implies that $\sum_{k=1}^{\infty} \eta_k \sigma_k(1) < \infty$ almost surely. Hence $\sum_{k=1}^{\infty} \eta_k \sigma_k(1) x_k$ in (3.30) converges in B almost surely since it is equivalent to the almost surely convergence of $\sum_{k=1}^{\infty} \eta_k \sigma_k(1)$ (Theorem 1.1.8). Thus S is a well defined $K_{(x_n)}$-valued random variable. Let $\sigma_{k,n}$ be positive random variable with characteristic function $\hat{\mu}_{\sigma_k(1)}^{1/n}$ for $k \geq 1$ and $n \geq 1$. Define $S_n = \sum_{k=1}^{\infty} \eta_k \sigma_{k,n} x_k$, for $n \geq 1$. Similarly as before we get, for each n, $Ee^{-\sum_{k=1}^{\infty} \eta_k \sigma_{k,n}} = e^{-\frac{1}{n}\sum_{k=1}^{\infty} \Phi_k(\eta_k)} > 0$. Therefore S_n is a $K_{(x_n)}$-valued random variable. Next, for every $f \in B^*$

$$Ee^{if(S)} = \prod_{k=1}^{\infty} Ee^{i\eta_k f(x_k)\sigma_k(1)} = \left\{ \prod_{k=1}^{\infty} \hat{\mu}_{\sigma_k(1)}^{1/n}(\eta_k f(x_k)) \right\}^n, \qquad n \geq 1.$$

On the other hand, for each $n \geq 1$,

$$Ee^{if(S_n)} = \prod_{k=1}^{\infty} Ee^{i\eta_k f(x_k)\sigma_{k,n}} = \prod_{k=1}^{\infty} \hat{\mu}_{\sigma_k}^{1/n}(\eta_k f(x_k)) = \left\{ Ee^{if(S)} \right\}^{1/n}$$

which proves that S is an infinitely divisible random element.

We can repeat the above arguments to observe that $\sum_{k=1}^{\infty} \eta_k \sigma_k(t)$ is an infinitely divisible nonnegative random variable, for each $t \geq 0$. Moroever, its Laplace transform is given by $Ee^{-u\sum_{k=1}^{\infty} \eta_k \sigma_k(t)} = e^{-t\sum_{k=1}^{\infty} \Phi_k(u\eta_k)}$. Hence

$$\sum_{k=1}^{\infty} \Phi_k(u\eta_k) < \infty \qquad \text{for } u \geq 0, \tag{3.33}$$

since the Laplace transform of an infinitely divisible random variable never vanishes.

Next, we shall prove that $Z(t)$ in (3.32) is a $K_{(x_n)}$-valued subordinator. For $0 \leq t_1 < t_2$ we have that $Z(t_2) - Z(t_1) = \sum_{k=1}^{\infty} \eta_k [\sigma_k(t_2) - \sigma_k(t_1)] x_k \in K_{(x_n)}$ almost surely since $\sigma_n(t_2) - \sigma_n(t_1) \geq 0$ a.s. for every $n \geq 1$. Then $\{Z_t\}$ is a $K_{(x_n)}$-increasing process.

It remains to prove that $\{Z_t\}$ is a Lévy process. It is clear that almost surely $Z(0) = 0$. For $0 \leq t_1 < t_2 < ... < t_m$, we have that $\sigma_n(t_2) - \sigma_n(t_1), \sigma_n(t_3) - \sigma_n(t_2), ..., \sigma_n(t_m) - \sigma_n(t_{m-1})$ are independent for every $m \geq 1$. Then $Z(t_2) - Z(t_1), Z(t_3) - Z(t_2), ..., Z(t_m) - Z(t_{m-1})$ are independent. This proves that $\{Z_t\}$ has independent increments. Next, for $0 \leq t < s$ we have that $Z(t+s) - Z(s) = \sum_{k=1}^{\infty} \eta_k [\sigma_k(t+s) - \sigma_k(s)] x_k$ and $Z(t) = \sum_{k=1}^{\infty} \eta_k \sigma_k(t) x_k$

are well defined almost surely by condition (3.31). Let $S_m(t,s)$ and $S_m(t)$ be the partial sums of $Z(t+s)-Z(s)$ and $Z(t)$ respectively. Notice that $Ee^{if(Z(t+s)-Z(s))} = Ee^{if(Z(t))}$ for every $f \in B^*$ since $Ee^{if(S_m(t,s))} \to Ee^{if(Z(t+s)-Z(s))}$, $Ee^{if(S_m(t))} \to Ee^{if(Z(t))}$ as $m \to \infty$ and $S_m(t,s) \stackrel{d}{=} S_m(t)$ for every $m \geq 1$. Therefore $Z(t+s)-Z(s) \stackrel{d}{=} S(t)$.

We claim that $\{Z_t\}$ is right-continuous. Let $t_0 \in [0,\infty)$ and let $t_n \downarrow 0$. For $Z(t_n)-Z(t_0) = \sum_{k=1}^{\infty} \eta_k [\sigma_k(t_n) - \sigma_k(t_0)] x_k$ we have almost surely

$$\|Z(t_n) - Z(t_0)\| \leq \left(\sup_{k\geq 1} \|x_k\|\right) \sum_{k=1}^{\infty} \eta_k [\sigma_k(t_n) - \sigma_k(t_0)]. \qquad (3.34)$$

Let $Y_n = \sum_{k=1}^{\infty} \eta_k [\sigma_k(t_n) - \sigma_k(t_0)]$. We show that Y_n converges to 0 a.s. From (3.33) and the dominated convergence theorem we have $Ee^{-uY_n} = e^{-(t_n-t_0)\sum_{k=1}^{\infty}\Phi_k(u\eta_k)}$ for $u \geq 0$. By continuity theorem $Y_n \to 0$ in distribution as $n \to \infty$ (Feller (1971, Th. 13.1.2)). Then from (3.34) $\|Z(t_n) - Z(t_0)\| \to 0$ almost surely. The existence of lef-limits of $\{Z_t\}$ are handled similarly. The stochastic continuity of $\{Z_t\}$ follows now from the fact that $Z(0) = 0$ a.s., the stationarity of the increments and the right continuity of the paths. Indeed, $P(\|Z(t)-Z(s)\| > \varepsilon) = P(\|Z(t-s)\| > \varepsilon) \to 0$ as $t \downarrow s$ or $s \uparrow t$. This concludes the proof. ∎

Remark 3.3.8 Theorem 3.3.7 implies that, for each $f \in B^*$, $f(S) = \sum_{k=1}^{\infty} \eta_k \sigma_k(1) f(x_k)$ is an infinitely divisible random variable on $\mathbb{R}$. If $\{Z_t : t \geq 0\}$ is a $K_{(x_n)}$-regular subordinator, Proposition 3.2.7 gives the Laplace transform of $f(S)$ for any $f \in K^*_{(x_n)}$. Moreover, if the Lévy measure ν_k of σ_k has Lévy density u_k, then the Lévy measures of the one dimensional nonnegative distributions of S (that is, $f(S) \geq 0$ a.s. for each $f \in K^*_{(x_n)}$) have the form

$$\nu \circ f^{-1}(dx) = \sum_{k=1}^{\infty} (f(x_k)\eta_k)^{-1} u_k\left((f(x_k)\eta_k)^{-1} x\right) dx \qquad f \in K^*_{(x_n)}, \qquad (3.35)$$

where ν denotes the Lévy measure of S. This provides a rich class of one dimensional positive infinitely divisible distributions and their associated subordinators.

We next provide concrete examples of $K_{(x_n)}$-subordinators of the form (3.32) arising from (3.30) under the condition (3.31). For the sake of completeness, we also provide, in each case, the Lévy densities u_k corresponding to (3.35).

Example 3.3.9 Let us consider the Gamma random variable $\sigma_k(1)$ with law $\Gamma(p_k, q_k)$. Then the random element S in (3.30) is called the $K_{(x_n)}$-Gamma random variable. We can get the convergence of (3.31) by choosing the convergent series $\sum_{k=1}^{\infty} p_k$ and the bounded sequence $\{\log(1+\frac{\eta_k}{q_k})\}$. As a special case we have $\eta_k = q_k$ for any k. For (3.35) we have $u_k(x) = p_k x^{-1} e^{-q^k x} 1_{(0,\infty)}(x)$.

Example 3.3.10 Let $0 < \alpha < 1$. Let $\sigma_k(1)$ be an α-stable positive random variable whose Laplace exponent is $\Phi_k(u) = c_k' u^{\alpha} + \gamma^{0,k} u$ where $\gamma^{0,k}$ is the drift, $c_k' = c_k \alpha^{-1}\Gamma(1-\alpha)$ and $c_k > 0$ is the constant appearing in the expression for the Lévy measure of $\sigma_k(1)$. Then (3.30) is a $K_{(x_n)}$-valued α-stable random variable.

Choosing the sequences (η_k), (c_k') and $(\gamma^{0,k})$ such that $\sum_{k=1}^{\infty} \eta_k^{\alpha} < \infty$ and the last two sequences being bounded, we obtain the convergence of (3.31). The corresponding $u_k(x)$ for (3.35) is $c_k x^{-1-\alpha} 1_{(0,\infty)}(x)$.

Example 3.3.11 Let $\sigma_k(1)$ be the one-sided 1/2-stable random variable with probability density function $c_k\left(\sqrt{2\pi}\right)^{-1/2} e^{-c_k^2/2x} x^{-3/2} 1_{(0,\infty)}(x)$. Then S is a $K_{(x_n)}$-valued one-sided 1/2-stable random variable. The series (3.31) converges if we select a bounded sequence (c_k) and a convergent series $\sum_{k=1}^{\infty} \eta_k^{1/2}$, or viceversa. Here $u_k(x) = c_k (2\pi)^{-1/2} x^{3/2} 1_{(0,\infty)}(x)$ is the Lévy density in (3.35).

Example 3.3.12 Take in (3.30) the Inverse Gaussian distribution for $\sigma_k(1)$ with parameters δ_k and γ_k whose probability density function is $(2\pi)^{-1/2} \delta_k e^{\delta_k \gamma_k} x^{-3/2} e^{-(\delta_k^2 x^{-1} + \gamma_k^2 x)/2} 1_{(0,\infty)}(x)$. Then S is called the $K_{(x_n)}$-inverse Gaussian random variable. Taking $\sum_{k=1}^{\infty} \delta_k \gamma_k < \infty$ and a bounded sequence (η_k/γ_k^2) we get (3.31). For (3.35) we have

$$u_k(x) = \delta_k (2\pi)^{-1/2} x^{-3/2} e^{-\gamma_k^2 x/2} 1_{(0,\infty)}(x).$$

Example 3.3.13 Consider the Bessel distribution for $\sigma_k(1)$ in (3.30) with parameter $p_k > 0$. Its probability density function is $p_k e^{-x}x^{-1}I_{p_k}(x)1_{(0,\infty)}(x)$. Here $I_p(x)$, for $p \geq 0$, is the modified Bessel function (Feller (1971, pp. 58, 437 and 451)). We call S the $K_{(x_n)}$-Bessel random variable. The series (3.31) converges for $\sum_{k=1}^{\infty} p_k < \infty$ and (η_k) bounded. We have that $u_k(x) = q_k x^{-1}e^{-x}I_0(x)1_{(0,\infty)}(x)$ is the Lévy density of $\sigma_k(1)$.

Example 3.3.14 Let $0 < \alpha_k < 1$. Let $\left\{\sigma_k^{(1)}(t) : t \geq 0\right\}$ and $\left\{\sigma_k^{(2)}(t) : t \geq 0\right\}$ be independent real subordinators where $\left\{\sigma_k^{(1)}\right\}$ is strictly α_k-stable and $\left\{\sigma_k^{(2)}\right\}$ has the Gamma law $\Gamma(p_k, q_k)$. Consider in (3.32) the subordinator $\sigma_k(t) = \sigma_k^{(1)}\left(\sigma_k^{(2)}(t)\right)$ whose Laplace exponent is given by

$$\Phi_k(u) = p_k \log\{1 + \frac{u^{\alpha_k}}{q_k}\} \qquad u \geq 0. \tag{3.36}$$

Then (3.31) is satisfied by choosing a convergent series $\sum_{k=1}^{\infty} p_k$ and a bounded sequence $(\log\{1 + \frac{\eta_k^{\alpha_k}}{q_k}\})$.

If, in particular, $\left\{\sigma_k^{(2)}\right\}$ has law $\Gamma(1,1)$ then the distribution of $\sigma_k(1)$ is $P(\sigma_k(1) \leq x) = 1 - E_{\alpha_k}(-x^{\alpha_k})$, $x \geq 0$, where $E_{\alpha_k}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(n\alpha_k+1)}$ is the Mittag-Leffler function (Pillai (1990)). The one dimensional subordinator $\{\sigma_k(t)\}$ is called the Mittag-Leffler subordinator. Then (3.30) is called a $K_{(x_n)}$-valued Mittag-Leffler random element. The Lévy density u_k in (3.35) is given by

$$u_k(x) = \alpha_k x^{-1} \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{\Gamma(n\alpha_k + 1)} 1_{(0,\infty)}(x).$$

Chapter 4

Application: Subordinators in Birkhoff-Kakutani Spaces

Results of the previous chapters are now applied to study Birkhoff-Kakutani space-valued subordinators. It is shown that these processes constitute a class of cone-increasing Lévy processes that behave very similar to the one dimensional subordinators. More generally, it is seen that to each cone-valued additive process, its corresponding norm is a one dimensional nonnegative additive process that provides relevant information about the Banach space valued process. A special application on the study of sample path behavior of cone-valued subordinators is done. Specifically, laws of large numbers and laws of iterated logarithm are easily obtained from the corresponding ones for the associated (one dimensional) norm subordinators presented in Bertoin (1996).

4.1 One dimensional similarities

Cone-additive processes in Birkhoff-Kakutani spaces have many properties similar to nonnegative additive processes, subordinators and infinitely divisible laws, as shown by the following three results.

Proposition 4.1.1 *Let $(B, \|\cdot\|, K)$ be a separable Birkhoff-Kakutani space.*

a) Every K-additive process is such that for each $t \geq 0$, $A_t = 0$, the Lévy measure ν_t satisfies

$$\int (1 \wedge \|x\|)\, \nu_t(dx) < \infty \tag{4.1}$$

and $\gamma_t - \int_{0<\|x\|\leq 1} x\nu_t(dx) \in K$, where the last integral is in the sense of Bochner.

b) All K-valued subordinators are K-regular subordinators.

Proof. From Proposition 1.3.2 K is a LK-cone. Then, Theorem 3.1.11 (b) gives the special Lévy Khintchine representation. Let $f_0 \in B^*$ be such that it agrees with the norm in K, i.e. $f_0(x) = \|x\|$ for $x \in K$. Hence finiteness in (4.1) follows from (3.1) since ν_t is concentrated on K. Finally, (b) follows from Proposition 3.2.6, since K is of LK-type. ■

Let $(\nu_t)_{t\geq 0}$ be a family of σ- finite measures concentrated on the cone K of a separable Birkhoff-Kakutani space B. Assume that $\nu_t(C)$ is continuous and increasing in t for each $C \in \mathcal{B}(K\backslash\{0\})$ and (4.1) is satisfied. Since the condition (4.1) implies that ν_t is a Lévy measure (Araujo and Giné (1980)) we apply Proposition 2.1.8 to get an additive process $\{Z_t : t \geq 0\}$ in B having the system of Lévy measures $(\nu_t)_{t\geq 0}$. Then $\{Z_t\}$ is a K-valued additive process by Proposition 3.1.1.

As a consequence of the above result and Proposition 3.2.7, we have the following characterization of infinitely divisible laws in cones of Birkhoff-Kakutani spaces.

Corollary 4.1.2 *Let $(B, \|\cdot\|, K)$ be a separable Birkhoff-Kakutani space. In order that a K-valued random element S has an infinitely divisible law it is necessary and sufficient that S has the Laplace transform (3.18), where γ^0 is in K and the Lévy measure ν is concentrated on K satisfying*

$$\int (1 \wedge \|x\|)\, \nu(dx) < \infty. \tag{4.2}$$

The above corollary answers in an affirmative way a question of Dettweiler (1976, Remark 2) of whether there are Banach spaces other than AL-spaces such that the condition (4.2) is satisfied for infinitely divisible cone valued elements. Indeed, from Section 1.3 we have

seen that the Banach spaces of trace-class operators and duals of C^*-algebras are examples of Birkhoff-Kakutani Banach spaces which are not lattices.

Similar to the one dimensional case, subordinators in Birkhoff-Kakutani spaces are of bounded variation.

Proposition 4.1.3 *Let $(B, \|\cdot\|, K)$ be a separable Birkhoff-Kakutani space. Then every K-valued subordinator has bounded variation on each interval $[0,t]$, $t > 0$, almost surely.*

Proof. Let $\{Z_t : t \geq 0\}$ be a K-valued subordinator and recall that $Z_t - Z_s \in K$ a.s. whenever $0 \leq s \leq t$. Using the additivity of the norm we have almost surely

$$var_{[0,t]} Z_s = \sup_{0=t_0<t_1<\ldots<t_n=t} \left\{ \sum_{k=1}^{n} \|Z_{t_k} - Z_{t_{k-1}}\| \right\}$$

$$= \sup_{0=t_0<t_1<\ldots<t_n=t} \left\{ \left\| \sum_{k=1}^{n} (Z_{t_k} - Z_{t_{k-1}}) \right\| \right\} = \|Z_t - Z_0\| < \infty.$$

■

From Section 3.3, several examples of additive processes and subordinators in Birkhoff-Kakutani spaces can be constructed. Specifically, Theorem 3.3.7 and Proposition 1.3.4 provide a class of $K_{(x_n)}$-regular subordinators. Examples 3.3.9 through 3.3.14 give the corresponding Gamma, α-stable, one-sided 1/2-stable, inverse Gaussian, Bessel and Mittag-Leffler $K_{(x_n)}$-subordinators, respectively.

4.2 The associated norm additive process

An important property of any cone-additive process in a Birkhoff-Kakutani space is that the associated norm process is a one dimensional (nonnegative) additive process.

Theorem 4.2.1 *Let $(B, \|\cdot\|, K)$ be a separable Birkhoff-Kakutani space and let $\{Z_t : t \geq 0\}$ be a K-valued additive process with drift γ_t^0 and Lévy measure ν_t, for $t \geq 0$. Then the process*

$\{r_t = \|Z_t\| : t \geq 0\}$ is an $\mathbb{R}_+$-valued additive process with Lévy-Khintchine representation

$$Ee^{iur_t} = exp\left\{\int_{\mathbb{R}_+} \left(e^{iur} - 1\right) \nu_t \circ \|\cdot\|^{-1}(dr) + iur_t^0\right\}, \tag{4.3}$$

where $r_t^0 = \|\gamma_t^0\|$ and

$$\int_{0<r\leq 1} r\nu_t \circ \|\cdot\|^{-1}(dr) < \infty.$$

In particular, if $\{Z_t : t \geq 0\}$ is a K-valued subordinator with Lévy measure ν_Z and drift γ_Z^0, the Laplace transform of $\{\|Z_t\| : t \geq 0\}$ is given by

$$Ee^{-u\|Z_t\|} = exp\{-t\Phi(u)\} \qquad u \in \mathbb{R}_+,$$

where the Laplace exponent has the expression

$$\Phi(u) = \int_{\mathbb{R}_+} \left(1 - e^{-ur}\right) \nu_Z \circ \|\cdot\|^{-1}(dr) + ur^0 \tag{4.4}$$

with $r^0 = \|\gamma_Z^0\|$.

Proof. For each $u \in \mathbb{R}$ take the continuous linear functional uf_0 on B where f_0 is a continuous linear functional on B such that $f_0(x) = \|x\|$ for all $x \in K$. Then, from (3.3) in Proposition 3.1.1 the characteristic functional of $\{Z_t\}$ evaluated in the linear functional uf_0 gives

$$Ee^{iu\|Z_t\|} = exp\left\{t\left(\int_K \left(e^{iu\|x\|} - 1\right)\nu_t(dx) + iu\left\|\gamma_t^0\right\|\right)\right\} \qquad u \in \mathbb{R},$$

from which (4.3) is obtained. Using the Laplace transform (3.18), a similar argument as above gives (4.4). ■

Definition 4.2.2 The one dimensional subordinator $\{\|Z_t\| : t \geq 0\}$ in Theorem 4.2.1 is called *the norm subordinator of* $\{Z_t : t \geq 0\}$ and it is denoted by $\{\sigma_t : t \geq 0\}$, i.e., $\sigma_t = \|Z_t\|$, $t \geq 0$.

In the sequel we will assume that $\{Z_t : t \geq 0\}$ is a K-subordinator in a Birkhoff-Kakutani space with associated norm subordinator $\{\sigma_t : t \geq 0\}$. From Example 3.3.3 we have the following.

Proposition 4.2.3 *Let $0 < \alpha < 1$. Let $\{Z_t : t \geq 0\}$ be an α-stable K-subordinator. Then $\{\sigma_t : t \geq 0\}$ is an α-stable real subordinator.*

Proof. The distribution of σ_1 is a one dimensional distribution of Z_1 since the norm comes from (Definition 1.3.1) a continuous linear functional. Since any one dimensional distribution $f(Z_1)$ of Z_1 is α-stable for $f \in B^*$, then in particular σ_1 is α-stable. ■

The proposition below introduces an interesting class of one dimensional subordinators. We recall that K-tempered subordinators are introduced in Example 3.3.4.

Proposition 4.2.4 *Let $\{Z_t : t \geq 0\}$ be a K-tempered subordinator and let $\{\sigma_t : t \geq 0\}$ be its norm subordinator. Then $\{\sigma_t : t \geq 0\}$ has Lévy density of the form*

$$u(r) = a\frac{h(r)}{r^{1+\alpha}} \qquad r > 0, \tag{4.5}$$

where $0 < \alpha < 1$ and $a > 0$ are parameters and $h(r)$ is a nonnegative, non-increasing function satisfying

$$\int_{(0,1]} \frac{h(r)}{r^{\alpha}} dr < \infty. \tag{4.6}$$

Proof. Suppose that Z_1 has the law of a tempered K-valued random variable of Example 3.3.4 with Lévy measure ν given by (3.27) where $0 < \alpha < 1$, the finite measure λ is concentrated on the set S_K and $p \in K^*$. Then

$$\begin{aligned} Ee^{-v\sigma_t} &= \exp\left\{t\int_K \left(e^{-v\|x\|} - 1\right)\nu(dx)\right\} \\ &= \exp\left\{tc_\alpha^{-1}\int_{(0,\infty)}\int_{S_K} \left(e^{-vr} - 1\right) e^{-rp(y)}\lambda(dy)\frac{dr}{r^{1+\alpha}}\right\} \\ &= \exp\left\{t\int_{(0,\infty)} \left(e^{-vr} - 1\right)\frac{h_{p,\lambda}(r)}{r^{1+\alpha}}dr\right\} \end{aligned}$$

where

$$h_{p,\lambda}(r) = c_\alpha^{-1} \int_{S_K} e^{-rp(y)} \lambda(dy). \tag{4.7}$$

It is clear that $h_{p,\lambda}(r)$ is nonnegative and non-increasing as function of $r > 0$. Notice that ν satisfies, for all $f \in B^*$,

$$\int_{0<\|x\|\leq 1} |f(x)|\, \nu_t(dx) = c_\alpha^{-1} \int_{(0,1]} \int_{S_K} |f(y)|\, e^{-rp(y)} \frac{dr}{r^\alpha} \lambda(dy) < \infty$$

and hence, taking in particular the functional f_0 which coincides with norm in K, we get condition (4.6) for the function (4.7). This concludes the proof. ■

One dimensional subordinators with Lévy densities (4.5) may be called *generalized tempered subordinators.* The reason for this terminology is the following. Notice that when λ in (4.7) is concentrated at one point, the family of Lévy densities of the form (4.5) becomes $a\frac{e^{-br}}{r^{1+\alpha}} 1_{(0,\infty)}(r)$, $a > 0$ and $b > 0$, which have the form of the family of Lévy densities of tempered nonnegative infinitely divisible random variables studied recently by Barndorff-Nielsen and Levendorskiĭ (2001) (see also Barndorff-Nielsen, Mikosh and Resnik (2001, pp. 24-27)). Those, generalized tempered laws may be thought as mixtures of tempered distributions. It is noticed that these one dimensional generalized tempered distributions describe new examples of laws of Bondesson class (Sato (1999, Th. 51.10)). We lack a more concrete interpretation of (4.7).

4.3 Norm-inheritance sample path properties of subordinators

For Birkhoff-Kakutani cones, some asymptotic sample path properties of the associated one dimensional norm subordinators are inherited by the cone-valued subordinators. In the next two sections we point out results on rates of growth and laws of iterated logarithm in both small and large times. The results are new, even in the finite dimensional cone $\mathbb{R}_+^d$, $d > 2$,

and for the cone of nonnegative definite $d \times d$ matrices.

4.3.1 Rates of growth

Throughout the present section $(B, \|\cdot\|, K)$ will be a separable Birkhoff-Kakutani space. The next law of large numbers holds.

Proposition 4.3.1 *Let $\{Z_t : t \geq 0\}$ be K-valued subordinator with drift γ_Z^0. Then*

$$P\left(\lim_{t\to 0+} \left\| \frac{Z_t}{t} - \gamma_Z^0 \right\| = 0\right) = 1.$$

Proof. Corollary 3.1.2 and Proposition 3.2.2 imply that $\{Z_t - t\gamma_Z^0 : t \geq 0\}$ is a K-valued subordinator with zero drift. Let $\{\sigma_t : t \geq 0\}$ be the corresponding norm subordinator which also has zero drift (take $\sigma_t = \|Z_t - t\gamma_Z^0\|$). Then Proposition 3.8 in Bertoin (1999) implies $\lim_{t\to 0+} \sigma_t/t = 0$ almost surely. Hence $\lim_{t\to 0+} \left\| \frac{Z_t}{t} - \gamma_Z^0 \right\| = 0$ a.s. ■

The following zero-one laws for the limsup of the rate of growth of a subordinator in a Birkhoff-Kakutani space, is characterized in terms of the behavior of the Lévy measure and the Laplace exponent of the associated norm subordinator.

For small times we have the following.

Proposition 4.3.2 *Let $\{Z_t : t \geq 0\}$ be a K-subordinator and let $\{\sigma_t : t \geq 0\}$ be its norm subordinator with Lévy measure ν_σ and Laplace exponent Φ_σ. Define the integral $I(t) := \int_0^t \bar{\nu}_\sigma(x)dx$ where $\bar{\nu}_\sigma(x) := \nu_\sigma((x,\infty))$, $x > 0$. Suppose that*

$$\liminf_{x\to 0+} I(2x)/I(x) > 1 \tag{4.8}$$

and let $h : [0,\infty) \to [0,\infty)$ be an increasing function. Then the following are equivalent:

a) $P\left(\limsup_{t\to 0+} \|Z_t/h(t)\| = \infty\right) = 1$

b) $\int_0^1 \bar{\nu}_\sigma(h(t))dt = \infty$

c) $\int_0^1 \Phi_\sigma(1/h(t))dt = \infty$.

In the case when the above conditions fail we have that $P\left(\lim_{t\to 0+}\|Z_t/h(t)\|=0\right)=1$.

Proof. Note that the one dimensional subordinator $\{\sigma_t : t \geq 0\}$ satisfies the condition (4.8) for its Lévy measure ν_σ. Then we apply Proposition 3.10 of Bertoin (1996) to the process $\sigma_t = \|Z_t\|$ to obtain the result for $\{Z_t\}$. ■

We next give an example of subordinators which satisfy condition (4.8). We first observe that this condition is equivalent to the following condition given by Bertoin (1966, p.100), in terms of the Laplace exponent, for the case that the real subordinator σ has zero drift:

$$\limsup_{u\to\infty}\frac{\Phi_\sigma(2u)}{\Phi_\sigma(u)} < 2. \tag{4.9}$$

Example 4.3.3 Consider a Birkhoff-Kakutai space B with a bounded basis in its cone K. Then $K_{(x_n)}$ is a cone of type l_+ by Proposition 1.3.4. Assume without loss of generality that the bounded basis is normalized, *i.e.,* $\|x_n\| = 1$ for $n \geq 1$. Let $\{Z_t : t \geq 0\}$ be the $K_{(x_n)}$-valued subordinator in (3.32) of Theorem 3.3.7. Then the norm subordinator $\{\|Z_t\| : t \geq 0\}$ is given by $\|Z_t\| = \sum_{k=1}^{\infty}\eta_k\sigma_k(t)$, where η_k and σ_k are given by Theorem 3.2.9 and satisfy condition (3.31). The Laplace exponent of the norm subordinator has the form

$$\Phi_{\|Z\|}(u) = \sum_{k=1}^{\infty}\Phi_k(u\eta_k)$$

by (3.33). Now, if we take σ_k, $k \geq 1$, as the α-stable subordinators in Example 3.3.10 with drift $\gamma^{0,k} = 0$, then the norm subordinator has zero drift and

$$\frac{\sum_{k=1}^{\infty}\Phi_k(2u\eta_k)}{\sum_{k=1}^{\infty}\Phi_k(u\eta_k)} = \frac{\sum_{k=1}^{\infty}c_k^{'}(2u\eta_k)^\alpha}{\sum_{k=1}^{\infty}c_k^{'}(u\eta_k)^\alpha} = 2^\alpha,$$

where $c_k^{'} > 0$ is the constant in Example 3.3.10. Hence condition (4.9) is satisfied for the norm subordinator, since $0 < \alpha < 1$.

The preceding zero-one law shows that, under the condition (4.8), the upper envelope of the rate of growth of the subordinator Z_t with respect to any increasing function $h(t)$ has

not regular behavior, since either $\|Z_t/h(t)\|$ converges to zero or its limsup goes to infinity, almost surely. As an application, we consider the case of K-tempered subordinators in a Birkhoff-Kakutani space.

Example 4.3.4 Take $\{Z_t : t \geq 0\}$ in Proposition 4.3.2 as a K-valued tempered subordinator. Suppose that its corresponding generalized tempered norm subordinator given by Proposition 4.2.4 and denoted by $\{\sigma_t : t \geq 0\}$ satisfies condition (4.8). From (4.5) and (4.7) we get the Lévy measure $\nu_\sigma(dr) = h_{p,\lambda}(r)/r^{1+\alpha}dr$ of $\{\sigma_t\}$ and hence the tail integral in (b) of Proposition 4.3.2 becomes

$$\int_0^1 \bar{\nu}_\sigma(h(t))dt = \int_0^1\int_{h(t)}^{\infty}\int_{\mathcal{S}_K} c_\alpha^{-1}\frac{e^{-rp(y)}}{r^{1+\alpha}}\lambda(dy)drdt. \tag{4.10}$$

Then, by the last proposition, either $\limsup_{t\to 0+}\|Z_t/h(t)\| = \infty$ almost surely or 0 almost surely according to the integral in (4.10) is infinite or finite.

When the finite measure λ is concentrated at one point $y_0 \in \mathcal{S}_K$, the tail integral reduces to

$$c_\alpha^{-1}\int_0^1\int_{h(t)}^{\infty}\lambda(\{y_0\})e^{-rp(y_0)}/r^{1+\alpha}drdt$$

which corresponds to the case of one dimensional tempered subordinators.

Furthermore, if $p \in K^*$ is the null functional, we obtain the α-stable case and therefore (4.10) is given by

$$c_\alpha^{-1}\lambda(\mathcal{S}_K)\int_0^1 h(t)^{-\alpha}dt. \tag{4.11}$$

Thus, for the α-stable case $\limsup_{t\to 0+}\|Z_t/h(t)\| = \infty$ or 0 almost surely according to the integral (4.11) diverges or converges respectively. We notice this is exactly the same condition as for the one dimensional α-stable subordinators, (see Bertoin (1996, p. 85)).

For large times, the following asymptotic rate of growth of sample paths is derived.

Proposition 4.3.5 *Let* $\{Z_t : t \geq 0\}$ *be a* K*-subordinator such that* $E\|Z_1\| = \infty$. *Let* $\{\sigma_t : t \geq 0\}$ *be its norm subordinator with Lévy measure* ν_σ *and Laplace exponent* Φ_σ. *Let*

$h : [0,\infty) \to [0,\infty)$ *be an increasing function such that the function* $t \to h(t)/t$ *is increasing as well. Then the following are equivalent:*

a) $P\left(\limsup_{t\to\infty} \|Z_t/h(t)\| = \infty\right) = 1$

b) $\int_1^\infty \bar{\nu}_\sigma(h(t))dt = \infty$

c) $\int_1^\infty \{\Phi_\sigma(1/h(t)) - (1/h(t))\,\Phi'_\sigma(1/h(t))\}\,dt = \infty.$

If the above conditions fail then $P\left(\lim_{t\to\infty} \|Z_t/h(t)\| = 0\right) = 1.$

Proof. Observe that the one dimensional subordinator $\{\sigma_t\}$ has infinite mean. Then, from Theorem 3.13 in Bertoin (1996) we have the equivalence between $\limsup_{t\to\infty} \sigma_t/h(t) = \infty$ a.s. and (b)-(c). Moreover, if $\limsup_{t\to\infty} \sigma_t/h(t)$ fails to be infinite a.s. then $\lim_{t\to\infty} \sigma_t/h(t) = 0$ a.s. The above shows the equivalence of (a), (b) and (c), and if one of these fails we have that $P\left(\lim_{t\to\infty} \|Z_t/h(t)\| = 0\right) = 1$, since $\sigma_t = \|Z_t\|$. This concludes the proof. ■

4.3.2 Laws of iterated logarithm

The following laws of iterated logarithm for cone-valued subordinators in a Birkhoff-Kakutani space $(B, \|\cdot\|, K)$ are transferred from those corresponding to the norm subordinator.

Recall that a positive measurable function $\varphi : (0,\infty) \to (0,\infty)$ is *regularly varying at* ∞ (respectively, at $0+$) if for each $c > 0$, the ratio $\varphi(cx)/\varphi(x)$ converges in $(0,\infty)$ as x tends to ∞ (respectively, to $0+$). It can be shown that, in both cases for φ, there exists $\rho > 0$ such that $\varphi(cx)/\varphi(x)$ converges to c^ρ. The number ρ is called *the index of* φ.

For small times, we point out the following result.

Proposition 4.3.6 *Let* $\{Z_t : t \geq 0\}$ *be* K*-valued subordinator. If the Laplace exponent* Φ_σ *of the norm-subordinator* $\{\sigma_t : t \geq 0\}$ *is regularly varying at* ∞ *with index* $\rho \in (0,1)$ *then*

$$P\left(\liminf_{t\to 0+} \left\|\frac{Z_t}{\psi(t)}\right\| = \rho(1-\rho)^{(1-\rho)/\rho}\right) = 1$$

where

$$\psi(t) = \frac{\log|\log(t)|}{\varphi_\sigma(t^{-1}\log|\log(t)|)} \qquad 0 < t < e^{-1}$$

and φ_σ *is the inverse function of* Φ_σ.

Proof. Since the Laplace exponent Φ_σ of the one dimensional subordinator $\{\sigma_t : t \geq 0\}$ is regularly varying at ∞ with index $\rho \in (0,1)$ we can apply Theorem 3.11 in Bertoin (1996) to yield $\liminf_{t\to 0+} \sigma_t/\psi(t) = \rho(1-\rho)^{(1-\rho)/\rho}$ a.s. and hence from the fact that $\sigma_t = \|Z_t\|$ we obtain that $\liminf_{t\to 0+} \|Z_t/\psi(t)\| = \rho(1-\rho)^{(1-\rho)/\rho}$ almost surely. ■

For large times it is possible to obtain the analogue of Proposition 4.3.6.

Proposition 4.3.7 *Let* $\{Z_t : t \geq 0\}$ *be* K*-valued subordinator. If the Laplace exponent* Φ_σ *of the norm-subordinator is regularly varying at* $0+$ *with index* $\rho \in (0,1)$ *then*

$$P\left(\liminf_{t\to\infty} \left\|\frac{Z_t}{\psi(t)}\right\| = \rho(1-\rho)^{(1-\rho)/\rho}\right) = 1$$

where

$$\psi(t) = \frac{\log\log(t)}{\varphi_\sigma(t^{-1}\log\log(t))} \qquad e < t < \infty$$

and φ_σ *is the inverse function of* Φ_σ.

Proof. Let $\{\sigma_t : t \geq 0\}$ be the norm subordinator of $\{Z_t\}$ with Laplace exponent Φ_σ. Then $\liminf_{t\to\infty} \sigma_t/\psi(t) = \rho(1-\rho)^{(1-\rho)/\rho}$ a.s. by Theorem 3.14 in Bertoin (1996). Hence we get the assertion for $\{Z_t\}$ since $\sigma_t = \|Z_t\|$. ■

o

Bibliography

[1] Albeverio, S. and Rudiger, B. (2002) Stochastic integrals and Lévy-Itô decomposition on separable Banach spaces. *Proceedings of 2nd MaPhySto Conference on Lévy processes -Theory and Applications-*, University of Aarhus.

[2] Araujo, A. and Giné, E. (1980) *The Central Limit Theorem for Real and Banach Valued Random Variables*, Wiley, New York.

[3] Barndorff-Nielsen, O. E (1998) Processes of normal inverse Gaussian type, *Finance and Stochastics* **2**, 41-68.

[4] Barndorff-Nielsen, O. E. (2002) *Proceedings of 2nd MaPhySto Conference on Lévy processes -Theory and Applications-*, University of Aarhus, Denmark.

[5] Barndorff-Nielsen, O. E. and Levendorskiĭ, S.Z. (2001) Feller processes of normal inverse Gaussian type, *Quantitative Finance* **1**, 318-331.

[6] Barndorff-Nielsen, O. E., Mikosh, T. and Resnick, S. I. (2001) *Lévy Processes -Theory and Applications-*, Birkhäuser, Boston.

[7] Barndorff-Nielsen, O. E., Pedersen, J. and Sato, K. (2001) Multivariate subordination, self-decomposability and stability, *Advances in Applied Probability* **33**, 160–187.

[8] Barndorff-Nielsen, O. E. and Pérez-Abreu V. (2002) Extensions of type G and marginal infinitely divisibility, *Teoriya Veroyatnostej i ee primeneniya* **47**, 2, 301-319.

[9] Barndorff-Nielsen, O. E. and Shepard, N. (2002) Normal modified stable processes, *Theory of Probability and Mathematical Statistics* ***65**, 1-20.*

[10] Bertoin, J. (1996) *Lévy Processes,* Cambridge University Press, Cambridge.

[11] Bertoin, J. (1999) Subordinators: examples and applications, *Lectures Notes in Mathematics* **1717**, Springer, Berlin, 1–91.

[12] Birkhoff, G. (1938) Dependent probabilities and the space (L), *Proceedings of the National Academy of Sciences* **24**, 154-159.

[13] Bochner, S. (1955) *Harmonic Analysis and the Theory of Probability,* University of California Press, Berkeley and Los Angeles.

[14] Brown, S. W. (1995) Weak sequential convergence in the dual of an algebra of compact operators, *Journal of Operator Theory* **33**, 33-42.

[15] de Acosta, A. (1980) Exponential moments of vector valued random series and triangular arrays, *Annals of Probability* **8**, 2, 381-389.

[16] Dettweiler, E. (1976) Infinitely divisible measures on the cone of an oredered locally convex vector space, *Annals of Sciences of the University of Clermont* **14**, 61, 11-17.

[17] Dettweiler, E. (1982) Banach space valued processes with independent increments and stochastic integration, *Lecture Notes in Mathematics* **990**, Springer, Berlin.

[18] Diestel, J. (1984) *Sequences and Series in Banach Spaces*, Springer, New York.

[19] Dynkin, E. B. (1961) *Die Grundlagen der Theorie der Markoffschen Prozesse*, Berlin-Göttingen-Heidelberg: Springer.

[20] Feller, W. (1971) *An Introduction to Probability Theory and Its Applications*, Vol. II, Second Edition, Wiley, New York

[21] Gihman, I. I. and Skorohod, A. V. (1975) *The Theory of Stochastic Processes*, Vol. II, Springer, Berlin.

[22] Giné, E. and Hahn, M. (1985) The Lévy-Khinchin representation for random compact convex sets which are infinitely divisible under Minkowski addition, *Zeitschrift fur Wahrscheinlichkeitstheorie Verw. Gebiete* **70**, 271-287.

[23] Guerre-Delabrière, S. (1992) *Classical Sequences in Banach Spaces*, Marcel Dekker, New York.

[24] Hoffmann-Jorgensen, J. (1974) Sums of independent Banach valued random variables, *Studia Mathematica* **52**, 159-186.

[25] Hoffmann-Jorgensen, J. (1977) *Probability in Banach Spaces*, Lecture Notes in Mathematics ***598**,* Springer, Berlin.

[26] Itô, K. (1942) On Stochastic Processes, I (Infinitely Divisible Laws of Probability), *Japan Journal of Mathematics* **18**, 261-301. (Reprinted in: *Kiyoshi Itô Selected Papers*, Springer, New York, 1987)

[27] Jonasson, J. (1998) Infinite divisibility of random objects in locally compact positive convex cones, *Journal of Multivariate Analysis* **65**, 129-138.

[28] Jurek, Z. J. (1983a) Limit distributions and one-parameter groups of linear operators on Banach spaces, *Journal of Multivariate Analysis* **13**, 578-604.

[29] Jurek, Z. J. (1983b) Polar coordinates in Banach spaces **32**, Bulletin of the Polish Academy of Sciences, 61-66.

[30] Kamthan, P. K. and Gupta, M. (1985) *Theory of Bases and Cones*, Pitman Advances Publishing Proceedings, Research Notes in Mathematics **117**.

[31] Kakutani, S. (1941) Concrete representation of Abstract (L)-spaces and the mean Ergodic Theorem, *Annals of Mathematics* **42**, 523-537.

[32] Kallianpur, G. and Xiong, J. (1995) *Stochastic Differential Equations in Infinite Dimensional Spaces,* Lectures Notes-Monograph Series **26**, Institute of Mathematical Statistics, Hayward, California.

[33] Kruglov, V. M. (1970) A note on infinitely divisible distributions, *Theory of Probabability and Its Applications* **15**, 319-324.

[34] Kruglov, V.M. (1972) Integrals with respect to infinitely divisible distributions in a Hilbert space, *Mathematical Notes* **11**, 407-411.

[35] Kuelbs, J. (1979) Rates of growth of Banach space valued independent increment processes, *Lectures Notes in Mathematics* **709**, Springer, Berlin, 151-169.

[36] Kumar, A and Schreiber B. M. (1975) Self-decomposable probability measures on Banach spaces, *Studia Mathematica* **53**, 55-71.

[37] Kwapień, S. (1974) On Banach spaces containing c_0, *Studia Mathematica* **52**, 187-188.

[38] Lévy, P. (1934) Sur les intégrales dont les éléments sont des variables aléatoires indépendantes, *Annali della Scuola Normale Superiore di Pisa* (2) **3**, 337-366; **4**, 217-218. (Reprinted in: *Œuvre de Paul Lévy* **4**, Gauthier-Villars, Paris 1980)

[39] Linde, W. (1986) *Probability in Banach Spaces -Stable and Infinitely Divisible Distributions-*, John Wiley & Sons, Berlin.

[40] Mase, S. (1979) Random compact convex sets which are infinitely divisible with respect to the Minkowski addition, *Advances in Applied Probability* **11**, 834-850.

[41] Parthasarathy, K. R. (1967) *Probability Measures on Metric Spaces*, Academic Press, New York.

[42] Pedersen, J. (2003) The Lévy-Itô decomposition of an independently scattered random measure, Preprint.

[43] Pedersen, J. and Sato, K. (2003a) Cone-parameter convolution semigroups and their subordination, *Tokyo Journal of Mathematics*, to appear.

[44] Pedersen, J. and Sato, K. (2003b) Semigroups and processes with parameter in a cone, Preprint.

[45] Pérez-Abreu, V. and Rocha-Arteaga, A. (2003) Covariance-parameter Lévy processes in the space of trace-class operators, Preprint.

[46] Pillai, R. N. (1990) On Mittag-Leffler functions and related distributions, *Annals of the Institute of Statistical Mathematics* **42**, 157–161.

[47] Reed, M. and Simon, B. (1975) *Fourier Analysis, Self-Adjoiness,* Vol. II, Academic Press, New York.

[48] Rocha-Arteaga, A. and Sato, K. (2003) *Topics in Infinitely Divisible Distributions and Lévy Processes*, Aportaciones Matemáticas, Notas de Investigación, Sociedad Matemática Mexicana, to appear.

[49] Rosinski, J. (1990) Infinitely divisible random vectors, *Annals of Probability* **18**, 405-430.

[50] Sato, K. (1999) *Lévy Processes and Infinitely Divisible Distributions,* Cambridge University Press, Cambridge.

[51] Sato, K. (2001) Subordination and self-decomposability, *Statistics and Probability Letters 54, 317-324.*

[52] Schaefer, H. H. (1974) *Banach Lattices and Positive Operators*, Springer, New York.

[53] Schaefer, H. H. (1999) *Topological Vector Spaces*, Springer, New York, Second edition.

[54] Singer, I. (1970) *Bases in Banach spaces I*, Springer, New York.

[55] Skorohod, A. V. (1991) *Random Processes with Independent Increments,* Kluwer Academic Publisher, Dordrecht, Netherlands (Russian original 1986).

[56] Tomiyama, J. (1963) A characterization of C*-algebras whose conjugate space are separable, *Tôhoku Mathematical Journal* **15**, 96-102.

[57] Ülger, A. (1997) Subspaces and subalgebras of $K(H)$ whose duals have the Schur property, *Journal of Operator Theory* **37**, 371-378.

[58] Wnuk, W. (1993) Banach lattices with properties of the Schur type - a survey, *Conferenze del Seminario di Matematica, Università di Bari* **249**, 25 pp.

[59] Woyczyński, W. (1978) Geometry and martingales in Banach spaces: Independent increments, *Advances in Probability and Related Topics* **4**, Marcel Dekker, 267-517.

[60] Yosida, K. (1980) *Functional Analysis*, Springer, Berlin, Sixth edition.

Index