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A nonstandard difference-integral method for the viscous Burgers' equation

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Abstract

We develop a nonstandard difference-integral method based on a nonstandard finite difference method coupled with a CE–SE scheme. We use the viscous Burgers' equation with preestablished conditions as a benchmark for testing our method. Numerical results obtained show that this new method is more robust and efficient than the associated standard difference-integral method.

Keywords: Nonstandard difference method; CE–SE scheme; Viscous Burgers' equation

1. Introduction

The phenomenon of the transport generated by a viscous fluid can be modeled by the viscous Burgers' equation, which is given by

$$\frac{\partial u}{\partial t}+\frac{1}{2}\frac{\partial u^2}{\partial x}=\tau\frac{\partial^2 u}{\partial x^2},\quad (x,t)\in[b_1,b_2]\times[0,T]\subseteq\mathbb{R}\times\mathbb{R}, \tag{1}$$

where x and t are the spatial and time variables, respectively, $u=u(x,t)$ is the velocity of the fluid and τ is the viscosity constant of the fluid.

Eq. (1) has been the subject of intense numerical scrutiny, as a result, several high resolution methods have been adapted and developed to solve it. Many of these methods use standard difference finite methods, for example: flux corrected techniques (FCT) [1–3], total variation diminishing schemes (TVD) [1,4], CE–SE method [5–7], and so on. Of the previous high resolution numerical methods, the CE–SE numerical scheme (method of space–time conservation element and solution element) was developed in the nineties [5]. This scheme has given very good results for problems with oscillating solutions which is a consequence of its properties: conservation of the physical characteristics of the solution, accuracy of the constructed numerical solution and low computational cost [7]. The CE–SE method uses two discretizations of the space–time

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domain, one for approximating the solution of (1) and the second for approximating an integral form obtained from (1).

The goals of this paper are: (i) to develop a nonstandard finite difference method which maintains the simplicity of the calculations and improves the results obtained by a standard finite difference scheme for handling the diffusion part of Eq. (1), (ii) to couple our nonstandard difference method with the CE–SE numerical solver for the Burgers' equation.

The finite difference algorithm that we develop in this paper is based in the nonstandard difference method introduced by Mickens [8,9]. Mickens uses the generalized form of the discretization of the second derivative,

$$\frac{\partial^2 u}{\partial x^2} = \lim_{h\to 0} \frac{u(x+h,t) - 2u(x,t) + u(x-h,t)}{\varphi(h)}, \tag{2}$$

and obtains the nonstandard function $\varphi(h)$ by calculating the exact solution of a second order differential equation. Mickens' method produces discrete solutions that are exact for particular equations and therefore have the same properties as their analytical solutions. In order to use this idea, we consider that the spatial second derivative of the solution of Eq. (1) is a spatial second order polynomial in the function u. The developed difference method is then coupled with a CE–SE numerical scheme.

The paper is organized as follows: In Section 2, a nonstandard difference finite method is developed. In the following section, the nonstandard CE–SE method for Burgers' equation is constructed. Finally, for the validation of the nonstandard CE–SE method we consider the exact solution of Burgers' equation with preestablished conditions. In the numerical results, we show the improvement of the nonstandard approximation over the approximation obtained with a associated standard CE–SE method.

2. Nonstandard finite difference method

The first stage of our work is to find a nonstandard function, $\varphi(h)$, assuming that our approximated solution, $w(x,t) = w_t(x)$ of (1), satisfies the following differential equation:

$$\frac{\partial^2 w_t}{\partial x^2} = a w_t^2 + b w_t, \tag{3}$$

where a and b are constant coefficients. The exact solution of (3) is

$$w_t(x) = \frac{6b}{a} \frac{k e^{\sqrt{b}x}}{(1 - k e^{\sqrt{b}x})^2}, \tag{4}$$

where k is a constant of integration. Substituting (4) in (2) we get,

$$\frac{\partial^2 w_t}{\partial x^2} = \lim_{h\to 0} \frac{w_t(x+h) - 2w_t(x) + w_t(x-h)}{\varphi(h)}, \tag{5}$$

and after some algebraic manipulations, a nonstandard function $\varphi(h)$ is obtained as it is indicated in the following proposition.

Proposition 2.1. The function $\varphi(h) = \frac{1}{b}(e^{\sqrt{b}h} - 2 + e^{-\sqrt{b}h})$ satisfies the following properties:

(i)

$$\varphi(h) \neq h^2,$$

(ii)

$$\frac{\partial^2 w_t}{\partial x^2} = \lim_{h\to 0} \frac{w_t(x+h) - 2w_t(x) + w_t(x-h)}{\varphi(h)},$$

(iii) $\varphi(h)$ is a nonstandard function associated to the Eq. (3).

Proof

(i) Straightforward by definition.
(ii) It follows from

$$\lim_{h\to 0}\frac{\varphi(h)}{h^2}=\lim_{h\to 0}\frac{1}{b}\frac{(\mathrm{e}^{\sqrt{b}h}-2+\mathrm{e}^{-\sqrt{b}h})}{h^2}=1. \tag{6}$$

(iii) To get a nonstandard function $\varphi(h)$, we use the calculated solution in (4) obtaining:

$$w_t(x+h)-2w_t(x)+w_t(x-h)=\frac{6b}{a}z\left[\frac{y}{(1-zy)^2}-\frac{2}{(1-z)^2}+\frac{y^{-1}}{(1-zy^{-1})}\right], \tag{7}$$

where $z=k\mathrm{e}^{\sqrt{b}x}$ and $y=\mathrm{e}^{\sqrt{b}h}$. After simplifications of (7), we have

$$w_t(x+h)-2w_t(x)+w_t(x-h)=\frac{(y-1)^2}{y}\left[\frac{6b(z^5+2z^4+2z^3+2z^2+z)-12b(y+2+y^{-1})z^3}{a(1-z)^2(1-zy)(1-zy^{-1})}\right]. \tag{8}$$

Finally, we divide both sides of (8) by $\frac{(y-1)^2}{by}$ obtaining,

$$\frac{w_t(x+h)-2w_t(x)+w_t(x-h)}{\frac{1}{b}(y-2+y^{-1})}=\left[\frac{6b^2(z^5+2z^4+2z^3+2z^2+z)-12b^2(y+2+y^{-1})z^3}{a(1-z)^2(1-zy)(1-zy^{-1})}\right]. \tag{9}$$

According to the generalized form (2), we take

$$\varphi(h)=\frac{1}{b}(\mathrm{e}^{\sqrt{b}h}-2+\mathrm{e}^{-\sqrt{b}h}). \tag{10}$$

So we have obtained a nonstandard function $\varphi(h)$, which gives an approximation to the spatial second derivative of $u(x,t)$, which is the base for our nonstandard finite difference method.

3. Nonstandard CE–SE method

Our second stage is to couple our previous method with the CE–SE method [5] for the viscous Burgers' equation. The domain $[b_1,b_2]\times[0,T]$ is discretized by a mesh formed by a set of points or nodes, (x_j,t^n) for $j=\frac{1}{2},1,\frac{3}{2}\ldots,M$ and $n=\frac{1}{2},1,\frac{3}{2},\ldots,N$. For simplicity, we take equally spaced nodes, with spatial variation $h=x_{j+\frac{1}{2}}-x_{j-\frac{1}{2}}$ and with step of time $k=t^{n+\frac{1}{2}}-t^{n-\frac{1}{2}}$ with n and j alternatively an integer number and a semi-integer one or vice versa, see Fig. 1. Fixed a spatial variation, h, we calculate the step of time, k, verifying the CFL condition defined in [6].

The CE–SE method uses two discretizations, defined in [5], of the space–time domain: solution elements (SE(j,n)), non-overlapping opened rhombus centered in (x_j,t^n) where the numerical approximation is expressed by a quadratic Taylor expansion and conservation elements (CE(j,n)), non-overlapping rectangles

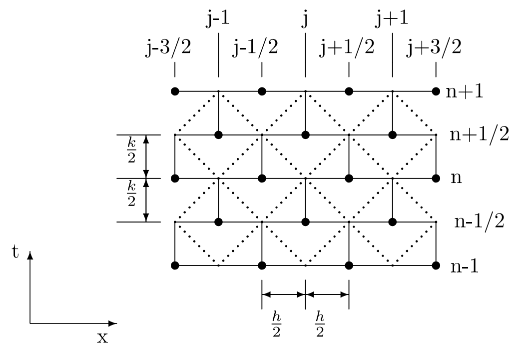

Fig. 1. Discretization of the space–time domain.

where an integral form of the Eq. (1) is required, in each rectangle take part three different SE's, $\left(\mathrm{SE}(j,n),\mathrm{SE}\left(j-\frac{1}{2},n-\frac{1}{2}\right),\mathrm{SE}\left(j+\frac{1}{2},n-\frac{1}{2}\right)\right)$, as it is shown in Fig. 1.

In each solution element, $\mathrm{SE}(j,n)$, we approximate the function $u(x,t)$ by means of a second order Taylor polynomial in the spatial variable, given by

$$w(x,t;j,n)=\sigma_j^n+\alpha_j^n(x-x_j)+\epsilon_j^n(x-x_j)^2+\beta_j^n(t-t^n), \tag{11}$$

where

$$\begin{cases}\sigma_j^n=w(x_j,t^n;j,n),\\ \alpha_j^n=\frac{\partial w}{\partial x}(x_j,t^n;j,n),\\ \epsilon_j^n=\frac{1}{2}\frac{\partial^2 w}{\partial x^2}(x_j,t^n;j,n),\\ \beta_j^n=\frac{\partial w}{\partial t}(x_j,t^n;j,n)=-\frac{1}{2}\frac{\partial w^2}{\partial x}(x_j,t^n;j,n)+\tau\frac{\partial^2 w}{\partial x^2}(x_j,t^n;j,n).\end{cases} \tag{12}$$

To obtain the numerical solution described in (11), it is necessary to compute the coefficients $\sigma_j^n,\alpha_j^n,\epsilon_j^n$ and $\beta_j^n,\forall j,n$. We assume that the coefficients of the time step n are known and we show how to calculate the coefficients of the next step, $n+\frac{1}{2}$. For the calculation of $\sigma_j^{n+\frac{1}{2}}$, we use the resolution of the integral form of the differential equation (1) in each $\mathrm{CE}\left(j,n+\frac{1}{2}\right)$, see Fig. 1,

$$\underbrace{\iint_{\mathrm{CE}\left(j,n+\frac{1}{2}\right)}\frac{\partial w}{\partial t}+\frac{1}{2}\frac{\partial w^2}{\partial x}\,\mathrm{d}x\,\mathrm{d}t}_{(13a)}=\tau\underbrace{\iint_{\mathrm{CE}\left(j,n+\frac{1}{2}\right)}\frac{\partial^2 w}{\partial x^2}\,\mathrm{d}x\,\mathrm{d}t}_{(13b)} \tag{13}$$

Applying the Green's theorem to (13a), we obtain the following path integral defined in the boundary of $\mathrm{CE}\left(j,n+\frac{1}{2}\right)$,

$$\begin{aligned}\oint_{\overline{\mathrm{CE}(j,n+\frac{1}{2})}}-w\mathrm{d}x+\frac{1}{2}w^2\mathrm{d}t={}&\sigma_j^{n+\frac{1}{2}}h+\epsilon_j^{n+\frac{1}{2}}\frac{h^3}{12}-\left[\sigma_{j+\frac{1}{2}}^n+\sigma_{j-\frac{1}{2}}^n\right]\frac{h}{2}+\left[\alpha_{j+\frac{1}{2}}^n-\alpha_{j-\frac{1}{2}}^n\right]\frac{h^2}{8}-\left[\epsilon_{j+\frac{1}{2}}^n+\epsilon_{j-\frac{1}{2}}^n\right]\frac{h^3}{24}\\&+\left[\left(\sigma_{j+\frac{1}{2}}^n\right)^2-\left(\sigma_{j-\frac{1}{2}}^n\right)^2\right]\frac{k}{4}+\left[\left(\sigma_{j+\frac{1}{2}}^n\beta_{j+\frac{1}{2}}^n\right)-\left(\sigma_{j-\frac{1}{2}}^n\beta_{j-\frac{1}{2}}^n\right)\right]\frac{k^2}{8}.\end{aligned} \tag{14}$$

We decompose the integral (13b) as the sum of three integrals defined in each SE,

$$\begin{aligned}\iint_{\mathrm{CE}\left(j,n+\frac{1}{2}\right)}\frac{\partial^2 w}{\partial x^2}(x,t;j,n+1/2)\mathrm{d}x\,\mathrm{d}t={}&\iint_{\mathrm{SE}\left(j-\frac{1}{2},n\right)}\frac{\partial^2 w}{\partial x^2}(x,t;j-1/2,n)\mathrm{d}x+\iint_{\mathrm{SE}\left(j+\frac{1}{2},n\right)}\frac{\partial^2 w}{\partial x^2}(x,t;j+1/2,n)\mathrm{d}x\\&+\iint_{\mathrm{SE}\left(j,n+\frac{1}{2}\right)}\frac{\partial^2 w}{\partial x^2}(x,t;j,n+1/2)\mathrm{d}x.\end{aligned} \tag{15}$$

By (12), we use the following approximation

$$\epsilon_j^{n+\frac{1}{2}}\approx\frac{1}{2}\frac{\partial^2 w}{\partial x^2}(x,t;j,n+1/2)\qquad\forall j,n,$$

then we consider two options to approximate the second spatial derivative of the function w in (x_j,t^n):

1. A standard finite difference method:

$$\begin{aligned}\left[\epsilon_j^{n+\frac{1}{2}}\right]_{\mathrm{S}}&=\frac{w\left(x_{j+\frac{1}{2}},t^{n+\frac{1}{2}};j+\frac{1}{2},n\right)-2w\left(x_j,t^{n+\frac{1}{2}};j,n+\frac{1}{2}\right)+w\left(x_{j-\frac{1}{2}},t^{n+\frac{1}{2}};j-\frac{1}{2},n\right)}{\frac{h^2}{2}}\\&=\frac{\left[\sigma_{j+\frac{1}{2}}^n+\beta_{j+\frac{1}{2}}^n\left(\frac{k}{2}\right)\right]-2\sigma_j^{n+\frac{1}{2}}+\left[\sigma_{j-\frac{1}{2}}^n+\beta_{j-\frac{1}{2}}^n\left(\frac{k}{2}\right)\right]}{\frac{h^2}{2}}.\end{aligned} \tag{16}$$

2. Our nonstandard finite difference method (Section 2):

$$\left[\epsilon_j^{n+\frac{1}{2}}\right]_{\mathrm{NS}} = \frac{w\left(x_{j+\frac{1}{2}}, t^{n+\frac{1}{2}}; j+\frac{1}{2}, n\right) - 2w\left(x_j, t^{n+\frac{1}{2}}; j, n+\frac{1}{2}\right) + w\left(x_{j-\frac{1}{2}}, t^{n+\frac{1}{2}}; j-\frac{1}{2}, n\right)}{\frac{1}{b}\left(\mathrm{e}^{\sqrt{b}h} - 2 + \mathrm{e}^{-\sqrt{b}h}\right)}$$
$$= \frac{\left[\sigma_{j+\frac{1}{2}}^n + \beta_{j+\frac{1}{2}}^n\left(\frac{k}{2}\right)\right] - 2\sigma_j^{n+\frac{1}{2}} + \left[\sigma_{j-\frac{1}{2}}^n + \beta_{j-\frac{1}{2}}^n\left(\frac{k}{2}\right)\right]}{\frac{1}{b}\left(\mathrm{e}^{\sqrt{b}h} - 2 + \mathrm{e}^{-\sqrt{b}h}\right)}, \tag{17}$$

where b is a real constant.

In the boundary of the domain, we calculate $\epsilon_0^{n+\frac{1}{2}}$ by using a second order forward differences and $\epsilon_{\mathrm{M}}^{n+\frac{1}{2}}$ by means of second order backward differences.

Substituting (16) or (17) in (15), the integrands of three integrals defined in each SE are constants, and Eq. (13b) is approximated by

$$\iint_{\mathrm{CE}\left(j,n+\frac{1}{2}\right)} \frac{\partial^2 w}{\partial x^2}(x,t;j,n+1/2)\,\mathrm{d}x\,\mathrm{d}t \cong \frac{hk}{2}\left[\frac{\epsilon_{j+\frac{1}{2}}^n + \epsilon_{j-\frac{1}{2}}^n}{2} + \epsilon_j^{n+\frac{1}{2}}\right]. \tag{18}$$

By replacing (13a) and (13b) by (14) and (18), respectively, we obtain the following equation:

$$\sigma_j^{n+\frac{1}{2}} + \epsilon_j^{n+\frac{1}{2}}\left[\frac{h^2}{12} - \tau\frac{k}{2}\right] = \frac{1}{2}\left[\sigma_{j+\frac{1}{2}}^n + \sigma_{j-\frac{1}{2}}^n\right] - \left[\alpha_{j+\frac{1}{2}}^n - \alpha_{j-\frac{1}{2}}^n\right]\frac{h}{8} + \left[\epsilon_{j+\frac{1}{2}}^n + \epsilon_{j-\frac{1}{2}}^n\right]\frac{h^2}{24}$$
$$- \left[\left(\sigma_{j+\frac{1}{2}}^n\right)^2 - \left(\sigma_{j-\frac{1}{2}}^n\right)^2\right]\frac{k}{4h} + \left[\left(\sigma_{j+\frac{1}{2}}^n\beta_{j+\frac{1}{2}}^n\right) - \left(\sigma_{j-\frac{1}{2}}^n\beta_{j-\frac{1}{2}}^n\right)\right]\frac{k^2}{8h}$$
$$+ \tau\left[\epsilon_{j+\frac{1}{2}}^n + \epsilon_{j-\frac{1}{2}}^n\right]\frac{k}{4}. \tag{19}$$

We solve this resulting equation by substituting $\epsilon_j^{n+\frac{1}{2}}$ by $[\epsilon_j^{n+\frac{1}{2}}]_{\mathrm{S}}$, or $[\epsilon_j^{n+\frac{1}{2}}]_{\mathrm{NS}}$ and thus we obtain $\sigma_j^{n+\frac{1}{2}}$.

In order to obtain the coefficients $\alpha_j^{n+\frac{1}{2}}$, we use the expression described in [5],

$$\alpha_k = \begin{cases} \frac{|\alpha_+|^c\alpha_- + |\alpha_-|^c\alpha_+}{|\alpha_+|^c + |\alpha_-|^c}, & |\alpha_+|^c + |\alpha_-|^c \neq 0, \\ 0, & |\alpha_+|^c + |\alpha_-|^c = 0, \end{cases} \tag{20}$$

where α_+ and α_- are the forward and backward standard differences of the first order, respectively, and c is a positive real constant. The aim of the constant c is to smoother the approximation of the spatial partial derivative of ω. In the boundary of the domain we calculate $\alpha_0^{n+1/2}$ using α_+, and $\alpha_{\mathrm{M}}^{n+1/2}$ by using α_-.

Finally, to obtain the coefficients $\beta_j^{n+\frac{1}{2}}$ by (11), we approximate the term $-\frac{\partial w^2}{\partial x}(x,t;j,n+1/2)$ by developing a Taylor series of w^2 and truncating the terms of higher order. Therefore,

$$\beta_j^{n+\frac{1}{2}} = -\sigma_j^{n+\frac{1}{2}}\alpha_j^{n+\frac{1}{2}} + \left[\left(\alpha_j^{n+\frac{1}{2}}\right)^2 + 2\sigma_j^{n+\frac{1}{2}}\epsilon_j^{n+\frac{1}{2}}\right]x_j. \tag{21}$$

Once the coefficients $\sigma_j^{n+\frac{1}{2}}, \alpha_j^{n+\frac{1}{2}}, \beta_j^{n+\frac{1}{2}}$ and $\epsilon_j^{n+\frac{1}{2}}$ are known, the solution (11) is defined in each $\mathrm{SE}\left(j, n+\frac{1}{2}\right)$ and we have completed half time step of integration.

If $\epsilon_j^{n+\frac{1}{2}} = [\epsilon_j^{n+\frac{1}{2}}]_{\mathrm{NS}}$, we get a nonstandard integro-differential method (nonstandard CE–SE scheme) and if $\epsilon_j^{n+\frac{1}{2}} = [\epsilon_j^{n+\frac{1}{2}}]_{\mathrm{S}}$, we obtain the associated standard integro-differential method (standard CE–SE scheme).

4. Numerical results

In this section we will make use of the viscous Burgers' equation to demonstrate the effectiveness of our nonstandard CE–SE scheme in solving nonlinear problems. We choose this equation for testing our method by two reasons: the first one is that an exact solution with preestablished conditions is known, namely

$$u(x,t) = -\frac{2\sinh(x)}{\cosh(x) - \mathrm{e}^{-t}}, \tag{22}$$

for $\tau = 1$ and boundary conditions, $u(-9,t) = 2$, $u(9,t) = -2$ for $t \in [0,T]$, and the second reason is that the solution (22) presents very fast changes for initial times and several methods are inaccurate and unstable in these changing regions. The numerical CE–SE scheme is a high resolution method for problems with oscillat-

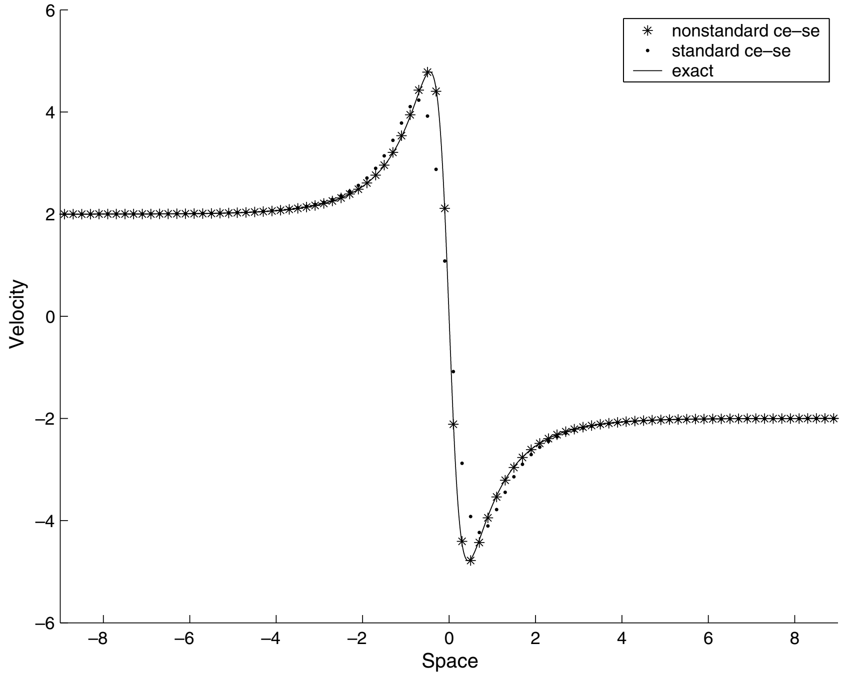

Fig. 2. Solution Burgers' equation in $T = 0.1$ for $h = 0.2$.

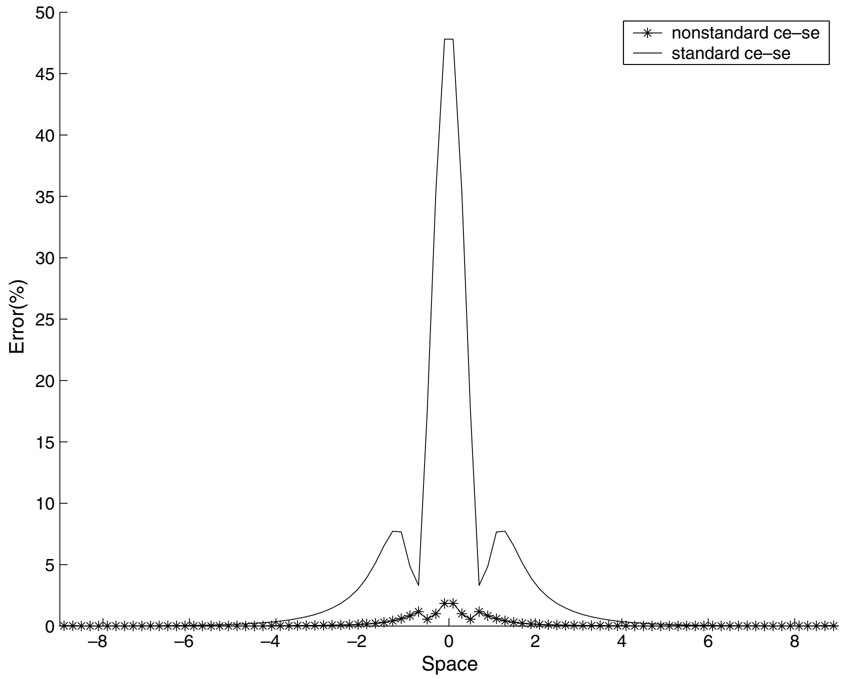

Fig. 3. Error Burgers' equation in $T = 0.1$ for $h = 0.2$.

ing solutions, therefore we consider important to compare the numerical results of our method with an associated standard CE–SE method.

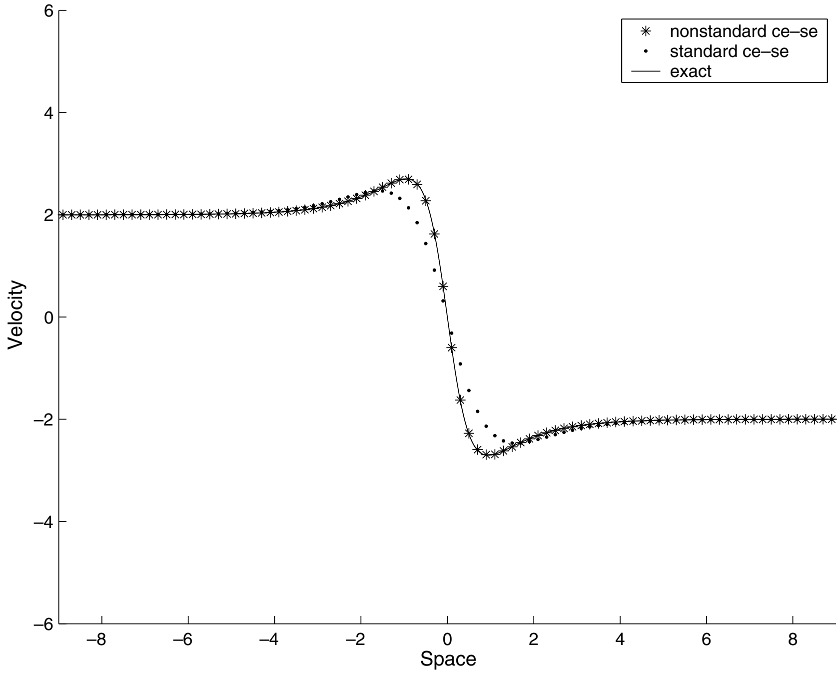

Fig. 4. Solution Burgers' equation in $T = 0.4$ for $h = 0.2$.

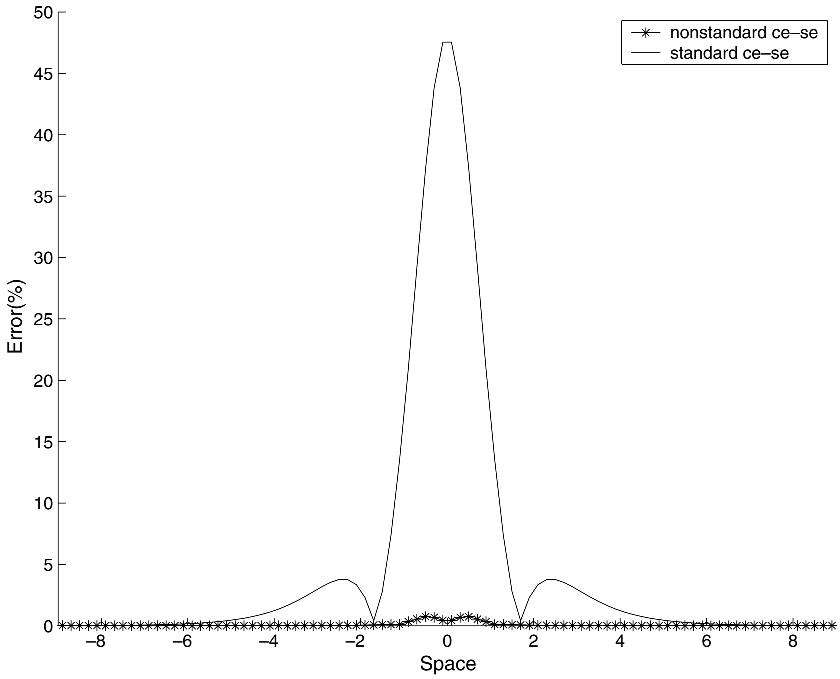

Fig. 5. Error Burgers' equation in $T = 0.4$ for $h = 0.2$.

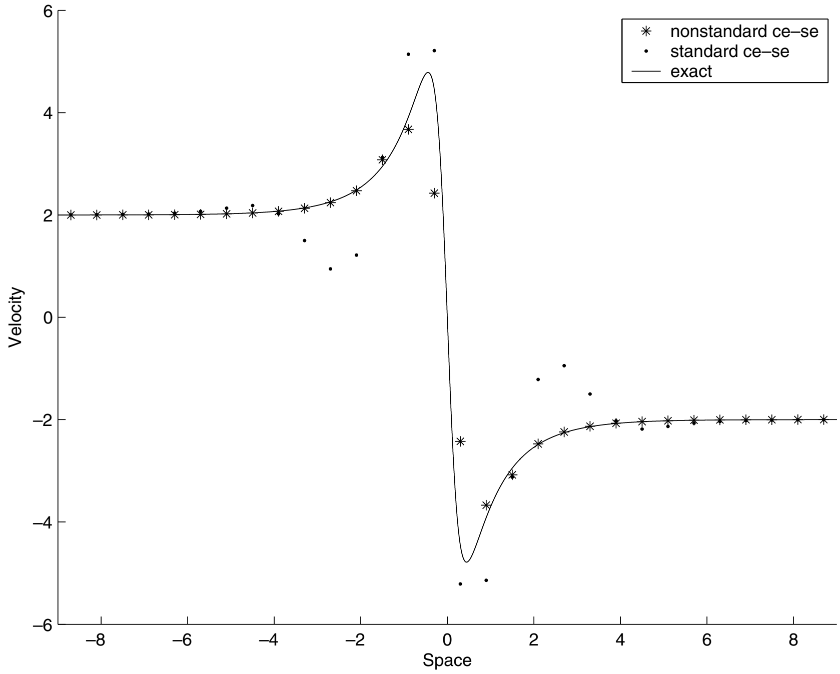

Fig. 6. Solution Burgers' equation in $T = 0.1$ for $h = 0.6$.

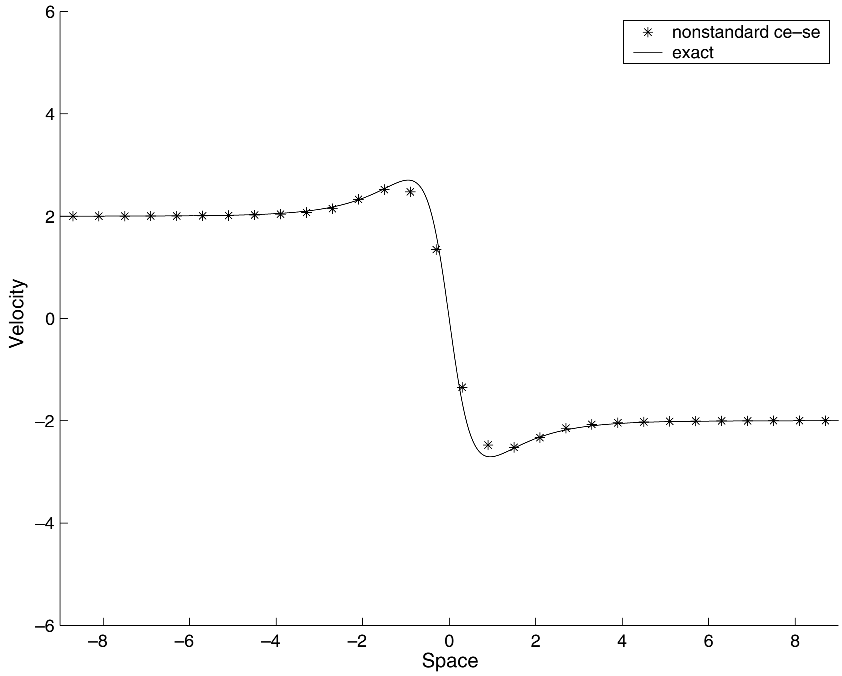

Fig. 7. Solution Burgers' equation in $T = 0.4$ for $h = 0.6$.

The numerical tests were realized for increments $0.2 \leqslant h \leqslant 0.65$ and $k = 0.01$, with $\text{CFL} = 0.95$, $c = 1.5$ and b was adjusted inside the scheme in every time step by

$$b^{n+\frac{1}{2}} = \delta\left[\max_{0\leqslant j\leqslant M}|\omega_j^n| - \min_{0\leqslant j\leqslant M}|\omega_j^n|\right], \quad n = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots, N,$$

where the parameter δ was taken depending of the size of the oscillation of the initial condition, in general we considered $\delta = 3.8$. A very important feature that is obtained from the numerical results is that in regions with fast and slow changes of the exact solution, the nonstandard CE–SE method is very accurate with an absolute error less than 5%, (see Figs. 3 and 5), whereas the standard CE–SE method approximates accurately the solution only away from the changes and there the error is less than 1%. The absolute error for the standard approximation grows up to 45% in neighborhoods close to changes of the solution. Figs. 2 and 4 show the behavior of the exact solution and their corresponding approximations from both methods for $T = 0.1$ and 0.4. Here one may observe the high precision of the nonstandard CE–SE method. It is of paramount importance to emphasize that for times smaller than 0.1, our method keeps its accuracy even though the solution presents steeper changes than the ones shown in the previous figures.

For spatial increments relatively large, the standard CE–SE scheme becomes unstable at initial times and diverges at later times as is shown in Figs. 6 and 7. In these figures, the robustness of the nonstandard CE–SE method is shown, that is, this method is stable and converge to the exact solution.

5. Conclusions

In this work we have developed a nonstandard integro-differential method of high accuracy in regions with fast or slow changes of the solution with good stability properties even for large spatial variations. Therefore, our method has a low computational cost which is an important feature sought in order to solve many problems of fluid dynamics that appear in engineering. A future work is to establish the exact region of stability of our nonstandard method, since we have numerical evidence that it is bigger than the region of the standard method.

Acknowledgements

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