THE PALAIS-SMALE CONDITION ON CONTACT TYPE ENERGY LEVELS FOR CONVEX LAGRANGIAN SYSTEMS

Gonzalo Contreras

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THE PALAIS-SMALE CONDITION AND MAÑÉ'S CRITICAL VALUES

GONZALO CONTRERAS, RENATO ITURRIAGA,
GABRIEL P. PATERNAIN, AND MIGUEL PATERNAIN

Abstract. Let $\mathbb{L}$ be a convex superlinear autonomous Lagrangian on a closed connected manifold N. We consider critical values of Lagrangians as defined by R. Mañé in [23]. We define energy levels satisfying the Palais-Smale condition and we show that the critical value of the lift of $\mathbb{L}$ to any covering of N equals the infimum of the values of k such that the energy level t satisfies the Palais-Smale condition for every $t > k$ provided that the Peierls barrier is finite. When the static set is not empty, the Peierls barrier is always finite and thus we obtain a characterization of the critical value of $\mathbb{L}$ in terms of the Palais-Smale condition.

We also show that if an energy level without conjugate points has energy strictly bigger than $c_u(\mathbb{L})$ (the critical value of the lift of $\mathbb{L}$ to the universal covering of N), then two different points in the universal covering can be joined by a unique solution of the Euler-Lagrange equation that lives in the given energy level. Conversely, if the latter property holds, then the energy of the energy level is greater than or equal to $c_u(\mathbb{L})$. In this way, we obtain a characterization of the energy levels where an analogue of the Hadamard theorem holds. We conclude the paper showing other applications such as the existence of minimizing periodic orbits in every non-trivial homotopy class with energy greater than $c_u(\mathbb{L})$ and homologically trivial periodic orbits such that the action of $\mathbb{L} + k$ is negative if $c_u(\mathbb{L}) < k < c_a(\mathbb{L})$, where $c_a(\mathbb{L})$ is the critical value of the lift of $\mathbb{L}$ the abelian covering of N. We also prove that given an Anosov energy level, there exists in each non-trivial free homotopy class a unique closed orbit of the Euler-Lagrange flow in the given energy level.

1. Introduction

In this paper we study geometric and dynamical properties of convex and superlinear Lagrangians, and it can be considered as a continuation of our previous paper [9]. This time we study the action functional from the viewpoint of Morse theory and we show, among other results, that for a compact manifold the critical value as defined by R. Mañé in [23] can be characterized by the Palais-Smale condition. This will follow from more general results to be precisely stated below.

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Gabriel Paternain is on leave from Centro de Matemática, Facultad de Ciencias, Iguá 4225, 11400 Montevideo, Uruguay.

It is well known that the action functional of the Lagrangian

$$L(x,v)=\frac{1}{2}|v|_x^2$$

arising from a Riemannian metric satisfies the Palais-Smale condition (cf. [29]). This condition ensures that the minimax principle holds and from the latter many standard properties of geodesics easily follow, namely, The Hopf-Rinow theorem, the Hadamard theorem and the existence of closed geodesics in each homotopy class. As we explain below, these and other properties as well as the approach in [29] and [31] hold for energy levels of convex and superlinear autonomous Lagrangians if the energy is greater than the critical value.

In order to describe precisely our results let us recall some preliminaries.

Let N be a closed connected smooth manifold and let $\mathbb{L} : TN \to \mathbb{R}$ be a smooth convex superlinear Lagrangian. This means that $\mathbb{L}$ restricted to each T_xN has positive definite Hessian and that for some Riemannian metric we have that

$$\lim_{|v|\to\infty}\frac{\mathbb{L}(x,v)}{|v|}=\infty,$$

uniformly on $x \in N$.

Since N is compact, the extremals of $\mathbb{L}$ give rise to a complete flow $\phi_t : TN \to TN$ called the *Euler-Lagrange flow* of the Lagrangian.

Recall that the *energy* $\mathbb{E}_{\mathbb{L}} : TN \to \mathbb{R}$ is defined by

$$\mathbb{E}_{\mathbb{L}}(x,v)=\frac{\partial\mathbb{L}}{\partial v}(x,v).v-\mathbb{L}(x,v).$$

Since $\mathbb{L}$ is autonomous, $\mathbb{E}_{\mathbb{L}}$ is a first integral of the flow ϕ_t.

Recall also that the action of the Lagrangian $\mathbb{L}$ on an absolutely continuous curve $\gamma : [a,b] \to N$ is defined by

$$S_{\mathbb{L}}(\gamma)=\int_a^b\mathbb{L}(\gamma(t),\dot\gamma(t))\,dt.$$

Given two points, q_1 and q_2 in N and $T > 0$ denote by $\mathcal{C}(q_1, q_2; T)$ the set of absolutely continuous curves $\gamma : [0,T] \to N$, with $\gamma(0) = q_1$ and $\gamma(T) = q_2$. For each $k \in \mathbb{R}$ we define the *action potential* $\Phi_k : N \times N \to \mathbb{R}$ by

$$\Phi_k(q_1,q_2)=\inf\{S_{\mathbb{L}+k}(\gamma):\ \gamma\in\cup_{T>0}\mathcal{C}(q_1,q_2;T)\}.$$

The *critical value of* $\mathbb{L}$, which was introduced by Mañé in [23], is the real number $c(\mathbb{L})$ defined as the infimum of $k \in \mathbb{R}$ such that for some $q \in N$, $\Phi_k(q,q) > -\infty$. Since $\mathbb{L}$ is convex and superlinear and N is compact such a number exists and it has various important properties [23, 6]. We briefly mention a few of them since we shall need them below. For any $k \geq c(\mathbb{L})$, the action potential Φ_k is a Lipschitz function that satisfies a triangle inequality. In general the action potential is not symmetric but if we define $d_k : N \times N \to \mathbb{R}$ by setting

$$d_k(q_1,q_2)=\Phi_k(q_1,q_2)+\Phi_k(q_2,q_1),$$

then d_k is a distance function for all $k > c(\mathbb{L})$ and a pseudo-distance for $k = c(\mathbb{L})$. In [23, 6] the critical value is characterized in other ways relating it to minimizing measures or to the existence of Tonelli minimizers with fixed energy between two points.

We can also consider the critical value of the lift of the Lagrangian $\mathbb{L}$ to a covering of the compact manifold N. Suppose that $p : M \to N$ is a covering space and consider the Lagrangian $L : TM \to \mathbb{R}$ given by $L := \mathbb{L} \circ dp$. For each $k \in \mathbb{R}$ we can define an action potential Φ_k in $M \times M$ just as above and similarly we obtain a critical value $c(L)$ for L. It can be easily checked that if M_1 and M_2 are coverings of N such that M_1 covers M_2, then

$$c(L_1) \leq c(L_2), \tag{1}$$

where L_1 and L_2 denote the lifts of the Lagrangian $\mathbb{L}$ to M_1 and M_2 respectively.

Among all possible coverings of N there are two distinguished ones; the universal covering which we shall denote by $\tilde{N}$, and the abelian covering which we shall denote by $\overline{N}$. The latter is defined as the covering of N whose fundamental group is the kernel of the Hurewicz homomorphism $\pi_1(N) \mapsto H_1(N, \mathbb{R})$. When $\pi_1(N)$ is abelian, $\tilde{N}$ is a finite covering of $\overline{N}$.

The universal covering of N gives rise to the critical value

$$c_u(\mathbb{L}) \stackrel{\text{def}}{=} c(\text{lift of } \mathbb{L} \text{ to } \tilde{N}),$$

and the abelian covering of N gives rise to the critical value

$$c_a(\mathbb{L}) \stackrel{\text{def}}{=} c(\text{lift of } \mathbb{L} \text{ to } \overline{N}).$$

From inequality (1) it follows that

$$c_u(\mathbb{L}) \leq c_a(\mathbb{L}),$$

but in general the inequality may be strict as it was shown in [30].

The critical values have another important feature: they single out those energy levels in which relevant globally minimizing objects (orbits or measures) live [10, 23, 6]. The study of these globally minimizing objects has a long history that goes back to M. Morse [27] and G.A. Hedlund [19]. Recent work on this subject has been done by V. Bangert [3, 4], M.J. Dias Carneiro [10], A. Fathi [14, 15, 16, 17], R. Mañé [23, 24] and J. Mather [25, 26]. We refer to [8, 18] for comprehensive accounts of the theory. Static and semistatic curves are the paradigms of what we mean by globally minimizing orbits and since they will play an important role in our results we give now their definition (cf. [23, 26]).

Set $c = c(L)$. We say that $x : [a, b] \to M$ is a *semistatic curve* if it is absolutely continuous and:

$$S_{L+c}\left(x|_{[t_0,t_1]}\right) = \Phi_c\left(x(t_0), x(t_1)\right), \tag{2}$$

for all $a < t_0 \leq t_1 < b$; and that it is a *static* curve if

$$S_{L+c}\left(x|_{[t_0,t_1]}\right) = -\Phi_c(x(t_1), x(t_0)) \tag{3}$$

for all $a < t_0 \leq t_1 < b$. Observe that since

$$\Phi_c(x(t_0), x(t_1)) + \Phi_c(x(t_1), x(t_0)) = d_c(x(t_0), x(t_1)) \geq 0$$

a static curve is a semistatic curve for which $d_c(x(t_0), x(t_1)) = 0$ for all $a < t_0 \leq t_1 < b$. Let $\widehat{\Sigma}(L)$ be the set of vectors $v \in TM$ such that the solutions $\gamma : \mathbb{R} \to M$ of the Euler-Lagrange equation satisfying $\dot{\gamma}(0) = v$ are static. We call $\widehat{\Sigma}(L)$ *the static set*.

As we mentioned before, one of our aims in this paper is to relate the Morse theory of the action functional to the critical values. Let $H^1(\mathbb{R}^k)$ be the set of absolutely continuous curves $x : [0,1] \to \mathbb{R}^k$ such that

$$\int_0^1 |\dot{x}(t)|^2 \, dt < \infty.$$

It is well known that $H^1(\mathbb{R}^k)$ is a Hilbert space with the inner product defined by

$$\langle x, y \rangle_1 = \langle x(0), y(0) \rangle_{\mathbb{R}^k} + \int_0^1 \langle \dot{x}(t), \dot{y}(t) \rangle_{\mathbb{R}^k} \, dt,$$

where $\langle \cdot \,, \, \cdot \rangle_{\mathbb{R}^k}$ is the standard inner product of $\mathbb{R}^k$.

Given any Riemannian metric of M we may assume, on account of the Nash embedding theorem, that M is isometrically embedded in some $\mathbb{R}^k$. Take q_1 and q_2 in M and let $\Omega(q_1, q_2)$ be the set of elements of $H^1(\mathbb{R}^k)$ such that $x([0,1]) \subset M$, $x(0) = q_1$ and $x(1) = q_2$. It follows from the arguments in [29] that $\Omega(q_1, q_2)$ inherits a Hilbert manifold structure compatible with the Riemannian metric on M. We now define another action A_L closely related to the action S_L we defined before. Given the Lagrangian $L : TM \to \mathbb{R}$ define

$$A_L : \mathbb{R}^+ \times \Omega(q_1, q_2) \to \mathbb{R}$$

by

$$A_L(b, x) = \int_0^1 b \, L(x(t), \dot{x}(t)/b) \, dt.$$

Observe that

$$A_L(b, x) = S_L(y),$$

where $y(t) = x(t/b)$.

We now recall the definition of the Palais-Smale condition. In fact, this is a rather stronger version of the condition in [29] and [31] that we borrow from [20] and [22].

Definition 1. Let $f : X \to \mathbb{R}$ be a C^1 map where X is an open set of a Hilbert manifold. We say that f satisfies the *Palais-Smale* condition if every sequence $\{x_n\}$ such that $\{f(x_n)\}$ is bounded and $||d_{x_n} f|| \to 0$ as $n \to \infty$ has a converging subsequence.

We remark that the manifold $\mathbb{R}^+ \times \Omega(q_1, q_2)$ is not complete, however if $q_1 \neq q_2$ then the set $\{A_{L+k} \leq a\}$ is complete when k is strictly bigger that the critical value. (see Lemma 15). This is important when we apply the minimax principle (see Proposition 21).

In order to show that the action functional A_{L+k} is C^2 and satisfies the Palais-Smale condition we need the Lagrangian to be *quadratic at infinity*:

Definition 2. We say that $\mathbb{L} : TN \to \mathbb{R}$ is *quadratic at infinity* if there exists a Riemannian metric on N and $R > 0$ such that for each $x \in N$ and $|v|_x > R$, $\mathbb{L}(x, v)$ has the form

$$\mathbb{L}(x, v) = \frac{1}{2}|v|_x^2 + \theta_x(v) - V(x),$$

where θ is a smooth 1-form on N and $V : N \to \mathbb{R}$ a smooth function.

A lifted Lagrangian $L : TM \to \mathbb{R}$ is said to be *quadratic at infinity* if it is the lift of a Lagrangian quadratic at infinity on N .

In Section 3 (cf. Proposition 18) we shall show that given a Lagrangian L and $k \in \mathbb{R}$ there is a Lagrangian L_0 quadratic at infinity such that $L(x, v) = L_0(x, v)$ for all (x, v) with $E(x, v) \leq k + 1$. We shall also show (cf. Lemma 19) that given two Lagrangians L and L_0 which agree for any (x, v) with $E(x, v) \leq c(L) + 1$, then $c(L) = c(L_0)$. These properties motivate the following definition:

Definition 3. We say that the energy level $E^{-1}(k)$ of a convex and superlinear Lagrangian L satisfies *the Palais-Smale condition* if there is a Lagrangian L_0 quadratic at infinity such that L and L_0 agree for any (x, v) with $E(x, v) \leq k + 1$ and the action functional A_{L_0+k} satisfies the Palais-Smale condition on $\mathbb{R}^+ \times \Omega(q_1, q_2)$ for every $q_1 \neq q_2$.

In Section 3 we shall prove:

Theorem A. *Let M be any covering of the closed manifold N and let $L : TM \to \mathbb{R}$ be the lifted Lagrangian. If the static set $\widehat{\Sigma}(L)$ is not empty, then*

$$c(L) = \inf\left\{t \in \mathbb{R} :\ E_L^{-1}(k) \text{ satisfies the Palais-Smale condition for every } k > t\right\}.$$

When M is compact the set $\widehat{\Sigma}(L)$ is not empty [6, 23] and therefore we obtain:

Corollary. *Suppose that M is compact. Then*

$$c(L) = \inf\left\{t \in \mathbb{R} :\ E_L^{-1}(k) \text{ satisfies the Palais-Smale condition for every } k > t\right\}.$$

S. Bolotin in [5] explores ideas which are similar to the ones we develop here. Using a somewhat different language he also notes that the Palais-Smale condition holds for the action A_{L+k} for large values of k but he does not give a characterization of $c(L)$ as before.

Theorem A will follow from Theorems B and C below. To state these theorems we need two more definitions. For $k \in \mathbb{R}$, let

$$\begin{aligned}\Phi_k(q_1, q_2; T) &:= \inf_{\gamma \in \mathcal{C}(q_1, q_2; T)} S_{L+k}(\gamma)\\ h_k(q_1, q_2) &:= \liminf_{T \to +\infty} \Phi_k(q_1, q_2; T).\end{aligned}$$

The function h_c is known as *the Peierls barrier* [15, 26].

Theorem B. *Assume that L is quadratic at infinity and $h_k(q_1,q_2)=+\infty$. Then the action functional*

$$A_{L+k}:\mathbb{R}^+\times\Omega(q_1,q_2)\to\mathbb{R}$$

satisfies the Palais-Smale condition provided $q_1\neq q_2$.

Theorem C. *Assume that L is quadratic at infinity and that for some pair (q_1,q_2), $h_c(q_1,q_2)<+\infty$. Then the action functional*

$$A_{L+c}:\mathbb{R}^+\times\Omega(q_1,q_2)\to\mathbb{R}$$

does not satisfy the Palais-Smale condition.

At the end of Section 3 we explain in detail how to derive Theorem A from Theorems B and C. Two extra ingredients needed for this derivation are given by Corollary 12 in Section 3, which states that the Peierls barrier is finite for every pair (q_1,q_2) in M when the static set $\widehat{\Sigma}(L)$ is not empty and Lemma 10 which states that $h_k(q_1,q_2)=+\infty$ for every pair (q_1,q_2) provided that $k>c(L)$. We remark that if $h_c(q_1,q_2)$ is finite for some pair (q_1,q_2) then it is finite for all pairs (q_1,q_2).

In the appendix we give an example of a Lagrangian on $\mathbb{R}^2$ for which the static set is empty and the Peierls barrier h_c is infinite (and hence the Palais-Smale condition holds at critical energy). Even though this Lagrangian is not the lift of a Lagrangian on a compact manifold, it shows that most likely Theorems A, B and C are optimal.

It is unknown whether the energy level $E_L^{-1}(k)$ satisfies the Palais-Smale condition for $k<c(L)$ (some authors have assumed that the Palais-Smale condition holds at subcritical energies for magnetic Lagrangians and this gap has been pointed out by S. Bolotin, see [33] for a discussion of the problem).

In Section 4 we prove an analogue of the Hadamard theorem on fixed energy levels. A pair of points $(x_1,v_1),(x_2,v_2)$ in TM are said to be *conjugate* if $(x_2,v_2)=\phi_t(x_1,v_1)$ for $t\neq 0$ and $d\phi_t(V(x_1,v_1))$ intersects $V(x_2,v_2)$ non-trivially. Here, $V(x,v)$ is the vertical fibre at (x,v) defined as usual as the kernel of $d\pi_{(x,v)}:T_{(x,v)}TN\to T_xN$ where $\pi:TN\to N$ is the canonical projection. We have:

Theorem D. *Let $\mathbb{L}:TN\to\mathbb{R}$ be a convex and superlinear Lagrangian. Assume that there is $q_0\in N$ such that q_0 has no conjugate points in $E_{\mathbb{L}}^{-1}(k)$. Let $\tilde{q}_0$ be a lift of q_0 to $\tilde{N}$, the universal covering of N and let L be the lift of $\mathbb{L}$ to $\tilde{N}$. Denote the following statement by (H):*

(H) For every $\tilde{q}$ in $\tilde{N}$ there is a unique solution of the Euler-Lagrange equation of L with energy k, joining $\tilde{q}_0$ to $\tilde{q}$.

Then,

$$\begin{aligned} k>c_u(\mathbb{L}) &\implies (H)\\ (H) &\implies k\geq c_u(\mathbb{L}). \end{aligned}$$

We note that when $k = c_u(\mathbb{L})$ there are examples in [7] where (H) does not hold. Also there are examples in [7] of multivalued Lagrangians, that become honests Lagrangians in the universal covering for which (H) does hold.

In Section 5 we recall some results on Morse theory that we will use in the last section. In Section 6 we give applications such as the existence of minimizing periodic orbits in every non-trivial free homotopy class with energy greater than $c_u(\mathbb{L})$ and homologically trivial periodic orbits such that the action of $\mathbb{L} + k$ is negative if $c_u(\mathbb{L}) < k < c_a(\mathbb{L})$, where $c_a(\mathbb{L})$ is the critical value of the lift of $\mathbb{L}$ to the abelian covering of N. These results should be compared with the work of S.P. Novikov, I. Taimanov and A. Bahri and I. Taimanov on the existence of closed orbits for magnetic Lagrangians. See [2, 28, 33, 34] and the extensive references therein. We also prove in Section 6 that given an Anosov energy level, there exists in each non-trivial free homotopy class a unique closed orbit of the Euler-Lagrange flow in the given energy level.

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2. First and Second Variations

In this section we calculate the first and second variations of the action functional A_{L+k}. These computations do not need any assumptions on the Lagrangian. However, if we want them to be the first and second derivative of the action functional A_{L+k} we need the Lagrangian to be quadratic at infinity.

Take a curve $s \to (b_s, x_s) \in \mathbb{R}^+ \times \Omega(q_1, q_2)$ and set $b := b_0$, $x := x_0$, $\xi(t) := \frac{\partial x_s}{\partial s}|_{s=0}(t)$, $\alpha := \frac{db_s}{ds}|_{s=0}$ and $g(s) := A_{L+k}(b_s, x_s)$. A straightforward calculation in local coordinates gives:

Lemma 4.

$$\begin{aligned} d_{(b,x)}A_{L+k}(\alpha, \xi) := g'(0) = \alpha \int_0^1 \big\{k - E_L(x(t), \dot{x}(t)/b)\big\}\, dt \\ + \int_0^1 \big\{bL_x(x(t), \dot{x}(t)/b)\xi(t) + L_v(x(t), \dot{x}(t)/b)\dot{\xi}(t)\big\}\, dt. \end{aligned}$$

Remark 5. If (b, x) is a critical point of A_{L+k}, then $y : [0, b] \to M$ given by $y(t) = x(t/b)$ is a solution of the Euler-Lagrange equation of L with energy k (see [1] and [6]). Indeed, the second term in the last equation is equal to

$$\int_0^b \big(L_x(y, \dot{y}) - \tfrac{d}{dt} L_v(y, \dot{y})\big)\, \eta\, dt\,,$$

where $\eta(t) := \xi(t/b)$. Since it is zero for all variations η, then y satisfies the Euler-Lagrange equation. Since the first term is zero for $\alpha = 1$, then $E_L(y, \dot{y}) \equiv k$.

Remark 6.

$$\frac{\partial A_{L+k}}{\partial b}(\alpha) := \frac{dA_{L+k}(b_s, x)}{ds}|_{s=0} = \alpha \int_0^1 \{k - E_L(x(t), \dot{x}(t)/b)\}\, dt,$$
$$= \frac{\alpha}{b} \int_0^b \{k - E_L(y(s), \dot{y}(s))\}\, ds,$$

where $y(s) = x(s/b)$.

Lemma 7.

$$\frac{\partial^2 A_{L+k}}{\partial^2 x}(\xi, \xi) := \frac{d^2 A_{L+k}(b, x_s)}{ds^2}|_{s=0}$$
$$= \int_0^b \{\eta L_{xx}(y, \dot{y})\eta + \eta L_{xv}(y, \dot{y})\dot{\eta} + \dot{\eta} L_{vx}(y, \dot{y})\eta + \dot{\eta} L_{vv}(y, \dot{y})\dot{\eta}\}\, dt,$$

where $\eta(t) = \xi(t/b)$ *and* $y(t) = x(t/b)$.

Proof. Calculate $g''(0)$ where $g(s) = A_{L+k}(b, x_s)$.

□

The following lemma is an immediate consequence of the Morse Index Theorem for convex Lagrangians (cf. [11]).

Lemma 8. *Let* y, (b, x) *be as in the previous lemma. If in addition* y *is a solution of the Euler-Lagrange equation with no conjugate points, then* $\frac{\partial^2 A_{L+k}}{\partial^2 x}$ *is positive definite.*

The following theorem is a particular case of a theorem due to Smale [31].

Theorem 9. *If a Lagrangian* L *is quadratic at infinity then the corresponding action functional* $A_{L+k} : \mathbb{R}^+ \times \Omega(q_1, q_2) \to \mathbb{R}$ *is a* C^2 *function with respect to the canonical Hilbert structure of* $\mathbb{R}^+ \times \Omega(q_1, q_2)$. *Moreover, the differential of* A_{L+k} *evaluated at the tangent vector* (α, ξ) *is given precisely by the number* $d_{(b,x)}A_{L+k}(\alpha, \xi)$ *defined in Lemma 4.*

Lemma 8 motivates the following:

Question: Is it true that (b, x) is a local minimum of A_{L+k} provided that $x(t/b)$ is a solution of the Euler-Lagrange equation with no conjugate points?

The next example shows that the answer to this question is negative. This example was motivated by the referee who pointed out a mistake in a previous version of the mansucript. We thank him or her for this observation. On the other hand we shall show in Lemma 30 that the answer to the question is affirmative in the case of Anosov energy levels and the space of closed paths with a fixed non-trivial homotopy class.

Let L be the Lagrangian on $T\mathbb{R}^2$ given by:

$$L(x, y, \dot{x}, \dot{y}) = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2) - x\dot{y}.$$

We take $k = 1/2$, $q_1 = (1, 0)$ and $q_2 = (-1, 0)$. The orbits of L with energy $1/2$ are circles of radius one oriented counterclockwise. Let c_0 be the half circle of radius one connecting

q_1 to q_2. Let $(-\varepsilon,\varepsilon) \ni s \mapsto c_s$ be a small variation of c_0 by circles with center at $(0,s)$ and radius $\sqrt{1+s^2}$. We parametrize these circles by arc length, hence c_s connects q_1 to q_2 in time $b(s) = \sqrt{1+s^2}(\pi + 2\,a(s))$, where $a(s)$ is the angle in $(-\pi/2,\pi/2)$ whose tangent is s. By observing that

$$(L + \tfrac{1}{2})(x,y,\dot{x},\dot{y}) = 1 - x\,\dot{y},$$

when $\dot{x}^2 + \dot{y}^2 = 1$ one obtains:

$$A(s) := A_{L+1/2}(b_s, x_s) = S_{L+1/2}(c_s) = b(s) - (1+s^2)\left(\tfrac{\pi}{2} + a(s) + \tfrac{1}{2}\sin(2a(s))\right).$$

A somewhat tedious computation shows that:

$$A'(0) = 0, \qquad A''(0) = 0, \qquad A'''(0) = -2.$$

Hence $s \mapsto A(s)$ does not have a local minumum at $s = 0$. On the other hand the piece of orbit $[0,\pi] \ni s \mapsto c_0(s)$ does not have conjugate points [8, Example A.3]. Finally we observe that if we consider a ball in $\mathbb{R}^2$ with radius, let us say, four, then L restricted to this ball can be embedded into a convex superlinear Lagrangian on a closed surface.

3. The Palais-Smale condition for Lagrangians quadratic at infinity

3.1. **The Peierls barrier.** For $k \in \mathbb{R}$, let

$$\begin{aligned}
\Phi_k(q_1,q_2;T) &:= \inf_{\gamma \in \mathcal{C}(q_1,q_2,T)} S_{L+k}(\gamma) \\
h_k(q_1,q_2) &:= \liminf_{T\to+\infty} \Phi_k(q_1,q_2;T).
\end{aligned}$$

The function h_c is called the Peierls barrier [15].

Lemma 10. *Set $c := c(L)$.*
If $k > c$ then $h_k(q_1,q_2) = +\infty$ for all q_1, $q_2 \in M$.
If $k < c$ then $h_k(q_1,q_2) = -\infty$ for all q_1, $q_2 \in M$.

Proof. If $k > c$, we have that

$$\Phi_k(q_1,q_2;T) \geq \Phi_c(q_1,q_2) + (k-c)\,T.$$

Hence $h_k(q_1,q_2) = +\infty$.

If $k < c$, since $\Phi_k(q_2,q_2) = -\infty$, there is a curve $\gamma \in \mathcal{C}(q_2,q_2;T)$ with $T > 0$ and $S_{L+k}(\gamma) < 0$. Then

$$\Phi_k(q_1,q_2;1+n\,T) \leq \Phi_k(q_1,q_2;1) + n\,S_{L+k}(\gamma) \xrightarrow{n} -\infty.$$

Thus $h_k(q_1,q_2) = -\infty$.

□

Proposition 11. *$h_c(p,p) = 0$ iff $p \in \pi\widehat{\Sigma}(L)$.*

Proof. First take $p \in \pi\widehat{\Sigma}(L)$ and let $\gamma : \mathbb{R} \to M$ be a static curve with $\gamma(0) = p$. Let $\varepsilon > 0$ be given. For any $t > 0$ there is a curve $\gamma_t : [0,T_t] \to M$ such that $\gamma_t(0) = \gamma(t)$, $\gamma_t(T_t) = \gamma(0)$ and

$$S_{L+c}(\gamma_t) \leq \Phi_c(\gamma(t),p) + \varepsilon.$$

Then

$$\begin{aligned} 0 \le h_c(p,p) &\le \liminf_{t\to+\infty} S_{L+c}(\gamma|_{[0,t]} * \gamma_t) \\ &\le \liminf_{t\to+\infty} \big(S_{L+c}(\gamma|_{[0,t]}) + S_{L+c}(\gamma_t)\big) \\ &\le \liminf_{t\to\infty} \left(\Phi_c(p,\gamma(t)) + \Phi_c(\gamma(t),p)\right) + \varepsilon \\ &= \varepsilon, \end{aligned}$$

where the last equality holds because γ is a static curve.

Now assume that $h_c(p,p) = 0$. Then there is $T_n \to \infty$ and Tonelli minimizers $\gamma_n : [0,T_n] \to M$ with $\gamma_n(0) = \gamma_n(T_n) = p$ and $S_{L+c}(\gamma_n) \to 0$. By Lemma 16 the speed of the Tonelli minimizers γ_n is bounded and hence there is a subsequence such that $\dot\gamma_n(0) \to v$. Let γ be the solution of the Euler-Lagrange equation with initial conditions (p,v). We are going to show that γ is static. Take $0 < t_1 < t_2$ and $\varepsilon > 0$. Since Φ_c is continuous we can take n so big that $T_n > t_2$,

$$S_{L+c}(\gamma_n) < \varepsilon,$$

$$\Phi_c(\gamma(t_1),\gamma(t_2)) < \Phi_c(\gamma_n(t_1),\gamma_n(t_2)) + \varepsilon$$

and

$$\Phi_c(\gamma(t_2),\gamma(t_1)) < \Phi_c(\gamma_n(t_2),\gamma_n(t_1)) + \varepsilon.$$

Next observe that

$$\Phi_c(\gamma_n(t_1),\gamma_n(t_2)) \le S_{L+c}(\gamma_n|_{[t_1,t_2]})$$

and

$$\Phi_c(\gamma_n(t_2),\gamma_n(t_1)) \le S_{L+c}(\gamma_n|_{[t_2,T_n]}) + S_{L+c}(\gamma_n|_{[0,t_1]}).$$

Hence

$$\Phi_c(\gamma(t_1),\gamma(t_2)) + \Phi_c(\gamma(t_2),\gamma(t_1)) \le 3\varepsilon,$$

and since ε was arbitrary we deduce that $\gamma|_{(0,\infty)}$ is static. By Proposition 3.5.5 in [8], $\gamma : \mathbb{R} \to M$ is a static curve as desired.

$\square$

Corollary 12. *If the static set $\widehat\Sigma(L)$ is not empty, then for any pair (q_1,q_2) we have*

$$h_c(q_1,q_2) \le \Phi_c(q_1,p) + \Phi_c(p,q_2),$$

for any $p \in \pi\widehat\Sigma(L)$.

Proof. From the definition of the Peierls barrier h_c we have:

$$h_c(q_1,q_2) \le \Phi_c(q_1,p) + h_c(p,q) + \Phi_c(q,q_2).$$

Set $p = q \in \pi\widehat\Sigma(L)$ and use Proposition 11.

$\square$

Even though we do not need the next result for the proof of the theorems in the introduction, we include it here because it gives an interesting characterization of the Peierls barrier in terms of the action potential. This result should be compared with Fathi's results in [15], in which he characterizes the Peierls barrier in terms of conjugate weak KAM solutions.

Proposition 13. *If M is compact then*

$$h_c(x,y) = \inf_{p\in\pi(\widehat{\Sigma}(L))} \left\{\Phi_c(x,p) + \Phi_c(p,y)\right\}.$$

Proof. Recall that when M is compact the set $\widehat{\Sigma}(L)$ is not empty [6, 23]. Using Corollary 12 we get that

$$h_c(x,y) \le \inf_{p\in\pi\widehat{\Sigma}(L)} \big[\,\Phi_c(x,p) + \Phi_c(p,y)\,\big].$$

Now let $\gamma_n \in \mathcal{C}(x,y;T_n)$ be Tonelli minimizers with $T_n \to +\infty$ and $S_{L+c}(\gamma_n) \to h_c(x,y) < +\infty$. Then $\frac{1}{T_n} S_{L+c}(\gamma_n) \to 0$. Observe that by Lemma 16 the speed of the Tonelli minimizers γ_n is bounded. Let μ be a weak limit of a subsequence of the probability measures μ_{γ_n} supported on the piece of orbits

$$[0,T_n] \ni t \mapsto (\gamma_n(t), \dot\gamma_n(t)).$$

Then μ is a minimizing measure (see [6, 23]). Let $q \in \pi\big(\mathrm{supp}(\mu)\big)$ and $q_n \in \gamma_n([0,T_n])$ be such that $\lim_n q_n = q$. Then,

$$\begin{aligned}\Phi_c(x,q) + \Phi_c(q,y) &\le \Phi_c(x,q_n) + \Phi_c(q_n,y) + \Phi_c(q_n,q) + \Phi_c(q,q_n)\\ &\le S_{L+c}(\gamma_n) + \Phi_c(q_n,q) + \Phi_c(q,q_n).\end{aligned}$$

Letting $n \to \infty$, we get that

$$\Phi_c(x,q) + \Phi_c(q,y) \le h_c(x,y).$$

□

3.2. **Proof of Theorem B.** We begin with the following lemma:

Lemma 14. *Suppose that L is quadratic at infinity. Given any coordinate chart there are positive numbers δ, B, C and D such that in the given chart we have:*

$$C|v-w|^2 \le (L_v(x,v) - L_v(y,w))\cdot(v-w) + [B\,(|v|+|w|) + D]\ |v-w|\, d(x,y)$$

provided that $d(x,y) < \delta$.

Proof. Take δ such that if $d(x,y) < \delta$ then x, y belong to the same coordinate chart of M. Next we work in local coordinates as if L were defined in $\mathbb{R}^{2n}$.

$$\begin{aligned}&\int_0^1 (v-w)\cdot L_{vv}\big(t\,(x,v) + (1-t)\,(y,w)\big)\cdot(v-w)\,dt =\\ &\qquad = \big(L_v(x,v) - L_v(y,w)\big)\cdot(v-w)\\ &\qquad\qquad - \int_0^1 (v-w)\cdot L_{xv}\big((x,v)t + (y,w)(1-t)\big)\cdot(x-y)\,dt.\end{aligned}$$

Now, since L is quadratic at infinity $|L_{xv}(x,v)| \leq B_1\,|v|_x + D_1$ for some positive constants B_1 and D_1. On the other hand, since L_{vv} is positive definite, there is $C > 0$ such that

$$u\,L_{vv}\,u \geq C\,|u|^2$$

for every u. If we now take into account the equivalence between d and the euclidean metric in the given coordinate chart we can easily obtain the statement of the lemma for appropriate constants B and D.

□

The next lemma will ensure that if $q_1 \neq q_2$ then the set $\{A_{L+k} \leq a\}$ is complete when $h_k(q_1,q_2) = \infty$. This observation is needed when we apply the minimax principle given by Proposition 21.

Lemma 15. *If $q_1 \neq q_2$ and $A_{L+k}(b_n, x_n) \leq D$ then b_n is bounded away from zero.*

Proof. Since our Lagrangian is quadratic at infinity, there are positive constants D_1 and D_2 such that

$$L(x,v) \geq D_1\,|v|^2 - D_2,$$

consequently

$$D \geq A_{L+k}(b_n, x_n) \geq \frac{D_1}{b_n} \int_0^1 |\dot{x_n}|^2\,dt \; + (k - D_2)\,b_n,$$

and thus there are positive numbers D_3 and D_4 such that for all n,

$$\int_0^1 |\dot{x}_n|^2\,dt \leq D_3\,b_n + D_4\,b_n^2. \tag{4}$$

Observe that if $b_n \to 0$ then $||x_n||_1 \to 0$ and then the length of x_n goes to 0 which is absurd provided $q_1 \neq q_2$.

□

Let us begin now with the proof of Theorem B.

Take $\{(b_n, x_n)\}$ such that $\{A_{L+k}(b_n,x_n)\}$ is bounded and $||d_{(b_n,x_n)}A_{L+k}||_1 \to 0$. Let $y_n(t) = x_n(t/b_n)$. Then $\{b_n\}$ is bounded, for if not, we may assume that $b_n \to +\infty$ and then

$$A_{L+k}(b_n,x_n) = S_{L+k}(y_n) \geq \Phi_k(q_1,q_2;b_n) \xrightarrow{n} +\infty.$$

So, we can assume that $\{b_n\}$ converges. Let $b = \lim_n b_n$. Let

$$w_n(s) = \begin{cases} y_n(s) & \text{if } s \leq b_n, \\ q_2 & \text{if } b_n \leq s \leq b+1. \end{cases}$$

Then $w_n \in \mathcal{C}(q_1,q_2;b+1)$ and

$$\begin{aligned} S_L(w_n) &= S_L(y_n) + L(q_2,0)\,(b+1-b_n) \\ &\leq A_{L+k}(b_n,x_n) - k\,b_n + 2\,L(q_2,0), \end{aligned}$$

if $b_n \geq b-1$. By the same arguments in the proof of Tonelli's Theorem and by the Arzela-Ascoli Theorem there is a convergent subsequence of w_n in the C^0 topology (cf.

[8, 25]). This implies that also $\{y_n\}$ and $\{x_n\}$ have convergent subsequences in the C^0 topology. For the sequel we work with a convergent subsequence of $\{x_n\}$. We shall assume without loss of generality that the limit point of the sequence $\{x_n\}$ is contained in a coordinate chart, for if not we can cover it with a finite number of charts and we do the argument below on each chart. Hence for n large enough x_n has its image contained in the same local chart as the limit point.

By Lemma 15 we may assume that $b_n \to b \neq 0$. Write $z_n = x_n/b_n$. Also by Lemma 15 (see inequality (4)) there is $K > 0$ such that

$$||x_n||_1 \leq K \quad \text{and} \quad ||z_n||_1 \leq K$$

(recall that $||\ .\ ||_1$ is the norm in $H^1(\mathbb{R}^k)$). Now we follow the lines of Lemma 7.1 in [31]. Since $||d_{(b_n,x_n)}A_{L+k}||_1 \to 0$, given ε there is N such that

$$||d_{(b_n,x_n)}A_{L+k}(\eta) - d_{(b_m,x_m)}A_{L+k}(\eta)||_1 < \varepsilon$$

for every n, $m \geq N$ and $||\eta||_1 \leq 2K$. We can take in particular $\eta = x_n - x_m$ and therefore using Lemma 4 we have,

$$\Big| \int_0^1 \{b_n L_x(x_n, \dot{z}_n) - b_m L_x(x_m, \dot{z}_m)\}\ (x_n - x_m)\ dt + \\ + \int_0^1 \{L_v(x_n, \dot{z}_n) - L_v(x_m, \dot{z}_m)\}\ (\dot{x}_n - \dot{x}_m)\ dt \Big| < \varepsilon$$

for m, $n > N$. Since our Lagrangian is quadratic at infinity, then there are positive constants a and c such that $||L_x|| < a\ |v|_x^2 + c$. Using (4) in the proof of Lemma 15 we have that the first term is bounded by $(2\,a\,D_3 + (b_n + b_m)(a\,D_4 + c))||x_n - x_m||_\infty$. Consequently the second integral is small for big m and n.

Now we apply Lemma 14 to obtain

$$C \int_0^1 \big|\dot{z}_n - \dot{z}_m\big|^2\ dt \leq \int_0^1 \{\ L_v(x_n, \dot{z}_n) - L_v(x_m, \dot{z}_m)\ \} \cdot \big(\dot{z}_n - \dot{z}_m\big)\ dt + \\ +\ B \int_0^1 \big(|\dot{z}_n| + |\dot{z}_m|\big)\ \big|\dot{z}_n - \dot{z}_m\big|\ |x_n - x_m|\ dt \\ + D \int_0^1 \big|\dot{z}_n - \dot{z}_m\big|\ |x_n - x_m|\ dt.$$

Now

$$\int_0^1 \big(|\dot{z}_n| + |\dot{z}_m|\big)\ \big|\dot{z}_n - \dot{z}_m\big|\ dt \leq \big(||z_n||_1 + ||z_m||_1\big)\ \|z_n - z_m\|_1$$

by the Cauchy-Shwartz inequality. Therefore

$$C\ \big\|\dot{z}_n - \dot{z}_m\big\|_{L^2}^2 \leq \int_0^1 \{\ L_v(x_n, \dot{z}_n) - L_v(x_m, \dot{z}_m)\ \} \cdot \big(\dot{z}_n - \dot{z}_m\big)\ dt + \\ + (4\,K^2\,B + 2KD)\ ||x_n - x_m||_\infty.$$

Since x_n converges in the C^0 topology and the integral is small, we conclude that z_n converges in the H^1-norm. Again, since $b \neq 0$, x_n also converges in the H^1-norm, finishing the proof of the theorem.

$\square$

3.3. Proof of Theorem C.

Lemma 16 ([25]). *For $B > 0$ there exists $C = C(B) > 0$ such that if $x, y \in M$ and $\gamma \in \mathcal{C}(x, y; T)$ is a solution of the Euler-Lagrange equation with $A_L(\gamma) \leq BT$, then $|\dot\gamma(t)| < C$ for all $t \in [0, T]$.*

Proof. By the superlinearity there is $D > 0$ such that $L(x, v) \geq |v| - D$ for all $(x, v) \in TM$. Since $S_L(\gamma) \leq BT$, the mean value theorem implies that there is $t_0 \in (0, T)$ such that

$$|\dot\gamma(t_0)| \leq D + B.$$

The conservation of the energy implies that there is $C = C(B) > 0$ such that $|\dot\gamma| \leq C$.

$\square$

Lemma 17. *For all $x, y \in M$ and $\varepsilon > 0$, the function $t \mapsto \Phi_k(x, y; t)$ is Lipschitz on $\varepsilon < t < +\infty$.*

Proof. Fix $\varepsilon > 0$. If $T > \varepsilon$, let $\gamma \in \mathcal{C}(x, y; T)$ be a Tonelli minimizer. Let $\tau : [0, T] \to M$ be a geodesic with speed $d(x, y)/T < d(x, y)/\varepsilon$ connecting x to y. Let

$$B = \max_{\{(x,v) \colon |v| \leq d(x,y)/\varepsilon\}} L(x, v).$$

Then since γ is a Tonelli minimizer we have $S_L(\gamma) \leq S_L(\tau) \leq BT$. On account of Lemma 16 there exists $C = C(\varepsilon) > 0$ such that $|E(\gamma, \dot\gamma) - k| \leq C(\varepsilon) + |k|$. Denote $h(s) := \Phi_k(x, y; s)$. If $\gamma_s(t) := \gamma(Tt/s)$ with $t \in [0, s]$, then $h(s) \leq S_{L+k}(\gamma_s) =: \mathcal{B}(s)$. Using Remark 6 we have that

$$\begin{aligned} f(T) :&= \limsup_{\delta \to 0} \frac{h(T + \delta) - h(T)}{\delta} \\ &\leq \mathcal{B}'(T) = \frac{1}{T} \int_0^T \big[k - E(\gamma, \dot\gamma) \big]\, dt \\ &\leq C(\varepsilon) + |k|. \end{aligned}$$

If S, $T > \varepsilon$ we have that

$$\begin{aligned} \Phi_k(x, y; S) &\leq \Phi_k(x, y; T) + \left| \int_T^S f(t)\, dt \right| \\ &\leq \Phi_k(x, y; T) + C(\varepsilon) + |k|\, |T - S|. \end{aligned}$$

Since we can reverse the roles of S and T, this implies the Lipschitz condition for $T \mapsto \Phi_k(x, y; T)$.

$\square$

We begin now with the proof of Theorem C. Let $f(t) := \Phi_c(x, y; t)$.

Case 1: Suppose first that there is $T_0 > 0$ such that f is monotonous on $[T_0, +\infty)$. Since by Lemma 17 f is Lipschitz on $[T_0, +\infty)$ by Rademacher's theorem [13], f is differentiable almost everywhere and

$$f(t) - f(T_0) = \int_{T_0}^{t} f'(s)\, ds. \tag{5}$$

Since f is monotonous $f' \geq 0$ or $f' \leq 0$ for all $t \geq T_0$ and $\lim\limits_{t\to+\infty} f(t) = \liminf\limits_{t\to+\infty} f(t) = h_c(x, y) < +\infty$. This implies that there is a sequence of differentiability points $t_n \to +\infty$ such that $f'(t_n) \to 0$, for otherwise there would exist $K > 0$ and $R > 0$ such that $f'(s) \geq K > 0$ for $s \geq R$ or $f'(s) \leq -K$ for $s \geq R$. Consequently equation (5) would imply that $\lim_{t\to+\infty} f(t)$ is infinite.

Let γ_n be a Tonelli minimizer in $\mathcal{C}(x, y; t_n)$ and $\eta_s(t) = \gamma_n(\frac{t_n}{s}\, t)$. Then $S_{L+k}(\eta_s) \geq f(s)$ for s in an open interval contaning t_n. This implies that

$$f'(t_n) = \tfrac{d}{ds}\big|_{t_n} S_{L+k}(\eta_s) = \frac{1}{t_n} \int_0^{t_n} \big[\, k - E(\gamma_n, \dot\gamma_n) \,\big]\, dt,$$

where the second equality follows from Remark 6. Since γ_n is a solution of the Euler-Lagrange equation, by Remark 5 and Remark 6, if $x_n(s) = \gamma_n(s\, t_n)$

$$dA_{L+k}(t_n, x_n)(\xi, \alpha) = \alpha\, f'(t_n) \to 0.$$

Observe also that $A_{L+k}(t_n, x_n) \to h_c(x, y) < +\infty$. On the other hand (t_n, x_n) does not converge.

Case 2: Suppose that the set of local minima of f is unbounded. We claim that there exists a sequence $t_n \to +\infty$ of local minima of f such that $\lim_n f(t_n) = \liminf_{t\to+\infty} f(t)$. For let s_n be an increasing sequence of local minima of f such that $s_n \to +\infty$. There exists a sequence $r_n > s_n$ such that $\lim_n f(r_n) = \liminf_{t\to+\infty} f(t)$. Excluding some s_n's if necessary we can assume that $s_n < r_n < s_{n+1}$. Minimizing f on the interval $[s_n, s_{n+1}]$, we obtain a local minimum $t_n \in [s_n, s_{n+1}]$ such that $\lim_n t_n = \lim_n s_n = +\infty$ and $f(t_n) \leq f(r_n)$ so that $\lim_n f(t_n) = \liminf_{t\to+\infty} f(t)$.

Let γ_n be a Tonelli minimizer in $\mathcal{C}(x, y; t_n)$ and $\eta_s(t) = \gamma_n(\frac{t_n}{s}\, t)$. Then $S_{L+k}(\eta_s) \geq f(t_n)$ for s in a neighbourhood of t_n. In particular, t_n is also a local minimum of $s \mapsto S_{L+k}(\eta_s)$. Since $s \mapsto S_{L+k}(\eta_s)$ is differentiable, $\frac{d}{ds}|_{t_n} A_{L+k}(\eta_s) = 0$. By Remark 5 and Remark 6, if $x_n(s) = \gamma_n(s\, t_n)$

$$dA_{L+k}(t_n, x_n)(\xi, \alpha) = \alpha\, \tfrac{d}{ds}|_{t_n} A_{L+k}(\eta_s) = 0.$$

Observe also that $A_{L+k}(t_n, x_n) \to h_c(x, y) < +\infty$. On the other hand (t_n, x_n) does not converge.

$\square$

3.4. **Proof of Theorem A.**

Proposition 18. *Given a convex and superlinear Lagrangian $\mathbb{L} : TN \to \mathbb{R}$ and $k \in \mathbb{R}$ there is a Lagrangian $\mathbb{L}_0$, convex and quadratic at infinity such that $\mathbb{L}_0(x,v) = \mathbb{L}(x,v)$ for every (x,v) such that $\mathbb{E}_{\mathbb{L}}(x,v) \le k+1$.*

Proof. Without loss of generality we may assume that $\mathbb{L} \ge 0$.

Choose $R > 0$ such that

$$\mathbb{E}_{\mathbb{L}}(x,v) \le k+1 \quad \text{implies} \quad |v|_x \le R.$$

Define $\psi : \mathbb{R} \to \mathbb{R}$ by

$$\psi(s) = \max_{x,|v|=1} \mathbb{L}(x, sv).$$

Let $\varphi_1 : \mathbb{R} \to \mathbb{R}$ be an even smooth strictly convex function such that $\varphi_1(2R) = \varphi_1(-2R) = 0$ and such that $\varphi_1(s) > \psi(s)$ for $|s| > R_1$ where R_1 is a sufficiently large positive number bigger than $2R$. Define $\mathbb{L}_1(x,v) := \varphi_1(|v|_x)$ and $\mathbb{L}_2 := \max\{\mathbb{L}, \mathbb{L}_1\}$. Then $\mathbb{L}_2$ coincides with $\mathbb{L}$ for those (x,v) with $|v|_x \le 2R$ and coincides with $\mathbb{L}_1$ for those (x,v) with $|v|_x > R_1$. The Lagrangian $\mathbb{L}_2$ is strictly convex and may be approximated by a smooth strictly convex Lagrangian $\mathbb{L}_3$ such that $\mathbb{L}_3$ coincides with $\mathbb{L}$ for those (x,v) with $|v|_x \le R$ and coincides with $\mathbb{L}_1$ for those (x,v) with $|v|_x > R_1$. We briefly explain how to achieve this approximation given a strictly convex function $f : \mathbb{R}^n \to \mathbb{R}$. This approximation can be done on each tangent space T_xN. The idea is to smooth out f using a convolution. Let $\eta \in C^\infty(\mathbb{R}^n)$ be the function that equals $C \exp\left(\frac{1}{|x|^2-1}\right)$ if $|x| < 1$ and 0 if $|x| \ge 1$. The constant C is selected so that $\int_{\mathbb{R}^n} \eta\, dx = 1$. For each $\varepsilon > 0$ set

$$\eta_\varepsilon(x) := \frac{1}{\varepsilon^n} \eta\left(\frac{x}{\varepsilon}\right).$$

The functions η_ε are C^∞, their integrals equal one and their support is inside the ball of radius ε around the origin. The function η is called the *standard mollifier*. We define the *mollification* of f by $f^\varepsilon := \eta_\varepsilon * f$. That is,

$$f^\varepsilon(x) = \int_{\mathbb{R}^n} \eta_\varepsilon(y-x) f(y)\, dy = \int_{\mathbb{R}^n} \eta_\varepsilon(y) f(x+y)\, dy.$$

It is straightforward to verify that if f is strictly convex then f^ε is also strictly convex. Moreover f^ε is C^∞ and approximates f as $\varepsilon \to 0$ uniformly on compact subsets [12, Appendix C]. It follows that the Hessian of f^ε is positive definite.

Suppose in addition that f is C^∞ in some open set U of the form,

$$U = \{x \in \mathbb{R}^n :\ |x| < a + \delta \text{ and } |x| > b - \delta\},$$

with $0 < a < b$ and δ very small. Now let r_1 and r_2 be positive numbers such that $r_1 < a < b < r_2$. Choose a smooth function $\alpha : \mathbb{R}^n \to \mathbb{R}$ such that

1. $0 \le \alpha(x) \le 1$ for all $x \in \mathbb{R}^n$;
2. $\alpha(x) = 0$ for $a < |x| < b$;
3. $\alpha(x) = 1$ for $|x| < r_1$ and $|x| > r_2$.

Now let $h^\varepsilon := (1-\alpha) f^\varepsilon + \alpha f = f^\varepsilon + \alpha(f - f^\varepsilon)$. We have:

1. h^ε is C^∞;
2. h^ε coincides with f for $|x| < r_1$ and $|x| > r_2$;
3. for any ε sufficiently small the function h^ε is strictly convex since the derivatives of f^ε approximate the derivatives of f uniformly on the set $r_1 \le |x| \le a$ and $b \le |x| \le r_2$ as $\varepsilon \to 0$.

Then h^ε gives the desired approximation of f.

Let us complete now the proof of the proposition. Let $\varphi_2 : \mathbb{R} \to \mathbb{R}$ be a smooth strictly convex function such that

- $\varphi_2(s) = \varphi_1(s)$ for $|s| \le R_1 + 1$;
- $\varphi_2(s) = s^2$ for $|s| > R_2$,

where R_2 is a sufficiently large positive number bigger than $R_1 + 1$. Finally let $\mathbb{L}_0$ be the Lagrangian which coincides with $\mathbb{L}_3$ for those (x, v) with $|v|_x \le R_1$ and coincides with $\varphi_2(|v|_x)$ for those (x, v) with $|v|_x \ge R_1$. Then, $\mathbb{L}_0$ is smooth, strictly convex, quadratic at infinity and coincides with $\mathbb{L}$ for those (x, v) with $|v|_x \le R$.

□

Lemma 19. *Assume that L and L_0 agree on the set of those (x, v) satisfying $E_L(x, v) \le c(L) + 1$. Then $c(L_0) = c(L)$.*

Proof. We use the following characterization of the critical value taken from [9]

$$c(L) = \inf_{u \in C^\infty(M,\mathbb{R})} \sup_{x \in M} H(x, d_x u), \tag{6}$$

where $H : T^*M \to \mathbb{R}$ is the Hamiltonian associated to L. Let $0 < \varepsilon < 1$. Observe that $H(x, p) < c(L) + \varepsilon$ implies $H_0(x, p) = H(x, p) < c(L) + \varepsilon$, where H_0 is the Hamiltonian associated to L_0. By (6) there is $u \in C^\infty(M, \mathbb{R})$ such that $H(x, d_x u) < c(L) + \varepsilon$ and hence $H_0(x, d_x u) = H(x, d_x u) < c(L) + \varepsilon$. Since ε is arbitrary, we obtain $c(L_0) \le c(L)$. Changing the roles of H_0, H, $c(L_0)$, $c(L)$ we obtain $c(L) \le c(L_0)$, concluding the proof.

□

Corollary 20. *Given a convex superlinear Lagrangian $L : TM \to \mathbb{R}$ and $k \ge c(L)$ there is a Lagrangian L_0 convex and quadratic at infinity, such that L and L_0 agree on the set of those (x, v) satisfying $E_L(x, v) \le k + 1$ and $c(L_0) = c(L)$.*

Proof. It follows from Proposition 18 and Lemma 19.

□

Let us prove Theorem A. Take L_0 such that $c(L_0) = c(L)$ according to the preceding corollary. Since L_0 and L agree on a neighbourhood of $E^{-1}(c(L))$, then L and L_0 have the same static set since the latter must be contained in $E^{-1}(c(L))$. Now if $k > c(L_0) = c(L)$, the barrier h_k of L_0 is $+\infty$, and then A_{L_0+k} satisfies the Palais-Smale condition by Theorem B. This means that $E_L^{-1}(k)$ is a Palais-Smale level. On the other hand, since the static set $\widehat{\Sigma}(L)$ is not empty, $h_c < +\infty$, and then Theorem C completes the proof.

□

4. Proof of Theorem D

The statement $k > c_u(\mathbb{L}) \Rightarrow (H)$ was proved in [7] and could also be obtained using the corollary of Theorem A in [9]: we reparametrize the energy level in the universal covering to obtain a complete Finsler metric to which we apply Morse theory which is known to hold for Finsler geometry.

Now we prove that $(H) \implies k \geq c_u(\mathbb{L})$. Let L be the lift of $\mathbb{L}$ to $T\tilde{N}$. Recall that the Hamiltonian H associated to L is given by

$$H(x,p) = \sup\{p(v) - L(x,v) \,|\, v \in T_x\tilde{N}\}, \tag{7}$$

and the supremum is achieved at v such that $p = L_v(x,v)$. Thus $H(x, L_v(x,v)) = E(x,v)$.

Given $\tilde{q} \in \tilde{N}$, let (b_q, x_q) be the unique critical point of A_{L+k} in $\mathbb{R}^+ \times \Omega(\tilde{q}_0, \tilde{q})$. Write $y_{\tilde{q}}(t) := x_q(t/b_{\tilde{q}})$ and define $f : \tilde{N} \to \mathbb{R}$ by $f(\tilde{q}) = A_{L+k}(b_{\tilde{q}}, x_{\tilde{q}}) = S_{L+k}(y_{\tilde{q}})$. The uniqueness of $y_{\tilde{q}}$ implies that f is of class C^1 because it is a composition of the action functional with an analogue of the exponential map $\exp_{\tilde{q}}$ on $E_L^{-1}(k)$. Let $\alpha : (-\varepsilon, \varepsilon) \to \tilde{N}$ be a smooth curve such that $\alpha(0) = \tilde{q}$ and $\dot{\alpha}(0) = w \in T_{\tilde{q}}\tilde{N}$. If we differentiate $f(\alpha(s))$ at $s = 0$ using the first variation given by Lemma 4 and integration by parts we obtain:

$$d_{\tilde{q}} f(w) = L_v(\tilde{q}, \dot{y}_{\tilde{q}}(b_{\tilde{q}})) \cdot w.$$

This implies that

$$H(\tilde{q}, d_{\tilde{q}} f) = E(\tilde{q}, \dot{y}_{\tilde{q}}(b_{\tilde{q}}))) = k.$$

In [9] we showed that

$$c_u(\mathbb{L}) = \inf\nolimits_{f \in C^\infty(\tilde{N},\mathbb{R})} \sup\nolimits_{x \in \tilde{N}} \; H(x, d_x f)\,.$$

However the same proof in [9] shows that one can replace in the above equality $C^\infty(\tilde{N}, \mathbb{R})$ by $C^k(\tilde{N}, \mathbb{R})$ for any $k \geq 1$ and we obtain the same critical value. We see right away that $k \geq c_u(\tilde{\mathbb{L}})$.

□

5. Some results on Morse theory

In this section we recall some results on Morse theory (cf. [29, 31]) that we shall use in the next section.

Let X be an open set in a Hilbert manifold and $f : X \to \mathbb{R}$ is a C^2 map. The following version of the minimax principle (Proposition 21 below) is a modification of that of [20] (see also [32]). The only (minor) difference with the usual minimax principle is that our manifold X is not necessarily complete, but instead each set $[f \leq b] \subseteq X$ is complete.

Observe that if the vector field $Y = -\nabla f$ is not globally Lipschitz, the gradient flow ψ_t of $-f$ is a priori only a local flow. Given $p \in X$, $t > 0$ define

$$\alpha(p) := \sup\{\, a > 0 \,|\, s \mapsto \psi_s(p) \text{ is defined on } s \in [0,a] \,\}.$$

We say that a function $\tau : X \to [0, +\infty)$ is an *admisible time* if τ is differentiable and $0 \leq \tau(x) < \alpha(x)$ for all $x \in X$. Given and admisible time τ, and a subset $F \subset X$. define

$$F_\tau := \{\, \psi_{\tau(p)}(p) \,|\, p \in F \,\}.$$

Let $\mathcal{F}$ be a family of subsets $F \subset X$. We say that $\mathcal{F}$ is *forward invariant* if $F_\tau \in \mathcal{F}$ for all $F \in \mathcal{F}$ and any admisible time τ. Define

$$c(f, \mathcal{F}) = \inf_{F\in\mathcal{F}} \sup_{p\in F} f(p).$$

Proposition 21. *Let f be a C^1 function satisfying the Palais-Smale condition. Assume also that $\mathcal{F}$ is forward invariant under the gradient flow of $-f$. Suppose that there is b such that $-\infty < c(f, \mathcal{F}) < b < +\infty$ and such that the subset $[f \le b] \subseteq X$ is complete. Then $c(f, \mathcal{F})$ is a critical value of f.*

Proof. We borrow the following lemma from [31],

Lemma 22. *Suppose that $f : X \to \mathbb{R}$ is C^1, ψ_t is the gradient flow of $-f$ and the subset $[a \le f \le b] \subset X$ is complete. Then the flow ψ_t is relatively complete on $[a \le f \le b]$, that is, if $a \le f(p) \le b$, then either $\alpha(p) = +\infty$ or $f(\psi_\beta(p)) \le a$ for some $0 \le \beta < \alpha(p)$.*

Proof. Let ψ_t be the flow of $Y = -\nabla f$. Then

$$f(\psi_{t_1}(p)) - f(\psi_{t_2}(p)) = -\int_{t_1}^{t_2} \nabla f(\psi_s(p)) \cdot Y(\psi_s(p))\, ds = \int_{t_1}^{t_2} \|Y(\psi_s(p))\|^2\, ds. \tag{8}$$

Moreover, using the Cauchy-Schwartz inequality, we have that

$$d(\psi_{t_1}(p), \psi_{t_2}(p))^2 \le \Big[\int_{t_1}^{t_2} \|Y(\psi_s(p))\|\, ds\Big]^2 \le |t_2 - t_1| \int_{t_1}^{t_2} \|Y(\psi_s(p))\|^2\, ds.$$

Thus

$$d(\psi_{t_1}(p), \psi_{t_2}(p))^2 \le |t_2 - t_1|\ \ | f(\psi_{t_1}(p)) - f(\psi_{t_2}(p)) |. \tag{9}$$

Let $I = [0, \alpha[$ a maximal interval of definition of $t \mapsto \psi_t(p)$. Suppose that $a \le f(\psi_t(p)) \le b$ for $0 \le t < \alpha < \infty$. Let $t_n \uparrow \alpha$. By inequality (9), $n \mapsto \psi_{t_n}(p)$ is a Cauchy sequence on $[a \le f \le b]$. Since $[a \le f \le b]$ is complete, it has a limit $q = \lim_n \psi_{t_n}(p) = \psi_\alpha(p)$. Since f is C^1, we can extend the solution $t \mapsto \psi_t(p)$ at $t = \alpha$. This contradicts the definition of α.

□

Write $c = c(f, \mathcal{F})$. We shall prove that for all $\varepsilon > 0$ there is a critical value c_ε such that $c - \varepsilon < c_\varepsilon < c + \varepsilon$. Then, using $\varepsilon = \frac{1}{n}$, the Palais-Smale condition implies that c is a critical value.

Suppose that there are no critical points on $A(\varepsilon) := [c - \varepsilon \le f \le c + \varepsilon]$. The Palais-Smale condition implies that there is $\delta > 0$ such that $\|\nabla f(p)\| > \delta$ for all $p \in A(\varepsilon)$.

If $f(p) \le c + \varepsilon$, let

$$\tau(p) := \inf\{\, t > 0 \mid s \mapsto \psi_s(p) \text{ is defined on } [0, t] \text{ and } f(\psi_t(p)) \le c - \varepsilon \,\}.$$

Since $s \mapsto f(\psi_s(p))$ is decreasing, by Lemma 22, either $\tau(p) = +\infty$ or $\tau(p) < \alpha(p)$ and $f(\psi_{\tau(p)}(p)) = c - \varepsilon$. Since $\|\nabla f\| > \delta$ on $A(\varepsilon)$, by equation (8),

$$c - \varepsilon \le f(\psi_t(p)) \le f(p) - t\,\delta^2 \le c + \varepsilon - t\,\delta^2 \qquad \text{for} \quad 0 \le t \le \tau(p).$$

Thus $\tau(p) \leq 2\varepsilon/\delta^2$ for all $p \in A(\varepsilon)$; in particular this shows that $\tau(p)$ cannot be $+\infty$. Observe that the implicit function theorem applied to the function $F(x,t) = f(\psi_t(x))$ implies that $p \mapsto \tau(p)$ is differentiable because $\frac{\partial}{\partial t} F(x,t) = \nabla f \cdot X = -\|\nabla f\| < -\delta$.

By the definition of $c(f,\mathcal{F})$ there exists $F \in \mathcal{F}$ such that

$$\sup_{x\in F} f(x) \leq c + \varepsilon.$$

Then

$$\sup_{x\in F_\tau} f(x) = \sup_{x\in F} f(\psi_{\tau(p)}(p)) \leq c - \varepsilon.$$

Since τ is an admisible time, this contradicts the definition of $c(f,\mathcal{F})$.

□

From Proposition 21 we derive, taking $\mathcal{F}$ to be the family of sets of the form $\{p\}$ with $p \in X$, the following

Corollary 23. *Let $f : X \to \mathbb{R}$ be a C^2 function for which $[f \leq b]$ is complete for every b. Suppose that f is bounded from below and satisfies the Palais-Smale condition. Then f has a global minimum.*

It is convenient to obtain a further refinement of the corollary above which will be useful in the next section.

Corollary 24. *Let X be a connected manifold. Let $f : X \to \mathbb{R}$ be a C^2 function for which $[f \leq b]$ is complete for every b, satisfying the Palais-Smale condition. Suppose that p_1 is a relative minimizer of f and suppose that f admits a second relative minimizer $p_2 \neq p_1$. Then,*

1. *either there exists a critical point p of f which is not a relative minimum or*
2. *p_1 and p_2 can be connected in any neighborhood of the set of relative minimizers p of f with $f(p) = f(p_1)$. Necessarily then $f(p_1) = f(p_2)$.*

Proof. A detailed proof can be found in [32, Theorem 10.3]. The idea is to apply again the minimax principle; this time $\mathcal{F}$ is the family of subsets of the form $x([0,1])$ where x is a curve joining p_1 to p_2.

□

We conclude this section with the following suggestive remark. Let $\mathcal{F}$ be the family of all subsets F of T^*M of the form $F = \{(x, d_x u) :\ x \in M\}$ where $u \in C^\infty(M,\mathbb{R})$. Then

$$\begin{aligned} c(H,\mathcal{F}) &= \inf_{F\in\mathcal{F}} \sup_{(x,p)\in F} H(x,p) \\ &= \inf_{u\in C^\infty(M,\mathbb{R})} \sup_{x\in M} H(x, d_x u) \\ &= c(L), \end{aligned}$$

where the last equality is proved in [9]. Hence Mañé's critical value resembles a critical value of H as a smooth function even though in general it is *not* a critical value of H as a smooth function. This resemblance shows that the name "critical value" for $c(L)$ is appropriate and explains the (intentional) similarities in our notation.

6. Applications to the reduced action functional

6.1. **Periodic Orbits.** In this subsection we show the existence of periodic orbits in every nontrivial free homotopy class and every energy level above $c_u(\mathbb{L})$, where $\mathbb{L} : TN \to \mathbb{R}$ is a convex Lagrangian quadratic at infinity and the manifold N is compact. Let σ be a nontrivial free homotopy class of closed loops in a compact manifold N. As before, define Ω_σ as the set of elements of $\mathbb{R}^+ \times H^1(\mathbb{R}^k)$ of the form (b, x) where $x([0, 1]) \subset N$, $x(0) = x(1)$ and $x \in \sigma$. Let $\mathbb{L} : TN \to \mathbb{R}$ be a convex and superlinear Lagrangian.

The action $A_{\mathbb{L}} : \mathbb{R}^+ \times \Omega_\sigma \to \mathbb{R}$ is defined by

$$A_{\mathbb{L}}(b, x) = \int_0^1 b(\mathbb{L}(x(t), \dot{x}(t)/b))\, dt.$$

The previous discussion about the Palais-Smale condition translates to the case of free loops in a non trivial free homotopy class with only minor changes. In particular if $\mathbb{L}$ is quadratic at infinity and $k > c_u(\mathbb{L})$, then $A_{\mathbb{L}+k}$ satisfies the Palais-Smale condition on $\mathbb{R}^+ \times \Omega_\sigma$. However one has to use the compactness of N in the following lemma.

Lemma 25. *Let $k > c_u(\mathbb{L})$ and $(b_n, x_n) \in \mathbb{R}^+ \times \Omega_\sigma$ such that $A_{\mathbb{L}+k}(b_n, x_n)$ is bounded. Then $\{(b_n, x_n)\}$ has a converging subsequence in the C^0 topology.*

Let us prove first:

Lemma 26. *If $k \geq c_u(\mathbb{L})$, then*

$$\inf_{\mathbb{R}^+ \times \Omega_\sigma} A_{\mathbb{L}+k} > -\infty.$$

Proof. Fix $x_0 \in \sigma$ and let R be twice the diameter of N . Take $C > 0$ such that

$$S_{\mathbb{L}+k}(z) \leq C$$

for all curves $z : [0, R] \to N$ such that $|\dot{z}| \leq 1$. Let (b, x) be an arbitrary element of $\mathbb{R}^+ \times \Omega_\sigma$. Let $x_1(t) := x(t/b)$. Then there is a curve z, parametrized by arc length, with length not greater than R such that $\gamma = x_0^{-1} * z * x_1 * z^{-1}$ is homotopic to zero.

Since γ lifts as a closed curve we have

$$S_{\mathbb{L}+k}(\gamma) \geq 0.$$

Then

$$S_{\mathbb{L}+k}(x_1) \geq -S_{\mathbb{L}+k}(x_0) - 2C.$$

Since $S_{\mathbb{L}+k}(x_1) = A_{\mathbb{L}+k}(x)$, we are done.

□

Proof of Lemma 25. Observe that if $k > c := c_u(\mathbb{L})$ then,

$$A_{\mathbb{L}+k}(b_n, x_n) = A_{\mathbb{L}+c}(b_n, x_n) + (k - c)b_n,$$

hence if $A_{\mathbb{L}+k}(b_n, x_n)$ is bounded it follows that $\{b_n\}$ is also bounded. Since the manifold N is compact by the same arguments in the proof of Tonelli's Theorem in Ω_σ and by the Arzela-Ascoli Theorem there is a convergent subsequence of x_n in the C^0 topology (cf. [8, 6, 25]) and hence Lemma 25 follows.

□

Using Corollary 23 of the previous section we can obtain right away the following theorem:

Theorem 27. *Let N be a closed manifold and let $\mathbb{L} : TN \to \mathbb{R}$ be a convex Lagrangian quadratic at infinity. Then for every $k > c_u(\mathbb{L})$ and every nontrivial free homotopy class σ there is a periodic orbit of $\mathbb{L}$ in σ having energy k that minimizes the action of $A_{\mathbb{L}+k}$ on $\mathbb{R}^+ \times \Omega_\sigma$.*

We conclude this subsection by showing:

Proposition 28. *Let $c_u(\mathbb{L}) < k < c_a(\mathbb{L})$. Then there is a periodic orbit γ of $\mathbb{L}$ which is homologically trivial and also has $S_{\mathbb{L}+k}(\gamma) < 0$.*

Proof. Since $k < c_a(\mathbb{L})$ there is a closed curve α such that

$$S_{\mathbb{L}+k}(\alpha) < 0.$$

Such a curve cannot be homotopically trivial, otherwise, we lift it to the universal covering as a closed curve having negative action, contradicting the condition $c_u(\mathbb{L}) < k$. Let σ be the (non trivial) homotopy class of α.

By the previous theorem

$$A_{\mathbb{L}+k} : \mathbb{R}^+ \times \Omega_\sigma \to \mathbb{R}$$

has a global minimum (b, x) which is a periodic orbit with energy k. If we set $y(t) := x(t/b)$, then

$$A_{\mathbb{L}+k}(b, x) = S_{\mathbb{L}+k}(y) \leq S_{\mathbb{L}+k}(\alpha) < 0,$$

as desired.

□

6.2. **Periodic orbits of Anosov energy levels.** In this last subsection we show the following theorem. It was proved for geodesic flows by W. Klingenberg [21].

Theorem 29. *Let N be a closed manifold and let $\mathbb{L} : TN \to \mathbb{R}$ be a convex superlinear Lagrangian. Suppose that the Euler-Lagrange flow of $\mathbb{L}$ restricted to the regular energy level $\mathbb{E}^{-1}(k)$ is Anosov. Then in any non-trivial free homotopy class there is a unique closed orbit of $\mathbb{L}$ with energy k.*

Proof. We proved in [9] that if the Euler-Lagrange flow of $\mathbb{L}$ restricted to the regular energy level $\mathbb{E}^{-1}(k)$ is Anosov then $k > c_u(\mathbb{L})$. Let σ be a non-trivial free homotopy class. Without loss of generality we can assume that $\mathbb{L}$ is quadratic at infinity and hence $A_{\mathbb{L}+k}$ satisfies the Palais-Smale condition on $\mathbb{R}^+ \times \Omega_\sigma$. By Theorem 27 we know that there exists a closed orbit of $\mathbb{L}$ with energy k that minimizes $A_{\mathbb{L}+k}$ on $\mathbb{R}^+ \times \Omega_\sigma$. The next lemma shows in fact that *every* closed orbit with energy k in the homotopy class σ has this minimizing property. We will postpone its proof until we complete the proof of the theorem.

Lemma 30. *Every closed orbit of $\mathbb{L}$ with energy k in the homotopy class σ is a minimum of $A_{\mathbb{L}+k}$ on $\mathbb{R}^+ \times \Omega_\sigma$.*

Now suppose that we have two *geometrically* different closed orbits γ_1 and γ_2 of $\mathbb{L}$ with energy k in the free homotopy class σ. By Lemma 30 all the critical points of $A_{\mathbb{L}+k}$ on $\mathbb{R}^+ \times \Omega_\sigma$ are minimizers and hence in Corollary 24 the second alternative holds. This contradicts that γ_1 (or γ_2) is hyperbolic.

□

Proof of Lemma 30. Let γ be a closed orbit of $\mathbb{L}$ with energy k in the free homotopy class σ. Let $W^s(\gamma)$ be the weak stable leaf of γ for the corresponding Hamiltonian flow and let $\widetilde{W}^s(\gamma)$ be its lift to the universal covering. We proved in [9] that $\widetilde{W}^s(\gamma)$ is the graph of an exact 1-form. This means that there exists $u : \widetilde{N} \to \mathbb{R}$ such that $\widetilde{W}^s(\gamma) = \{(x, d_x u) : x \in \widetilde{N}\}$ and since $\widetilde{W}^s(\gamma)$ is contained in the energy level k we have that $H(x, d_x u) = k$ for all $x \in \widetilde{N}$. By the relation between H and L we have

$$L(x, v) - d_x u(v) + k \geq 0, \tag{10}$$

and equality holds if and only if $L_v(x, v) = d_x u$, i.e. when (x, v) belongs to the inverse image of $\widetilde{W}^s(\gamma)$ under the Legendre transform which is the same as the lift of the weak stable leaf for the Euler-Lagrange flow on TN.

Let $D \subset \widetilde{N}$ be a fundamental domain for the action of $\pi_1(N)$. Let $\widetilde{\gamma}$ be a lift of the closed curve $\gamma : [0, T] \to N$ to $\widetilde{N}$ with initial point in D. Let $\eta : [0, T_1] \to N$ be a closed curve in the free homotopy class σ and let $\widetilde{\eta}$ be a lift to $\widetilde{N}$ with initial point in D. Let $a : [0, 1] \to \widetilde{N}$ be a curve such that $a(0) = \widetilde{\gamma}(0)$ and $a(1) = \widetilde{\eta}(0)$. Let $\varphi : \widetilde{N} \to \widetilde{N}$ be the covering transformation that takes $\widetilde{\gamma}(0)$ to $\widetilde{\gamma}(T)$. Let $b_n : [0, 1] \to \widetilde{N}$ be the curve $\varphi^n a^{-1}$. Using (10) we get:

$$\int_{a * \widetilde{\eta} * \cdots * \varphi^{n-1} \widetilde{\eta} * b_n} L - du + k \geq 0 = \int_{\widetilde{\gamma} * \cdots * \varphi^{n-1} \widetilde{\gamma}} L - du + k.$$

Observe that the curves $a * \widetilde{\eta} * \cdots * \varphi^{n-1} \widetilde{\eta} * b_n$ and $\widetilde{\gamma} * \cdots * \varphi^{n-1} \widetilde{\gamma}$ have the same end points, hence

$$\int_a (L + k) + \int_{b_n} (L + k) + n \int_{\widetilde{\eta}} (L + k) \geq n \int_{\widetilde{\gamma}} (L + k)$$

Since $\int_{b_n} (L + k)$ is independent of n, diving by n and letting $n \to \infty$ we obtain:

$$S_{\mathbb{L}+k}(\eta) \geq S_{\mathbb{L}+k}(\gamma).$$

□

7. Appendix: an example of a Lagrangian with $h_c = +\infty$.

Let $L : T\mathbb{R}^2 \to \mathbb{R}$ be $L(x, v) = \frac{1}{2} |v|^2 + \psi(x)$, where $|\cdot|$ is the euclidean metric on $\mathbb{R}^2$ and $\psi(x)$ is a smooth function with $\psi(x) = \frac{1}{|x|}$ for $|x| \geq 2$, $\psi > 0$ and $\psi(x) = 2$ for $0 \leq |x| \leq 1$.

Then

$$c(L) = -\inf \psi = 0,$$

because if $\gamma_n : [0, T_n] \to \mathbb{R}^2$ is a smooth closed curve with length $\ell(\gamma_n) = 1$, $|\gamma_n(t)| \geq n$ for $t \in [0, T_n]$ and energy $E(\gamma_n) = \frac{1}{2}\,|\dot\gamma_n|^2 - \psi(\gamma_n) \equiv 0$, then

$$\begin{aligned} c(L) &\geq -\inf_{n>0} A_L(\gamma_n) \geq -\int_0^{T_n} \tfrac{1}{2}\,|\dot\gamma_n|^2 + \psi(\gamma_n) \\ &= -\int_0^{T_n} |\dot\gamma_n|^2 \geq -\sqrt{\tfrac{2}{n}} \longrightarrow 0. \end{aligned}$$

On the other hand,

$$c(L) = -\inf\{A_L(\gamma)\,|\,\gamma \text{ closed}\} \leq 0,$$

because $L > 0$.

Observe that since $L > 0$ and on compact subsets of $\mathbb{R}^2$, $L > a > 0$, then we have that

$$d_c(x, y) = \Phi_c(x, y) > 0 \text{ for all } x, y \in \mathbb{R}^2 \text{ with } x \neq y.$$

Hence $\widehat{\Sigma}(L) = \emptyset$.

Suppose that $h_c(0, 0) < +\infty$. For every $x \in \mathbb{R}^2$ there exists a vector $v \in T_x\mathbb{R}^2$ such that the solution of the Euler-Lagrange equation with initial conditions (x, v) is semistatic in forward time [9]. Let v be such a vector in $T_0\mathbb{R}^2$ and write $x_v(t) = (r(t), \theta(t))$ in polar coordinates about the origin $0 \in \mathbb{R}^2$. Then $\liminf_{t\to+\infty} r(t) = +\infty$ because otherwise the orbit of v would lie on a compact subset of $E^{-1}(0)$ and then $\emptyset \neq \omega-\text{limit}(v) \subseteq \widehat{\Sigma}(L) = \emptyset$ (see [8] for a proof of the fact that the $\omega-$ limit set of a forward semistatic orbit is contained in the static set). Note that for any t with $r(t) \geq 2$ we have:

$$|\dot x_v(t)| = \sqrt{\tfrac{2}{r(t)}}$$

and

$$L(\phi_t\, v) = |\dot x_v(t)|^2 = \sqrt{\tfrac{2}{r(t)}}\; |\dot x_v(t)|.$$

Let $T_n \to +\infty$ be such that $r(T_n) \to +\infty$. Hence there is n_0 such that for all $n \geq n_0$, $r(T_n) \geq 2$. Since $L + c = L > 0$, then

$$\begin{aligned} h_c(0, 0) &\geq \int_0^{+\infty} (L(\phi_t(v)) + c)\, dt \\ &\geq \limsup_{T_n} \int_{T_{n_0}}^{T_n} \sqrt{\tfrac{2}{r(t)}}\; |\dot x_v(t)|\; dt \\ &\geq \limsup_{T_n} \int_{T_{n_0}}^{T_n} \sqrt{\tfrac{2}{r(t)}}\; |\dot r|\; dt \\ &\geq \limsup_{T_n} \int_{T_{n_0}}^{T_n} \sqrt{\tfrac{2}{r}}\; \dot r\; dt \\ &= \limsup_{n} \int_{T_{n_0}}^{r(T_n)} \sqrt{\tfrac{2}{r}}\; dr = +\infty. \end{aligned}$$

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CIMAT, A.P. 402, 36000, Guanajuato. Gto., México.
E-mail address: `gonzalo@fractal.cimat.mx`

CIMAT, A.P. 402, 36000, Guanajuato. Gto., México.
E-mail address: `renato@fractal.cimat.mx`

CIMAT, A.P. 402, 36000, Guanajuato. Gto., México.
E-mail address: `paternain@fractal.cimat.mx`

Centro de Matemática, Facultad de Ciencias, Iguá 4225, 11400 Montevideo, Uruguay.
E-mail address: `miguel@cmat.edu.uy`