COVARIANCE-PARAMETER LEVY PROCESSES IN THE SPACE OF TRACE-CLASS OPERATORS

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Abstract

The paper deals with Lévy processes with values in $L_1(H)$, the Banach space of trace-class operators in a Hilbert space H. Lévy processes with values and parameter in a cone K of $L_1(H)$ are defined and several properties are established. A family of $L_1(H)$-valued Lévy processes is obtained via the subordination of K- parameter, $L_1(H)$-valued Lévy processes.

Key words: Random covariance operator, subordination, multiparameter process.

AMS 1991 Subject Classification 60H25, 60G51.

1 Introduction

Subordination of a finite dimensional Lévy process $\{X_t : t \geq 0\}$ by an independent one dimensional positive Lévy process $\{Z_t : t \geq 0\}$ is a largely studied area, see for example Bochner (1955), Sato (1999, 2001). The subordinated process $Y_t = X_{Z_t}, t \geq 0$, is still a Lévy process, whose characteristic triplet can be obtained in terms of the characteristic triplets of the processes X and Z (Sato (1999, Th. 30.1).

Real valued processes with parameter in a finite dimensional cone K are considered by Bochner (1955) as multidimensional time variable. The Gaussian case has been studied by Lévy (1948), Chentsov (1957), McKean (1963), Orey and Pruitt (1973) and Khoshnevisan and Shi (1999). In general, subordination in higher dimensions is not done in a unique manner. Barndorff-Nielsen, Pedersen and Sato (2001) introduce K-parameter Lévy processes with values in a finite dimensional space and study a type of subordination by K-increasing Lévy process when $K = \mathbb{R}^n_+$. Other type of subordination of K-parameter convolution semigroups and K-parameter multivariate Lévy processes are studied in Pedersen and Sato (2001) for a proper cone K of $\mathbb{R}^n$. In the last two named papers the authors are able to identify the characteristic triplet of the subordinated process when K is a cone with a basis.

The present paper deals with Lévy processes, subordination and trace-class operators. The latter operators are important in probability theory and stochastic analysis in infinite dimensional spaces, since positive trace-class self-adjoint operators are the "*covariance operators*" of Gaussian measures in Banach spaces (Kuo (1975)). Therefore, it is natural to study random covariance operators and their corresponding Lévy process.

Let $L_1(H)$ be the Banach space of trace-class operators in a separable Hilbert space H and let K be a cone of covariance operators in $L_1(H)$. In this paper we study K-increasing Lévy processes $\{Z_t : t \geq 0\}$ and introduce the K-parameter Lévy processes $\{X(S); S \in K\}$ with values in $L_1(H)$. The subordinated $L_1(H)$-valued Lévy process $Y_t = X(Z_t),\ t \geq 0$, is constructed, identifying its characteristic triplet in terms of the corresponding characteristic triples of X and Z, in the case when the cone K has a basis. Our corresponding finite dimensional consequence do not recover the results of Barndorff-Nielsen, Pedersen and Sato (2001) and Pedersen and Sato (2001), but they are rather of different nature, giving new results for the finite dimensional case that includes symmetric real matrices. Our approach is such that the parameter set and the state of the process belong to the same space. This provides another approach to subordination in the finite dimensional case.

The paper is organized as follows. Section 2 recalls several facts and introduces notation about $L_1(H)$-valued Lévy processes and trace class operators. Section 3 studies the Lévy-Khintchine representation of subordinators taking values in a cone K of covariance operators. Examples and a detailed study of their corresponding Laplace transform is also presented. Section 4 introduces the K-parameter $L_1(H)$-valued Lévy processes and derives a representation theorem for them as well as some useful tail and moments estimates. Section 5 derives the characteristic triplet of a trace-class-valued Lévy process obtained via the subordination of a covariance-parameter $L_1(H)$-valued Lévy process by a covariance-valued subordinator. For the sake of completeness we include the corresponding result for the case of symmetric real matrices with basis.

2 Preliminaries and notation

In this section we recall well known facts and assemble some basic notation about trace-class operators in a separable Hilbert space as well as Lévy processes taking values in the Banach space of trace-class operators.

2.1 Trace-class operators

Throughout this work, $(L_1(H), \|\cdot\|_1)$ will denote the separable Banach space of trace-class operators of a separable Hilbert space H whose inner product is denoted by $< \cdot, \cdot >$. We recall that a compact operator S of H is said to be of trace-class if $\sum_{j=1}^{\infty} s_j < \infty$, where $s_j,\ j \geq 1$, are the eigenvalues of the positive compact operator $|S|$. The norm of a trace-class operator S is defined by $\|S\|_1 = \sum_{j=1}^{\infty} s_j$. The trace of S in $L_1(H)$ is defined by $tr\,(S) = \sum_{j=1}^{\infty} \langle Se_j\,,\, e_j\rangle$, where $\{e_j\}$ is a complete orthonormal set of H. The dual space of $L_1(H)$ will be denoted by $L_1^*(H)$ and $L_1^+(H)$ will denote the cone of positive trace-class (covariance) operators in $L_1(H)$. An important fact that will be used often is that $tr(S) = \|S\|_1$ when S belongs to $L_1^+(H)$, because the eigenvalues of a positive operator are nonnegative. It is well known (see for example Reed and Simon (1980, Th. VI.26)) that $L_1^*(H) = L(H)$, where $L(H)$ is the Banach space of bounded linear operators in H. More precisely, for every continuous linear functional $f \in L_1^*(H)$ there exists V in $L\,(H)$ such that

$$f(S) = tr(VS) \qquad \text{for every } S \in L_1\,(H)\,. \tag{1}$$

Often, f in (1) shall be denoted by f_V indicating that $f_V(S) = tr(VS)$. Linear functionals in (1) take values in the complex numbers $\mathbb{C}$, therefore V is no necessarily self-adjoint. They will be $\mathbb{R}$-valued linear functionals when V and S be self-adjoint operators. We recall that f is positive linear functional (with respect to $L_1^+(H)$) when V and S are positive operators.

When dealing with characteristic functionals we will assume in (1) that both, V and S are self-adjoint operators, unless otherwise stated.

A nonempty closed convex set K of a Banach space B is said to be a *cone* if $\lambda \geq 0$ and $x \in K$ imply $\lambda x \in K$. A cone K is said to be *generating* if $B = K - K$, that is, every $x \in B$ can be written as $x = x_1 - x_2$ for $x_1 \in K$ and $x_2 \in K$. A cone K is called a *proper cone* if $x = 0$ whenever x and $-x$ are in K. Let K be a cone of B and let B^* be the topological dual of B. The *dual cone* K^* of K is defined as the set of *positive linear functional* (with respect to K)

$$K^* = \{f \in B^* : f(s) \geq 0 \text{ for every } s \in K\}\,.$$

We observe that $L_1^+(H)$ is a proper generating cone for $L_1(H)$ (Reed and Simon (1980, p. 212)).

2.2 Trace-class-valued Lévy processes

A Lévy process $\{X_t : t \geq 0\}$ with values in $L_1(H)$ is a stochastically continuous process such that $X_0 = 0$, it has independent and stationary increments and it is a right-continuous with left-limit (càdlàg) process. From the Lévy-Khintchine representation of the characteristic functional of a separable Banach space valued Lévy processes (see Gihman and Skorohod (1975, Th. IV.5)) we obtain the corresponding one for trace-class valued Lévy processes.

Theorem 1 *Let $\{X_t : t \geq 0\}$ be a Lévy process with values in $L_1(H)$. Then, its characteristic functional $Ee^{if(X_t)}$ has the form*

$$exp\left\{t\left(-\frac{1}{2}A(f,f) + if(\gamma) + \int_{L_1(H)} \left[e^{if(x)} - 1 - if(x)1_{\{\|x\|_1 \leq 1\}}(x)\right]\nu(dx)\right)\right\}, \tag{2}$$

for $f \in L(H)$, where γ is in $L_1(H)$, $A(f,f)$ is a nonnegative quadratic functional in f, $\nu(A)$ is a finite measure in $A \in \mathcal{B}_\varepsilon$ for all $\varepsilon > 0$, such that for any continuous linear functional f

$$\int_{\|x\|_1 \leq 1} f^2(x)\, \nu(dx) < \infty. \tag{3}$$

Also, from Theorem IV.8 of Gihman and Skorohod (1975) we obtain the next result for bounded variation trace-class-valued Lévy processes.

Proposition 2 *Let $\{Z_t : t \geq 0\}$ be a $L_1(H)$-valued Lévy process. Then, $\{Z_t\}$ has bounded variation on each interval $[0,t]$, with probability 1, if and only if, it has characteristic functional is given by*

$$Ee^{if(Z_t)} = exp\left\{ t\left(\int_{L_1(H)} \left(e^{if(x)} - 1\right) \nu(dx) + if(\gamma) \right) \right\} \qquad f \in L_1^*(H),$$

where $\gamma \in L_1(H)$ and the Lévy measure ν satisfies

$$\int_{0<\|x\|_1 \leq 1} \|x\|_1\, \nu(dx) < \infty. \tag{4}$$

The triplet of parameters (A, ν, γ) in Theorem 1 is called generating triplet of X and it is unique. We recall from Araujo and Giné (1980) and Linde (1986) that a σ-finite measure ν on a separable Banach space B with $\nu(\{0\}) = 0$ is called *Lévy measure* if the function

$$f \longmapsto \exp\{\int \left[e^{if(x)} - 1 - if(x)1_{\{\|x\| \leq 1\}}(x)\right] \nu(dx)\} \qquad f \in B^*, \tag{5}$$

is characteristic functional of a probability measure on B.

In general, identification of Lévy measures in Banach spaces is not an easy problem. It is known (Linde (1986)) that a weak limit of a sequence of infinitely divisible probability measures in a separable Banach space is infinitely divisible, however, the corresponding sequence of generating triplets does not converge necessarily to the generating triplet of the weak limit. In this direction the following two results will be useful in the sequel. The first is a special case of Linde (1986, Prop. 5.7.4) and the second one follows straightforward.

Theorem 3 *Let μ_n, $n \geq 1$, and μ_0 be infinitely divisible probability measures on $L_1(H)$ with generating triplets (A_n, ν_n, γ_n) and (A_0, ν_0, γ_0) respectively. Suppose that μ_n converges weakly to μ_0. Then we have the following*

a) If $\nu_0(\{\|x\|_1 = 1\}) = 0$ then $\gamma_n \to \gamma_0$.

b) Let P_n and P_0 be the probability measures corresponding to ν_n and ν_0 whose characteristic functional are given by (5). Then P_n converge weakly to P_0 provided $\gamma_n \to \gamma_0$ and

$$\lim_{\delta \downarrow 0} \liminf_{n\to\infty} \int_{\|x\|_1 \leq \delta} |f(x)|^2\, \nu_n(dx) = 0 \qquad f \in L_1^*(H).$$

Proposition 4 *Let $g(f,x) = e^{if(x)} - 1 - if(x)1_{\{\|x\|_1 \le 1\}}(x)$ for $f \in L(H)$. Let ν be a measure on $L_1(H)$ satisfying $\nu(\|x\|_1 > 1) < \infty$ and*
$\int_{\|x\|_1 \le 1} |f(x)|^2 \nu(dx) < \infty$ for all continuous linear functional f in $L(H)$. Then
a) $\int_{L_1(H)} |g(f,x)| \nu(dx) < \infty$.
b) If ν_n is a sequence of measures such that $\nu_n \uparrow \nu$ then

$$\int_{L_1(H)} g(f,x)\nu_n(dx) \to \int_{L_1(H)} g(f,x)\nu(dx).$$

Proof. See the appendix. ■

Finally, we recall the following general result proved in Pérez-Abreu and Rocha-Arteaga (2002), which identifies generating triplets of Lévy processes obtained as linear transforms of Lévy processes.

Proposition 5 *Let $\{X_t : t \ge 0\}$ be a B-valued Lévy process with generating triplet (A, ν, γ). Let B_1 be a Banach space such that the map $V \longmapsto f_V$ is an isomorphism of B_1 onto B^*. Let $T : B \to B$ and let $T^{'} : B_1 \to B_1$ be continuous linear transformations with the property*

$$f_V(TS) = f_{T'V}(S), \tag{6}$$

for every $V \in B_1$ and $S \in B$. Then $\{T(X_t) : t \ge 0\}$ is a B-valued Lévy process with generating triplet (A_T, ν_T, γ_T) given by

$$\begin{gathered} A_T = TAT^{'}, \\ \nu_T = (\nu T^{-1}) \mid_{B\backslash\{0\}}, \\ \gamma_T = T\gamma + \int Tx \left[1_{\{\|Tx\| \le 1\}}(Tx) - 1_{\{\|x\| \le 1\}}(x)\right] \nu(dx). \end{gathered} \tag{7}$$

Here $\nu T^{-1}(C) = \nu(\{x : Tx \in C\})$ and $(\nu T^{-1}) \mid_{B\backslash\{0\}}$ denotes the restriction of the measure νT^{-1} to $B\backslash\{0\}$ and the last integral is a Bochner integral.

3 Subordinators of covariance operator type

In this section we study trace-class valued subordinators, i.e., Lévy processes taking values in a cone of covariance operators in $L_1^+(H)$. A class of general examples is given as well as a detailed study of the Laplace transform of this class of subordinators.

3.1 Trace-class increasing Lévy processes

A proper cone K of $L_1(H)$ introduces a partial order on $L_1(H)$ by defining $S_1 \leq_K S_2$ whenever $S_2 - S_1 \in K$ for any S_1, $S_2 \in L_1(H)$. This allows us to define the notions of increasingness and decreasingness in $L_1(H)$. Let $\{S_n\}$ be a sequence in $L_1(H)$. If $S_n \leq_K S_{n+1}$ for each n, the sequence is called K-*increasing*. If $S_{n+1} \leq_K S_n$, for each n, the sequence is called K-*decreasing*. A function $f : [0, \infty) \to L_1(H)$ is called K-*increasing* if $f(t_1) \leq_K f(t_2)$ for $t_1 \leq t_2$; and it is called K-*decreasing* if $f(t_2) \leq_K f(t_1)$ for $t_1 \leq t_2$.

The proof of the following proposition is standard.

Proposition 6 *Let $\{Z_t : t \geq 0\}$ be a Lévy process in $L_1(H)$. Let K be a proper cone in $L_1(H)$. Then the following are equivalent.*
a) For any fixed $t \geq 0$, $Z_t \in K$ almost surely.
b) Almost surely, $Z_t(\omega)$ is K-increasing in t.

A very useful tool in the study of one dimensional subordinators is the special form that takes their characteristic and Laplace transforms; see Bertoin (1996), Sato (1999). The following result extends to cones of covariance operators in $L_1^+(H)$ a result by Skorohod (1991, Th. 3.21), who derives the characteristic function of a Lévy process taking values in a cone of a (finite dimensional) Euclidean space. The first part of our proof follows some of the ideas in the Corollary to Theorem IV.7 in Gihmann and Skorohod (1975) (who deal with independent increments in a special cone of a Banach space). We make the proof shorter, more precise and taking advantage of the linearity of the trace norm $\|\cdot\|_1$ in the cone $L_1^+(H)$.

Proposition 7 *Let K be a proper cone of $L_1^+(H)$ such that the identity operator $I \in K^*$. An $L_1(H)$-valued Lévy process $\{Z_t : t \geq 0\}$ with generating triplet (A, ν, γ) given by (2) is K-increasing if and only if, its characteristic functional has the form*

$$Ee^{itr(VZ_t)} = exp\left\{t\left(\int_K \left(e^{itr(VS)} - 1\right)\nu(dS) + itr(V\gamma_0)\right)\right\} \qquad V \in L(H), \tag{8}$$

where the drift $\gamma_0 := \gamma - \int_{0<\|S\|_1\leq 1} S\nu\,(dS)$ belongs to K (Bochner integral), the Lévy measure ν is concentrated on K and satisfies

$$\int_{0<\|S\|_1\leq 1} \|S\|_1\,\nu\,(dS) < \infty. \tag{9}$$

Moreover, its Laplace transform is given by

$$Ee^{-tr(VZ_t)} = exp\left\{t\left(\int_K \left(e^{-tr(VS)} - 1\right)\nu(dS) - tr(V\gamma_0)\right)\right\} \qquad V \in K^*. \tag{10}$$

Proof. By using a consequence of the Hahn-Banach Theorem we can select a sequence of functionals f_k such that $K = \cap_{k=1}^{\infty} \{S : f_k(S) \geq 0\}$. Assume the K-increasingness of the process almost surely. Then, each one dimensional process $\{f_k(Z_t)\}$ has only nonnegative jumps since it is nonnegative. If A is contained in $\cup_{k=1}^{\infty} \{S : f_k(S) < 0\}$ then $\nu(A) = 0$. Thus, ν is concentrated in K.

Let $\triangle_\varepsilon = K \cap \{x : \|x\|_1 > \varepsilon\}$ which has positive distance from 0 and let $Z_t^{\triangle_\varepsilon} = \sum_{s<t} (Z_s - Z_{s-}) 1_{\triangle_\varepsilon}(Z_s - Z_{s-})$ which is the finite sum of jumps of the process for each $\varepsilon > 0$. Hence $\left\{Z_t^{\triangle_\varepsilon}\right\}$ belongs to K almost surely. Since for each k $f_k\left(Z_t - Z_t^{\triangle_\varepsilon}\right) \geq 0$ almost surely, then $\left\{Z_t - Z_t^{\triangle_\varepsilon}\right\}$ belongs to K almost surely. For the positive linear functional $tr(I\cdot) \in K^*$ we have that $\lim_{\varepsilon\downarrow 0}\left(tr(Z_t) - tr(Z_t^{\triangle_\varepsilon})\right)$ exists since $tr(Z_t^{\triangle_\varepsilon})$ is increasing as function of ε and is bounded by $tr(Z_t)$. Hence $tr(Z_t^{\triangle_{\varepsilon_{n+k}}} - Z_t^{\triangle_{\varepsilon_n}}) = \left\|Z_t^{\triangle_{\varepsilon_{n+k}}} - Z_t^{\triangle_{\varepsilon_n}}\right\|_1 \to 0$ as $n, k \to \infty$ for any subsequence $\varepsilon_n \downarrow 0$. Let $Z_t^0 \in L_1(H)$ the strong limit of $Z_t^{\triangle_\varepsilon}$. Therefore the process $\left\{Z_t - Z_t^0\right\}$ is continuous almost surely and from (2)

$$Ee^{itr(V(Z_t - Z_t^0))} = exp\left\{t\left(itr(V\gamma) - \frac{1}{2}A(tr(V\cdot), tr(V\cdot))\right)\right\} \quad V \in L(H). \tag{11}$$

Decompose $tr(V\cdot) = tr(V^+\cdot) - tr(V^-\cdot)$ where $V = V^+ - V^-$ and V^+ and $V^-$are positive linear operators in K^*. Notice that the process $\left\{tr(V^+(Z_t - Z_t^0))\right\}$ is nonnegative and continuous almost surely. Then $var\left(tr(V^+(Z_t - Z_t^0))\right) = tA(tr(V^+\cdot), tr(V^+\cdot)) = 0$ and

$$tA(tr(V\cdot), tr(V\cdot)) = var\left[tr(V^+(Z_t - Z_t^0)) + tr(V^-(Z_t - Z_t^0))\right] = 0$$

since $\left\{tr(V^+((Z_t - Z_t^0))\right\}$ and $\left\{tr(V^-(Z_t - Z_t^0))\right\}$ are constants almost surely. This shows that the covariance operator $A = 0$. Next, let $\gamma_0 \in K$ be such that

$$tr(tV\gamma_0) = tr(V(Z_t - Z_t^0)). \tag{12}$$

Observe that,

$$Ee^{itr(VZ_t^0)} = \lim_{\varepsilon\downarrow 0} Ee^{itr(VZ_t^{\triangle_\varepsilon})} = exp\left\{t\int_K \left(e^{itr(VS)} - 1\right)\nu(dS)\right\}, \tag{13}$$

then from (12) and (13) we get (8). Since

$$Ee^{itr(VZ_t^0)} = \lim_{\varepsilon\downarrow 0} exp\left\{t\left(\int_{\varepsilon<\|S\|_1\leq 1}\left[e^{itr(VS)} - 1 - itr(VS)\right]\nu(dS)\right.\right.$$
$$\left.\left.+\int_{\|S\|_1>1}\left(e^{itr(VS)} - 1\right)\nu(dS) + i\int_{\varepsilon<\|S\|_1\leq 1} tr(VS)\nu(dS)\right)\right\}$$

from Theorem 1 and (13) we have the convergence of $exp\left\{it\int_{\varepsilon<\|S\|_1\leq 1} tr(VS)\nu\,(dS)\right\}$ as $\varepsilon\downarrow 0$, which is equivalent to the convergence of the degenerate distribution at the point $\int_{\varepsilon<\|S\|_1\leq 1} tr(VS)\nu\,(dS)$ to the degenerate distribution at the point $\int_{0<\|S\|_1\leq 1} tr(VS)\nu\,(dS)$ and hence (9) follows from (1) choosing the positive linear functional $f_I(\cdot)=tr(I\cdot)$. From (2), (11), (12) and (13) we get $tr(V\gamma_0)=tr(V\gamma)-\int_{0<\|S\|_1\leq 1} tr(VS)\nu(dS)$. Now that γ_0 equals to $\gamma-\int_{0<\|S\|_1\leq 1} S\nu(dS)$ follows from (9).

Conversely, assume that the process has the characteristic functional (8). In view of (2) $tr(V\gamma_0)=tr(V\gamma)-\int_{0<\|S\|_1\leq 1} tr(VS)\nu(dS)$. We have used here that $A=0$ and (9). Since $\gamma_0\in K$ we show that $Z_t-t\gamma_0\in K$ almost surely. Let $J_t=Z_t-t\gamma_0$. Notice that J_t and Z_t have the same jumps and $J_t^{\triangle_\varepsilon}=\sum_{s<t}(Z_s-Z_{s-})\,1_{\triangle_\varepsilon}(Z_s-Z_{s-})$. Since ν is concentrated on K the jump measure of Z_t is concentrated on K. Therefore $J_t^{\triangle_\varepsilon}$ is concentrated on K for each $\varepsilon>0$. Then the Lévy process $\{J_t^{\triangle_\varepsilon}\}$ is K-increasing. Note that $J_t^{\triangle_{\varepsilon_2}}-J_t^{\triangle_{\varepsilon_1}}\in K$ for $\varepsilon_2<\varepsilon_1$. Hence $\lim_{\varepsilon\downarrow 0} f_k(J_t^{\triangle_\varepsilon})$ exists for each k. Since

$$Ee^{if_k(J_t-J_t^{\triangle_\varepsilon})}=exp\left\{t\left(\int_{K\cap\{x:\|x\|\leq\varepsilon\}}\left(e^{if_k(x)}-1\right)\nu(dx)\right)\right\}$$

tends to 1 as $\varepsilon\downarrow 0$ we have that $f_k(J_t)=\lim_{\varepsilon\downarrow 0} f_k(J_t^{\triangle_\varepsilon})$. Since $f_k\left(J_t^{\triangle_\varepsilon}\right)\geq 0$ for all k then $f_k(J_t-J_t^{\triangle_\varepsilon})\geq 0$. Hence $J_t-J_t^{\triangle_\varepsilon}$ and $J_t^{\triangle_\varepsilon}$ are in K. Then $J_t\in K$. Now, K-increasingness of $\{Z_t\}$ follows from Proposition 6. ■

Corollary 8 *The process $\{tr(Z_t)\}$ is an $\mathbb{R}$-valued subordinator with*

$$Ee^{-utr(Z_t)}=exp\left\{t\left(\int_K\left(e^{-utr(S)}-1\right)\nu(dS)-utr(\gamma_0)\right)\right\}\qquad u\in\mathbb{R}^+.$$

Proof. Let f_I be the positive linear functional in (1) corresponding to the identity operator I. Then (10) evaluated in the positive linear functional uf_I gives the result. ■

The following are two important examples of covariance operators that are subordinators.

Example 9 *(Stable positive covariance subordinator) Let S be and $\alpha/2$-stable random operator taking values in $L_1^+(H)$. Using (10) and the representation of the characteristic function of S after Remark of Proposition 6.3.3 in Linde (1986) we get*

$$Ee^{-tr(VS)}=\exp\left\{\frac{\Gamma(-\alpha/2)}{C_{\alpha/2}}\int_{\mathcal{S}_1^+(H)}\{tr(V\Theta)\}^{\alpha/2}\,\sigma(d\Theta)\right\}\quad V\in L^+(H),\tag{14}$$

where σ is the spectral measure of S, $\mathcal{S}_1^+(H)$ denotes the intersection of $L_1^+(H)$ and the unit sphere of $L_1(H)$ and $C_{\alpha/2}^{-1}$ is the constant appearing in such a representation. The Lévy measure ν is given *by* $\nu(d\Sigma) = C_{\alpha/2}^{-1} \frac{dt}{t^{1+\alpha/2}} \sigma(d\Theta)$ *where* $\Sigma = t\Theta$ *with* $0 < t < \infty$ *and* $\Theta \in \mathcal{S}_1^+(H)$. *The* $L_1(H)$-*valued Lévy process* S_t *such that* S_1 *has the law of* S *is called the* $\alpha/2$*-stable covariance operator subordinator.*

Example 10 *(Inverse Gaussian covariance subordinator). Let* S *be an* $\alpha/2$*-stable random operator taking values in* $L_1^+(H)$ *and let* p *be a positive linear functional of* $L_1(H)$. *Let us define the probability distribution on* $L_1^+(H)$ *by*

$$F(p; d\Theta) = \frac{e^{-p(\Theta)}}{Ee^{-p(S)}} F_S(d\Theta),$$

where F_S *is the distribution of* S. *Let* R *be a random operator having the probability distribution* $F(p; d\Theta)$. *Using again the representation of the characteristic function of* S, *(10) and (14) we obtain*

$$Ee^{-tr(VR)} = \exp\left\{ \int_{S_1^+(H)} \left(e^{-tr(V\Sigma)} - 1 \right) \nu(d\Sigma) \right\} \qquad V \in L^+(H),$$

where $\nu(d\Sigma) = \frac{C_{\alpha/2}^{-1}}{t^{1+\alpha/2}} e^{-tp(\Theta)} \sigma(d\Theta)$ *and* $\Sigma = t\Theta$. *Thus,* R *is an infinitely divisible random operator taking values in* $L_1^+(H)$ *with Lévy measure* ν. *This extends, to the infinite dimensional case, the concept of inverse Gaussian matrix introduced in Barndorff-Nielsen and Pérez-Abreu (2002). The* $L_1(H)$*-valued Lévy process* S_t *such that* S_1 *has the law of* R *is called the inverse gaussian trace-class subordinator.*

3.2 A class of examples

A natural class of infinitely divisible $L_1^+(H)$-valued random operators S can be obtained via a general method described in this section. Let $\{Z_j(t)\}$ $j = 1, 2, ...,$ be independent subordinators in $\mathbb{R}$ satisfying $\sum_{j=1}^{\infty} \Phi_j(\lambda_j) < \infty$, where λ_j is a positive number and Φ_j is the Laplace exponent of Z_j, i.e.

$$\Phi_j(u) = \int_{(0,\infty)} \left(1 - e^{-ux}\right) \nu_j(dx) + \gamma_{0,j} u, \quad u \in \mathbb{R}^+. \tag{15}$$

Then $R_t = \sum_{j=1}^{\infty} \lambda_j Z_j(t)$ is well defined as an infinitely divisible one-dimensional positive random variable for each $t \geq 0$.

Theorem 11 *Let* $\{Z_j\}$, *for* $j = 1, 2, ...,$ *be infinitely divisible subordinators in* $\mathbb{R}^+$ *which are independent. Define*

$$S = \sum_{j=1}^{\infty} \lambda_j Z_j(1) e_j \otimes e_j, \tag{16}$$

where $\{e_j\}$ is a complete orthonormal set of H, the linear operator $e_j \otimes e_j$ is defined by $\langle \cdot, e_j \rangle e_j$ and $\{\lambda_j\}$ is a sequence of positive real numbers. Assume that

$$\sum_{j=1}^{\infty} \Phi_j(\lambda_j) < \infty, \tag{17}$$

where Φ_j is the Laplace exponent for Z_j. Then S is an infinitely divisible random operator taking values in $L_1^+(H)$ and therefore

$$S_t = \sum_{j=1}^{\infty} \lambda_j Z_j(t) e_j \otimes e_j, \qquad t \geq 0$$

is a covariance subordinator such that S_1 has the law of S.

Proof. Let $Z_{n,j}$ be positive random variable with characteristic function $\hat{\mu}_{Z_j}^{1/n}$ for $j \geq 1$ and $n \geq 1$. Define $S_n = \sum_{j=1}^{\infty} \lambda_j Z_{n,j} e_j \otimes e_j$, for $n \geq 1$. From the dominated convergence theorem we get, for each n,

$$Ee^{-tr(S_n)} = Ee^{-\sum_{j=1}^{\infty} \lambda_j Z_{n,j}} = e^{-\frac{1}{n}\sum_{j=1}^{\infty} \Phi_j(\lambda_j)} > 0.$$

This proves that S_n is a $L_1^+(H)$-valued random operator. Next, for every self-adjoint V in $L(H)$

$$Ee^{itr(VS)} = \prod_{j=1}^{\infty} Ee^{i\lambda_j v_j Z_j} = \left\{ \prod_{j=1}^{\infty} \hat{\mu}_{Z_j}^{1/n}(\lambda_j v_j) \right\}^n, \qquad n \geq 1,$$

where the real numbers $v_j = \langle Ve_j, e_j \rangle$, for $j = 1, 2, ...$, satisfy $|tr(VS)| = \left| \sum_{j=1}^{\infty} \lambda_j v_j Z_j \right| < \infty$ almost surely. Then for each $n \geq 1$,

$$Ee^{itr(VS_n)} = \prod_{j=1}^{\infty} Ee^{i\lambda_j v_j Z_{n,j}} = \prod_{j=1}^{\infty} \hat{\mu}_{Z_j}^{1/n}(\lambda_j v_j) = \left\{ Ee^{itr(VS)} \right\}^{1/n}.$$

■

Remark 12 *From (10), one obtains the Laplace transform of $tr(VS)$. Theorem 11 implies that $tr(VS) = \sum_{j=1}^{\infty} \lambda_j v_j Z_j$ is an infinitely divisible random variable on $\mathbb{R}^+$, where $v_j = \langle Ve_j, e_j \rangle$. Moreover, if Z_j has Lévy density $l_j(x)$, the Lévy measure of $tr(VS)$ is given by*

$$\nu_{tr(VS)}(dx) = \sum_{j=1}^{\infty} (v_j \lambda_j)^{-1} l_j\left((v_j \lambda_j)^{-1} x \right) dx.$$

Therefore the Lévy measures of the one dimensional distributions of S have the form

$$\nu \circ tr^{-1}(V\cdot) = \sum_{j=1}^{\infty} (v_j\lambda_j)^{-1} l_j \left((v_j\lambda_j)^{-1} x \right) dx,$$

which provides a rich class of one dimensional positive infinitely divisible distributions and their associated subordinators.

We now provide examples of $L_1^+(H)$-random operators of the form (16) satisfying condition (17).

Example 13 *Let us consider the Gamma random variable Z_j with parameters p_j and q_j and probability density function $q_j^{p_j}\Gamma(p_j)x^{p_j-1}e^{-q_jx}$. The random covariance operator S in (16) is called the Gamma random operator. We can get the convergence of (17) by choosing the convergent series $\sum_{j=1}^{\infty} p_j$ and the bounded sequence $\{\log(1+\frac{\lambda_j}{q_j})\}$. As an special case we have $\lambda_j = q_j$ for any j.*

Example 14 *Let $0 < \alpha < 1$. If Z_j is an α-stable random variable with Laplace exponent $\Phi_j(u) = c_j' u^{\alpha} + \gamma_{0j}u$ where γ_{0j} is the drift, $c_j' = c_j\alpha^{-1}\Gamma(1-\alpha)$ and $c_j > 0$ is the constant appearing in the Lévy measure of Z_j (see Sato (1999, Ex. 24.12)). Then (16) is called the α-stable random covariance operator. Choosing $\{\lambda_j\}, \left\{c_j'\right\}$ and $\{\gamma_{0j}\}$ such that $\sum_{j=1}^{\infty} \lambda_j^{\alpha} < \infty$ and the last two sequences be bounded we obtain the convergence of (17).*

Example 15 *Take the random variable Z_j in (16) as the inverse Gaussian distribution with parameters δ_j and γ_j and whose probability density function is $(2\pi)^{-1}\delta_j e^{-\delta_j\gamma_j}x^{-3/2}e^{-(\delta_j^2x^{-1}+\gamma_j^2x)/2}$. Then S is called the inverse Gaussian random covariance operator. Taking $\sum_{j=1}^{\infty}\delta_j\gamma_j < \infty$ and a bounded sequence $\left\{\lambda_j/\gamma_j^2\right\}$ we have (17).*

3.3 The Laplace transform

For special cones of covariance operators, one can deduce a useful property for the Laplace transform of trace-class valued subordinators evaluated in complex linear functionals. Let $\{e_j\}$ be a fix complete orthonormal set in H. Let K_e the cone generated by $e = \{e_j\}$, that is

$$K_e = \left\{ S \in L_1^+(H) : S = \sum_{j=1}^{\infty} s_j e_j \otimes e_j \right\} \tag{18}$$

the subcone of $L_1^+(H)$ of all covariance operators on H having the same system of eigenvectors $\{e_j\}$. Recall that the linear operator $e_j \otimes e_j$ in H is defined by $\langle \cdot, e_j\rangle e_j$. In this section we consider subordinators with values in K_e, that correspond to Lévy processes taking values in covariance operators with random eigenvalues but with the same nonrandom eigenvectors. They include the class of examples given in Section 3.2. The one dimensional result can be seen in Sato (1999, Th. 25.17) and a multivariate (finite dimensional) case is due to Barndorff-Nielsen, Pedersen and Sato (2001) for arbitrary cones of $\mathbb{R}^n$. Our result is only proved for the cone with basis K_e, being the proof not straightforward even in this case.

Proposition 16 *Let $\{Z_t : t \geq 0\}$ be a K_e-valued subordinator with Lévy measure ν and drift γ_0 given by Proposition 7. Then*

$$Ee^{f(Z_t)} = e^{t\Psi_Z(f)} \qquad f \in L(H), \tag{19}$$

where

$$\Psi_Z(f) = \int_{K_e} (e^{f(S)} - 1)\nu(dS) + f(\gamma_0) \tag{20}$$

and $\mathrm{Re}(f(S)) \leq 0$ *for every* $S \in K_e$.
Here $f : L_1(H) \to \mathbb{C}$ *is a complex-valued continuous linear functional.*

For the proof of Proposition 16 we need the following three technical lemmas whose proofs are straightforward (see the appendix).

Lemma 17 *Let U be a self-adjoint operator in $L(H)$. Define the mappings $T'_U : L(H) \to L(H)$ and $T_U : L_1(H) \to L_1(H)$ by*

$$T'_U(V) = \sum_{j=1}^{\infty} e_j \otimes e_1 V e_1 \otimes e_j U, \quad \textit{for } V \in L(H) \textit{ and} \tag{21}$$

$$T_U(S) = \sum_{j=1}^{\infty} e_1 \otimes e_j U S e_j \otimes e_1 \quad \textit{for } S \in L_1(H),$$

respectively. Then T'_U and T_U are linear transformations.

Lemma 18 *Let U be a self-adjoint operator in $L(H)$. Then, the linear transformations in (21) satisfy*

$$f_V(T_U S) = f_{T'_U V}(S), \tag{22}$$

for $V \in L(H)$ and $S \in L_1(H)$.

Lemma 19 *Let $U \in L(H)$. Define the mappings $T'_U : L(H) \to L(H)$ and $T_U : L_1(H) \to L_1(H)$ by*

$$T'_U(V) = T'_{U^1}(V^1) + iT'_{U^2}(V^2), \quad \textit{for } V = V^1 + iV^2 \in L(H) \textit{ and} \tag{23}$$

$$T_U(S) = T_{U^1}(S^1) + iT_{U^2}(S^2) \quad \textit{for } S = S^1 + iS^2 \in L_1(H),$$

where T'_{U^k} and T_{U^k}, $k \in \{1, 2\}$, are the continuous linear transformations in (21). Then, T'_U and T_U are continuous linear transformations.
Note that (23) reduces to (21) when $U = U^1$ is self-adjoint.

Proof of Proposition 16. Let $f \in L_1^*(H)$ be a complex-valued continuous linear functional. From (1) we identify f with $V \in L(H)$ via $f_V(\cdot) = tr(V \cdot)$ where V is no necessarily self-adjoint. Let $V = V^1 + iV^2$ where V^1 and V^2 are self-adjoint operators in $L(H)$. Denote $\operatorname{Re} V = V^1$ and $Im\ V = V^2$. We prove the proposition in three steps.

Step I. Let $V \in L(H)$. For $\operatorname{Re}\, V = V^1$ assume that $f_{V^1}(e_1 \otimes e_1) \in [0,1]$ and $f_{V^1}(e_j \otimes e_j) = 0$ for $j \geq 2$. Since $Z\,(t) = \sum_{j=1}^{\infty} Z_j(t) e_j \otimes e_j \in K_e$ then

$$f_V(Z_t) = \sum_{j=1}^{\infty} Z_j(t) f_V(e_j \otimes e_j).$$

We prove the formula (19) for this particular case of f_V. Let $v_j = f_V(e_j \otimes e_j) \in \mathbb{C}$, $j \geq 1$, and note that $v_j = f_{V^1}(e_j \otimes e_j) + i f_{V^2}(e_j \otimes e_j)$ where $\operatorname{Re} v_1 = f_{V^1}(e_1 \otimes e_1) \in [0,1]$ and $\operatorname{Re} v_j = f_{V^1}(e_j \otimes e_j) = 0$, for $j \geq 2$.

We proceed as in the proof of continuous extensions of the Laplace transform in the finite dimensional case (see Sato (1999, Th. 25.17)), so we only sketch the proof. Consider v_1 as a variable and let $A = \{v_1 \in \mathbb{C} : \operatorname{Re} v_1 \in [0,1]\}$ and define $\Phi_1(v_1) = Ee^{f_V(Z_t)}$ and
$\Phi_2(v_1) = \exp\left\{ t \left(\int_{K_e} (e^{f_V(s)} - 1)\nu(ds) + f_V(\gamma_0) \right) \right\}$ for $v_1 \in A$. One can prove that $\Phi_1(v_1)$ and $\Phi_2(v_1)$ are continuous in A and analytic in the interior of A. Thus, by the Schwarz 's principle of reflection, $\Phi_1(v_1) - \Phi_2(v_1)$ extends to an analytic function on the domain $\{v_1 \in \mathbb{C} : \operatorname{Re} v_1 \in (-1,1)\}$. When $\operatorname{Re} v_1 = 0$, we have $\Phi_1(v_1) - \Phi_2(v_1) = 0$ which corresponds to the Lévy-Khintchine representation of the subordinator Z_t. By the uniqueness Theorem for analytic functions $\Phi_1(v_1) - \Phi_2(v_1) = 0$ in this domain. Then we get (19).

Step II. Let $T^{'} : L(H) \to L(H)$ and $T : L_1(H) \to L_1(H)$ be continuous linear transformations as in Proposition 5 satisfying (6). Then, the process $\{Y_t : t \geq 0\}$ defined by $T(Z_t)$ is a Lévy process with generating triplet (A_T, ν_T, γ_T) given by (7), where $\gamma_T = T\gamma$ (*i.e.* the integral in (7) does not appear).

Given $U \in L(H)$ with $\operatorname{Re} f_U(S) \leq 0$ for every $S \in K_e$, assume that there exists $V \in L(H)$ such that

$$T^{'} V = U. \tag{24}$$

From (6) and (24) we obtain $\operatorname{Re} f_V(TS) = \operatorname{Re} f_{T^{'}V}(S) = \operatorname{Re} f_U(S) \leq 0$ for every $S \in K_e$. Therefore we can define

$$\Psi_T(f_V) = \int_{K_e} (e^{f_V(S)} - 1)\nu_T(dS) + f_V(\gamma_T). \tag{25}$$

We claim that if

$$Ee^{f_V(Z_t)} = e^{t\Psi_T(f_V)} \tag{26}$$

then

$$Ee^{f_U(Z_t)} = e^{t\Psi_Z(f_U)}. \tag{27}$$

In fact, from (6) and (24) we have that $f_V(Y_t) = f_V(TZ_t) = f_{T'V}(Z_t) = f_U(Z_t)$ and then, by Proposition 5,

$$\Psi_T(f_V) = \int_{K_e} (e^{f_V(TS)} - 1)\nu_T(dS) + f_{T'V}(\gamma) = \Psi_Z(f_U).$$

This proves that (26) and (27) coincide.

Step III. Let $U = U^1 + iU^2$ be a linear operator in $L(H)$ such that $\operatorname{Re} f_U(S) \leq 0$ for every $S \in K_e$. Consider the continuous linear transformations of Lemma 17, $T'_{U^k} : L(H) \to L(H)$ and $T_{U^k} : L_1(H) \to L_1(H)$, $k \in \{1, 2\}$, defined by (21). Also, consider the linear transformations of Lemma 19, $T'_U : L(H) \to L(H)$ and $T_U : L_1(H) \to L_1(H)$ given by (23). Note that $T'_U(e_1 \otimes e_1) = T'_{U^1}(e_1 \otimes e_1) = U^1$ since $e_1 \otimes e_1$ is self-adjoint operator. Let us take a linear operator V in $L(H)$ defined by $V = e_1 \otimes e_1 + iV^2$ with the property

$$T'_U(V) = U \tag{28}$$

(we can choose $V^2 = e_1 \otimes e_1$ for instance).

Notice that T'_U and T_U satisfy condition (22) in Lemma 18. Then the process defined by $Y_t = T_U(Z_t)$ is a Lévy process. Next, since Z_t lies in the cone K_e, we have that $f_{e_1 \otimes e_1}(Y_t) = f_{T'_U e_1 \otimes e_1}(Z_t) = f_{U^1}(Z_t) \leq 0$. Then, equation (25) is definable for

$$\Psi_{T_U}(f_{e_1 \otimes e_1}) = \int_{K_e} (e^{f_{e_1 \otimes e_1}(S)} - 1)\nu_{T_U}(dS) + f_{e_1 \otimes e_1}(\gamma_{T_U})$$

and the linear operator V whose $\operatorname{Re} V = e_1 \otimes e_1$ satisfies conditions of Step I. This yields

$$Ee^{f_V(Y_t)} = e^{t\Psi_{T_U}(f_V)}. \tag{29}$$

From (23) and (28) we get $\operatorname{Re} f_V(Y_t) = f_{e_1 \otimes e_1}(Y_t) \leq 0$ (recall that $T_U(Z_t)$ does not contain complex part). Next, we apply Step II to the operators U and V which satisfy (28) and (29) to get (19). This ends the proof. ■

4 Covariance-parameter Lévy processes with values in $L_1(H)$

4.1 Definition and a representation theorem

Lévy processes with parameter in a proper cone K of $L_1^+(H)$ and taking values in $L_1(H)$ are considered in this section. The concept of Lévy process is extended to a process $\{X(S) : S \in K\}$ with a proper cone as the time parameter set. The case of a cone $K \subset \mathbb{R}^n$ and $\mathbb{R}^d$-valued random variables is considered in Pedersen and Sato (2001).

Let $f : K \to L_1(H)$ be a mapping. It is said that f is K-*right continuous* at $S \in K$, if for every K-decreasing sequence $\{S_n\}$ in K with $\|S_n - S\|_1 \to 0$, $\|f(S_n) - f(S)\|_1 \to 0$. It is said that f

has K-left limit at $S \in K\backslash\{0\}$, if for every K-increasing sequence $\{S_n\}$ in K with $\|S_n - S\|_1 \to 0$, $\lim_{n\to\infty} f(S_n)$ exists in $L_1(H)$. The function f is càdlàg if it is K-right continuous in K and has K-left limits in $K\backslash\{0\}$. The K-left limit $\lim_{n\to\infty} f(S_n)$ of f at point $S \in K$ depends on the sequence $\{S_n\}$. In fact, it can be shown that a function may has infinitely K-left limits at one point.

We now introduce the concept of covariance-parameter Lévy process with values in trace-class operators.

Definition 20 *A collection of $L_1(H)$-valued random variables $\{X(S) : S \in K\}$ with parameter in a cone K and defined on a probability space $(\Omega, \mathcal{F}, P)$ is called a cone-parameter Lévy process if it satisfies the following:*
a) The random variables $X_{S_n} - X_{S_{n-1}}, X_{S_{n-1}} - X_{S_{n-2}}, ..., X_{S_1} - X_{S_0}$ are independent for any K-increasing sequence $\{S_j\}_{j=1,2,...,n}$ in K.
b) The increments $X_{S_3} - X_{S_2}$ and $X_{S_1} - X_{S_0}$ has the same law whenever $S_3 - S_2 = S_1 - S_0$ for S_3, S_2, S_1 and S_0 in K.
c) $X(0) = 0$ almost surely.
d) It is stochastically continuous, i.e., for every $\epsilon > 0$, $P(\|X(S_n) - X(S)\|_1 > \epsilon) \to 0$ whenever $S \in K$ and $\{S_n\}$ be a sequence such that $\|S_n - S\|_1 \to 0$.
e) It is K-càdlàg, that is, $X_S(\omega)$ is K-càdlàg in S, ω-almost surely.

Examples that show the existence of such processes are the following.

Example 21 *Let $\{Y(t) : t \geq 0\}$ be a Lévy process in $L_1(H)$ and let $f \in K^*$. Then $X(S) = Y(f(S))$ $S \in K$, is a K-parameter Lévy process in $L_1(H)$.*

Example 22 *Let $\{X^j(S) : S \in K\}$ be K-parameter Lévy process in $L_1(H)$, for $j = 1, 2, ..., n$. Then*

$$X(S) = X^1(S) + ... + X^n(S) \qquad S \in K,$$

is a K-parameter Lévy process in $L_1(H)$.

Example 23 *Let K_e be the cone of $L_1(H)$ of positive trace-class operators of the form $S = \sum_{j=1}^{\infty} s_j e_j \otimes e_j$ introduced in (18). Let $\{V(t) : t \geq 0\}$ be a Lévy process on $L_1(H)$ and let $\{c_j\}$ be a positive bounded sequence in $\mathbb{R}$. Define*

$$X(S) = V(\sum_{j=1}^{\infty} c_j s_j) \qquad S \in K_e.$$

Then $\{X(S) : S \in K_e\}$ is a K_e-parameter Lévy process in $L_1(H)$.

Example 24 *Let $\{V^j(t) : t \geq 0\}$, $j = 1, 2, ...,$ be independent Lévy process on $L_1(H)$ which are symmetric and identically distributed. Define*

$$X(S) = \sum_{j=1}^{\infty} V^j(s_j) \qquad S \in K_e.$$

Then $\{X(S) : S \in K_e\}$ is a K_e-parameter Lévy process in $L_1(H)$.

A useful result is the following representation theorem for covariance-parameter trace-class-valued Lévy processes in the cone K_e.

Proposition 25 *Let $\{X(S) : S \in K_e\}$ be a K_e-parameter Lévy process in $L_1(H)$. Let us denote $X^j(t) = X(te_j \otimes e_j)$, for $j = 1, 2, ...,$ where $\{X(te_j \otimes e_j) : t \geq 0\}$, $j = 1, 2, ...,$ is a sequence of Lévy processes on $L_1(H)$. Let $\{U^j(t) : t \geq 0\}$, $j = 1, 2, ...,$ be a sequence of independent Lévy processes such that*

$$\{X^j(t)\} \stackrel{d}{=} \{U^j(t)\}. \tag{30}$$

For every $S = \sum_{j=1}^{\infty} s_j e_j \otimes e_j \in K_e$ define $U(S) = \sum_{j=1}^{\infty} U^j(s_j)$.
Then

$$X(S) \stackrel{d}{=} U(S) \qquad \text{for every } S \in K_e. \tag{31}$$

Moreover

$$Ee^{if(X(S))} = \prod_{j=1}^{\infty} Ee^{if(X^j(s_j))} \qquad \text{for every } f \in L_1^*(H). \tag{32}$$

Proof. Let $S = \sum_{j=1}^{\infty} s_j e_j \otimes e_j \in K_e$. We observe that $\{X^j(t) : t \geq 0\}$ is a Lévy process for each $j \geq 1$. From independence of $\{X^j(t)\}$ and $\{U^j(t)\}$ and stationarity of increments of $\{X(S)\}$ we have that

$$\begin{aligned}
U^1(s_1) &\stackrel{d}{=} X^1(s_1) = X(s_1 e_1 \otimes e_1), \\
U^2(s_2) &\stackrel{d}{=} X(s_2 e_2 \otimes e_2) = X\left(\sum_{j=1}^{2} s_j e_j \otimes e_j\right) - X(s_1 e_1 \otimes e_1), \\
U^3(s_3) &\stackrel{d}{=} X(s_3 e_3 \otimes e_3) = X\left(\sum_{j=1}^{3} s_j e_j \otimes e_j\right) - X\left(\sum_{j=1}^{2} s_j e_j \otimes e_j\right), ...,
\end{aligned}$$

and so on. Then

$$\sum_{j=1}^{n} U^j(s_j) \stackrel{d}{=} X\left(\sum_{j=1}^{n} s_j e_j \otimes e_j\right) \qquad \text{for every } n \geq 1. \tag{33}$$

Let $S_n = \sum_{j=1}^{n} s_j e_j \otimes e_j$ and note that $\|S_n - S\|_1 \to 0$ as $n \to \infty$. By the stochastic continuity $P\left(\|X_{S_n} - X_S\|_1 > \varepsilon\right) = P\left(\left\|\sum_{j=1}^{n} U^j(s_j) - X_S\right\|_1 > \varepsilon\right) \to 0$ as $n \to \infty$. Then, by Itô-Nisio Theorem we have $\sum_{j=1}^{n} U^j(s_j) \to X(S)$ as $n \to \infty$ almost surely. This proves (31).
Let $f \in L_1^*(H)$. Using (30) and (31)

$$\prod_{j=1}^{n} Ee^{if(X^j(s_j))} = \prod_{j=1}^{n} Ee^{if(U^j(s_j))} = Ee^{if\left(\sum_{j=1}^{n} U^j(s_j)\right)} \to Ee^{if(X(S))},$$

as $n \to \infty$, which gives (32) ■

Remark 26 *Let $f \in L_1^*(H)$ and let $\psi_{X,j}(f) = \log Ee^{if(X(e_j \otimes e_j))}$. It can be shown that there exists a (no self-adjoint) bounded linear operator in $L(H)$ denoted by $\psi_X(f)$ such that $\langle \psi_X(f)e_j\, , e_j\rangle = \psi_{X,j}(f)$ for $j \geq 1$. Then from (32)*

$$Ee^{if(X(S))} = \prod_{j=1}^{\infty} Ee^{if(X^j(s_j))} = \prod_{j=1}^{\infty} e^{\psi_{X,j}(f)s_j} = e^{\sum_{j=1}^{\infty} \psi_{X,j}(f)s_j} = e^{tr(\psi_X(f)S)}. \tag{34}$$

4.2 Distributional properties

Our following result gives tail estimates and moment inequalities for covariance-parameter, trace-class-valued Lévy processes. They are infinite dimensional analogous of Lemma 30.3 in Sato (1999) for one-dimensional time Lévy processes and Lemma 102 in Rocha-Arteaga and Sato (2001) for cone-parameter Lévy processes, when the cone is finite dimensional.

Lemma 27 *Let $\{X(S) : S \in K_e\}$ be a K_e-parameter Lévy process in $L_1(H)$. Let $\{X_t^j : t \geq 0\}$ and $\{U_t^j : t \geq 0\}$, for $j \geq 1$, be as in Proposition 25 which satisfy (30).*
Let (A_j, ν_j, γ_j) be the generating triplet of $\left\{X_t^j\right\}$ for each j. Let ν_j^0 and ν_j^1 be the restrictions of ν_j to the sets $\{\|x\|_1 \leq 1\}$ and $\{\|x\|_1 > 1\}$ respectively. Let $\left\{X_t^{0,j}\right\}$ and $\left\{X_t^{1,j}\right\}$, $j \geq 1$, be independent Lévy processes with generating triplets (A_j, ν_j^0, γ_j) and $(0, \nu_j^1, 0)$ such that $\left\{X_t^j\right\} \overset{d}{=} \left\{X_t^{0,j} + X_t^{1,j}\right\}$ for each j. Assume that

$$\sum_{j=1}^{\infty} \left(\nu_j^1\left(\{\|x\|_1 > 1\}\right) + E\left\|X_1^{0,j}\right\|_1^2\right) < \infty. \tag{35}$$

Then
a) There exist positive constants $C(\varepsilon)$, C_1 and C_2 such that, for every $S \in K_e$,

$$P(\|X_S\|_1 > \varepsilon) \leq C(\varepsilon)\, \|S\|_1 \qquad \text{for } \varepsilon > 0, \tag{36}$$

$$E\left[\|X_S\|_1^2 ; \|X_S\|_1 \leq 1\right] \leq C_1 \|S\|_1\,, \tag{37}$$

$$E\left[\|X_S\|_1 ; \|X_S\|_1 \leq 1\right] \leq C_2 \|S\|_1^{1/2}\,. \tag{38}$$

b) For each continuous linear functional f on $L_1(H)$ there are constants $C_1(f)$ and $C_2(f)$ such that, for every $S \in K_e$,

$$E\left[|f(X_S)|^2 ; \|X_S\|_1 \leq 1\right] \leq C_1(f) \|S\|_1 , \tag{39}$$

$$|E[f(X_S); \|X_S\|_1 \leq 1]| \leq C_2(f) \|S\|_1 . \tag{40}$$

c) There exists a constant $C_3 > 0$ such that, for every $S \in K_e$,

$$\|E[X_S; \|X_S\|_1 \leq 1]\|_1 \leq C_3 \|S\|_1 . \tag{41}$$

Proof. Let $S = \sum_{j=1}^{\infty} s_j e_j \otimes e_j \in K_e$. In view of (31) we prove the assertions for $U(S)$. We shall keep in mind that $tr(S) = \|S\|_1 = \sum_{j=1}^{\infty} s_j$ and $U(S) = \sum_{j=1}^{\infty} U^j(s_j)$.

a) Let $\epsilon > 0$. For each Lévy process $\{U_t^j\}$ we apply Lemma 8 in Pérez-Abreu and Rocha-Arteaga (2002) to yield positive constants $C_j(\varepsilon)$ such that $P\left(\left\|U_{s_j}^j\right\|_1 > \varepsilon\right) \leq C_j(\varepsilon)s_j$. In fact, one can obtain the expressions $C_j(\varepsilon) = \nu_j(\{\|x\|_1 > 1\}) + \frac{E\left\|X_1^{0,j}\right\|_1^2}{\varepsilon^2}$. Then

$$\begin{aligned} P(\|U_S\|_1 > \varepsilon) &= P\left(\left\|U_{s_1}^1 + U_{s_2}^2 + ...\right\|_1 > \varepsilon\right) \\ &= P\left(\left\|U_{s_1}^1 + ... + U_{s_N}^N\right\|_1 > \varepsilon; \text{ for some } N\right) \\ &= \sum_{j=1}^{\infty} P\left(\left\|U_{s_j}^j\right\|_1 > \varepsilon\right) \leq \sum_{j=1}^{\infty} C_j(\varepsilon)s_j \leq \left\{\sum_{j=1}^{\infty} C_j(\varepsilon)\right\} \|S\|_1 , \end{aligned}$$

where $\sum_{j=1}^{\infty} C_j(\varepsilon)$ is finite due to (35). This proves (36).

Next we show (37). Note that

$$\begin{aligned} \{\|U_S\|_1 \leq 1\} = &\left\{\left\|\sum_{j=1}^{\infty} U_{s_j}^j\right\|_1 \leq 1; \left\|U_{s_j}^j\right\|_1 \leq 1 \text{ for all } j\right\} \\ &\cup \left\{\left\|\sum_{j=1}^{\infty} U_{s_j}^j\right\|_1 \leq 1; \left\|U_{s_j}^j\right\|_1 > 1 \text{ for some } j\right\}. \end{aligned}$$

Then

$$\begin{aligned} E\left[\|U_S\|_1^2 ; \|U_S\|_1 \leq 1\right] &= E\left[\left\|\textstyle\sum_{j=1}^{\infty} U_{s_j}^j\right\|_1^2 ; \|U_S\|_1 \leq 1\right] \\ \leq E\left[\|U(S)\|_1^2 ; \left\|U_{s_j}^j\right\|_1 \leq 1 \text{ for all } j\right] &+ P\left(\left\|U_{s_j}^j\right\|_1 > 1 \text{ for some } j\right). \end{aligned} \tag{42}$$

We show the assertion for the first term in the sum in (42). Since $\left\{U_t^j\right\}$ is a independent sequence, we use a estimation for the second moment of a series of independent random variables on Banach spaces, (Borovskikh (1996, Cor. 2.1.1),

$$
\begin{aligned}
E\left[\|U(S)\|_1^2 ; \left\|U_{s_j}^j\right\|_1 \leq 1 \text{ for all } j\right] &\leq E\left\|\sum_{j=1}^{\infty} U_{s_j}^j 1_{\left\{\ \left\|U_{s_j}^j\right\|_1 \leq 1\right\}}\right\|_1^2 \\
&\leq \left(E\left\|\sum_{j=1}^{\infty} U_{s_j}^j 1_{\left\{\ \left\|U_{s_j}^j\right\|_1 \leq 1\right\}}\right\|_1\right)^2 + 2^3 E \sum_{j=1}^{\infty}\left[\left\|U_{s_j}^j\right\|_1 1_{\left\{\ \left\|U_{s_j}^j\right\|_1 \leq 1\right\}}\right]^2 \\
&\leq \left(\sum_{j=1}^{\infty} C_{2,j} s_j^{1/2}\right)^2 + 2^3 \sum_{j=1}^{\infty} C_{1,j} s_j \leq \left(\sum_{j=1}^{\infty} C_{2,j}^2 + 2^3 \sum_{j=1}^{\infty} C_{1,j}\right)\|S\|_1 .
\end{aligned}
$$

Again, we have applied Lemma 8 in Pérez-Abreu and Rocha-Arteaga (2002) to each Lévy process $\left\{U_t^j\right\}$ as follows. Constants $C_{1,j}$ are obtained which satisfies $E\left[\left\|U_{s_j}^j\right\|_1^2 1_{\left\{\ \left\|U_{s_j}^j\right\|_1 \leq 1\right\}}\right] \leq C_{1,j} s_j$ and constants $C_{2,j}$ satisfying the inequality $E\left[\left\|U_{s_j}^j\right\|_1 1_{\left\{\ \left\|U_{s_j}^j\right\|_1 \leq 1\right\}}\right] \leq C_{2,j} s_j^{1/2}$. Moreover, from (30) one can obtain the expressions $C_{1,j} = \nu_j^1\left(\{\|x\|_1 > 1\}\right) + E\left\|X_1^{0,j}\right\|_1^2$ and $C_{2,j} = \sqrt{C_{1,j}}$. Then by (35) $\sum_{j=1}^{\infty} C_{2,j}^2$ and $\sum_{j=1}^{\infty} C_{1,j}$ are finite.

For the second term of the sum in (42) we have

$$
\begin{aligned}
P\left(\left\|U_{s_j}^j\right\|_1 > 1 \text{ for some } j \geq 1\right) &\leq \sum_{j=1}^{\infty} P\left(\left\|U_{s_j}^j\right\|_1 > 1\right) \\
&\leq \sum_{j=1}^{\infty} C_j(1) s_j \leq \left(\sum_{j=1}^{\infty} C_j(1)\right)\|S\|_1 ,
\end{aligned}
$$

where $C_j(1)$ are as in a) for $\varepsilon = 1$. Thus, (37) holds.

Finally, apply Cauchy, Bunyakowski and Schwarz's inequality to get

$$
E\left[\|X_S\|_1 ; \|X_S\|_1 \leq 1\right] \leq \left\{E\left[\|X_S\|_1^2 ; \|X_S\|_1 \leq 1\right]\right\}^{1/2} \leq C_1^{1/2}\|S\|_1^{1/2} ,
$$

where C_1 is the constant in (37). This proves (38).

b) Let f be a continuous linear functional on $L_1(H)$. Inequality (39) follows from $E\left[|f(X_S)|^2 ; \|X_S\|_1 \leq 1\right] \leq |f|^2 E\left[\|X_S\|_1^2 ; \|X_S\|_1 \leq 1\right]$ and (37).

Next,

$$
\begin{aligned}
&\left|E\left[f(X_S); \|X_S\|_1 \le 1\right]\right| \\
&= \Big| E\left[e^{if(X_S)} - 1\right] - E\left[e^{if(X_S)} - 1; \|X_S\|_1 > 1\right] \\
&-E\left[e^{if(X_S)} - 1 - if(X_S); \|X_S\|_1 \le 1\right] \Big| \\
&\le \left| E\left[e^{if(X_S)} - 1\right] \right| + 2P\left(\|X_S\|_1 > 1\right) + \frac{1}{2} E\left[f^2(X_S); \|X_S\|_1 \le 1\right].
\end{aligned}
$$

Inequalities (36) and (39) provide constant multiples of $\|S\|_1$ as bounds for the last two terms of the former sum, respectively. Next, from (34) it follows that $Ee^{if(X_S)} = e^{tr(\psi_X(f)S)}$ where $\psi_X(f) \in L(H)$ is no necessarily self-adjoint operator. Then $E\left|e^{if(X_t)} - 1\right| = \left|e^{t\Psi_X(f)} - 1\right|$. It is known that $|tr(\psi_X(f)S)| \le \|\psi_X(f)\| \, \|S\|_1$. If $|tr(\psi_X(f)S)| \le 1$, $\left|e^{t\Psi_X(f)} - 1\right| \le \frac{7}{4} \|\psi_X(f)\| \, \|S\|_1$ and if $|tr(\psi_X(f)S)| > 1$ then $\left|e^{t\Psi_X(f)} - 1\right| \le 2 \|\psi_X(f)\| \, \|S\|_1$. This proves (40).

c) By (38) $E\left[\|X_S\|_1 ; \|X_S\|_1 \le 1\right]$ is finite for every $S \in K_e$ and hence $E\left[X_S; \|X_S\|_1 \le 1\right]$ is a Bochner integral. Let us denote $V = X_S 1_{\|X_S\|_1 \le 1}$. Since V is Pettis integrable it satisfies

$$
f\,(EV) = Ef\,(V) \quad \text{for every } f \in L_1^*(H). \tag{43}
$$

On the other hand, since $EV \in L_1(H)$ we use the polar decomposition of a compact operator $EV = T\,|EV|$ where $|EV|$ denotes the positive compact operator $|EV| = \{(EV)\,(EV)^*\}^{1/2}$. This is equivalent to $|EV| = T^*EV$,where $T \in L(H)$ is a isometry and T^* denotes the adjoint operator of T which is also in $L(H)$. Then

$$
\|EV\|_1 = tr(|EV|) = tr(T^*EV) = f_{T^*}\,(EV) = Ef_{T^*}\,(V)\,.
$$

Here we have used (43) with $f_{T^*} \in L_1^*(H)$. Finally, from (40)

$$
\left\|E\left[X_S; \|X_S\|_1 \le 1\right]\right\|_1 = Ef_{T^*}\left(X_S; \|X_S\|_1 \le 1\right) \le C_2(f_{T^*})\,\|S\|_1\,.
$$

■

5 Covariance-parameter subordination

Let $\{Z(t) : t \ge 0\}$ be a K-valued subordinator and let $\{X(S) : S \in K\}$ be an independent K-parameter Lévy process with values in $L_1(H)$. Define the process $\{Y(t) : t \ge 0\}$ by $Y(t) = X(Z_t)$. This procedure of getting $\{Y(t)\}$ from $\{X(S)\}$ and $\{Z(t)\}$ is called (Bochner´s) subordination. For a general cone K one can show that Y is also a Lévy process in $L_1(H)$. The proof of this fact is similar to the one dimensional case in Sato (1999) and to the multivariate case in Barndorff-Nielsen, Pedersen and Sato (2001) (one should use the fact that if U and V are independent random variables in $L_1(H)$ and if $f : L_1(H) \times L_1(H) \to \mathbb{R}$ is a bounded and measurable function, then $g(v) = Ef(U, v)$ is also bounded and measurable and $Ef(U, V) = Eg(V)$).

The identification of the characteristic triplet of the Lévy process Y is not an easy problem, even in the multivariate finite dimensional case. For example, Barndorff-Nielsen, Pedersen and Sato (2001) and Pedersen and Sato (2001) obtain the generating triplet of their subordinated processes only in cases of cones of $\mathbb{R}^n$ with basis (see also Rocha-Arteaga and Sato (2001)).

In this section we obtain the generating triplet of Y in terms of the generating triplets of X and Z in the case of the cone with basis K_e. Our result is of different nature that the one obtained in the last three named works, in the sense that our K-parameter $L_1(H)$-valued Lévy process is such that K and $L_1(H)$ are in the same space. In this direction our corresponding finite dimensional results for real symmetric matrices are new.

Theorem 28 *Let $\{X(S) : S \in K_e\}$ be a K_e-parameter Lévy process in $L_1(H)$ and let $\{Z(t) : t \geq 0\}$ be a K_e-valued subordinator. Then the process $\{Y(t) : t \geq 0\}$ defined by $Y(t) = X(Z(t))$ is a $L_1(H)$-valued Lévy process such that*

a) The characteristic functional is given by

$$Ee^{if(Y_t)} = e^{t\Psi_Z(\psi_X(f))} \qquad \text{for every real-valued } f \in L_1^*(H),$$

where Ψ_Z is given by (20) and $\psi_X(f) \in L(H)$ is no necessarily self-adjoint and satisfies $\langle \psi_X(f)e_j\,,\, e_j\rangle = \psi_{X,j}(f)$ where $\psi_{X,j}(f) = \log Ee^{if(X(te_j \otimes e_j))}$.

b) Let ν_Z and $\gamma_Z^0 = \sum_{j=1}^{\infty} \gamma_{Z,j}^0 e_j \otimes e_j$ be the Lévy measure and the drift of $\{Z_t\}$. Let $(A_{X,j}, \nu_{X,j}, \gamma_{X,j})$ be the generating triplet of $\{X(te_j \otimes e_j)\}$ satisfying

$$\sum_{j=1}^{\infty} \|A_{X,j}\| < \infty, \tag{44}$$

$$\sup_j\{\nu_{X,j}(C)\} < \infty \quad \text{for every } C \in \mathcal{B}(L_1(H)\backslash\{0\}), \tag{45}$$

$$\sup_j\{\int_{\|x\|_1 \leq 1} |f(x)|^2 \, \nu_{X,j}(dx)\} < \infty \quad \text{for every } f \in L_1^*(H), \tag{46}$$

$$\sum_{j=1}^{\infty} \|\gamma_{X,j}\|_1 < \infty. \tag{47}$$

Let $\left\{X_t^{0,j}\right\}$ and $\left\{X_t^{1,j}\right\}$, $j \geq 1$, be independent Lévy processes with triplets $(A_{X,j}, \nu_{X,j}^0, \gamma_{X,j})$ and $(0, \nu_{X,j}^1, 0)$ and $\{X(te_j \otimes e_j)\} \overset{d}{=} \left\{X_t^{0,j} + X_t^{1,j}\right\}$ where $\nu_{X,j}^0$ and $\nu_{X,j}^1$ denotes the restrictions of ν_j to the sets $\{\|x\|_1 \leq 1\}$ and $\{\|x\|_1 > 1\}$ respectively. Assume that

$$\sum_{j=1}^{\infty} \left(\nu_{X,j}^1 \left(\{\|x\|_1 > 1\}\right) + E \left\| X_1^{0,j} \right\|_1^2 \right) < \infty. \tag{48}$$

Let $\mu_S = \mathcal{L}(X(S))$. *Then the generating triplet* (A_Y, ν_Y, γ_Y) *of* $\{Y_t\}$ *is given by*

$$A_Y = \sum_{j=1}^{\infty} \gamma^0_{Z,j} A_{X,j}, \tag{49}$$

$$\nu_Y(C) = \int_{K_e} \mu_S(C)\nu_Z(dS) + \sum_{j=1}^{\infty} \gamma^0_{Z,j}\nu_{X,j}(C), \ C \in \mathcal{B}(L_1(H)\backslash\{0\}) \tag{50}$$

$$\gamma_Y = \int_{K_e} \int_{\|x\|_1 \le 1} x\mu_S(dx)\nu_Z(dS) + \sum_{j=1}^{\infty} \gamma^0_{Z,j}\gamma_{X,j}. \tag{51}$$

c) If $\int_{\|S\|_1 \le 1} \|S\|_1^{1/2} \nu_Z(dS) < \infty$ *and* $\gamma^0_Z = 0$, *then* $\{Y_t\}$ *has bounded variation on each interval* $[0, t]$.

The corresponding result for the case of symmetric real matrices is as follows.

Corollary 29 *Let* M_m *be the Banach space of* $m \times m$ *real symmetric matrices and let* M_m^+ *be the cone of symmetric nonnegative definite matrices in* M_m. *Fix an orthogonal matrix* $O \in M_m$ *with* $\{e_j\}$, $j = 1, 2, ..., n$, *as its system of eigenvectors, i.e,* $O = (e_1, ..., e_n)$ *where the vectors* e_j *are column vectors. Let* K_O *be the proper subcone of* M_m^+ *defined by*

$$K_O = \left\{ S \in M_m^+ : S = OD_SO' \right\},$$

where $D_S = diag\,(\lambda_1(S), ..., \lambda_n(S))$ *is the diagonal matrix containing the eigenvalues* $\lambda_j(S)$ *of* S. *Let* $\{Z(t) : t \ge 0\}$ *be a subordinator in* K_O *with characteristic function*

$$Ee^{-itr(\Sigma Z_t)} = exp\left\{ t\left(\int_{K_O} \left(e^{-itr(\Sigma S)} - 1 \right) \nu_Z(dS) - tr(\Sigma\gamma^0_Z) \right) \right\} \qquad \Sigma \in M_m^+,$$

where $\gamma^0_Z = OD_{\gamma^0_Z}O^T$ *and* ν_Z *is concentrated on* K_e.

Let $\{X(S) : S \in K_O\}$ *be a matrix cone-parameter Lévy process in* M_m *and let* $\mu_S = \mathcal{L}(X(S)$. *For each* j *fix the matrix* $e_je_j^T$ *in* K_O. *Then* $\{X(e_je_j^Tt) : t \ge 0\}$ *is a Lévy process and let us denote by* $(A_{X,j}, \nu_{X,j}, \gamma_{X,j})$ *its generating triplet.*

Then the generating triplet of the subordinated matrix valued process $\{Y(t) : t \ge 0\}$ *defined by* $Y_t = X(Z(t))$, *is given by*

$$A_Y = \lambda_1\left(\gamma^0_Z\right) A_{X,1} + ... + \lambda_n\left(\gamma^0_Z\right) A_{X,n},$$

$$\nu_Y(B) = \int_{K_e} \mu_S(B)\nu_Z(dS) + \sum_{j=1}^{n} \lambda_j\left(\gamma^0_Z\right)\nu_{X,j}(B),$$

$$\gamma_Y = \int_{K_e} \int_{\|x\| \le 1} x\mu_S(dx)\nu_Z(dS) + \sum_{j=1}^{n} \lambda_j\left(\gamma^0_Z\right)\gamma_{X,j}.$$

Before the proof of the Theorem 28 we need the following technical Lemma.

Lemma 30 *Let $\{X(S) : S \in K_e\}$ be a K_e-parameter Lévy process in $L_1(H)$ and let $\nu_{X,j}$ be the Lévy measure of $\{X(te_j \otimes e_j)\}$ for each j. Assume the condition (46) of the Theorem 28. Then*

$$\lim_{\delta \downarrow 0} \liminf_{n \to \infty} \int_{\|x\|_1 \leq \delta} |f(x)|^2 \nu_n(dx) = 0,$$

where $\nu_n = \sum_{j=1}^n a_j \nu_{X,j}$, $\sum_{j=1}^\infty a_j < \infty$ and $a_j \geq 0$.

Proof. Let $g_n(\delta) = \inf_{m \geq n} \left\{ \sum_{j=1}^m a_j \int_{\|x\|_1 \leq \delta} |f(x)|^2 \nu_{X,j}(dx) \right\}$.
Then $g_n(\delta) = \sum_{j=1}^n a_j \int_{\|x\|_1 \leq \delta} |f(x)|^2 \nu_{X,j}(dx)$ increases to the function $g(\delta) := \sum_{j=1}^\infty a_j \int_{\|x\|_1 \leq \delta} |f(x)|^2 \nu_{X,j}(dx)$. Let $\delta \in (0,1]$. Then $|g_n(\delta) - g(\delta)| \leq \sum_{j=n+1}^\infty a_j \int_{\|x\|_1 \leq \delta} |f(x)|^2 \nu_{X,j}(dx) \to 0$ as $n \to \infty$. Thus, $g_n(\delta) \to g(\delta)$ uniformly in $(0,1]$ and hence $\lim_{\delta \downarrow 0} \liminf_{n \to \infty} g_n(\delta) = \liminf_{n \to \infty} \lim_{\delta \downarrow 0} g_n(\delta)$. The assertion follows from the fact that $\lim_{\delta \downarrow 0} \int_{\|x\|_1 \leq \delta} |f(x)|^2 \nu_{X,j}(dx) = 0$ for each j. ■

Proof of Theorem 28. a) Let f be a real-valued continuous linear functional in $L_1^*(H)$. From (19) and (34) we get

$$\begin{aligned} Ee^{if(Y(t))} &= E\left[\left(Ee^{itr(\psi_X(f)S)} \right)_{S=Z_t} \right] \\ &= Ee^{tr(\psi_X(f)Z_t)} = e^{t\Psi_Z(\psi_X(f))}. \end{aligned}$$

b) We have that

$$\begin{aligned} Ee^{if(Y(t))} &= e^{t\Psi_Z(\psi_X(f))} \\ &= \exp\left\{ t \left(\int_{K_e} (e^{tr(\psi_X(f)S)} - 1)\nu_Z(dS) + tr(\psi_X(f)\gamma_Z^0) \right) \right\} \end{aligned} \tag{52}$$

by (20) since $\operatorname{Re} tr(\psi_X(f)S) = \lim_{n \to \infty} \sum_{j=1}^n s_j \operatorname{Re} \psi_{X,j}(f) \leq 0$. Here we have used that $S = \sum_{j=1}^\infty s_j e_j \otimes e_j$ and that $\operatorname{Re} \psi_{X,j}(f) = \log \left| Ee^{if(X(te_j \otimes e_j))} \right| \leq 0$ for each j. Let $g(f,x) = e^{f(x)} - 1 - if(x)1_{\{\|x\|_1 \leq 1\}}(x)$. Note that

$$\begin{aligned} tr(\psi_X(f)\gamma_Z^0) &= \sum_{j=1}^\infty \gamma_{Z,j}^0 \psi_{X,j}(f) = -\frac{1}{2} \sum_{j=1}^\infty \gamma_{Z,j}^0 (A_{X,j}f, f) + i \sum_{j=1}^\infty \gamma_{Z,j}^0 f(\gamma_{X,j}) \\ &+ \int_{L_1(H)} g(f,x) \left(\sum_{j=1}^\infty \gamma_{Z,j}^0 \nu_{X,j} \right)(dx) \end{aligned}$$

and hence

$$tr(\psi_X(f)\gamma_Z^0) = -\frac{1}{2} f\left(\sum_{j=1}^{\infty} \gamma_{Z,j}^0 A_{X,j}(f)\right) + if\left(\sum_{j=1}^{\infty} \gamma_{Z,j}^0 \gamma_{X,j}\right) + \int_{L_1(H)} g(f,x)\left(\sum_{j=1}^{\infty} \gamma_{Z,j}^0 \nu_{X,j}\right) dx. \tag{53}$$

We have passed to the limit since the mapping $\sum_{j=1}^{\infty} \gamma_{Z,j}^0 A_{X,j} : L(H) \to L_1(H)$ and $\sum_{j=1}^{\infty} \gamma_{Z,j}^0 \gamma_{X,j} \in L_1(H)$ are well defined due to condition (44) and (47), respectively. That $\sum_{j=1}^{\infty} \gamma_{Z,j}^0 A_{X,j}$ is a covariance operator follows from nonnegativeness of $\gamma_{Z,j}^0$ and that $A_{X,j}$ is a covariance operator for each j.

Next, let $\mu_S = \mathcal{L}(X(S))$. Notice that $\int_{K_e} \|S\|_1 1_{\{\|S\|_1 \le 1\}} \nu_Z(dS) < \infty$ by (9) and that $\int_{L_1(H)} x 1_{\{\|x\|_1 \le 1\}}(x)\mu_S(dx)$ is Bochner integrable by Lemma 27 relation (41). Then $\int_{K_e} \left\| \int_{L_1(H)} x 1_{\{\|x\|_1 \le 1\}}(x)\mu_S(dx) \right\| \nu_Z(dS)$ is finite and hence $\int_{K_e} \int_{L_1(H)} x 1_{\{\|x\|_1 \le 1\}}(x)\mu_S(dx)\nu_Z(dS)$ is a Bochner integral. These Bochner integrals are, in particular, Pettis integrals and therefore

$$\int_{K_e} \int_{L_1(H)} f(x) 1_{\{\|x\|_1 \le 1\}}(x)\mu_S(dx)\nu_Z(dS) = f\left(\int_{K_e} \int_{L_1(H)} x 1_{\{\|x\|_1 \le 1\}}(x)\mu_S(dx)\nu_Z(dS)\right). \tag{54}$$

It follows from (34) and (54) that

$$\begin{aligned}
&\int_{K_e} (e^{tr(\psi_X(f)S)} - 1)\nu_Z(dS) \\
&= \int_{K_e} (Ee^{if(X(S))} - 1)\nu_Z(dS) = \int_{K_e} \int_{L_1(H)} (e^{if(x)} - 1)\mu_S(dx)\nu_Z(dS) \\
&= \int_{K_e} \int_{L_1(H)} g(f,x)\mu_S(dx)\nu_Z(dS) \\
&+ if\left(\int_{K_e} \int_{L_1(H)} x 1_{\{\|x\|_1 \le 1\}}(x)\mu_S(dx)\nu_Z(dS)\right).
\end{aligned} \tag{55}$$

From (52), (53) and (55) we arrive at

$$\begin{aligned} Ee^{if(Y_t)} = \exp\Bigg\{ t \Bigg[&-\frac{1}{2} f \left(\sum_{j=1}^{\infty} \gamma_{Z,j}^{0} A_{X,j}\,(f) \right) \\ &+ if \left(\int_{K_e} \int_{\{\|x\|_1 \le 1\}} x \mu_S(dx) \nu_Z(dS) + \sum_{j=1}^{\infty} \gamma_{Z,j}^{0} \gamma_{X,j} \right) \\ &+ \int_{L_1(H)} g(f,x) \left(\int_{K_e} \mu_S(\cdot) \nu_Z(dS) + \sum_{j=1}^{\infty} \gamma_{Z,j}^{0} \nu_{X,j}(\cdot) \right) dx \Bigg] \Bigg\}. \end{aligned} \tag{56}$$

Next we prove (49), (50) and (51). Let $\mu = \mathcal{L}(Y_1)$. It has been shown that the characteristic functional $\hat{\mu}$ of Y_1 has the form $\hat{\mu} = \hat{\rho}_{\tilde{A}} \cdot h_{\tilde{\nu}} \cdot \hat{\delta}_{\tilde{\gamma}}$ where $\hat{\rho}_{\tilde{A}}$ is a characteristic functional of a zero-mean Gaussian probability measure with covariance $\tilde{A}$, $\hat{\delta}_{\tilde{\gamma}}$ is a characteristic functional of a degenerating probability distribution at the point $\tilde{\gamma}$ and the function $h_{\tilde{\nu}}$ is given by

$$h_{\tilde{\nu}}\,(f) = \int_{L_1(H)} g(f,x) \tilde{\nu}\,(dx) \quad f \in L_1^*(H), \tag{57}$$

where

$$\begin{aligned} \tilde{A} &= \sum_{j=1}^{\infty} \gamma_{Z,j}^{0} A_{X,j} \\ \tilde{\nu}\,(dx) &= \int_{K_e} \mu_S(dx) \nu_Z(dS) + \sum_{j=1}^{\infty} \gamma_{Z,j}^{0} \nu_{X,j}(dx) \\ \tilde{\gamma} &= \int_{K_e} \int_{\{\|x\|_1 \le 1\}} x \mu_S(dx) \nu_Z(dS) + \sum_{j=1}^{\infty} \gamma_{Z,j}^{0} \gamma_{X,j}. \end{aligned} \tag{58}$$

We shall prove $A_Y = \tilde{A}$, $\nu_Y = \tilde{\nu}$ and $\gamma_Y = \tilde{\gamma}$. In view of the uniqueness of the generating triplet, it is enough to show that $\tilde{\nu}$ in (58) is a Lévy measure. Thus, we claim that $h_{\tilde{\nu}}$ in (57) is a characteristic functional (see (5)).

Let $\tilde{\nu}\,(dx) = \tilde{\nu}_{(1)}\,(dx) + \tilde{\nu}_{(2)}\,(dx)$ where

$$\tilde{\nu}_{(1)}\,(dx) = \int_{K_e} \mu_S(dx) \nu_Z(dS), \tag{59}$$

$$\tilde{\nu}_{(2)}\,(dx) = \sum_{j=1}^{\infty} \gamma_{Z,j}^{0} \nu_{X,j}(dx). \tag{60}$$

First, we show that $\tilde{\nu}_{(2)}$ is a Lévy measure. Let $\{U^j : t \geq 0\}$ be the sequence of independent Lévy processes appearing in Proposition 25 and satisfying (30) and (31). Let $U_n = \sum_{j=1}^{n} U^j\left(\gamma_{Z,j}^0\right)$, $n \geq 1$. Let $\mu_n = \mathcal{L}(U_n)$ with generating triplet (A_n, ν_n, γ_n) which are given by

$$A_n = \sum_{j=1}^{n} \gamma_{Z,j}^0 A_{X,j} \quad \nu_n = \sum_{j=1}^{n} \gamma_{Z,j}^0 \nu_{X,j} \quad \text{and} \quad \gamma_n = \sum_{j=1}^{n} \gamma_{Z,j}^0 \gamma_{X,j}. \tag{61}$$

Using (31) we have that $U_n \to X\left(\gamma_Z^0\right)$ almost surely. Let (A_0, ν_0, γ_0) be the generating triplet of $X\left(\gamma_Z^0\right)$ and let $\mu_0 = \mathcal{L}(X\left(\gamma_Z^0\right))$. Then, by Itô-Nisio Theorem

$$\mu_n \text{ converges weakly to } \mu_0. \tag{62}$$

We assume that $\nu_0\left(\{\|x\|_1 = 1\}\right) = 0$. This is not a restriction, since not more than countably many circles of the form $\{x : \|x\|_1 = 1\}$ has positive ν_0-measure. Then, from (62) and Theorem 3 we have that

$$\gamma_n \to \gamma_0. \tag{63}$$

Let f be any continuous linear functional f in $L_1^*(H)$. Use condition (46) and Lemma 30 to obtain

$$\lim_{\delta \downarrow 0} \liminf_{n \to \infty} \int_{\|x\|_1 \leq \delta} |f(x)|^2 \nu_n(dx) = 0. \tag{64}$$

Now, from (62), (63) and (64), we are in position to apply Theorem 3 to yield that the probability measures P_n corresponding to the Lévy measures ν_n in (61) converge to the probability measure P_0 corresponding to the Lévy measure ν_0. That is,

$$P_n \text{ converges weakly to } P_0. \tag{65}$$

On the other hand, recall that $g(f,x) = e^{f(x)} - 1 - if(x)1_{\{\|x\|_1 \leq 1\}}(x)$. It is clear from (61) that ν_n converges to $\tilde{\nu}_{(2)}$ which satisfies

$$\tilde{\nu}_{(2)}(\{0\}) = 0, \quad \tilde{\nu}_{(2)}\left(\|x\|_1 > 1\right) < \infty \quad \text{and} \quad \int_{\|x\|_1 \leq 1} |f(x)|^2 \tilde{\nu}_{(2)}(dx) < \infty.$$

These properties of $\tilde{\nu}_{(2)}$ are immediate from the fact that $\nu_{X,j}(\{0\}) = 0$, $j \geq 1$ and conditions (45) and (46), respectively. Then $\int_{L_1(H)} |g(f,x)|\, \tilde{\nu}_{(2)}(dx) < \infty$ and $\int_{L_1(H)} g(f,x)\, \nu_n(dx) \to \int_{L_1(H)} g(f,x)\, \tilde{\nu}_{(2)}(dx)$ by Proposition 4. But from (65) we have that the characterictic functional

$$\hat{P}_n(f) = \int_{L_1(H)} g(f,x)\, \nu_n(dx) \to \hat{P}_0(f)$$

and hence $\hat{P}_0(f) = \int_{L_1(H)} g(f,x)\, \tilde{\nu}_{(2)}(dx)$. This proves that $\tilde{\nu}_{(2)}$ is a Lévy measure.

We show that $\tilde{\nu}_{(1)}(dx) = \int_{K_e} \mu_S(dx)\nu_Z(dS)$ in (59) is a Lévy measure by proving that $\exp\left\{\int_{L_1(H)} g\,(f,x)\,\tilde{\nu}_{(1)}(dx)\right\}$ is a characteristic functional. In view of (34) and (54) we have

$$e^{\int_{L_1(H)} g(f,x)\tilde{\nu}_{(1)}(dx)} = e^{\int_{K_e}(e^{tr(\psi_X(f)S)}-1)\nu_Z(dS)} \cdot e^{-if\ \int_{K_e}\int_{\{\|x\|_1\leq 1\}} x\mu_S(dx)\nu_Z(dS)}.$$

Notice that the last two factors are characteristic functionals, since the first factor corresponds to the characteristic functional of a subordinated process at time 1 obtained by subordination of X_S by $Z_t - t\gamma_Z^0$ and the second one corresponds to the characteristic functional of a degenerated distribution.

We have shown that (58) is a Lévy measure and hence by uniqueness of the generating triplet we get $A_Y = \tilde{\nu}$, $\nu_Y = \tilde{\nu}$ and $\gamma_Y = \tilde{\gamma}$, where $\tilde{A}$, $\tilde{\nu}$ and $\tilde{\gamma}$ are defined in (58). This proves (49), (50) and (51).

c) Assume that $\int_{\|S\|_1\leq 1} \|S\|_1^{1/2}\,\nu_Z(dS) < \infty$ and $\gamma_Z^0 = 0$. Then $A_Y = 0$ by (49) and $\nu_Y(dx) = \int_{K_e} \mu_S(dx)\nu_Z(dS)$ by (50). We have

$$\int_{\|x\|_1\leq 1} \|x\|_1\,\nu_Y(dx) = \int_{K_e}\int_{\|x\|_1\leq 1} \|x\|_1\,\mu_S(dx)\nu_Z(dS) < \infty$$

by (38). Now, from (54) and (56)

$$\begin{aligned} Ee^{if(Y(t))} &= \exp\left\{t\left[\int_{L_1(H)} g(f,x)\nu_Y(dx)\right.\right. \\ &\left.\left.+i\int_{K_e}\int_{\|x\|_1\leq 1} f\,(x)\,\mu_S(dx)\nu_Z(dS)\right]\right\} \\ &= \exp\left\{t\int_{L_1(H)} (e^{if(x)}-1)\nu_Y(dx)\right\}. \end{aligned}$$

It follows from Proposition 2 that $\{Y_t\}$ has bounded variation on each interval $[0,t]$. ■

6 Appendix

Here we present the proofs of some the technical results used in the paper.

Proof of Proposition 4. a) Note that $|g(f,x)| \leq \frac{1}{2}|f(x)|^2$ if $\|x\|_1 \leq 1$ and $|g(f,x)| \leq 2\nu(\|x\|_1 > 1)$ if $\|x\|_1 > 1$.
b) It is enough to prove that $\int_{L_1(H)} h(x)\nu_n(dx) \to \int_{L_1(H)} h(x)\nu(dx)$ for every nonnegative measurable function h on $L_1(H)$. There exists a sequence $\{h_k\}$ of simple functions such that $0 \leq h_k \uparrow h$

and $\int h_k d\nu \uparrow \int h d\nu$. Since for each n we have $\int h_k d\nu_n \uparrow \int h d\nu_n$ as $k \to \infty$ and for each k we have $\int h_k d\nu_n \uparrow \int h_k d\nu$ as $n \to \infty$ then

$$\lim_{k\to\infty} \int h_k d\nu = \lim_{k\to\infty}\lim_{n\to\infty} \int h_k d\nu_n \leq \lim_{k\to\infty}\lim_{n\to\infty} \int h d\nu_n = \lim_{n\to\infty} \int h d\nu_n.$$

This proves that $\int h d\nu \leq \lim\limits_{n\to\infty} \int h d\nu_n$. On the other hand, since ν_n is dominated by ν, $\int h_k d\nu_n \leq \int h_k d\nu \leq \int h d\nu$ for all n and all k. Now, for each n, $\int h_k d\nu_n \uparrow \int h_k d\nu_n$ as $k \to \infty$. Hence $\int h d\nu_n \leq \int h d\nu$ for all n. This proves that $\lim\limits_{n\to\infty} \int h d\nu_n \leq \int h d\nu$. ■

Proof of Lemma 17. Let $h \in H$ and let $V \in L(H)$. Then

$$\begin{aligned} \left\| T_U^{'}(V)(h) \right\|_H^2 &= \sum_{j=1}^{\infty} \left| \sum_{k=1}^{\infty} \langle V e_1 \otimes e_k U(h), e_1 \rangle \langle e_k, e_j \rangle \right|^2 \\ &= \sum_{j=1}^{\infty} \left| \langle e_1, V^* e_1 \rangle \langle U(h), e_j \rangle \right|^2 = \left| \langle V e_1, e_1 \rangle \right|^2 \| U(h) \|_H^2 < \infty. \end{aligned} \tag{66}$$

Next, let $h \in H$ and let $S \in L_1(H)$. Then

$$\begin{aligned} \| T_U(S)(h) \|_H^2 &= \sum_{j=1}^{\infty} \left| \sum_{k=1}^{\infty} \langle U S e_k \otimes e_1(h), e_k \rangle \langle e_1, e_j \rangle \right|^2 \\ &= \left| \sum_{k=1}^{\infty} \langle h, e_1 \rangle \langle e_k, (US)^* e_k \rangle \right|^2 = \left| \langle h, e_1 \rangle \right|^2 \left| tr\,(US) \right|^2 < \infty. \end{aligned} \tag{67}$$

From (66) and (67) the transformations $T_U^{'}$ and T_U are well defined for each U. It is easy to check that they are linear and continuous.

It remains to show that $T_U(S)$ belongs to $L_1(H)$ whenever S belongs to $L_1(H)$. Let $\{\phi_k\}$ be any complete orthonormal set of H. Then

$$\begin{aligned} \sum_{k=1}^{\infty} \left| \langle T_U(S) \phi_k, \phi_k \rangle \right| &\leq \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \left| \langle \left(e_1 \otimes e_j U S e_j \otimes e_1 \right) \phi_k \,,\, \phi_k \rangle \right| \\ &\leq \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \left| \langle e_j, (US)^* e_j \rangle \right| \left| \langle e_1, \phi_k \rangle \right|^2 \\ &\leq \| e_1 \|_H^2 \sum_{j=1}^{\infty} \left| \langle U S e_j, e_j \rangle \right| < \infty. \end{aligned}$$

This proves that $T_U(S) \in L_1(H)$. We have used here a characterization for a linear operator be of trace-class (see Remark after Theorem VI. 24 of Reed and Simon (1980)). ■

Proof of Lemma 18. Let $V \in L(H)$ and let $S \in L_1(H)$. Then

$$\begin{aligned} f_V(T_U S) &= tr\left(\sum_{j=1}^{\infty} V e_1 \otimes e_j U S e_j \otimes e_1\right) = \sum_{j=1}^{\infty} tr\left(V e_1 \otimes e_j U S e_j \otimes e_1\right) \\ &= \sum_{j=1}^{\infty} tr\left(e_j \otimes e_1 V e_1 \otimes e_j U S\right) = tr\left(\sum_{j=1}^{\infty} e_j \otimes e_1 V e_1 \otimes e_j U S\right) \\ &= f_{T_U' S}(V). \end{aligned}$$

■

Proof of Lemma 19. Let $V = V^1 + V^2$ and $W = W^1 + W^2$ be both in $L(H)$ and let $c \in \mathbb{R}$. Then

$$\begin{aligned} T_U'(V + cW) &= T_{U^1}'(V^1 + cW^1) + iT_{U^2}'(V^2 + cW^2) \\ &= T_{U^1}'\left(V^1\right) + iT_{U^2}'\left(V^2\right) + c\left[T_{U^1}'\left(W^1\right) + iT_{U^2}'\left(W^2\right)\right]. \end{aligned}$$

Thus, T_U' is linear. Linearity of T_U is proved similarly. Continuity of T_U' and T_U follows from continuity of T_{U^k}' and T_{U^k}, $k \in \{1, 2\}$, and the fact that $\|V\|$ tends to zero is equivalent to $\left\|V^1\right\|$ and $\left\|V^2\right\|$ tend to zero both at time. ■

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