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# MORSE THEORY AND APPLICATIONS

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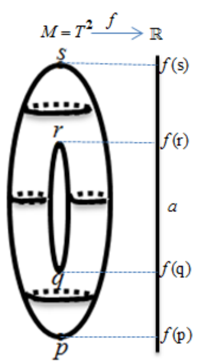
# Introduction

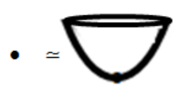
Morse theory is a powerful method to study the topological structure of a smooth manifold $M$ by examining the critical points of a Morse function defined on it. For example, let $M = \mathbb{T}^2 \subset \mathbb{R}^3$ be the two dimensional torus and $f : M \to \mathbb{R}$ the height function $f((x, y, z)) = z$. The functon $f$ has four critical points $p, q, r$ and $s$ on $M$ with indices $0, 1, 1$ and $2$ respectively. Let $M^a$ denote the set of all points $x \in M$ such that $f(x) \leq a$, and " $\simeq$ " denote homotopy equivalence. We can describe the change in homeomorphism and homotopic types of $M^a$ as $a$ passes through each critical value of $f$ as follows:

**Case** $a < f(p)$**:**



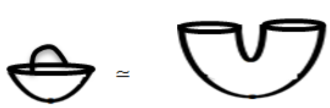

$M^a$ is the empty set.

**Case** $f(p) < a < f(q)$**:**



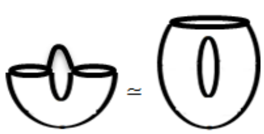
* $M^a$ is homeomorphic to a 2-cell or a disk.
* The homotopy type of $M^a$ is a single 0-cell since the index of $p$ is 0.

**Case** $f(q) < a < f(r)$ :



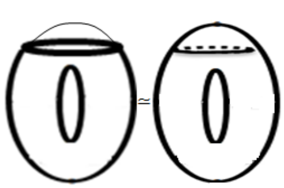
* $M^a$ is homeomorphic to a cylinder.
* The homotopy type of $M^a$ is a disk with a 1-cell attached since the index of $q$ is 1.

**Case** $f(r) < a < f(s)$ :



* $M^a$ is homeomorphic to a torus with a disk removed.
* The homotopy type of $M^a$ is a cylinder with a 1-cell attached since the index of $r$ is 1.

**Case** $f(s) < a$ :



* $M^a$ is homeomorphic to the full torus.
* The homotopy type of $M^a$ is a torus minus a disk with a 2-cell attached since the index of $s$ is 2.

In this thesis, we will present Morse theory on smooth finite-dimensional manifolds and one application based on the books [10, 13].

# Chapter 1

# Basic Definitions and Examples

We shall use the words "**differentiable**" and "**smooth**" and "**differentiable of class** $C^\infty$" as synonyms. Before the main chapter of this thesis, let us recall some definitions and examples from differential geometry and topology.

## 1.1 Differential Geometry of Manifolds

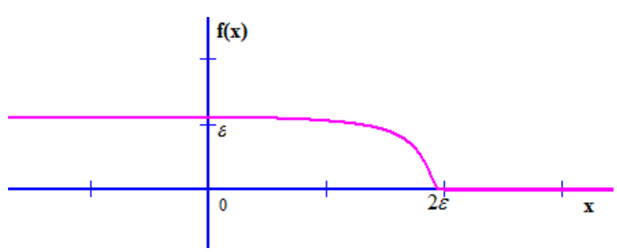
### 1.1.1 Smooth Functions in Euclidean space

**Definition 1.1.1:** *Let $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ be open subsets. We say that a function $f : U \to V$ is* ***smooth*** *if it has derivatives of all orders everywhere in $U$. The map $f$ is called a diffeomorphism from $U$ to $V$ if it is a smooth bijection and its inverse $f^{-1} : V \to U$ is again smooth. We denote by $C^\infty(U, V)$ the set of smooth functions from $U$ to $V$.*

**Example 1.1.1:** *The function $f : \mathbb{R} \to \mathbb{R}$ defined by*

$$f(x) = \begin{cases} 1.1\epsilon & \textit{if } x \le 0 \\ \dfrac{1.1\epsilon(1 + e^{1/4\epsilon^2})}{1 + e^{1/(4\epsilon^2 - x^2)}} & \textit{if } x \in (0, 2\epsilon) \\ 0 & \textit{if } x \ge 2\epsilon \end{cases}$$

*is smooth for any $\epsilon > 0$.*



### 1.1.2 Smooth manifolds

To formalize the definition of a smooth manifold we need the following notions.

**Definition 1.1.2:** *Let $M$ be a Hausdorff and second countable topological space.*

*(1) A **coordinate chart** or just **chart** of $M$ is a pair $(U, X)$ where $U$ is an open subset of $M$ and $X : U \to \mathbb{R}^n$ is a map such that $X(U)$ is an open subset of $\mathbb{R}^n$ and $X$ is a homeomorphism from $U$ to $X(U)$.*

*(2) Two charts $(U, X)$ and $(V, Y)$ are called **compatible** if the subsets $X(U \cap V)$ and $Y(U \cap V)$ are open subsets of $\mathbb{R}^n$, and the transition map*

$$X \circ Y^{-1} : Y(U \cap V) \to X(U \cap V)$$

*is a diffeomorphism.*

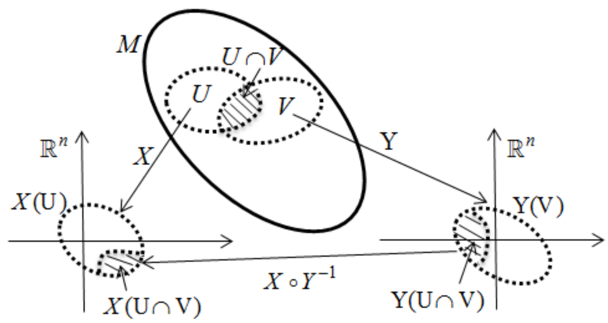



Figure 1.1: Compatible charts

*(3) A collection of charts $\mathcal{A} = \{(U_i, X_i)\}$ is an $n$-dimensional **atlas** on $M$ if any two charts are compatible and $\cup_i U_i = M$.*

*(4) Two atlases $\mathcal{A}_1$ and $\mathcal{A}_2$ are **equivalent** if $\mathcal{A}_1 \cup \mathcal{A}_2$ is again an atlas.*

*(5) **A differentiable manifold structure** on $M$ is an equivalence class of atlases.*

*(6) **A smooth manifold of dimension** $n$ is a topological space $M$ together with a differentiable manifold structure on it.*

**Remark 1.1.1:** *From Definition 1.1.2:*

*(a) If $p \in U \subset M$, then $X(p) = (x_1(p), x_2(p), \cdots, x_n(p)) \in \mathbb{R}^n$.*

*(b) Since $X$ is continuous, so $x_i : U \to \mathbb{R}$ is a real valued continuous function for each $i = 1, 2, \ldots, n$.*

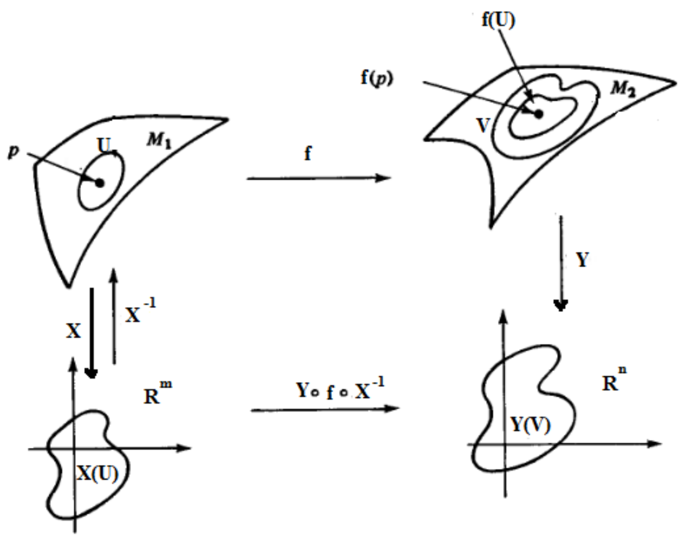
*(c) The pair $(U, X)$ is called a **coordinate neighborhood** (or a **coordinate chart** or a **chart**) of $M$.*

*(d) $(x_1, x_2, \cdots, x_n)$ is called the **local coordinate system** (or **local coordinate** ) on $(U, X)$.*

**Example 1.1.2:** *The $n$-sphere $S^n = \{x = (x_1, \cdots, x_{n+1}) \ : ||x|| = 1\} \subset \mathbb{R}^{n+1}$ is a smooth manifold.*

### 1.1.3 Smooth Maps between Smooth Manifolds

**Definition 1.1.3:** *Let $M_1$ and $M_2$ be smooth manifolds of dimension $m$ and $n$ respectively. A map $f : M_1 \to M_2$ is* ***smooth*** *at $p \in M_1$ if given a chart $(V, Y)$ at $f(p) \in M_2$ there exists a chart $(U, X)$ at $p \in M_1$ such that $f(U) \subseteq V$ and the mapping $Y \circ f \circ X^{-1} : X(U) \subset \mathbb{R}^m \to Y(V) \subset \mathbb{R}^n$ is smooth at $X(p)$. A map $f$ is* ***smooth*** *if it is* ***smooth*** *at every point of $M_1$. The set of smooth functions from $M_1$ to $M_2$ is denoted $C^\infty(M_1, M_2)$.*




In particular, a map $f : M \to \mathbb{R}$ on a smooth manifold $M$ is called **smooth** if for all $p \in M$ there is a chart $(U, X)$ about $p$ such that the map $f \circ X^{-1} : X(U) \to \mathbb{R}$ is **smooth**. We denote by $C^\infty(M, \mathbb{R}) = C^\infty(M)$ the set of real valued smooth functions on $M$.

**Definition 1.1.4:** *Let $M$ and $N$ be two smooth manifolds. We say that a mapping*

$$\varphi : M \to N$$

*(1) is a* ***diffeomorphism*** *if it is bijection, and the maps $\varphi$ and $\varphi^{-1}$ are smooth;*

*(2) is a* ***local diffeomorphism*** *at $p \in M$ if there exist neighborhoods $U$ of $p$ and $V$ of $\varphi(p)$ such that the map $\varphi_{|_U} : U \to V$ is a diffeomorphism.*

### 1.1.4 Tangent Vectors and Tangent Spaces

**Definition 1.1.5:** *Let $M$ be a smooth manifold. For any $p \in M$, choose a smooth curve $\alpha : (-\epsilon, \epsilon) \to M$ with $\alpha(0) = p$. Let $\mathcal{D}$ be the set of all real valued functions on $M$ that* *are smooth at $p$. The* ***tangent vector*** *to the curve $\alpha$ at $t = 0$ (or the* ***tangent vector*** *to $M$ at $p$) is a function $\alpha'(0) : \mathcal{D} \to \mathbb{R}$ given by*

$$\alpha'(0)f = \frac{d(f \circ \alpha)}{dt} \,|_{t=0}, \; f \in \mathcal{D}.$$

*The* ***tangent space*** *of $M$ at $p$, denoted by $T_pM$, is the set of all tangent vectors to $M$ at $p$.*

**Definition 1.1.6:** *Let $M$ and $N$ be smooth manifolds of dimension $m$ and $n$ respectively, and let $g : M \to N$ be a smooth map. For any $p \in M$ and for each $v \in T_pM$, choose a smooth curve $\alpha : (-\epsilon, \epsilon) \to M$ with $\alpha(0) = p, \alpha'(0) = v$. The* ***differential*** *of $g$ at $p$ is the linear map $dg_p : T_pM \to T_{g(p)}N$ given by $dg_p(v) = \beta'(0)$, where $\beta = g \circ \alpha$ is independent of the choice of $\alpha$.*

### 1.1.5 Hessian, Regular points, Critical Points of a Function

**Definition 1.1.7:** *Let $M$ be a smooth manifold of dimension $n$, and let $f : M \to \mathbb{R}$ be a smooth map of $M$. For each point $p \in M$, we choose a chart about $p$, $X : U \to V \subset \mathbb{R}^n$ such that $X(p) = (x_1(p), \cdots, x_n(p)) \in V$. Let*

$$F = f \circ X^{-1} : \mathbb{R}^n \to \mathbb{R},$$

*and the derivative*

$$dF_{X(p)} : T_{X(p)}\mathbb{R}^n \to T_{F(X(p))}\mathbb{R}.$$

*Then*

*(1) The* ***Hessian*** *of $f$ with respect to $X$ is defined as the symmetric matrix of second order partial derivatives:*

$$H_F = H(f \circ X^{-1}) = \left( \frac{\partial^2 F}{\partial x_i \partial x_j} \right)_{1 \leq i,j \leq n}.$$

*(2) $p$ is a* ***critical point or singular point*** *of $f$ if $dF_{X(p)}$ is not surjective, this means that the partial derivatives*

$$\frac{\partial F}{\partial x_1}(X(p)) = 0, \cdots, \frac{\partial F}{\partial x_n}(X(p)) = 0.$$

*The real value $f(p) = F(X(p))$ is then called a* ***critical value*** *of $f$.*

*(3) Any point which is not* ***a critical point*** *of $f$ is called a* ***regular point*** *of $f$, and any real value which is not a* ***critical value*** *of $f$ is called a* ***regular value*** *of $f$.*

*(4) $p$ is a* ***non-degenerate critical point*** *of $f$ if the Hessian is* ***non-singular****, that is, $\det\Big(H_F(X(p))\Big) \neq 0$.*

*(5) Any critical point whose Hessian is singular is called a* ***degenerate critical point****.*

*(6) The* ***index*** *of a non-degenerate critical point $p$ with respect to $f$ is the number of negative eigenvalues of the Hessian $H_F(X(p))$.*

**Example 1.1.3:** *Let $M = S^2 = \{(x, y, z) \in \mathbb{R}^3 \ : \ x^2 + y^2 + z^2 = 1\}$ be the unit sphere in $\mathbb{R}^3$. The function $f : M \to \mathbb{R}$ by $(x, y, z) \mapsto z$ is a Morse function.*

*Proof.* Let

$$\phi_1(x_1, x_2, x_3) = (\frac{x_1}{1-x_3}, \frac{x_2}{1-x_3}) \qquad \text{and} \qquad \phi_2(x_1, x_2, x_3) = (\frac{x_1}{1+x_3}, \frac{x_2}{1+x_3})$$

be two charts of $S^2$. The inverses of $\phi_1$ and $\phi_2$ are

$$\phi_1^{-1}(y_1, y_2) = \left(\frac{2y_1}{y_1^2+y_2^2+1}, \frac{2y_2}{y_1^2+y_2^2+1}, \frac{y_1^2+y_2^2-1}{y_1^2+y_2^2+1}\right)$$

and

$$\phi_2^{-1}(x_1, x_2) = \left(\frac{2x_1}{1+x_1^2+x_2^2}, \frac{2x_2}{1+x_1^2+x_2^2}, \frac{1-x_1^2-x_2^2}{1+x_1^2+x_2^2}\right)$$

respectively. In order to determine the critical points of $f$, consider the map $f \circ \phi_i^{-1} : \mathbb{R}^2 \to \mathbb{R}$ for each $i = 1, 2$. Note that $(S^2 \setminus \{S\}, \phi_2)$ is the coordinate chart around $(0, 0, 1)$ and define a map $g = f \circ \phi_2^{-1} : \mathbb{R}^2 \to \mathbb{R}$ by

$$g(x_1, x_2) := f \circ \phi_2^{-1}(x_1, x_2) = \frac{1-x_1^2-x_2^2}{1+x_1^2+x_2^2}.$$

Since

$$dg_{(x_1,x_2)} = \left(\frac{-4x_1}{(1+x_1^2+x_2^2)^2}, \frac{-4x_2}{(1+x_1^2+x_2^2)^2}\right),$$

we have

$$dg_{(x_1,x_2)} = 0 \text{ if and only if } x_1 = x_2 = 0.$$

Hence $\phi_2^{-1}(0, 0) = (0, 0, 1)$ is the only critical point of $f$ in $S^2 \setminus \{S\}$. We will now find the Hessian of $f$ at $(0, 0, 1)$. By Definition 1.1.7,

$$\begin{aligned} H_g(\phi_2(0,0,1)) = H_g(0,0) &= \left(\frac{\partial^2 g}{\partial x_i \partial x_j}(0,0)\right)_{1 \le i,j \le 2} \\ &= \begin{pmatrix} \frac{-4(1-3x_1^2+x_2^2)}{(1+x_1^2+x_2^2)^3}\Big|_{(0,0)} & \frac{16x_1x_2}{(1+x_1^2+x_2^2)^3}\Big|_{(0,0)} \\ \frac{16x_1x_2}{(1+x_1^2+x_2^2)^3}\Big|_{(0,0)} & \frac{-4(1+x_1^2-3x_2^2)}{(1+x_1^2+x_2^2)^3}\Big|_{(0,0)} \end{pmatrix} \\ &= \begin{pmatrix} -4 & 0 \\ 0 & -4 \end{pmatrix} \end{aligned}$$

This shows that $(0, 0, 1)$ is a non-degenerate critical point of $f$ with index 2. For the point $(0, 0, -1)$, we use the chart $(S^2 \setminus \{N\}, \phi_1)$, and a similar calculation shows that $(0, 0, -1)$ is the only critical point of $f$ in $S^2 \setminus \{N\}$ with index 0. $\square$

**Example 1.1.4:** *Let $r$ and $R$ be real numbers satisfying $0 < r < R$, and let*

$$M = \mathbb{T}^2 = \{(x, y, z) \ : x^2 + (\sqrt{y^2 + z^2} - R)^2 = r^2\}$$

*be a two dimensional torus. The function $f : \mathbb{T}^2 \to \mathbb{R}$ defined by $f((x, y, z)) = z$ is a Morse funtion which has four non-degenerate critical points*

$$(0, 0, -(R + r)),\ (0, 0, -(R - r)),\ (0, 0, R - r)\ \text{ and } (0, 0, R + r)$$

*with indices $0, 1, 1$ and $2$ respectively.*

**Proposition 1.1.1:** *The notions* (2), (3), (4) *defined in Definition 1.1.7 do not depend on the choice of chart.*

*Proof.* Let $(U_1, \varphi_1)$ and $(U_2, \varphi_2)$ be coordinate charts of $M$ around a critical point $p$ of $f$ such that $\varphi_1(p) = \Big(x_1(p), \cdots, x_n(p)\Big) = \Big(y_1(p), \cdots, y_n(p)\Big) = \varphi_2(p)$.We note that

$$f \circ \varphi_1^{-1} = (f \circ \varphi_2^{-1}) \circ (\varphi_2 \circ \varphi_1^{-1}) \tag{1.1.1}$$

and

$$f \circ \varphi_2^{-1} = (f \circ \varphi_1^{-1}) \circ (\varphi_1 \circ \varphi_2^{-1}). \tag{1.1.2}$$

(2) We will prove that $\dfrac{\partial(f \circ \varphi_1^{-1})}{\partial x_i}(\varphi_1(p)) = 0$ if and only if $\dfrac{\partial(f \circ \varphi_2^{-1})}{\partial y_i}(\varphi_2(p)) = 0$, for all $i = 1, 2, \cdots, n$.

Suppose that for all $i$, we have $\dfrac{\partial(f \circ \varphi_2^{-1})}{\partial y_i}(\varphi_2(p)) = 0$, and let $(\varphi_2 \circ \varphi_1^{-1})_j$ be the $j$th coordinate function of $\varphi_2 \circ \varphi_1^{-1}$. By equation (1.1.1) and $\varphi_2 \circ \varphi_1^{-1}(\varphi_1(p)) = \varphi_2(p)$, using the chain rule we obtain

$$\left.\frac{\partial(f \circ \varphi_1^{-1})}{\partial x_i}\right|_{\varphi_1(p)} = \sum_{j=1}^{n} \frac{\partial(f \circ \varphi_2^{-1})}{\partial y_j}(\varphi_2 \circ \varphi_1^{-1})\Big|_{\varphi_1(p)} \left.\frac{\partial(\varphi_2 \circ \varphi_1^{-1})_j}{\partial x_i}\right|_{\varphi_1(p)}. \tag{1.1.3}$$

Hence

$$\frac{\partial(f \circ \varphi_1^{-1})}{\partial x_i}(\varphi_1(p)) = \sum_{j=1}^{n} \frac{\partial(f \circ \varphi_2^{-1})}{\partial y_j}(\varphi_2(p)) \frac{\partial(\varphi_2 \circ \varphi_1^{-1})_j}{\partial x_i}(\varphi_1(p)).$$

By hypothesis, $\dfrac{\partial(f \circ \varphi_1^{-1})}{\partial x_i}(\varphi_1(p)) = 0$. Similarly, by equation (1.1.2) and the chain rule, if we have $\dfrac{\partial(f \circ \varphi_1^{-1})}{\partial x_i}(\varphi_1(p)) = 0$, for all $i$, then we have $\dfrac{\partial(f \circ \varphi_1^{-1})}{\partial y_i}(\varphi_2(p)) = 0$.

(3) It follows from the previous point.

(4) We will prove that $\det\Big(H_{f\circ\varphi_1^{-1}}(\varphi_1(p))\Big) \neq 0$ if and only if $\det\Big(H_{f\circ\varphi_2^{-1}}(\varphi_2(p))\Big) \neq 0$.

From equation (1.1.3), for $1 \leq j \leq n$, we have

$$\frac{\partial(f\circ\varphi_1^{-1})}{\partial x_j}\bigg|_{\varphi_1(p)} = \sum_{k=1}^{n} \frac{\partial(f\circ\varphi_2^{-1})}{\partial y_k}(\varphi_2\circ\varphi_1^{-1})\bigg|_{\varphi_1(p)} \frac{\partial(\varphi_2\circ\varphi_1^{-1})_k}{\partial x_j}\bigg|_{\varphi_1(p)}.$$

By applying the chain rule again and $\varphi_2\circ\varphi_1^{-1}(\varphi_1(p)) = \varphi_2(p)$,

$$\begin{aligned}
\frac{\partial^2(f\circ\varphi_1^{-1})}{\partial x_i\partial x_j}\bigg|_{\varphi_1(p)} &= \frac{\partial}{\partial x_i}\Bigg(\sum_{k=1}^{n} \frac{\partial(f\circ\varphi_2^{-1})}{\partial y_k}(\varphi_2\circ\varphi_1^{-1})\bigg|_{\varphi_1(p)} \frac{\partial(\varphi_2\circ\varphi_1^{-1})_k}{\partial x_j}\bigg|_{\varphi_1(p)}\Bigg)\\
&= \sum_{k=1}^{n}\frac{\partial}{\partial x_i}\Bigg(\frac{\partial(f\circ\varphi_2^{-1})}{\partial y_k}(\varphi_2\circ\varphi_1^{-1})\bigg|_{\varphi_1(p)} \frac{\partial(\varphi_2\circ\varphi_1^{-1})_k}{\partial x_j}\bigg|_{\varphi_1(p)}\Bigg)\\
&= \sum_{k=1}^{n}\Bigg(\sum_{l=1}^{n}\frac{\partial^2(f\circ\varphi_2^{-1})}{\partial y_l\partial y_k}(\varphi_2\circ\varphi_1^{-1})\bigg|_{\varphi_1(p)} \frac{\partial\left(\varphi_2\circ\varphi_1^{-1}\right)_l}{\partial x_i}\bigg|_{\varphi_1(p)}\Bigg)\frac{\partial(\varphi_2\circ\varphi_1^{-1})_k}{\partial x_j}\bigg|_{\varphi_1(p)}\\
&+ \sum_{k=1}^{n}\frac{\partial(f\circ\varphi_2^{-1})}{\partial y_k}(\varphi_2\circ\varphi_1^{-1})\bigg|_{\varphi_1(p)}\frac{\partial^2(\varphi_2\circ\varphi_1^{-1})_k}{\partial x_i\partial x_j}\bigg|_{\varphi_1(p)}\\
&= \sum_{k=1}^{n}\Bigg(\sum_{l=1}^{n}\frac{\partial^2(f\circ\varphi_2^{-1})}{\partial y_l\partial y_k}(\varphi_2(p))\frac{\partial\left(\varphi_2\circ\varphi_1^{-1}\right)_l}{\partial x_i}(\varphi_1(p))\Bigg)\frac{\partial(\varphi_2\circ\varphi_1^{-1})_k}{\partial x_j}(\varphi_1(p)).
\end{aligned}$$

Since $p$ is a critical point of $f$,

$$\frac{\partial(f\circ\varphi_2^{-1})}{\partial y_k}(\varphi_2\circ\varphi_1^{-1})\bigg|_{\varphi_1(p)} = \frac{\partial(f\circ\varphi_2^{-1})}{\partial y_k}(\varphi_2(p)) = 0,\ \forall k.$$

Now, for $1 \leq k, l \leq n$, the above expression can be written as:

$$\frac{\partial^2(f\circ\varphi_1^{-1})}{\partial x_i\partial x_j}(\varphi_1(p)) = \left(\frac{\partial(\varphi_2\circ\varphi_1^{-1})_1}{\partial x_i},\cdots,\frac{\partial(\varphi_2\circ\varphi_1^{-1})_n}{\partial x_i}\right)_{\varphi_1(p)} H_2\begin{pmatrix}\frac{\partial(\varphi_2\circ\varphi_1^{-1})_1}{\partial x_j}\\ \vdots\\ \frac{\partial(\varphi_2\circ\varphi_1^{-1})_n}{\partial x_j}\end{pmatrix}_{\varphi_1(p)},$$

where $H_2 = H_{f\circ\varphi_2^{-1}}(\varphi_2(p))$. Hence, for all $1 \leq i, j \leq n$, we obtain

$$H_{f\circ\varphi_1^{-1}}(\varphi_1(p)) = J^t H_{f\circ\varphi_2^{-1}}(\varphi_2(p))J, \tag{1.1.4}$$

where

$$J = J(\varphi_2\circ\varphi_1^{-1})(\varphi_1(p)) = \begin{pmatrix}\frac{\partial(\varphi_2\circ\varphi_1^{-1})_1}{\partial x_1} & \cdots & \frac{\partial(\varphi_2\circ\varphi_1^{-1})_1}{\partial x_n}\\ \vdots & \ddots & \vdots\\ \frac{\partial(\varphi_2\circ\varphi_1^{-1})_n}{\partial x_1} & \cdots & \frac{\partial(\varphi_2\circ\varphi_1^{-1})_n}{\partial x_n}\end{pmatrix}_{\varphi_1(p)}$$

is the Jacobian of $\varphi_2 \circ \varphi_1^{-1}$ at $\varphi_1(p)$ and $J^t$ is its transpose. Since $\varphi_2 \circ \varphi_1^{-1}$ is a smooth map with smooth inverse, the matrix $J = J(\varphi_2 \circ \varphi_1^{-1})(\varphi_1(p))$ is non-singular. Therefore, equation (1.1.4) implies $\det\left(H_{f\circ\varphi_1^{-1}}(\varphi_1(p))\right) \neq 0$ if and only if $\det\left(H_{f\circ\varphi_2^{-1}}(\varphi_2(p))\right) \neq 0$.

□

**Proposition 1.1.2:** *The index of a non-degenerate critical point is independent of the chart.*

*Proof.* According to equation (1.1.4), $H_{f\circ\varphi_1^{-1}}(\varphi_1(p))$ and $H_{f\circ\varphi_2^{-1}}(\varphi_2(p))$ are congruent. Therefore, by Sylvester's Law, $H_{f\circ\varphi_1^{-1}}(\varphi_1(p))$ and $H_{f\circ\varphi_2^{-1}}(\varphi_2(p))$ have the same index. □

We need to state, without proof, the Morse-Sard-Federer theorem (see Theorem 3.4.3 of [4] or Theorem 4, p. 10 of [2] or p.16 of [12]).

**Theorem 1.1.1:** ***(Morse-Sard-Federer theorem)*** *Let $f : M \to N$ be a smooth map between smooth finite dimensional manifolds.*

*(1) The set of critical values of $f$ has measure zero in $N$.*

*(2) If $f(M)$ has nonempty interior, then the set of regular values is dense in the image $f(M)$.*

□

### 1.1.6 Vector Fields and One-Parameter Tranformation Groups

**Definition 1.1.8:** *A* ***smooth vector field*** *on a smooth manifold $M$ is a smooth map $X : M \to TM$, such that for each $p \in M$ we assign a vector $X_p \in T_pM$, $X : p \longmapsto (p, X_p)$.*

**Definition 1.1.9:** *Let $c : I \to M$ be a smooth curve. A* ***smooth vector field*** *$V$ along $c$ is a smooth map that associates to every $t \in I$ a tangent vector $V(t) \in T_{c(t)}M$. A* ***velocity vector*** *(or* ***tangent vector field****), $\frac{dc}{dt} \in T_{c(t)}M$, is defined by*

$$\frac{dc}{dt}(f) = \lim_{h\to 0} \frac{f(c(t+h)) - f(c(t))}{h}, \; f \in \mathcal{D}.$$

**Definition 1.1.10:** *A one-parameter group of diffeomorphisms of a smooth manifold $M$ is a smooth map $\phi : \mathbb{R} \times M \to M$ satisfying the following properties:*

*(a) For each $t \in \mathbb{R}$, the map $\phi_t : M \to M$ defined by $\phi_t(q) = \phi(t, q)$ is a diffeomorphism of $M$ onto itself.*

*(b) For all $s, t \in \mathbb{R}$, we have $\phi_{s+t} = \phi_s \circ \phi_t$.*

Next, given a one-parameter group of diffeomorphisms $\phi$ on a smooth manifold $M$, we define a vector field $X$ on $M$ by

$$X_q(f) = \lim_{h\to 0} \frac{f(\phi_h(q)) - f(q)}{h}, \tag{1.1.5}$$

where $f$ is any smooth real valued function on $M$. This smooth vector field is said to **generate** the group $\phi$.

**Example 1.1.5:** *Let $M$ be the 1-sphere $S^1$. The map*

$$\begin{aligned} \phi : \mathbb{R}\times S^1 &\longrightarrow S^1 \\ \big(t,(x,y)\big) &\mapsto \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \end{aligned}$$

*is a one-parameter group of diffeomorphisms.*

**Definition 1.1.11:** *A* ***Riemannian manifold*** *is a smooth manifold endowed with an inner product on each tangent space which vary smoothly.*

**Definition 1.1.12:** *Let $M$ be a Riemannian manifold. Let $\langle X, Y\rangle$ denote the inner product of two tangent vectors, as defined by this metric, and let $f \in \mathcal{D}$. The* ***gradient*** *of $f$ as a vector field* ***grad****$f$ on $M$ defined by*

$$\langle X, gradf\rangle = X(f).$$

*In other words,*

$$\langle v, gradf(p)\rangle = df_p(v),\ p\in M,\ \forall v \in T_pM.$$

**Remark 1.1.2:**

*(a) The vector field $gradf(p) = 0$ if $p$ is a critical point of $f$.*

*(b) If we have a curve $c : \mathbb{R} \to M$ with velocity vector $\dfrac{dc}{dt}$, then*

$$\frac{d(f\circ c)}{dt} = df_{c(t)}(\frac{c(t)}{dt}) = \left\langle \frac{dc}{dt}, gradf(c(t))\right\rangle.$$

**Lemma 1.1.1:** *A smooth vector field $X$ on $M$ which vanishes outside of a compact subset $K$ of $M$ generates a unique one-parameter group of diffeomorphisms $\phi$ of $M$.* □

## 1.1.7 Jacobian of a map and coarea formula

Let $M_0$ and $M_1$ be smooth, connected, Riemannian manifolds of dimension $n$, equipped with Riemann metrics $g_0$ and $g_1$ respectively. Let $F : M_0 \to M_1$ be a smooth map. For any $x_0 \in M_0$, the differential map of $F$ at $x_0$ is a linear map

$$dF_{x_0} : T_{x_0}M_0 \to T_{F(x_0)}M_1.$$

If we choose an orthonormal basis $\{\vec{e}_i\}_{1\le i\le n}$ of $T_{x_0}M_0$ and let $\vec{f}_k = dF_{x_0}(\vec{e}_k)$, then we can form an $n\times n$ symmetric matrix

$$G_F(x_0) := \left(\left\langle \vec{f}_i, \vec{f}_j\right\rangle_{g_1}\right)_{1\le i,j\le n}.$$

The matrix $G_F(x_0)$ is non-negative because for any $y=(y_1,\cdots,y_n)\in\mathbb{R}^n$, we have

$$\begin{aligned}
yG_F(x_0)y^T &= \Big(\sum_{i=1}^n \langle f_i, f_1\rangle\, y_i, \cdots, \sum_{i=1}^n \langle f_i, f_n\rangle\, y_i\Big) y^T \\
&= \sum_{j=1}^n\sum_{i=1}^n \langle f_i, f_j\rangle\, y_i y_j \\
&= \sum_{j=1}^n\sum_{i=1}^n \langle f_i y_i, f_j y_j\rangle \\
&= \sum_{j=1}^n \left\langle \sum_{i=1}^n f_i y_i, f_j y_j\right\rangle \\
&= \left\langle \sum_{i=1}^n f_i y_i, \sum_{j=1}^n f_j y_j\right\rangle \\
&= ||\sum_{k=1}^n f_k y_k||^2 \\
&\ge 0
\end{aligned}$$

Since $G_F(x_0)$ is non-negative, all of its eigenvalues are non-negative so that

$$\det(G_F(x_0)) \ge 0.$$

The **Jacobian** of $F$ is the smooth non-negative function

$$\begin{aligned}
|J_F| : M_0 &\to [0,+\infty) \\
x_0 &\mapsto \sqrt{\det G_F(x_0)}.
\end{aligned}$$

Since $G_F(x_0)$ is a symmetric matrix, it can be expressed as

$$G_F(x_0) = Q_F D_F Q_F^T$$

(the spectral decomposition) where $Q$ is an orthogonal matrix and $D=\mathrm{diag}(\lambda_1,\cdots,\lambda_n)$ is a diagonal matrix formed with the egenvalues $\lambda_1,\cdots,\lambda_n$ of $G_F(x_0)$. By the non-negativity of the eigenvalues of $G_F(x_0)$, we have

$$G_F(x_0) = Q_F D_F Q_F^T = Q_F\sqrt{D_F}Q_F^T Q_F\sqrt{D_F}Q_F^T = B_F(x_0)B_F(x_0),$$

where $B_F(x_0) = Q_F\sqrt{D_F}Q_F^T$. Therefore,

$$\det G_F(x_0) = \det\left(B_F(x_0)B_F(x_0)\right) = \det B_F(x_0)\det B_F(x_0) = (\det B_F(x_0))^2.$$

According to Theorem 1.1.1, if $F : M_0 \to M_1$ is a smooth map between smooth finite dimensional manifolds, then almost every $x_1 \in M_1$ is a regular value of $F$. For such $x_1$'s, the fiber $F^{-1}(x_1)$ is a finite set and we denote by $N_F(x_1) \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ its cardinality.

Now we state the coarea formula theorem without proof.

**Theorem 1.1.2:** ***(Coarea formula)*** *Let $F : (M_0, g_0) \to (M_1, g_1)$ be a smooth map between two smooth, compact, connected, oriented, finite-dimensional Riemannian manifolds. Then the function*

$$M_1 \ni x_1 \longmapsto N_F(x_1) \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$$

*is measurable with respect to the Lebesgue measure defined by the volume form $dV_{g_1}$, and*

$$\int_{M_1} N_F(x_1) dV_{g_1}(x_1) = \int_{M_0} |J_F|(x_0) dV_{g_0}(x_0),$$

*where $x_1 = F(x_0)$.*

### 1.1.8 Frenet-Serret formulas

Let $\alpha : I \subset \mathbb{R} \to E = \mathbb{R}^3$ be a curve parametrized by arc length $s$. The tangent, normal, and binormal unit vectors, often called $T(s)$, $N(s)$, and $B(s)$ (or simply $T$, $N$, and $B$) form an orthonormal basis spanning $\mathbb{R}^3$ and are defined as follows:

$$\begin{aligned} T(s) &= \alpha'(s), \\ N(s) &= \frac{\alpha''(s)}{||\alpha''(s)||}, \\ B(s) &= T(s) \times N(s). \end{aligned}$$

The Frenet-Serret Formulas are the following

$$\begin{aligned} T' &= \kappa N \\ N' &= -\kappa T + \tau B \\ B' &= \tau N, \end{aligned}$$

where

$\kappa = \kappa(s) = ||\alpha''(s)||$ is called **the curvature or bending** of $\alpha$ at $s$,
$\tau = \tau(s)$ is called **the torsion or twisting** of $\alpha$ at $s$.

# 1.2 Topology of Manifolds

In this section we will assume that $X$ and $Y$ are topological spaces unless stated otherwise.

## 1.2.1 Homotopy

**Definition 1.2.1:** *A family of maps $h_t : X \to Y,\ t \in [0,1]$ is called a* ***homotopy*** *if the associated map $H : X \times [0,1] \to Y$ given by $H(x,t) = h_t(x)$ is continuous on $X \times [0,1]$.*

**Definition 1.2.2:** *Let $f, g : X \to Y$ be two continuous maps.Then the maps $f$ and $g$ are* ***homotopic*** *if there exists a homotopy $h_t : X \to Y$ such that $h_0(x) = f(x)$ and $h_1(x) = g(x)$ for all $x \in X$, and we write $f \simeq g$.*

**Definition 1.2.3:** *A continuous map $f : X \to Y$ is a* ***homotopy equivalence*** *if there exists a continuous map $g : Y \to X$ such that $f \circ g \simeq id_Y$ and $g \circ f \simeq id_X$. In this case, the spaces $X$ and $Y$ are said to be* ***homotopy equivalent*** *(or to have* ***the same homotopy type****).*

**Remark 1.2.1:** *The map $g$ mentioned in Definition 1.2.3 is called a* ***homotopy inverse*** *of $f$.*

**Example 1.2.1:** *Let $p \in \mathbb{R}^n$. The space $\mathbb{R}^n \setminus \{p\}$ is homotopy equivalent to $S^{n-1}$.*

**Definition 1.2.4:** *A subspace $A$ of $X$ is a* ***deformation retract*** *of $X$ if there exists a homotopy $h_t : X \to X,\ t \in [0,1]$ satisfying:*

*(i)* $h_0(x) = x$ *for all* $x \in X$,

*(ii)* $h_1(x) \in A$ *for all* $x \in X$,

*(iii)* $h_t(a) = a,$ *for all* $a \in A$ *and* $t \in [0,1]$.

## 1.2.2 CW-Complexes

**Definition 1.2.5:** ***(Attaching a $\lambda$-cell)***
*Let $Y$ be any topological space, and let $e^\lambda = \{x \in \mathbb{R}^\lambda\ : ||x|| \leq 1\}$ be the $\lambda$-cell with boundary $\partial(e^\lambda) = \{x \in \mathbb{R}^\lambda\ : ||x|| = 1\} = S^{\lambda-1}$. If $g : S^{\lambda-1} \to Y$ is a continuous map, then $Y$ with a $\lambda$-cell attached by $g$, denoted by $Y \cup_g e^\lambda$, is obtained by taking the disjoint union of $Y$ and $e^\lambda$, and identifying each $x \in S^{\lambda-1}$ with $g(x) \in Y$.*

**Remark 1.2.2:** *$e^0$ is a point and $\partial(e^0) = S^{-1}$ is the empty set.*

**Definition 1.2.6:** *Let $X$ be a Hausdorff space. $X$ is said to be a* ***CW-complex*** *(or* ***cell complex****) if there exists a sequence of subspaces $X^{(0)} \subset X^{(1)} \subset X^{(3)} \subset \cdots \subset X$ such that*

*(i)* $X^{(0)}$ *is a discrete (disjoint union of* $0$*-cells).*

*(ii)* $X^{(i+1)}$ *is obtained from* $X^{(i)}$ *by attaching* $(i+1)$*-cells.*

*(iii)* *The set* $X = \cup_n X^{(n)}$ *is endowed with the weak topology* ***(****if* $A \subset X$ *is open (or closed) if and only if* $A \cap X^{(n)}$ *is open (or closed) in* $X^{(n)}$ *for each* $n$***)****.*

**Definition 1.2.7:** *Let* $X$ *be a CW complex. The* $n$***-skeleton*** *of* $X$*, denoted by* $X^{(n)}$*, is the union of all the cells of dimensions less than or equal to* $n$ *in* $X$*.*

**Definition 1.2.8:** *Let* $X, Y$ *be two CW complexes. A map* $f : X \to Y$ *is a* ***cellular*** *map if* $f(X^{(n)}) \subseteq Y^{(n)}$ *for all* $n$*.*

# Chapter 2

# Morse Theory

In this chapter we will give the definition of Morse functions, prove their existence and describe their properties.

## 2.1 Morse Function

**Definition 2.1.1:** *Let $f$ be a smooth function on a smooth manifold $M$. $f$ is said to be a **Morse function** if every critical point of $f$ is non-degenerate.*

**Example 2.1.1:** *The **height functions** on the sphere $S^2$ and the torus $\mathbb{T}^2$ (Examples 1.1.3 and 1.1.4) are Morse functions.*

**Example 2.1.2:** *Let $[z_0, z_1, \cdots, z_n]$ be an equivalence class of $(n+1)$-tuples $(z_0, z_1, \cdots, z_n)$ of complex numbers, with $\sum_{j=0}^{n} |z_j|^2 = 1$, and let $M = \mathbb{C}P^n = \{[z_0, z_1, \cdots, z_n]\}$ be the complex projective $n$-space. Define $f : M \to \mathbb{R}$ by*

$$[z_0, z_1, \cdots, z_n] \mapsto \sum_{j=0}^{n} c_j |z_j|^2,$$

*where $c_0, c_1, \cdots, c_n$ are distinct real constants. Such a function $f$ is a Morse function.*

*Proof.* In order to determine the critical points of $f$ and their indices, we consider the following local coordinate system. For each $j \in \{0, 1, \cdots, n\}$, let $U_j$ be the set of equivalence classes of $(n+1)$-tuples $(z_0, z_1, \cdots, z_n)$ of complex numbers with $z_j \neq 0$. That is,

$$\begin{aligned}
U_j &= \{[z_0, z_1, \cdots, z_j, \cdots, z_n] \ : \ z_j \neq 0\} \\
&= \left\{\Big[\frac{z_0}{z_j}, \frac{z_1}{z_j}, \cdots, 1, \cdots, \frac{z_n}{z_j}\Big]\right\} \\
&= \left\{\Big[|z_j|\frac{z_0}{z_j}, |z_j|\frac{z_1}{z_j}, \cdots, |z_j|, \cdots, |z_j|\frac{z_n}{z_j}\Big]\right\} \\
&= \left\{\Big[x_0 + iy_0, \cdots, \sqrt{1 - \sum_{k\neq j}(x_k^2 + y_k^2)}, \cdots, x_n + iy_n\Big]\right\},
\end{aligned}$$

where

$$|z_j|\frac{z_k}{z_j} = x_k + iy_k$$

and

$$|z_j| = \sqrt{1 - \sum_{k\neq j}(x_k^2 + y_k^2)}.$$

Let $B_1(0)$ be the open unit ball in $\mathbb{R}^{2n}$. We now prove that $U_j$ is diffeomorphic to $B_1(0)$. We define $g_j : U_j \to B_1(0)$ by

$$g_j(u) = (x_0, y_0, x_1, y_1, \cdots, \cancel{x_j}, \cancel{y_j}, \cdots, x_n, y_n),$$

where $u = [x_0 + iy_0, x_1 + iy_1, \cdots, \sqrt{1 - \sum\limits_{k\neq j}(x_k^2 + y_k^2)}, \cdots, x_n + iy_n]$. The map $g_j$ is well defined since for any

$$u = [x_0 + iy_0, x_1 + iy_1, \cdots, \sqrt{1 - \sum_{k\neq j}(x_k^2 + y_k^2)}, \cdots, x_n + iy_n] \in U_j,$$

we have

$$\begin{aligned}
|g_j(u)|^2 &= |(x_0, y_0, x_1, y_1, \cdots, \cancel{x_j}, \cancel{y_j}, \cdots, x_n, y_n)| \\
&= \sum_{k=0}^{n}(x_k^2 + y_k^2) - (x_j^2 + y_j^2) \\
&< \sum_{k=0}^{n}(x_k^2 + y_k^2) && (\text{since } z_j \neq 0, \text{ so } x_j^2 + y_j^2 > 0) \\
&< 1 && (\text{since } \sum_{k=0}^{n}(x_k^2 + y_k^2) = 1).
\end{aligned}$$

This means that $Im(g_j) \subset B_1(0)$. In addition, it is clear that $g$ is bijective and smooth. Hence $(U_j, g_j)$ is a coordinate chart of $M$ around $[0, \cdots, 1_j, \cdots, 0]$. Note that for any

$$v = (x_0, y_0, x_1, y_1, \cdots, \cancel{x_j}, \cancel{y_j}, \cdots, x_n, y_n) \in B_1(0),$$

we have

$$g_j^{-1}(v) = [x_0 + iy_0, x_1 + iy_1, \cdots, \sqrt{1 - \sum_{k\neq j}(x_k^2 + y_k^2)}, \cdots, x_n + iy_n] \in U_j.$$

We now define $F := f \circ g_j^{-1} : B_1(0) \subset \mathbb{R}^{2n} \to \mathbb{R}$ by

$$\begin{aligned} F(v) &= \sum_{k\neq j} c_k(x_k^2 + y_k^2) + c_j\left(\sqrt{1 - \sum_{k\neq j}(x_k^2 + y_k^2)}^{\,2}\right) \\ &= c_j + \sum_{k\neq j}(c_k - c_j)(x_k^2 + y_k^2) \\ &= c_j + \sum_{k\neq j} b_k(x_k^2 + y_k^2), \end{aligned}$$

with $b_k = c_k - c_j \neq 0,\ \forall k \neq j$, where $v = (x_0, y_0, \cdots, \cancel{x_j}, \cancel{y_j}, \cdots, x_n, y_n) \in B_1(0)$.

To find the critical point of $f$, we have to solve the equation $dF_v = 0$. For any $v = (x_0, y_0, \cdots, \cancel{x_j}, \cancel{y_j}, \cdots, x_n, y_n) \in B_1(0)$, we have

$$dF_v = 2\Big(b_0x_0, b_0y_0, \cdots, \cancel{b_jx_j}, \cancel{b_jy_j}, \cdots, b_nx_n, b_ny_n\Big).$$

This shows that $dF_v = 0$ if and only if $v = 0$. Hence

$$p_j = g_j^{-1}(0) = [0, \cdots, 1_j, \cdots, 0]$$

is the only critical point in $U_j$. We next find the Hessian of $f$ at $p_j$. Let $t_{2s} = x_s$ and $t_{2s+1} = y_s$ for $s = 0, 1, \cdots, n$ and so by definition 1.1.7,

$$H_F(g_j(p_j)) = H_F(0) = \left(\frac{\partial^2 F}{\partial t_k \partial t_l}(0)\right)_{k,l\in\{0,1,\cdots,2n+1\}\setminus\{2j,2j+1\}} = \begin{pmatrix} 2b_0 & 0 & \cdots & 0 & 0 \\ 0 & 2b_0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 2b_n & 0 \\ 0 & 0 & \cdots & 0 & 2b_n \end{pmatrix}.$$

This shows that $p_j$, for each $j = 0, 1, \cdots, n$, is a non-degenerate critical point of $f$ since $b_k \neq 0,\ \forall k \neq j$, so that $H_F(g_j(p_j))$ is non-singular. The critical point $p_j$ has index equal to twice the number of $k$ with $b_k < 0$ (or $c_k < c_j$). Therefore, $f$ is a Morse function. □

## 2.2 Morse lemma

**Lemma 2.2.1:** ***(Morse lemma)*** *Let $p$ be non degenerate critical point of $f$ with index $\lambda$. Then there is a local coordinate system $Y : V \subset \mathbb{R}^n \to U_p$ in a neighborhood $U_P$ of $p$ with $0 \in V$ and $Y(0) = p$ such that the identity*

$$(f \circ Y)(y_1, y_2, \cdots, y_n) = f(p) - y_1^2 - \cdots - y_\lambda^2 + y_{\lambda+1}^2 + \cdots + y_n^2 \tag{2.2.1}$$

*holds throughout $V$.*

Before proving the Morse lemma we prove the following.

**Lemma 2.2.2:** *Let $f \in C^\infty$ be function in a convex neighborhood $V$ of $0$ in $\mathbb{R}^n$, with $f(0) = 0$. Then*

$$f(x_1, x_2, \cdots, x_n) = \sum_{i=1}^{n} x_i g_i(x_1, x_2, \cdots, x_n)$$

*for some suitable $C^\infty$ functions $g_i$ defined in $V$, with $g_i(0) = \frac{\partial f}{\partial x_i}(0)$.*

*Proof.* Let $(x_1, x_2, \cdots, x_n) \in V$. Since $V$ is convex, then $t(x_1, x_2, \cdots, x_n) + (1-t)0 \in V$, for all $0 \le t \le 1$. In other words, $(tx_1, tx_2, \cdots, tx_n) \in V$, for all $0 \le t \le 1$. Define

$$\begin{aligned} F : [0,1] &\rightarrow \mathbb{R} \\ F(t) &= f(tx_1, tx_2, \cdots, tx_n). \end{aligned}$$

By the Fundamental Theorem of Calculus

$$F(1) - F(0) = \int_0^1 \frac{dF(t)}{dt} dt.$$

Since $F(1) = f(x_1, x_2, \cdots, x_n)$ and $F(0) = f(0) = 0$,

$$\begin{aligned} f(x_1, x_2, \cdots, x_n) &= \int_0^1 \frac{df}{dt}(tx_1, tx_2, \cdots, tx_n) dt \\ &= \int_0^1 \left( \frac{\partial f(tx_1, tx_2, \cdots, tx_n)}{\partial x_1} x_1 + \cdots + \frac{\partial f(tx_1, tx_2, \cdots, tx_n)}{\partial x_n} x_n \right) dt \\ &= \int_0^1 \sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i}(tx_1, tx_2, \cdots, tx_n) dt \\ &= \sum_{i=1}^{n} x_i \int_0^1 \frac{\partial f}{\partial x_i}(tx_1, tx_2, \cdots, tx_n) dt. \end{aligned}$$

We define

$$\begin{aligned} g_i : V &\rightarrow \mathbb{R} \\ g_i(x_1, x_2, \cdots, x_n) &= \int_0^1 \frac{\partial f}{\partial x_i}(tx_1, tx_2, \cdots, tx_n) dt. \end{aligned}$$

Since $f \in C^\infty$, so is $g_i$ for each $i$. Furthermore, $g_i(0) = \frac{\partial f}{\partial x_i}(0) \int_0^1 dt = \frac{\partial f}{\partial x_i}(0)$. Therefore, $f(x_1, x_2, \cdots, x_n) = \sum_{i=1}^{n} x_i g_i(x_1, x_2, \cdots, x_n)$. ☐

*Proof.* (**of the Morse lemma**) Without loss of generality, assume that $f(p) = 0$, since we can replace $f$ by $f - f(p)$ if necessary. Choose a local coordinate system $X : V_0 \subset \mathbb{R}^n \rightarrow U_p$ in a neighborhood $U_p$ of $p$ such that $X(0) = p$. Since $f(p) = (f \circ X)(0) = 0$ and

$(f \circ X) \in C^\infty$, by Lemma 2.2.2, there exists $n$ suitable functions $g_i \in C^\infty$ defined in a convex neighborhood $V_1 \subset \mathbb{R}^n$ of 0 such that

$$(f \circ X)(x_1, x_2, \cdots, x_n) = \sum_{i=1}^{n} x_i g_i(x_1, x_2, \cdots, x_n)$$

and satisfy

$$g_i(0) = \frac{\partial (f \circ X)}{\partial x_i}(0), \quad \text{for any } i = 1,\ 2, \cdots, n.$$

Now we have $g_i(0) = \frac{\partial (f\circ X)}{\partial x_i}(0) = \frac{\partial f}{\partial x_i}(p) = 0$. Using Lemma 2.2.2 again, for every $i = 1,\ 2, \cdots, n$, we have

$$g_i(x_1, x_2, \cdots, x_n) = \sum_{j=1}^{n} x_j h_{ij}(x_1, x_2, \cdots, x_n),$$

where for each $1 \le j \le n,\ h_{ij}$ is a $C^\infty$ function defined in a convex neighborhood $V_2 \subseteq V_1$ of 0, with $h_{ij}(0) = \frac{\partial g_i}{\partial x_j}(0),$ for any $j = 1,\ 2, \cdots, n$. Hence

$$\begin{aligned}
(f \circ X)(x_1, x_2, \cdots, x_n) &= \sum_{i=1}^{n} x_i \left( \sum_{j=1}^{n} x_j h_{ij}(x_1, x_2, \cdots, x_n) \right) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j h_{ij}(x_1, x_2, \cdots, x_n) \\
&= \sum_{i=1}^{n} x_i^2 h_{ii}(x_1, x_2, \cdots, x_n) + \sum_{i<j} x_i x_j \left(h_{ij} + h_{ji}\right)(x_1, x_2, \cdots, x_n) \\
&= \sum_{i=1}^{n} x_i^2 H_{ii}(x_1, x_2, \cdots, x_n) + 2 \sum_{i<j} x_i x_j H_{ij}(x_1, x_2, \cdots, x_n),
\end{aligned}$$

where $H_{ij} = \frac{1}{2}\left(h_{ij} + h_{ji}\right) = H_{ji}$. Now let us compute $H_f^x(p) := \left(\frac{\partial f}{\partial x_i \partial x_j}(p)\right)_{1 \le i,j \le n}$, the Hessian matrix of $f$ at $p$. We know that $\frac{\partial f}{\partial x_i \partial x_j}(p) = \frac{\partial (f\circ X)}{\partial x_i \partial x_j}(0)$. So, let us compute the second order partial derivative of $f \circ X$ at the origin. From the computation above, we have defined $f \circ X$ in a convex neighborhood $V_2$ of 0 by

$$(f \circ X)(x_1, x_2, \cdots, x_n) = \sum_{i=1}^{n} x_i^2 H_{ii}(x_1, x_2, \cdots, x_n) + 2 \sum_{i<j} x_i x_j H_{ij}(x_1, x_2, \cdots, x_n). \quad (2.2.2)$$

Therefore,

$$\frac{\partial (f \circ X)}{\partial x_i \partial x_i}(0) = 2H_{ii}(0)$$

and

$$\frac{\partial (f \circ X)}{\partial x_i \partial x_j}(0) = \frac{\partial (f \circ X)}{\partial x_j \partial x_i}(0) = 2H_{ij}(0), \text{ for all } i < j.$$

Therefore,

$$\frac{\partial (f \circ X)}{\partial x_i \partial x_j}(0) = 2H_{ij}(0), \text{ for all } 1 \le i, j \le n.$$

By hypothesis, $p$ is a non degenerate critical point of $f$, so

$$0 \neq \det\left(H_f^x(p)\right) = \det\left(\frac{\partial f}{\partial x_i \partial x_j}(p)\right) = \det\left(\frac{\partial f \circ X}{\partial x_i \partial x_j}(0)\right) = \det\left(2H_{ij}(0)\right)_{1\le i,j\le n}.$$

We can assume that $H_{rr}^x(p) = 2H_{rr}(0) \neq 0$. If $H_{rr}(0) = 0$ and $\det\left(H_{ij}(0)\right)_{1\le i,j\le n} \neq 0$, then there exists $i > r$ such that $H_{ir} \neq 0$. Hence we choose a new suitable local coordinate system

$$(\widehat{x}_1, \cdots, \widehat{x}_{r-1}, \widehat{x}_r, \cdots, \widehat{x}_{i-1}, \widehat{x}_i, \cdots, \widehat{x}_n) = (x_1, \cdots, x_{r-1}, \frac{x_r + x_i}{2}, \cdots, x_{i-1}, \frac{x_r - x_i}{2}, \cdots, x_n).$$

Therefore, $\widehat{H}_{rr} = H_{rr} + H_{ir} \neq 0$. We wish to prove this lemma by induction. Now suppose that $H_{11}(0) \neq 0$ and by the continuity of $H_{ij}$ ($H_{ij} \in C^\infty$, for every $i, j$), there is a neighborhood $V_3 \subseteq V_2$ of 0 such that $H_{11} \neq 0$ on it. We define a new first coordinate $y_1$ near $V_3$ by

$$y_1 = \sqrt{|H_{11}|}\left(x_1 + \sum_{j=2}^n x_j \frac{H_{1j}}{H_{11}}\right)$$

and for each $2 \le j \le n$, we keep the $x_j$-coordinate as it is. Thus

$$x_1 = \frac{1}{\sqrt{|H_{11}|}} y_1 - \sum_{j=2}^n x_j \frac{H_{1j}}{H_{11}}$$

and

$$\det\begin{pmatrix} \frac{\partial x_1}{\partial y_1}(0) & \frac{\partial x_1}{\partial x_2}(0) & \cdots & \frac{\partial x_1}{\partial x_n}(0) \\ \frac{\partial x_2}{\partial y_1}(0) & \frac{\partial x_2}{\partial x_2}(0) & \cdots & \frac{\partial x_2}{\partial x_n}(0) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial y_1}(0) & \frac{\partial x_n}{\partial x_2}(0) & \cdots & \frac{\partial x_n}{\partial x_n}(0) \end{pmatrix} = \det\begin{pmatrix} \frac{1}{\sqrt{|H_{11}(0)|}} & -\frac{H_{12}(0)}{H_{11}(0)} & \cdots & -\frac{H_{1n}(0)}{H_{11}(0)} \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$
$$= \frac{1}{\sqrt{|H_{11}(0)|}} \neq 0.$$

Since the determinant of the Jacobian matrix of the transformation from $(y_1, x_2, \cdots, x_n)$ to $(x_1, x_2, \cdots, x_n)$ evaluated at 0 is not zero, $(y_1, x_2, \cdots, x_n)$ is a local coordinate system on the neighborhood $V_3$ of 0. In $V_3$, we square $y_1$ and

$$y_1^2 = |H_{11}|x_1^2 + 2|H_{11}|\sum_{j=2}^n x_1 x_j \frac{H_{1j}}{H_{11}} + |H_{11}|\left(\sum_{j=2}^n x_j \frac{H_{1j}}{H_{11}}\right)^2$$
$$= \begin{cases} H_{11}x_1^2 + 2\sum\limits_{j=2}^n x_1 x_j H_{1j} + \dfrac{\left(\sum\limits_{j=2}^n x_j H_{1j}\right)^2}{H_{11}} & \text{if } H_{11} > 0 \\ -H_{11}x_1^2 - 2\sum\limits_{j=2}^n x_1 x_j H_{1j} - \dfrac{\left(\sum\limits_{j=2}^n x_j H_{1j}\right)^2}{H_{11}} & \text{if } H_{11} < 0. \end{cases}$$

Hence

$$H_{11}x_1^2 + 2\sum_{j=2}^{n} x_1x_jH_{1j} = \begin{cases} y_1^2 - 2\sum_{2\leq i<j} x_ix_j\frac{H_{1i}H_{1j}}{H_{11}} - \sum_{j=2}^{n} x_j^2\frac{H_{1j}^2}{H_{11}} & \text{if } H_{11} > 0 \\ -y_1^2 - 2\sum_{2\leq i<j} x_ix_j\frac{H_{1i}H_{1j}}{H_{11}} - \sum_{j=2}^{n} x_j^2\frac{H_{1j}^2}{H_{11}} & \text{if } H_{11} < 0 \end{cases} \tag{2.2.3}$$

Therefore, by equations (2.5.10) and (2.2.3),

$$\begin{aligned} f\circ X &= \sum_{i=1}^{n} x_i^2H_{ii} + 2\sum_{i<j} x_ix_jH_{ij} \\ &= x_1^2H_{11} + 2\sum_{j=2}^{n} x_1x_jH_{1j} + \sum_{i=2}^{n} x_i^2H_{ii} + 2\sum_{2\leq i<j} x_ix_jH_{ij} \\ &= \begin{cases} y_1^2 + \sum_{j=2}^{n} x_j^2\left(H_{jj} - \frac{H_{1j}^2}{H_{11}}\right) + 2\sum_{2\leq i<j} x_ix_j\left(H_{ij} - \frac{H_{1i}H_{1j}}{H_{11}}\right) & \text{if } H_{11} > 0 \\ -y_1^2 + \sum_{j=2}^{n} x_j^2\left(H_{jj} - \frac{H_{1j}^2}{H_{11}}\right) + 2\sum_{2\leq i<j} x_ix_j\left(H_{ij} - \frac{H_{1i}H_{1j}}{H_{11}}\right) & \text{if } H_{11} < 0 \end{cases} \\ &= \begin{cases} y_1^2 + \sum_{j=2}^{n} x_j^2H_{jj}^{(1)} + 2\sum_{2\leq i<j} x_ix_jH_{ij}^{(1)} & \text{if } H_{11} > 0 \\ -y_1^2 + \sum_{j=2}^{n} x_j^2H_{jj}^{(1)} + 2\sum_{2\leq i<j} x_ix_jH_{ij}^{(1)} & \text{if } H_{11} < 0, \end{cases} \\ &= \pm y_1^2 + \sum_{j=2}^{n} x_j^2H_{jj}^{(1)} + 2\sum_{2\leq i<j} x_ix_jH_{ij}^{(1)} \end{aligned}$$

where $H_{ij} = \frac{1}{2}(h_{ij} + h_{ji}) = H_{ji}$. Suppose that there is $r > 1$ such that the following equation holds:

$$f\circ Y = \pm y_1^2 \pm y_2^2 \pm \cdots \pm y_{r-1}^2 + \sum_{j=r}^{n} x_j^2H_{jj}^{(r)} + 2\sum_{r\leq i<j} x_ix_jH_{ij}^{(r)}. \tag{2.2.4}$$

We will prove that the equation (2.2.4) holds for $r+1$. We have assumed that $H_{rr}^{(r)}(0) \neq 0$ and again by the continuity of $H_{ij}^{(r)}$, there is a neighborhood $V_{r+2} \subseteq V_{r+1} \subseteq \cdots \subseteq V_3$ of 0 such that $H_{rr}^{(r)} \neq 0$ on it. As in the base case, we define a new $r^{th}$ coordinate $y_r$ near $V_{r+2}$ by

$$y_r = \sqrt{|H_{rr}^{(r)}|}\left(x_r + \sum_{j=r+1}^{n} x_j\frac{H_{rj}^{(r)}}{H_{rr}^{(r)}}\right)$$

and for each $j \neq r$, we keep the $x_j$-coordinate as it is. We obtain that $(y_1, y_2, \cdots, y_{r-1}, y_r, x_{r+1}, \cdots, x_n)$ is a local coordinate system of $V_{r+2}$. By a similar calculation as that of equation (2.2.3), we have

$$x_r^2H_{rr}^{(r)} + 2\sum_{j=r+1} x_rx_jH_{rj}^{(r)} = \pm y_r^2 - 2\sum_{r+1\leq i<j} x_ix_j\frac{H_{ri}H_{rj}}{H_{rr}} - \sum_{j=r+1}^{n} x_j^2\frac{H_{rj}^2}{H_{rr}}.$$

This, together with equation (2.2.4), gives

$$\begin{aligned}
f \circ Y &= \sum_{i \leq r-1} \pm y_i^2 + \sum_{j=r}^{n} x_j^2 H_{jj}^{(r)} + 2 \sum_{r \leq i < j} x_i x_j H_{ij}^{(r)} \\
&= \sum_{i \leq r-1} \pm y_i^2 + x_r^2 H_{rr}^{(r)} + 2 \sum_{j=r+1} x_r x_j H_{rj}^{(r)} + \sum_{j=r+1}^{n} x_j^2 H_{jj}^{(r)} + 2 \sum_{r+1 \leq i < j} x_i x_j H_{ij}^{(r)} \\
&= \sum_{i \leq r} \pm y_i^2 - 2 \sum_{r+1 \leq i < j} x_i x_j \frac{H_{ri} H_{rj}}{H_{rr}} - \sum_{j=r+1}^{n} x_j^2 \frac{H_{rj}^2}{H_{rr}} + \sum_{j=r+1}^{n} x_j^2 H_{jj}^{(r)} + 2 \sum_{r+1 \leq i < j} x_i x_j H_{ij}^{(r)} \\
&= \sum_{i \leq r} \pm y_i^2 + \sum_{j=r+1}^{n} x_j^2 \left( H_{jj}^{(r)} - \frac{H_{rj}^2}{H_{rr}} \right) + 2 \sum_{r+1 \leq i < j} x_i x_j \left( H_{ij}^{(r)} - \frac{H_{ri} H_{rj}}{H_{rr}} \right) \\
&= \sum_{i \leq r} \pm y_i^2 + \sum_{j=r+1}^{n} x_j^2 H_{jj}^{(r+1)} + 2 \sum_{r+1 \leq i < j} x_i x_j H_{ij}^{(r+1)},
\end{aligned}$$

where $H_{jj}^{(r+1)} = H_{jj}^{(r)} - \frac{H_{rj}^2}{H_{rr}}$ and $H_{ij}^{(r+1)} = H_{ij}^{(r)} - \frac{H_{ri} H_{rj}}{H_{rr}}$. □

**Corollary 2.2.1:** *Let $f : M \to \mathbb{R}$ be a smooth function on a smooth manifold $M$. A non-degenerate critical point of a smooth function $f$ is isolated. In particular, if $f$ is a Morse function and $M$ is compact, then $f$ has a finite number of critical points.*

*Proof.* By Lemma 2.2.1, we observe that if $f$ has a non-degenerate critical point at $p$, then there is a coordinate chart $(U_p, Y^{-1})$ of $M$ about $p$ that satisfies equation (2.2.1). This chart contains no other critical point of $f$ other than $p$ since, by equation (2.2.1),

$$d(f \circ Y)_{(y_1, \cdots, y_n)} = (\pm 2y_1, \cdots, \pm 2y_n)$$

and $d(f \circ Y)_{(y_1, \cdots, y_n)} = 0$ if and only if $(y_1, \cdots, y_n) = 0$. Hence $Y(0) = p$ is the only critical point of $f$ in $U_p$. Therefore, $p$ is isolated.

Now suppose that $M$ is compact. If the set of critical points were infinite, it would have an accumulation point. By continuity of $df$, such a point would also be a critical point which is not isolated, which is a contradiction. □

## 2.3 Existence of Morse Functions

The goal of this section is to show the existence of Morse functions on any smooth manifold. Since the Whitney embedding theorem (see [14], Chapter IV) tells us that any smooth manifold is embedded in a suitable Euclidean vector space, let $M$ be a smooth $n$-dimensional manifold embedded in $E = \mathbb{R}^{n+k}$ for some $k \in \mathbb{N}$.

Let $\Lambda$ be a smooth finite dimensional manifold. We will consider the families of smooth functions $f_\lambda : M \to \mathbb{R}$, for all $\lambda \in \Lambda$, and investigate the conditions on $\lambda$ such that $f_\lambda$ has no degenerate critical points. To do this, we will produce a smooth map $\pi : Z \to \Lambda$ and then prove that $f_\lambda$ has no degenerate critical points for every $\lambda \in \Lambda$, which is a regular value of $\pi$. Moreover, Theorem 1.1.1 implies that $f_\lambda$ is a Morse function for almost all $\lambda \in \Lambda$.

Let us recall that $E^* = \{\alpha \mid \alpha : E \to \mathbb{R} \text{ is a linear map}\}$ is the dual space of real vector space $E$, and the following useful definitions:

**Definition 2.3.1:** *The dual of the tangent space $T_xM$ of a smooth manifold $M$ is called the cotangent space at $x$ denoted by*

$$T_x^*M = (T_xM)^*.$$

*An element of $T_x^*M$ is called cotangent vector or covector.*

**Definition 2.3.2:** *Let $f : M \to N$ be a smooth map between smooth finite dimensional manifolds. The differential map of $f$ at $x$ is the linear map $df_x : T_xM \to T_{f(x)}N$.*

*(1) $f$ is called **immersion** if $df_x$ is injective for every $x \in M$.*

*(2) $f$ is called **submersion** if $df_x$ is surjective for every $x \in M$*

**Definition 2.3.3:** *Let $f : M \to N$ be a smooth map and $x \in M$. We have the cotangent map*

$$d^*f_x := (df_x)^* : T^*_{f(x)}N \to T_x^*M$$

*defined as the dual to the tangent map (the differential map of $f$ at $x$)*

$$df_x : T_xM \to T_{f(x)}N.$$

*In particular, if $N = \mathbb{R}$, then $df_x$ is a covector (i.e. $df_x \in T_x^*M$).*

Let $F : \Lambda \times E \to \mathbb{R}$ be a smooth function. We associate to $F$ a smooth family of functions $F_\lambda : E \to \mathbb{R}$ given as $F_\lambda(x) = F(\lambda, x)$, for all $(\lambda, x) \in \Lambda \times E$. Let $f$ and $f_\lambda$, respectively, be the restriction of $F$ to $\Lambda \times M$ and of $F_\lambda$ to $M$. That is

$$F_{|_{\Lambda\times M}} := f : \Lambda \times M \subset \Lambda \times E \to \mathbb{R},$$

and

$$F_{\lambda|_M} := f_\lambda : \{\lambda\} \times M \cong M \to \mathbb{R}.$$

Let $x \in M$. Since $i : M \hookrightarrow E$ is an embedding, $di_x : T_xM \hookrightarrow T_xE = E$ is injective and there is a natural surjective linear map $(di_x)^* := P_x : E^* \to T_x^*M$ defined by

$$\alpha \mapsto \alpha(di_x).$$

In particular, we have the following identity

$$d(f_\lambda)_x = P_x d(F_\lambda)_x$$

since $T_xM \overset{di_x}{\hookrightarrow} T_xE \xrightarrow{d(F_\lambda)_x} \mathbb{R}$ determined $d(f_\lambda)_x : T_xM \longrightarrow \mathbb{R}$ by

$$d(f_\lambda)_x = d(F_\lambda)_x \circ di_x = (di_x)^*(d(F_\lambda)_x) = P_x d(F_\lambda)_x.$$

**Remark 2.3.1:** *The surjective linear map $P_x$ is a submersion since the differential of the linear map $P_x$ is $P_x$ and it is surjective.*

For every $x \in M$, we define a smooth partial differential map of $f$, $\partial^x f : \Lambda \to T_x^* M$ by

$$\partial^x f(\lambda) = d(f_\lambda)_x.$$

**Definition 2.3.4:** *Let $F : \Lambda \times E \to \mathbb{R}$ be a family of smooth functions. We say that*

*(1) $F$ is **sufficiently large relative to the submanifold** $M \hookrightarrow E$ if $\dim \Lambda \geq \dim M$ and for every $x \in M$, the point $0 \in T_x^* M$ is a regular value for $\partial^x f$.*

*(2) $F$ is **large** if for every $x \in E$ the partial differential map*

$$\partial^x F : \Lambda \to E^*$$

*defined by $\partial^x F(\lambda) = d(F_\lambda)_x$ is a submersion.*

**Example 2.3.1:** *Let $E$ be Euclidean space with the standard inner product $\langle \cdot, \cdot \rangle$.*

*(a) Suppose $\Lambda = E^*$ and let $H : E^* \times E \to \mathbb{R}$ be the function defined by*

$$H(\lambda, x) = \lambda(x), \quad \textit{for all } (\lambda, x) \in E^* \times E.$$

*(b) Suppose $\Lambda = E$ and let $R : E \times E \to \mathbb{R}$ be the function defined by*

$$R(\lambda, x) = \frac{1}{2} ||x - \lambda||^2, \quad \textit{for all } (\lambda, x) \in E \times E.$$

*(c) Let $\Lambda$ be the space of positive definite symmetric endomorphisms $A : E \to E$, and let $F : \Lambda \times E \to \mathbb{R}$ be the function defined by*

$$F(A, x) = \frac{1}{2} \langle Ax, x \rangle, \quad \textit{for all } (A, x) \in \Lambda \times E.$$

*The first two functions above are large and the last function is sufficiently large relative to any submanifold of $E$ not passing through the origin.*

*Proof.* (a) Let $x \in E$. We will prove that the differential of

$$\partial^x H : E^* \to E^*, \ \lambda \longmapsto d(H_\lambda)_x$$

is surjective for all $\lambda \in E^*$. Since $H_\lambda : E \to \mathbb{R}$ is given by $H_\lambda(y) = \lambda(y)$.

For every $v \in T_x E = E$, we choose $\alpha : (-\epsilon, \epsilon) \to E$ be the smooth curve on $E$ which is defined by

$$\alpha(t) = x + tv. \tag{2.3.1}$$

Then

$$d(H_\lambda)_x(v) = \frac{d}{dt}\Big(H_\lambda(\alpha(t))\Big)_{\big|t=0} = \frac{d}{dt}\Big(\lambda(x+tv)\Big)_{\big|t=0} = \frac{d}{dt}\Big(\lambda(x)+t\lambda(v)\Big)_{\big|t=0} = \lambda(v).$$

Then $\partial^x H$ is the identity and hence the differential of $\partial^x H$ is surjective. Therefore, $H$ is large.

(b) Let $x \in E$. We wish to show that the differential of

$$\partial^x R : E \to E^*,\ \lambda \longmapsto d(R_\lambda)_x$$

is surjective. We will first find $d(R_\lambda)_x$, where $R_\lambda : E \to \mathbb{R}$ is given by

$$R_\lambda(y) = \frac{1}{2}||y-\lambda||^2.$$

For every $v \in E$, using the smooth curve on $E$ as in (2.3.1), we have

$$\begin{aligned}
d(R_\lambda)_x(v) &= \frac{d}{dt}\Big(R_\lambda(\alpha(t))\Big)_{\big|t=0} \\
&= \frac{1}{2}\frac{d}{dt}\Big(||(x-\lambda)+tv||^2\Big)_{\big|t=0} \\
&= \frac{1}{2}\frac{d}{dt}\Big(\langle (x-\lambda)+tv, (x-\lambda)+tv\rangle\Big)_{\big|t=0} \\
&= \frac{1}{2}\frac{d}{dt}\Big(\langle (x-\lambda), (x-\lambda)\rangle + 2\langle (x-\lambda), tv\rangle + \langle tv, tv\rangle\Big)_{\big|t=0} \\
&= \frac{1}{2}\frac{d}{dt}\Big(||(x-\lambda)||^2 + 2t\langle (x-\lambda), v\rangle + t^2||v||^2\Big)_{\big|t=0} \\
&= \langle (x-\lambda), v\rangle .
\end{aligned}$$

Thus $\partial^x R$ is a linear function which is defined by

$$\partial^x R(\lambda) = d(R_\lambda)_x = \langle (x-\lambda), \cdot\rangle = (x-\lambda)^*,$$

where $(x-\lambda)^*$ is the metric dual.

Now we will prove that the differential of $\partial^x R$ is surjective. It suffices to prove that $\partial^x R$ is surjective since $\partial^x R$ is a linear function. To see this, let $e_1, \cdots, e_{n+k}$ be the orthonormal basis of $E$. Define $\{e_i^* = \langle e_i, \cdot\rangle\}$ as a basis of $E^*$. Let $\alpha \in E^*, \alpha : E \to \mathbb{R}$, and let

$$\lambda = x - \sum_{i=1}^{n+k} \alpha(e_i)e_i \in E.$$

Then $\langle x-\lambda,\cdot\rangle=\alpha$. Indeed, for every $v\in E$, $v=\sum\limits_{i=1}^{n+k}v_ie_i$, we have

$$\langle x-\lambda,v\rangle=\left\langle\sum_{i=1}^{n+k}\alpha(e_i)e_i,\sum_{i=1}^{n+k}v_ie_i\right\rangle=\sum_{i=1}^{n+k}\alpha(e_i)v_i=\alpha(\sum_{i=1}^{n+k}v_ie_i)=\alpha(v)$$

since $\{e_1,\cdots,e_{n+k}\}$ is the orthonormal basis of $E$ and $\alpha$ is linear function.

(c) Let $M$ be a submanifold of $E$ which does not pass through the origin. We will prove that $F:\Lambda\times E\to\mathbb{R}$ is sufficiently large relative to $M$. For every $x\in M$, we consider the partial differential map

$$\partial^x f:=g:\Lambda\to T_x^*M,$$

which is given by $A\mapsto d(f_A)_x$. Since $f_A:M\to\mathbb{R}$ is defined by $y\mapsto\frac{1}{2}\langle Ay,y\rangle$ for every $v\in T_xM$, take a smooth curve $\alpha:(-\varepsilon,\varepsilon)\to M$ : such that $\alpha(0)=x$ and $\frac{d\alpha}{dt}(0)=v$, so that

$$\begin{aligned}
d(f_A)_x(v)&=\frac{d}{dt}\Big(f_A(\alpha(t))\Big)_{\big|t=0}\\
&=\frac{1}{2}\left(\left\langle\frac{d}{dt}A(\alpha(t)),\alpha(t)\right\rangle+\left\langle A(\alpha(t)),\frac{d}{dt}\alpha(t)\right\rangle\right)_{\big|t=0}\\
&=\frac{1}{2}\left(\left\langle A\frac{d\alpha}{dt}(t),\alpha(t)\right\rangle+\left\langle A(\alpha(t)),\frac{d\alpha}{dt}(t)\right\rangle\right)_{\big|t=0}\\
&=\frac{1}{2}\Big(\langle Av,x\rangle+\langle Ax,v\rangle\Big)\\
&=\langle Ax,v\rangle\,.
\end{aligned}$$

Define $g(A)=\langle Ax,\cdot\rangle=(Ax)^*$. To prove that $0\in T_x^*M$ is a regular value of $\partial^x f$, it suffices to prove that

$$dg_B:T_B\Lambda\to T_{(Bx)^*}(T_x^*M)=T_x^*M$$

is surjective for every $B\in g^{-1}(0)$. Note that $g^{-1}(0)=\{B\in\Lambda\ :\ Bx\perp T_xM\}$. Let $B\in g^{-1}(0)$, $C\in T_B\Lambda$, and $\beta:(-\epsilon,\epsilon)\to\Lambda$ be a smooth curve with $\beta(0)=B$ and $\beta'(0)=C$. Then, for every $v\in T_xM$ we have

$$\begin{aligned}
dg_B(C)(v)&=\frac{d}{dt}\Big(g(\beta(t))(v)\Big)_{\big|t=0}\\
&=\frac{d}{dt}\Big(\langle\beta(t)x,v\rangle\Big)_{\big|t=0}\\
&=\left(\langle\beta'(t)x,v\rangle+\left\langle\beta(t)x,\frac{dv}{dt}\right\rangle\right)_{\big|t=0}\\
&=\langle Cx,v\rangle\,.
\end{aligned}$$

Since $\Lambda$ is an open set of the vector space of symmetric matrices, $T_B\Lambda$ is isomorphic to the space of symmetric matrices. Let $\rho \in T_x^*M,\ \rho : T_xM \to \mathbb{R}$ (linear). We will then find an element $C$ of $T_B\Lambda$ such that $\langle Cx, v\rangle = \rho(v),$ for all $v \in T_xM$. To see this, we claim that for every $\rho(v)$, there exists $w \in T_xM$ such that $\rho(v) = \langle w, z\rangle$ . Indeed, let $\alpha_1, \cdots, \alpha_{n+k}$ be orthonormal basis of $E$ such that

$$v = \sum_{i=1}^{n+k} v_i\alpha_i = \sum_{i=1}^{n+k} \alpha_i^*(v)\alpha_i$$

where $\{\alpha_i^*\}_{1\leq i\leq n+k}$ is the dual basis of $E$. Thus

$$\rho(v) = \rho\Big(\sum_{i=1}^{n+k} \alpha_i^*(v)\alpha_i\Big) = \sum_{i=1}^{n+k} \alpha_i^*(v)\rho(\alpha_i) = \left(\sum_{i=1}^{n+k} \rho(\alpha_i)\alpha_i^*\right)(v),$$

and we choose $w = \rho^* = \sum\limits_{i=1}^{n+k} \rho(\alpha_i)\alpha_i$. Therefore,

$$\langle w, v\rangle = \left\langle \sum_{i=1}^{n+k} \rho(\alpha_i)\alpha_i, \sum_{i=1}^{n+k} v_i\alpha_i \right\rangle = \sum_{i=1}^{n+k} v_i\rho(\alpha_i) = \rho(\sum_{i=1}^{n+k} v_i\alpha_i) = \rho(v).$$

Now, for any $x \in M,\ x \neq 0$, we can choose an orthonormal basis $\{\beta_i\}_{1\leq i\leq n+k}$ of $E$ such that $x = \sum\limits_{i=1}^{n+k} x_i\beta_i$ with $x_i \neq 0$ for all $i$ and we have

$$w = \rho^* = \sum_{i=1}^{n+k} \rho(\beta_i)\beta_i$$

such that $\rho(v) = \langle w, v\rangle$ . In this basis, we can find a symmetric matrix

$$C = \begin{pmatrix} \dfrac{\rho(\beta_1)}{x_1} & 0 & \\ & \ddots & \\ & 0 & \dfrac{\rho(\beta_{n+k})}{x_{n+k}} \end{pmatrix}$$

such that $Cx = w$. That is, there exists $C = \begin{pmatrix} \dfrac{\rho(\beta_1)}{x_1} & 0 & \\ & \ddots & \\ & 0 & \dfrac{\rho(\beta_{n+k})}{x_{n+k}} \end{pmatrix} \in T_B\Lambda$ such that $\langle Cx, v\rangle = \langle w, v\rangle = \rho(v),\ \forall v \in T_xM$. This proves that $dg_B$ is surjective. □

**Lemma 2.3.1:** *If $F : \Lambda \times E \to \mathbb{R}$ is large, then it is sufficiently large relative to any smooth submanifold $M \subset E$.*

*Proof.* Suppose that $F$ is large. Then, for every $x \in E$, we have

$$\partial^x F : \Lambda \to E^*,\ \lambda \mapsto d(F_\lambda)_x$$

is a submersion. We wish to prove that $0 \in T_x^*M$ is a regular value for

$$\partial^x f : \Lambda \to T_x^*M, \ \lambda \mapsto d(f_\lambda)_x.$$

Using the identity $d(f_\lambda)_x = P_x d(F_\lambda)_x$, we have

$$\partial^x f = P_x \partial^x F.$$

Hence $\partial^x f$ is a submersion since it is a composition of two submersions $P_x$ and $\partial^x F$. This means that the differential of $\partial^x f$ is surjective and so it has no critical values. This proves that $0 \in T_x^*M$ is a regular value for $\partial^x f$. $\square$

**Theorem 2.3.1:** *If $F : \Lambda \times E \to \mathbb{R}$ is sufficiently large relative to a smooth submanifold $M \subset E$, then there exists a subset $\Lambda_\infty \subset \Lambda$ of measure zero such that $f_\lambda = F_\lambda|_M : M \to \mathbb{R}$ is a Morse function for all $\lambda \in \Lambda \setminus \Lambda_\infty$.*

*Proof.* It will be convenient to divide it proof into various steps, claims and lemmas.

**Step 1.** First assume that $M$ is special, i.e. that there exist global coordinates

$$(x_1, \cdots, x_n, \cdots, x_{n+k})$$

on $E$ such that $M$ can be identified with an open subset $W \subset F = \mathbb{R}^n$ of the coordinate subspace

$$\{x_{n+1} = \cdots = x_{n+k} = 0\}.$$

For every $\lambda \in \Lambda$, we now consider the function $f_\lambda : M \to \mathbb{R}$ as a function

$$f_\lambda : M = W \to \mathbb{R}$$

and the differential of $f_\lambda$ at $w = (x_1, \cdots, x_n) \in W$,

$$d(f_\lambda)_w : T_w W = F = \mathbb{R}^n \to T_{f_\lambda(w)}\mathbb{R} = \mathbb{R}$$

is given by

$$v \mapsto \langle \mathrm{grad}(f_\lambda)(w), v \rangle,$$

and we have a function $\varphi_\lambda : W \to \mathbb{R}^n$,

$$\varphi_\lambda(w) = \mathrm{grad}(f_\lambda)(w) = \Big(\frac{\partial f_\lambda}{\partial x_1}|_w, \cdots, \frac{\partial f_\lambda}{\partial x_n}|_w\Big).$$

Thus a point $w \in W$ is a non-degenerate critical point of $f_\lambda$ if and only if $\varphi_\lambda(w) = 0$ and the map $d\varphi_\lambda : T_w W \to \mathbb{R}^n$ is bijective (i.e., the Hessian matrix of $f_\lambda$ is non-singular). Hence, we deduce that $f_\lambda$ is a Morse function if and only if for every $w \in W$ such that $\varphi_\lambda(w) = 0$, $w$ is not a critical point of $\varphi_\lambda$ (since $d\varphi_\lambda$ is surjective at the point $w$). Equivalently, $0 \in \mathbb{R}^n$ is a regular value of $\varphi_\lambda$.

We now consider the smooth function $\Phi : \Lambda \times W \to \mathbb{R}^n$ defined by

$$\Phi(\lambda, w) = \varphi_\lambda(w)$$

**Claim 2.3.1:** *$0 \in \mathbb{R}^n$ is a regular value of $\Phi$.*

It suffices to prove that for every $(\lambda_0, w_0) \in \Phi^{-1}(0) \subset \Lambda \times W$, the differential map $d\Phi_{(\lambda_0,w_0)} : T_{(\lambda_0,w_0)}(\Lambda \times W) \to \mathbb{R}^n$ is surjective (i.e. $(\lambda_0, w_0) \in \Phi^{-1}(0)$ is not a critical point of $\Phi$). Since $F$ is sufficiently large relative to $M$, by our definition of $F$, we have the differential map

$$\partial^{w_0} f : \Lambda \to T^*_{w_0}M,\ \lambda \mapsto d(f_\lambda)_{w_0} = \langle \mathrm{grad} f_\lambda(w_0), \cdot\rangle = \langle \varphi_\lambda(w_0), \cdot\rangle = (\varphi_\lambda(w_0))^*$$

is surjective for every $(\lambda, w_0) \in \Lambda \times \{w_0\} \subset \Lambda \times W$ such that

$$\partial^{w_0} f(\lambda) = 0$$

i.e. $\varphi_\lambda(w_0) = \Phi(\lambda, w_0) = 0$. We then have

$$d(\partial^{w_0} f)_{\lambda_0} : T_{\lambda_0}\Lambda \to T_{\partial^{w_0} f(\lambda_0)}(T^*_{w_0}M) = T^*_{w_0}M = (\mathbb{R}^n)^*$$

is surjective for every $(\lambda_0, w_0) \in \Lambda \times \{w_0\} \subset \Phi^{-1}(0)$. Next, we will prove that the partial differential map

$$\frac{\partial}{\partial\lambda}\Phi_{(\lambda_0,w_0)} : T_{\lambda_0}\Lambda \to \mathbb{R}^n$$

is surjective so that we can conclude that the differential

$$d\Phi_{(\lambda_0,w_0)} : T_{(\lambda_0,w_0)}(\Lambda \times W) \to \mathbb{R}^n$$

is surjective. To see $\dfrac{\partial}{\partial\lambda}\Phi_{(\lambda_0,w_0)}$ is surjective, we first note that for every $v \in T_{\lambda_0}\Lambda$ we have

$$d(\partial^{w_0} f)_{\lambda_0}(v) = \left\langle \frac{\partial}{\partial\lambda}\Phi_{(\lambda_0,w_0)}(v), \cdot \right\rangle = \Big(\frac{\partial}{\partial\lambda}\Phi_{(\lambda_0,w_0)}(v)\Big)^*, \tag{2.3.2}$$

since, if $\alpha$ is a smooth curve in $\Lambda$ with $\alpha(0) = \lambda_0$ and $\alpha'(0) = v$

$$\begin{aligned}
d(\partial^{w_0} f)_{\lambda_0}(v) &= \frac{d}{dt}\Big|_{t=0}(\partial^{w_0} f(\alpha(t))) = \frac{d}{dt}\Big|_{t=0}\big\langle \varphi_{\alpha(t)}(w_0), \cdot\big\rangle \\
&= \frac{d}{dt}\Big|_{t=0}\Big(\sum_{i=1}^n \big\langle \varphi_{\alpha(t)}(w_0), e_i\big\rangle e_i^*\Big), \text{ where } \{e_i\} \text{ is an orthonormal basis of } \mathbb{R}^n \\
&= \sum_{i=1}^n \Big(\frac{d}{dt}\Big|_{t=0}\big\langle \varphi_{\alpha(t)}(w_0), e_i\big\rangle e_i^*\Big) = \sum_{i=1}^n \big\langle \varphi'_{\alpha(0)}(w_0)(\alpha'(0)), e_i\big\rangle e_i^* \\
&= \sum_{i=1}^n \left\langle \frac{\partial}{\partial\lambda}\varphi_\lambda(w_0)\Big|_{\lambda_0}(v), e_i \right\rangle e_i^* = \left\langle \frac{\partial}{\partial\lambda}\varphi_\lambda(w_0)\Big|_{\lambda_0}(v), \cdot \right\rangle \\
&= \left\langle \frac{\partial}{\partial\lambda}\Phi_{(\lambda_0,w_0)}(v), \cdot \right\rangle = \Big(\frac{\partial}{\partial\lambda}\Phi_{(\lambda_0,w_0)}(v)\Big)^*
\end{aligned}$$

We now let $B \in \mathbb{R}^n$. Then $\langle B, \cdot\rangle \in (\mathbb{R}^n)^*$ and by surjectivity of $d(\partial^{w_0} f)_{\lambda_0}$, there exists $A \in T_{\lambda_0}\Lambda$ such that

$$d(\partial^{w_0} f)_{\lambda_0}(A) = \langle B, \cdot\rangle\,.$$

By (2.3.2),

$$\left\langle \frac{\partial}{\partial \lambda} \Phi_{(\lambda_0, w_0)}(A), \cdot \right\rangle = \langle B, \cdot \rangle .$$

This shows that $\frac{\partial}{\partial \lambda} \Phi_{(\lambda_0, w_0)}(A) = B$. Therefore, $\frac{\partial}{\partial \lambda} \Phi_{(\lambda_0, w_0)}$ is surjective.

Now, we wish to produce a smooth map $\pi : Z \to \Lambda$ as we planned at the beginning of this section. According to the **Regular Value Theorem** (see **Lemma** 1 on page 11 of [12]), we obtain

$$\Phi^{-1}(0) = \{(\lambda, w) \in \Lambda \times W \mid \varphi_\lambda(w) = 0\}$$

as a closed smooth submanifold of $\Lambda \times W$ of dimension

$$\dim(\Phi^{-1}(0)) = \dim(\Lambda \times W) - \dim(\mathbb{R}^n). \tag{2.3.3}$$

We set $Z = \Phi^{-1}(0)$. Since $Z$ is a smooth submanifold of $\Lambda \times W$, the smooth map $\pi : Z \to \Lambda$ is induced by the natural projection $p : \Lambda \times W \to \Lambda$. We have the condition on $\lambda$ as follows:

**Lemma 2.3.2:** *If $\lambda$ is a regular value of $\pi$, then $0$ is a regular value of $\varphi_\lambda$, which means that $f_\lambda$ is a Morse function.*

To prove this we need the following lemma from linear algebra:

**Lemma 2.3.3:** *Let $T_1, T_2$ and $V$ be finite dimensional real vector spaces. If*

$$D_i : T_i \to V, \ i = 1, 2$$

*are linear maps such that $D_1 + D_2 : T_1 \oplus T_2 \to V$ is surjective and the restriction of the natural projection $P : T_1 \oplus T_2 \to T_1$ to Ker$(D_1 + D_2)$ is surjective, then $D_2$ is surjective.*

*Proof.* Let $v \in V$. Since $D_1 + D_2$ is surjective, there exists $(t_1, t_2) \in T_1 \oplus T_2$ such that

$$(D_1 + D_2)(t_1, t_2) = D_1(t_1) + D_2(t_2) = v. \tag{2.3.4}$$

By the surjectivity of

$$P_{|\mathrm{Ker}(D_1+D_2)} : \mathrm{Ker}(D_1 + D_2) \to T_1,$$

for every $t_1 \in T_1$, there exists $(t_1', t_2') \in \mathrm{Ker}(D_1 + D_2) \subset T_1 \oplus T_2$ such that $P(t_1', t_2') = t_1$ and

$$D_1(t_1') + D_2(t_2') = 0. \tag{2.3.5}$$

But $P(t_1', t_2') = t_1'$ so that $t_1' = t_1$. Next, by (2.3.4), (2.3.5) and the linearity of $D_2$, we have the following

$$v = D_1(t_1) + D_2(t_2) = D_1(t_1) + D_2(t_2) - \Big(D_1(t_1) + D_2(t_2')\Big) = D_2(t_2) - D_2(t_2') = D_2(t_2 - t_2'),$$

so $v \in \mathrm{Im} D_2$. Therefore $D_2$ is surjective. $\square$

*Proof.* **(of Lemma 2.3.2)** Suppose that $\lambda$ is a regular value of $\pi$. If $\lambda \notin \pi(Z)$, then $(\lambda, w) \notin Z$ and hence $\varphi_\lambda(w) \neq 0$. This shows that $f$ has no critical points on $M$, and so it is a Morse function. If $\lambda \in \pi(Z)$, then the differential map $d\pi_{(\lambda,w)} : T_{(\lambda,w)}Z \to T_\lambda\Lambda$ is surjective for every $(\lambda, w) \in \pi^{-1}(\lambda) \subseteq Z$. We wish to prove that 0 is a regular value of $\varphi_\lambda$, i.e. for every $w \in W$ such that $\varphi_\lambda(w) = 0$ the differential map

$$d(\varphi_\lambda)_w = \frac{\partial}{\partial w}\Phi(\lambda, w) : T_w W \to \mathbb{R}^n$$

is surjective.

For every $(\lambda, w) \in \Lambda \times W$, let

$$\frac{\partial}{\partial \lambda}\Phi(\lambda, w) : T_\lambda\Lambda \to \mathbb{R}^n$$

and

$$\frac{\partial}{\partial w}\Phi(\lambda, w) : T_w W \to \mathbb{R}^n.$$

Then we observe that

$$d\Phi_{(\lambda,w)} = \frac{\partial}{\partial \lambda}\Phi(\lambda, w) + \frac{\partial}{\partial w}\Phi(\lambda, w) : T_\lambda\Lambda \oplus T_w W \to \mathbb{R}^n.$$

Since $d\Phi_{(\lambda,w)}$ is surjective for every $(\lambda, w) \in Z$ (as we saw the proof of Claim 2.3.1), then so is

$$\frac{\partial}{\partial \lambda}\Phi(\lambda, w) + \frac{\partial}{\partial w}\Phi(\lambda, w) : T_\lambda\Lambda \oplus T_w W \to \mathbb{R}^n.$$

Thus $\frac{\partial}{\partial w}\Phi(\lambda, w)$ is surjective by Lemma 2.3.3, since $\frac{\partial}{\partial \lambda}\Phi(\lambda, w)$ and $\frac{\partial}{\partial w}\Phi(\lambda, w)$ are linear maps, and $d\pi_{(\lambda,w)} : T_{(\lambda,w)}Z \to T_\lambda\Lambda$ is surjective with

$$T_{(\lambda,w)}Z = \operatorname{Ker}\Big(\frac{\partial}{\partial \lambda}\Phi(\lambda, w) + \frac{\partial}{\partial w}\Phi(\lambda, w)\Big)$$

for every $(\lambda, w) \in Z$. To prove the last assertion, let $z = (\lambda, w) \in Z = \Phi^{-1}(0), v \in T_z Z$, and $\gamma : (-\epsilon, \epsilon) \to Z$ be a smooth curve on $Z$ with $\gamma(0) = z$ and $\gamma'(0) = v$. Thus

$$\Phi(\gamma(t)) = 0.$$

Then we have

$$0 = \frac{d}{dt}\Phi(\gamma(t))\Big|_{t=0} = d\Phi_{\gamma(0)}(\gamma'(0)) = d\Phi_z(v).$$

This shows that $v \in \ker(d\Phi_z)$. Hence $T_{(\lambda,w)}Z \subseteq \ker(d\Phi_{(\lambda,w)})$. Since

$$\begin{aligned}
\dim(T_z Z) &= \dim(Z) = \dim(\Phi^{-1}(0)) \\
&= \dim(\Lambda \times W) - n \\
&= \dim(T_z(\Lambda \times W)) - n \\
&= \dim(\ker(d\Phi_z)) + \dim(\operatorname{Im}(d\Phi_z)) - n \\
&= \dim(\ker(d\Phi_z))
\end{aligned}$$

since $d\Phi_z$ is surjective, so $\dim(\mathrm{Im}(d\Phi_z)) = \dim(\mathbb{R}^n) = n$. Therefore, we conclude that

$$T_{(\lambda,w)}Z = \ker\left(d\Phi_{(\lambda,w)}\right) = \ker\Big(\frac{\partial}{\partial\lambda}\Phi(\lambda,w) + \frac{\partial}{\partial w}\Phi(\lambda,w)\Big).$$

□

Let $\Lambda_M \subset \Lambda$ be the set of critical values of $\pi : Z \to \Lambda$. Theorem 1.1.1 implies that $\Lambda_M$ has measure zero in $\Lambda$. Set $\Lambda_\infty = \Lambda_M$. Then, by Lemma 2.3.2, the function $f_\lambda : M \to \mathbb{R}$ is a Morse function for all $\lambda \in \Lambda \setminus \Lambda_\infty$.

**Step 2.** $M$ is a general manifold. We can cover $M$ by a countable open cover $(M_k)_{k\geq 1}$ such that $M_k$ is special. Thus, for every $k \geq 1$ there exists a subset $\Lambda_{M_k} \subset \Lambda$ of measure zero such that $f_\lambda : M_k \to \mathbb{R}$ is a Morse function for all $\lambda \in \Lambda \setminus \Lambda_{M_k}$ by **Step 1**. Let us set $\Lambda_\infty = \bigcup\limits_{k\geq 1} \Lambda_{M_k}$. Then $\Lambda_\infty$ a set of measure zero in $\Lambda$ since it is the union of the measure zero sets in $\Lambda$. Therefore, the function $f_\lambda : M \subseteq \bigcup\limits_{k\geq 1} M_k \to \mathbb{R}$ is a Morse function for all $\lambda \in \Lambda \setminus \Lambda_\infty$. □

From Example 2.3.1, Lemma 2.3.1 and Theorem 2.3.1, we have the following corollary.

**Corollary 2.3.1:** *Suppose that $M$ is a submanifold of the Euclidean space $E$. Thus*

*(1) For almost all $v \in E^*$ and $p \in E$, the functions $h_v, r_p : M \to \mathbb{R}$ defined by*

$$h_v(x) = v(x) \text{ and } r_p(x) = \frac{1}{2}||x-p||^2$$

*are Morse functions.*

*(2) If $M$ does not contain the origin, then the function $q_A : M \to \mathbb{R}$ defined by*

$$q_A(x) = \frac{1}{2}\langle Ax, x\rangle$$

*is a Morse function for almost all positive symmetric endomorphism $A$ of $E$.*

**Lemma 2.3.4:** *Let $M$ be a compact smooth manifold, and let $f : M \to \mathbb{R}$ be a Morse function on $M$. Then $f$ can be viewed as a height function $h_v$ with respect to some suitable embedding of $M$ in a Euclidean space.*

*Proof.* Let $\Phi : M \hookrightarrow E = \mathbb{R}^N$ be an embedding (inclusion). We define a new embedding relative to $f$ as follows:

$$\begin{array}{rcl} \Phi_f : M & \hookrightarrow & \mathbb{R} \times \mathbb{R}^N \\ x & \mapsto & (f(x), \Phi(x)). \end{array}$$

Let $\{\vec{e}_i\}_{1\leq i\leq N+1}$ be the canonical basis of $(\mathbb{R}\times\mathbb{R}^N)^* = \mathbb{R}\times\mathbb{R}^N$. According to Corollary 2.3.1, we have a Morse function as a height function $h_{\vec{e}_1} : \mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ which is given by

$$h_{\vec{e}_1}(z) = \vec{e}_1(z) = \langle \vec{e}_1, z\rangle .$$

Therefore, $f$ can be written as follows:

$$f(x) = \left\langle \vec{e}_1, \Big(f(x), \Phi(x)\Big)\right\rangle = \vec{e}_1(\Phi_f(x)) = h_{\vec{e}_1} \circ \Phi_f(x).$$

□

## 2.4 Fundamental Theorems of Morse Theory

In this section, we let $f : M \to \mathbb{R}$ be a real valued function on a smooth manifold $M$, and let

$$M^a = f^{-1}((-\infty, a]) = \{p \in M \ : f(p) \leq a\}.$$

### 2.4.1 First Fundamental Theorem

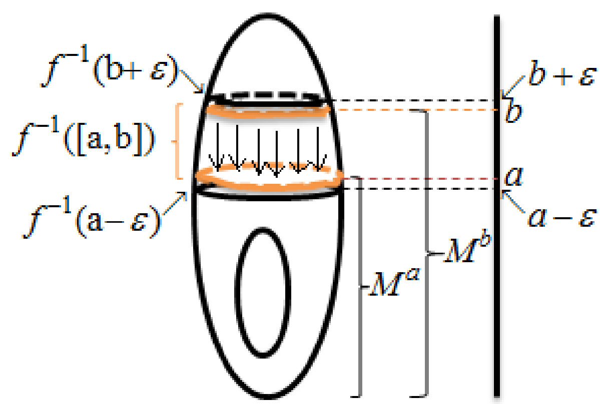
We first consider the region that $f$ has no critical points as follows:

**Theorem 2.4.1:** *Let $f : M \to \mathbb{R}$ be a smooth real valued function on a manifold $M$. Let $a$ and $b$ be regular values of $f$ with $a < b$ such that the set*

$$f^{-1}([a, b]) = \{p \in M \mid a \leq f(p) \leq b\}$$

*is compact and contains no critical points of $f$. Then $M^a$ is diffeomorphic to $M^b$. Furthermore, $M^a$ is a deformation retract of $M^b$, so that the inclusion map $M^a \hookrightarrow M^b$ is a homotopy equivalence.*

*Proof.* Since $f^{-1}([a, b])$ is compact and contains no critical points, there exists $\epsilon > 0$ small enough such that the set $f^{-1}((a-\epsilon, b+\epsilon))$ contains no critical points of $f$. Let $\rho : M \to \mathbb{R}$




be a smooth function defined by

$$\rho(x) = \begin{cases} \frac{1}{||\text{grad} f(x)||^2}, & \text{if } x \in f^{-1}((a-\epsilon, b+\epsilon)) \\ 0 & \text{otherwise} \end{cases}.$$

Now we can define a smooth vector field $X$ on $M$ by

$$X_x = \rho(x)\text{grad}f(x) \text{ for all } x \in M.$$

That is,

$$X_x = \begin{cases} \frac{1}{||\text{grad}f(x)||^2}\,\text{grad}f(x), & \text{if } x \in f^{-1}((a-\epsilon, b+\epsilon)) \\ 0 & \text{otherwise,} \end{cases} \tag{2.4.1}$$

which satisfies the conditions of Lemma 1.1.1. Thus $X$ generates a 1- parameter group of diffeomorphism $\phi : \mathbb{R} \times M \to M$. Then for each fixed $p \in M$ the map $c := \phi_p : \mathbb{R} \to M$ is a smooth curve in $M$ defined by $c(t) = \phi_t(p)$ and $c(0) = \phi_0(p) = p,$ because $\phi_0 = id_M$. Therefore, by Remark 1.1.2,

$$\begin{aligned}
\frac{d(f \circ \phi_t(p))}{dt} &= \frac{d(f \circ c)}{dt} \\
&= \left\langle \frac{dc(t)}{dt}, \text{grad}f(c(t)) \right\rangle \\
&= \left\langle \frac{d\phi_t(p)}{dt}, \text{grad}f(\phi_t(p)) \right\rangle \\
&= \left\langle X_{\phi_t(p)}, \text{grad}f(\phi_t(p)) \right\rangle
\end{aligned}$$

since$\frac{d\phi_t(p)}{dt} = X_{\phi_t(p)}$. Hence, the last equality together with equation (2.4.1) give us that

$$\frac{df(\phi_t(p))}{dt} = \begin{cases} 1 & \text{if } \phi_t(p) \in f^{-1}((a-\epsilon, b+\epsilon)) \\ 0 & \text{otherwise} \end{cases}.$$

We then have

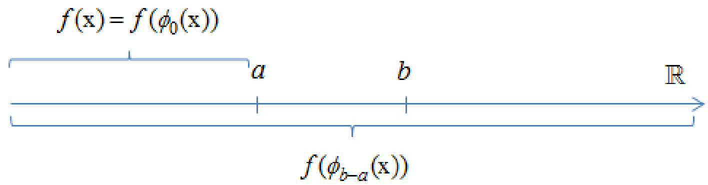
$$f(\phi_t(p)) = \begin{cases} t + f(p) & \text{if } \phi_t(p) \in f^{-1}((a-\epsilon, b+\epsilon)) \\ f(p) & \text{otherwise} \end{cases} \tag{2.4.2}$$

since $\phi_0(p) = p$. In addition, $f(\phi_t(p))$ is increasing since $\dfrac{df(\phi_t(p))}{dt} \geq 0$ for all $t \in \mathbb{R}$ and $p \in M$.

Consider the diffeomorphism $\phi_{b-a} : M \to M$. We claim that $\phi_{b-a}\Big|_{M^a} : M^a \to M^b$ is a diffeomorphism.

First, we prove that $\phi_{b-a}$ maps $M^a$ into $M^b$. We wish to prove that for every $x \in M^a$, then $f(\phi_{b-a}(x)) \leq b$ ($i.e$ $\phi_{b-a}(x) \in M^b$). Let $x \in M^a$. Since $f(\phi_t(p))$ is increasing,

$$f(\phi_0(x)) = f(x) < f(\phi_{b-a}(x)).$$




There are two cases:

- if $f(\phi_{b-a}(x)) \leq b$, then $\phi_{b-a}(x) \in M^b$.
- if $f(\phi_{b-a}(x)) > b$, then by (2.4.2), $f(x) - a > 0$ and $f(x) > b$ which is a contradiction.

Therefore, $\phi_{b-a}$ maps $M^a$ into $M^b$. Since $\phi_t : M \to M$ is a diffeomorphism for each $t$, then the restriction of $\phi_{b-a}$ to $M^a$ is also one to one. So, we only remain to prove that $\phi_{b-a}$ maps $M^a$ onto $M^b$. Let $y \in M^b$. There exist $x = \phi_{a-b}(y) \in M^a$ because, by (2.4.2), we have

$$f(x) = f(\phi_{a-b}(y)) \leq b$$

and if $f(\phi_{a-b}(y)) > a$, since $f(\phi_t(p))$ is increasing, we obtain

$$a < f(\phi_{a-b}(y)) < f(\phi_{(f(x)-b)}(y)) \leq f(\phi_0(y)) \leq b,$$

and this implies that

$$f(\phi_{a-b}(y)) = a - b + f(y) \leq a - b + b = a$$

which is contradiction. Therefore, the map $\phi_{b-a}$ is onto since

$$\phi_{b-a}(x) = \phi_{b-a}(\phi_{a-b}(y)) = \phi_0(y) = y.$$

Now we proceed to prove the second part: $M^a$ is a deformation retract of $M^b$. Consider the family of maps $r_t : M^b \to M^b$ defined by

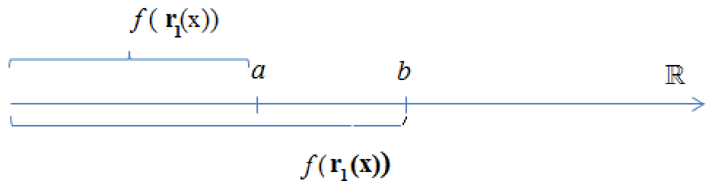
$$r_t(x) = \begin{cases} x & \text{if } x \in M^a \\ \phi_{(a-f(x))t}(x) & \text{if } a \leq f(x) \leq b \end{cases}, \quad t \in [0,1].$$

If $x \in M^a$, then $r_t(x) = x \in M^a \subset M^b$. If $a \leq f(x) \leq b$, then $(a - f(x))t \leq 0$ and by the monotonicity of $f(\phi_t(p))$, this implies that $f(\phi_{(a-f(x))t}(x)) \leq f(\phi_0(x)) = f(x) \leq b$. Thus $r_t(x) = \phi_{(a-f(x))t}(x) \in M^b$. This family also satisfies the following conditions:

- $r_t(x)$ is continuous on the product topology $M^b \times [0,1]$.
- $r_0(x) = x$ for all $x \in M^b$.
- $r_1(x) = \phi_{a-f(x)}(x) \in M^a$. Indeed, if $x \in M^a$, then $r_1(x) = x \in M^a$ and by the monotonicity of $f(\phi_t(p))$, if $a \leq f(x) \leq b$, then

  $$f(r_1(x)) = f(\phi_{a-f(x)}(x)) \leq f(\phi_0(x)) = f(x) \leq b.$$

  Case 1: if $f(r_1(x)) \leq a$, then $r_1(x) \in M^a$.
  Case 2: if $a \leq f(r_1(x)) \leq b$, then $f(r_1(x)) = a - f(x) + f(x) = a$. Hence $r_1(x) \in M^a$.




- It is clear that $r_1(x) = x$ for all $x \in M^a$.

Therefore, $M^a$ is a deformation retract of $M^b$, so that the inclusion map $M^a \hookrightarrow M^b$ is a homotopy equivalence. □

### 2.4.2 Second Fundamental Theorem

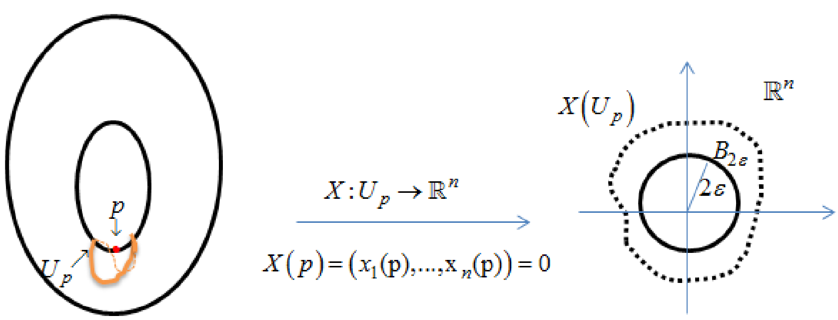
Now let us consider a region in which $f$ has one critical point.

**Theorem 2.4.2:** *Let $p$ be a non degenerate critical point of $f$ with index $\lambda$. Let $c = f(p)$ and assume $f^{-1}([c-\epsilon, c+\epsilon])$ is compact and contains no other critical point of $f$ for some $\epsilon > 0$. Then for all sufficiently small $\epsilon$, the set $M^{c+\epsilon}$ has the homotopy type of $M^{c-\epsilon}$ with a $\lambda$-cell attached.*

*Proof.* By the Morse lemma, there is a local coordinate system $X : U_p \to \mathbb{R}^n$ defined by $X = (x_1, x_2, \cdots, x_n)$ in a neighborhood $U_p$ of $p$ with $X(p) = (x_1(p), x_2(p), \cdots, x_n(p)) = 0$ and such that the identity

$$f = c - x_1^2 - \cdots - x_\lambda^2 + x_{\lambda+1}^2 + \cdots + x_n^2$$

holds throughout $U_p$.




Choose $\epsilon > 0$ sufficiently small such that the set $f^{-1}([c-\epsilon, c+\epsilon])$ is compact and contains no critical point of $f$ other than $p$ and the image $X(U_p)$ contains the closed ball $B_{2\epsilon} = \{(x_1, x_2, \cdots, x_n) \mid \sum_{i=1}^{n} x_i^2 \leq 2\epsilon\}$. We construct a smooth function $\rho : \mathbb{R} \to \mathbb{R}$ such that

$$\begin{aligned}
&\rho(t) \geq 0 \text{ for all } \ t \in \mathbb{R} \\
&\rho(0) > \epsilon \\
&\rho(t) = 0 \text{ for all } \ t \geq 2\epsilon \\
&-1 < \rho'(t) \leq 0 \text{ for all } \ t \in \mathbb{R}
\end{aligned}$$

(see Example 1.1.1). We define a new smooth function $F : M \to \mathbb{R}$ by

$$F(q) = \begin{cases} f(q), & \text{if } q \notin U_p \\ f(q) - \rho(x_1^2(q) + \cdots + x_\lambda^2(q) + 2x_{\lambda+1}^2(q) + \cdots + 2x_n^2(q)), & \text{if } q \in U_p \end{cases}$$

For convenience, we define functions $X_-, X_+ : U_p \to [0, +\infty)$ by $X_- = x_1^2 + \cdots + x_\lambda^2$ and $X_+ = x_{\lambda+1}^2 + \cdots + x_n^2$. In terms of these functions, we have

$$f(q) = c - X_-(q) + X_+(q) \text{ for all } q \in U_p,$$

and

$$F(q) = \begin{cases} f(q), & \text{if } q \notin U_p \\ f(q) - \rho(X_-(q) + 2X_+(q)), & \text{if } q \in U_p \end{cases}.$$

By definition of $F$, it is clear that $F$ is smooth on the interior and exterior of $U_p$. In order to verify that $F$ is smooth, it suffices to check that $F$ is smooth on the boundary of $U_p$, that is, on the set $\{q \in X^{-1}(B_{2\epsilon}) \ : \ \sum_{i=1}^{n}(x_i(q))^2 = X_-(q) + X_+(q) = 2\epsilon\}$. Let us prove that $F$ is continuous on the boundary of $U_p$. For any $q_0 \in \partial U_p$ (boundary of $U_p$), let $\{q_i\}$ be a sequence that converges to $q_0$. Then, there are subsequences $\{q_{i_j}\} \in X^{-1}(B_{2\epsilon})$ and $\{q_{i_k}\} \notin X^{-1}(B_{2\epsilon})$ of $\{q_i\}$ such that both sequences converge to $q_0$. If $\{q_{i_j}\} \in X^{-1}(B_{2\epsilon})$, then $F(q_{i_j}) = f(q_{i_j}) - \rho(X_-(q_{i_j}) + 2X_+(q_{i_j}))$ and hence $F(q_{i_j}) \to f(q_0) - \rho(X_-(q_0) + 2X_+(q_0))$ as $q_{i_j} \to q_0$. Since $X_-(q_0) + 2X_+(q_0) = 2\epsilon + X_+(q_0)$ and $\rho(t) = 0$ for all $t \geq 2\epsilon$, $\lim_{j\to\infty} F(q_{i_j}) \to f(q_0)$. If $\{q_{i_k}\} \notin X^{-1}(B_{2\epsilon})$, then $F(q_{i_k}) = f(q_{i_k})$ so that $\lim_{k\to\infty} F(q_{i_k}) = f(q_0)$ This implies that $F$ is continuous at $q_0 \in \partial U_p$. Next, we want to prove that $dF$ is continuous on the boundary of $U_p$. We note that the derivatives of all orders of $\rho$ are identically 0 for all $t \geq 2\epsilon$ since $\rho \equiv 0$ on this interval. If $\{q_{i_j}\} \in X^{-1}(B_{2\epsilon})$, then

$$\frac{\partial F}{\partial x_i}(q_{i_j}) = \begin{cases} -2x_i(q_{i_j}) - 2x_i(q_{i_j})\rho'(X_-(q_{i_j}) + 2X_+(q_{i_j})) & \text{if } i \leq \lambda \\ 2x_i(q_{i_j}) - 4x_i(q_{i_j})\rho'(X_-(q_{i_j}) + 2X_+(q_{i_j})) & \text{if } i \geq \lambda + 1 \end{cases}.$$

Since $q_{i_j} \to q_0$ and $\rho'(X_-(q_0) + 2X_+(q_0)) = \rho'(2\epsilon + X_+(q_0)) = 0$, we obtain

$$\frac{\partial F}{\partial x_i}(q_0) \to \begin{cases} -2x_i(q_0) & \text{if } i \leq \lambda \\ 2x_i(q_0) & \text{if } i \geq \lambda + 1 \end{cases}. \tag{2.4.3}$$

If $\{q_{i_k}\} \notin X^{-1}(B_{2\epsilon})$, then $\frac{\partial F}{\partial x_i}(q_{i_k}) = \begin{cases} -2x_i(q_{i_k}) & \text{if } i \leq \lambda \\ 2x_i(q_{i_k}) & \text{if } i \geq \lambda + 1 \end{cases}$. Similarly, since $q_{i_k} \to q_0$, we obtain

$$\frac{\partial F}{\partial x_i}(q_0) \to \begin{cases} -2x_i(q_0) & \text{if } i \leq \lambda \\ 2x_i(q_0) & \text{if } i \geq \lambda + 1 \end{cases}. \tag{2.4.4}$$

Therefore, by (2.4.3) and (2.4.4), we conclude that $dF$ is continuous at $q_0 \in \partial U_p$. Now, we prove that $d^2F$ is continuous on the boundary of $U_p$. If $\{q_{i_j}\} \in X^{-1}(B_{2\epsilon})$, then

$$\frac{\partial^2 F}{\partial x_i^2}(q_{i_j}) = \begin{cases} -2 - 2\rho'(X_-(q_{i_j}) + 2X_+(q_{i_j})) - 4x_i^2(q_{i_j})\rho''(X_-(q_{i_j}) + 2X_+(q_{i_j})) & \text{if } i \leq \lambda \\ 2 - 4\rho'(X_-(q_{i_j}) + 2X_+(q_{i_j})) - 16x_i^2(q_{i_j})\rho''(X_-(q_{i_j}) + 2X_+(q_{i_j})) & \text{if } i \geq \lambda + 1 \end{cases}.$$

Since $q_{i_j} \to q_0$ and $\rho'(2\epsilon + X_+(q_0)) = 0 = \rho''(2\epsilon + X_+(q_0))$, we have

$$\frac{\partial^2 F}{\partial x_i^2}(q_0) \to \begin{cases} -2 & \text{if } i \leq \lambda \\ 2 & \text{if } i \geq \lambda + 1 \end{cases}. \tag{2.4.5}$$

If $\{q_{i_k}\} \notin X^{-1}(B_{2\epsilon})$, then $\dfrac{\partial^2 F}{\partial x_i^2}(q_{i_k}) = \begin{cases} -2 & \text{if } i \leq \lambda \\ 2 & \text{if } i \geq \lambda + 1 \end{cases}$. Since $q_{i_k} \to q_0$, we then have

$$\frac{\partial^2 F}{\partial x_i^2}(q_0) \to \begin{cases} -2 & \text{if } i \leq \lambda \\ 2 & \text{if } i \geq \lambda + 1 \end{cases}. \tag{2.4.6}$$

By (2.4.5) and (2.4.6), we conclude that $d^2F$ is continuous at $q_0 \in \partial U_p$.
Similarly, it is easy to check that for all $n \geq 3$, we have $\dfrac{\partial^n F}{\partial x_i^n} = 0$ in the boundary of $U_p$.
In conclusion, $F$ is smooth on the boundary of $U_p$

**Claim 2.4.1:** $M^{c+\epsilon} = F^{-1}((-\infty, c+\epsilon])$.

*Proof.* Since $\rho(t) \geq 0$ for all $t \in \mathbb{R}$, $F(q) \leq f(q)$ for all $q \in M$.

- **Case**: $q \notin U_p$. We have $F = f$. Therefore

$$F^{-1}((-\infty, c+\epsilon]) = f^{-1}((-\infty, c+\epsilon]) = M^{c+\epsilon}.$$

- **Case**: $q \in U_p$. For any $q \in M^{c+\epsilon}$, then $f(q) \leq c+\epsilon$ and hence $F(q) \leq f(q) \leq c+\epsilon$. Thus $q \in F^{-1}((-\infty, c+\epsilon])$. Hence $M^{c+\epsilon} \subseteq F^{-1}((-\infty, c+\epsilon])$. For any $q \in F^{-1}((-\infty, c+\epsilon])$, then $F(q) = f(q) - \rho(X_-(q) + 2X_+(q)) \leq c+\epsilon$. If $X_-(q) + 2X_+(q) \geq 2\epsilon$, then $\rho(X_-(q) + 2X_+(q)) = 0$ and so $f(q) = F(q) \leq c+\epsilon$. If $X_-(q) + 2X_+(q) \leq 2\epsilon$ (or $\frac{X_-(q)}{2} + X_+(q) \leq \epsilon$), then $f(q) = c - X_-(q) + X_+(q) \leq c + \frac{X_-(q)}{2} + X_+(q) \leq c+\epsilon$. We then have $f(q) \leq c+\epsilon$ for all $q \in U_p$. Hence $q \in f^{-1}((-\infty, c+\epsilon]) = M^{c+\epsilon}$. That is, $F^{-1}((-\infty, c+\epsilon]) \subseteq M^{c+\epsilon}$. Therefore, $M^{c+\epsilon} = F^{-1}((-\infty, c+\epsilon])$.

□

**Claim 2.4.2:** $F^{-1}((-\infty, c-\epsilon])$ *is diffeomorphic to* $M^{c+\epsilon}$.

*Proof.* By Theorem 2.4.1 and Claim 2.4.1, we only prove that the set $F^{-1}([c-\epsilon, c+\epsilon])$ is compact and contains no critical point of $F$. First, we only show that the set $F^{-1}([c-\epsilon, c+\epsilon])$ is a closed subset of a compact set $f^{-1}([c-\epsilon, c+\epsilon])$. For any $q \in f^{-1}((-\infty, c-\epsilon))$, then $f(q) < c-\epsilon$. Hence $F(q) < c-\epsilon$ since $F(q) \leq f(q)$. That is, $q \in F^{-1}((-\infty, c-\epsilon))$. We then have

$$f^{-1}((-\infty, c-\epsilon)) \subset F^{-1}((-\infty, c-\epsilon)) \subset F^{-1}((-\infty, c+\epsilon]) = f^{-1}((-\infty, c+\epsilon]).$$

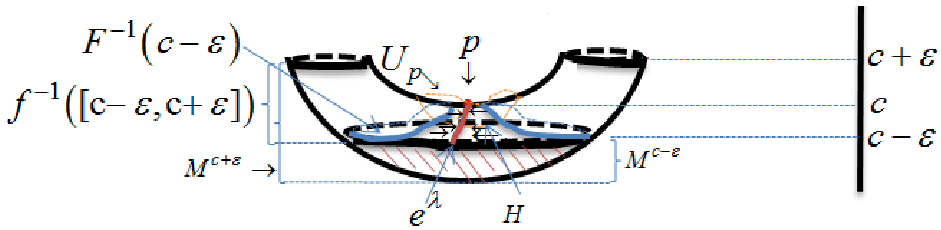



Figure 2.1:

It follows that $F^{-1}([c-\epsilon, c+\epsilon]) \subset f^{-1}([c-\epsilon, c+\epsilon])$. Since $F$ is a smooth function and the set $[c-\epsilon, c+\epsilon]$ is closed, the set $F^{-1}([c-\epsilon, c+\epsilon])$ is closed. Thus the set $F^{-1}([c-\epsilon, c+\epsilon])$ is compact.

Next, we show that the set $F^{-1}([c-\epsilon, c+\epsilon])$ contains no critical point of $F$. Before doing this, we prove that the functions $f$ and $F$ have the same critical points.

- **Case:** $q \notin U_p$. The functions $F$ and $f$ coincide. Then they have the same critical points in this region.
- **Case:** $q \in U_p$. We have $F(X_-, X_+) = f-\rho(X_-+2X_+) = c-X_-+X_+-\rho(X_-+2X_+)$ and $X^{-1} : X(U_p) \to U_p$ as the inverse of $X$. We then have

$$\begin{aligned} d(F\circ X^{-1}) &= \frac{\partial F}{\partial X_-} d(X_-\circ X^{-1}) + \frac{\partial F}{\partial X_+} d(X_+\circ X^{-1}) \\ &= (-1-\rho'(X_-+2X_+))d(X_-\circ X^{-1}) + (1-2\rho'(X_-+2X_+))d(X_+\circ X^{-1}) \\ &= Ad(X_-\circ X^{-1}) + Bd(X_+\circ X^{-1}) \end{aligned}$$

Since $-1 < \rho'(t) \leq 0$ for all $t$, then the coefficients $A = (-1-\rho'(X_-+2X_+))$ and $B = (1-2\rho'(X_-+2X_+))$ are nowhere zero. And also we have

$$d(X_-\circ X^{-1})(x) = (2x_1, 2x_2, \cdots, 2x_\lambda, 0_{\lambda+1}, \cdots, 0_n)$$

and

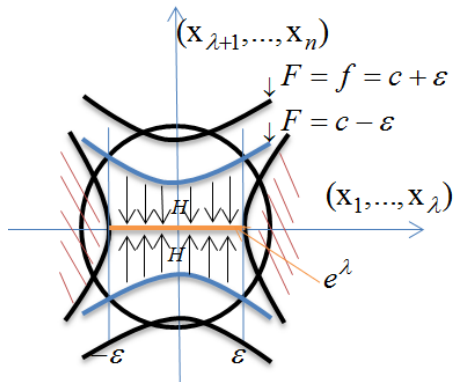
$$d(X_+\circ X^{-1})(x) = (0_1, \cdots, 0_\lambda, 2x_{\lambda+1}, 2x_{\lambda+2}, \cdots, 2x_n).$$

Therefore, $d(F\circ X^{-1})(x) = (2Ax_1, 2Ax_2, \cdots, 2Ax_\lambda, 2Bx_{\lambda+1}, 2Bx_{\lambda+2}, \cdots, 2Bx_n)$ and so $d(F\circ X^{-1})(x) = 0$ if and only if $x = 0$. Since there is only point $p$ in $U_p$ such that $(x_1(p), x_2(p), \cdots, x_n(p)) = 0$, then $x = 0$ only at the point $p \in U_p$. This proves that $p$ is the only critical point of $F$ within $U_p$.

Now, we return to prove that the region $F^{-1}([c-\epsilon, c+\epsilon])$ contains no critical points of $F$. By assumption, $f^{-1}([c-\epsilon, c+\epsilon])$ contains no critical points of $f$ other than $p$ and, since $F^{-1}([c-\epsilon, c+\epsilon]) \subset f^{-1}([c-\epsilon, c+\epsilon])$, then $F^{-1}([c-\epsilon, c+\epsilon])$ contains no critical points of $F$ with the possible exception of $p$. However, we note that $F(p) = f(p) - \rho(X_-(p) + 2X_+(p)) = c - \rho(0) < c - \epsilon$. That is, $p \in F^{-1}((-\infty, c-\epsilon))$ cannot be in $F^{-1}([c-\epsilon, c+\epsilon])$. By Theorem 2.4.1 and Claim 2.4.1, we see that $F^{-1}((-\infty, c-\epsilon])$ is diffeomorphic to $F^{-1}((-\infty, c+\epsilon]) = M^{c+\epsilon}$.

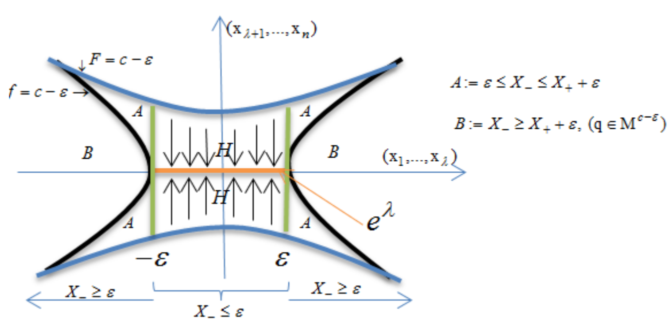
$\square$

Define the $\lambda$-cell by $e^\lambda := \{q \in U_p \mid X_-(q) \leq \epsilon \text{ and } X_+(q) = 0\}$ and denote the closure of the region $F^{-1}((-\infty, c-\epsilon]) \setminus M^{c-\epsilon}$ by $H$ (see Figure 2.1).




**Claim 2.4.3:** *$M^{c-\epsilon} \cup e^\lambda$ is a deformation retract of $M^{c-\epsilon} \cup H$.*

*Proof.* First, we wish to see that $e^\lambda \subset H$. For any $q \in e^\lambda$, we have $X_-(q) \leq \epsilon$ and $X_+(q) = 0$. Hence $f(q) = c - X_-(q) \geq c - \epsilon$. That is, $q$ is a point of the closure of the complement of $M^{c-\epsilon}$. Now consider a function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(t) = \rho(t) + t$. We know that $-1 < \rho'(t) \leq 0$ for all $t$, so $g$ is increasing. Then $\rho(X_-(q)) + X_-(q) > \rho(0)$ since $g(0) < g(X_-(q))$. We also know that $F(q) = c - X_-(q) - \rho(X_-(q)) < c - \rho(0) < c - \epsilon$, so $q \in F^{-1}((-\infty, c-\epsilon])$. Therefore, $q \in H$.




In the case $\epsilon \leq X_- \leq X_+ + \epsilon$,

$$0 \leq \frac{X_- - \epsilon}{X_+} \leq 1,$$

we define $s_t : [0,1] \to [0,1]$ by

$$s_t = t + (1-t)\sqrt{\frac{X_- - \epsilon}{X_+}}.$$

Thus the function $s_t x_i$ remains continuous for each $i > \lambda$ as $X_+ \to 0$ and $X_- \to \epsilon$. This is true since for each $i > \lambda$, we have $|x_i| \leq \sqrt{X_+}$ and

$$|s_t x_i| \leq \left(t + (1-t)\sqrt{\frac{X_- - \epsilon}{X_+}}\right)\sqrt{X_+} = t\sqrt{X_+} + (1-t)\sqrt{X_- - \epsilon} \to 0$$

as $X_+ \to 0,\ X_- \to \epsilon$. We define a family $r_t : M^{c-\epsilon} \cup H \to M^{c-\epsilon} \cup H$ by

$$r_t(x_1, \cdots, x_n) = \begin{cases} (x_1, \cdots, x_n) & \text{if } q \notin U_p \text{ or } q \in M^{c-\epsilon} \\ (x_1, \cdots, x_\lambda, tx_{\lambda+1}, \cdots, tx_n) & \text{if } q \in H \text{ and } X_-(q) \leq \epsilon \\ (x_1, \cdots, x_\lambda, s_t x_{\lambda+1}, \cdots, s_t x_n) & \text{if } \epsilon \leq X_-(q) \leq X_+(q) + \epsilon \end{cases}, \quad t \in [0,1].$$

For each $t \in [0,1]$, this family is well defined because for any $q \in M^{c-\epsilon} \cup H$, we obtain $r_t(q) \in M^{c-\epsilon} \cup H$. Indeed, if $q \notin U_p$ or $q \in M^{c-\epsilon}$, then it is clear that $r_t$ is the identity map. Thus $r_t(q) \in M^{c-\epsilon} \cup H$. If $q$ satisfies $X_-(q) \leq \epsilon$ or $\epsilon \leq X_-(q) \leq X_+(q) + \epsilon$, we then consider the following

$$F(r_t(q)) = c - X_-(r_t(q)) + X_+(r_t(q)) - \rho(X_-(r_t(q)) + X_+(r_t(q))). \tag{2.4.7}$$

By the proof of Claim 2.4.2, we recall that $\dfrac{\partial F}{\partial X_+} = 1 - 2\rho'(t) > 0$ since $-1 < \rho'(t) \leq 0$ for all $t \in \mathbb{R}$. We also note that $X_-(r_t) = X_-$ is independent of $t$. Therefore, it suffices to verify that $r_0(q)$ and $r_1(q)$ belong to $M^{c-\epsilon} \cup H$ since $F$ is increasing and depends smoothly on the variable $X_+$.

- **Case** $t = 1$ : If $q \in H$ and $X_-(q) \leq \epsilon$, then it is clear that $r_1$ is the identity map. If $\epsilon \leq X_-(q) \leq X_+(q) + \epsilon$, then $s_1 = 1$, which implies $r_1$ is the identity map. Hence $r_1 \in M^{c-\epsilon} \cup H$.
- **Case** $t = 0$ : If $q \in H$ and $X_-(q) \leq \epsilon$, then $r_0(x_1, \cdots, x_n) = (x_1, \cdots, x_\lambda, 0, \cdots, 0)$ and $X_+(r_0(x_1, \cdots, x_n)) = 0$. Thus $r_0(x_1, \cdots, x_n) \in e^\lambda \subset H$.

  If $\epsilon \leq X_-(q) \leq X_+(q) + \epsilon$, then $s_0 = \sqrt{\dfrac{X_- - \epsilon}{X_+}}$, so

$$X_+(r_0(x_1, \cdots, x_n)) = \sum_{i=\lambda+1}^{n} \left(\sqrt{\frac{X_- - \epsilon}{X_+}} x_i\right)^2 = X_- - \epsilon.$$

  Therefore, $r_0(x_1, \cdots, x_n) \in M^{c-\epsilon}$ since $X_-(r_t) = X_-$ and

$$f(r_0(x_1, \cdots, x_n)) = c - X_-(r_0(x_1, \cdots, x_n)) + X_+(r_0(x_1, \cdots, x_n)).$$

Now, we can conclude the following results:

- for each $t \in [0,1]$, the family $r_t$ is well defined.
- $r_1$ is the identity map.
- the image of $r_0$ is contained in $M^{c-\epsilon} \cup e^\lambda$.

Finally, It is easy to check that $r_0(q) = q$ for all $q \in M^{c-\epsilon} \cup e^\lambda$. Indeed, it is clear for $q \in M^{c-\epsilon}$. If $q \in e^\lambda$, then $q \in H$, $X_-(q) \leq \epsilon$ and $X_+(q) = 0$. Then we have

$$r_0(x_1(q), \cdots, x_n(q)) = r_0(x_1(q), \cdots, x_\lambda(q), 0, \cdots, 0) = (x_1(q), \cdots, x_\lambda(q), 0, \cdots, 0).$$

Therefore, we have proved the Claim 2.4.3. □

By Claims 2.4.2 and 2.4.3, the proof of Theorem 2.4.2 is complete. □

**Proposition 2.4.1:** *(**Generalization of Theorem 2.4.2**) Suppose that $p_1, \cdots, p_k$ are $k$ non-degenerate critical points with indices $\lambda_1, \cdots, \lambda_k$ in $f^{-1}(c)$. Then, $M^{c+\epsilon}$ has the homotopy type of $M^{c-\epsilon} \cup e^{\lambda_1} \cup \cdots \cup e^{\lambda_k}$.*

**Example 2.4.1:** *In the case $k = 2$, see Figures 2.2 and 2.3.*

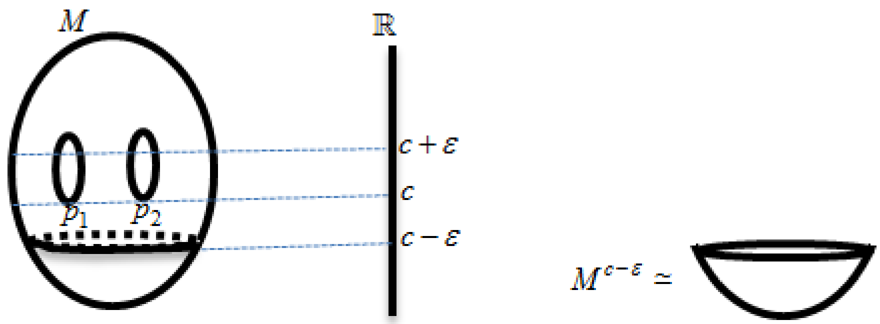



Figure 2.2: $p_1$ and $p_2$ are non-degenerate critical points with indices $\lambda_1 = \lambda_2 = 1$ in $f^{-1}(c)$.


Figure 2.3: $M^{c+\epsilon}$ has the homotopy type of $M^{c-\epsilon} \cup e^1 \cup e^1$.

### 2.4.3 Consequence of the Fundamental Theorems

**Theorem 2.4.3:** *If $f : M \to \mathbb{R}$ is a smooth function on a compact smooth manifold $M$ with no degenerate critical points and if each $M^a$ is compact, then $M$ has the homotopy type of a CW-complex, with one cell of dimension $\lambda$ for each critical point of index $\lambda$.*

To prove this theorem, we will need the following two lemmas.

**Lemma 2.4.1:** ***(Whitehead)*** *Let $\varphi_0$ and $\varphi_1$ be homotopic maps from the sphere $\partial(e^\lambda)$ to a topological space $X$. Then the identity map of $X$ extends to a homotopy equivalence*

$$k : X \cup_{\varphi_0} e^\lambda \to X \cup_{\varphi_1} e^\lambda.$$

*Proof.* Let $\varphi_t$ be a homotopy between $\varphi_0$ and $\varphi_1$. Define $k : X \cup_{\varphi_0} e^\lambda \to X \cup_{\varphi_1} e^\lambda$ by

$$k(x) = \begin{cases} x & \text{if } x \in X \\ 2ru & \text{if } x = ru,\ u \in \partial(e^\lambda),\ 0 \le r \le \frac{1}{2} \\ \varphi_{2-2r}(u) & \text{if } x = ru,\ u \in \partial(e^\lambda),\ \frac{1}{2} \le r \le 1, \end{cases}$$

and $\widetilde{k} : X \cup_{\varphi_1} e^\lambda \to X \cup_{\varphi_0} e^\lambda$ by

$$\widetilde{k}(x) = \begin{cases} x & \text{if } x \in X \\ 2su & \text{if } x = su,\ u \in \partial(e^\lambda),\ 0 \le s \le \frac{1}{2} \\ \varphi_{2s-1}(u) & \text{if } x = su,\ u \in \partial(e^\lambda),\ \frac{1}{2} \le s \le 1. \end{cases}$$

Since the functions $k$ and $\widetilde{k}$ are continuous, there are the compositions

$$\widetilde{k} \circ k(x) = \begin{cases} x & \text{if } x \in X \\ 4ru & \text{if } x = ru,\ u \in \partial(e^\lambda),\ 0 \le r \le \frac{1}{4} \\ \varphi_{4r-1}(u) & \text{if } x = ru,\ u \in \partial(e^\lambda),\ \frac{1}{4} \le r \le \frac{1}{2} \\ \varphi_{2-2r}(u) & \text{if } x = ru,\ u \in \partial(e^\lambda),\ \frac{1}{2} \le r \le 1, \end{cases}$$

and

$$k \circ \widetilde{k}(x) = \begin{cases} x & \text{if } x \in X \\ 4su & \text{if } x = su,\ u \in \partial(e^\lambda),\ 0 \le s \le \frac{1}{4} \\ \varphi_{2-4s}(u) & \text{if } x = su,\ u \in \partial(e^\lambda),\ \frac{1}{4} \le s \le \frac{1}{2} \\ \varphi_{2s-1}(u) & \text{if } x = su,\ u \in \partial(e^\lambda),\ \frac{1}{2} \le s \le 1. \end{cases}$$

We want to find a homotopy $h_t : X \cup_{\varphi_0} e^\lambda \to X \cup_{\varphi_0} e^\lambda,\ t \in [0,1]$ such that $h_0 = \widetilde{k} \circ k$ and $h_1 = id$. Consider a family of maps $h_t : X \cup_{\varphi_0} e^\lambda \to X \cup_{\varphi_0} e^\lambda$ defined by

$$h_t(x) = \begin{cases} x & \text{if } x \in X \\ \dfrac{4ru}{1+3t} & \text{if } x = ru,\ u \in \partial(e^\lambda),\ 0 \le r \le \frac{1+3t}{4} \\ \varphi_{(\frac{4r}{1+3t}-1)(1-t)}(u) & \text{if } x = ru,\ u \in \partial(e^\lambda),\ \frac{1+3t}{4} \le r \le \frac{t+1}{2} \\ \varphi_{\frac{(2-2r)}{1+3t}(1-t)}(u) & \text{if } x = ru,\ u \in \partial(e^\lambda),\ \frac{t+1}{2} \le r \le 1. \end{cases}$$

It is easy to check that $h_t$ is continuous, $h_0 = \widetilde{k} \circ k$ and $h_1 = id$.

We next consider a family of maps $h'_t : X \cup_{\varphi_1} e^\lambda \to X \cup_{\varphi_1} e^\lambda$ defined by

$$h'_t(x) = \begin{cases} x & \text{if } x \in X \\ \dfrac{4su}{1+3t} & \text{if } x = su,\ u \in \partial(e^\lambda),\ 0 \le s \le \frac{1+3t}{4} \\ \varphi_{1-(\frac{4s}{1+3t}-1)(1-t)}(u) & \text{if } x = su,\ u \in \partial(e^\lambda),\ \frac{1+3t}{4} \le s \le \frac{t+1}{2} \\ \varphi_{1-\frac{(2-2s)}{1+3t}(1-t)}(u) & \text{if } x = su,\ u \in \partial(e^\lambda),\ \frac{t+1}{2} \le s \le 1. \end{cases}$$

Again $h'_t$ is continuous and satisfies $h'_0 = k \circ \widetilde{k}$, $h'_1 = id$. □

**Lemma 2.4.2:** ***(Hilton)*** *Let $\varphi : \partial(e^\lambda) \to X$ be an attaching map. A homotopy equivalence $f : X \to Y$ can be extended to a homotopy equivalence*

$$F : X \cup_\varphi e^\lambda \to Y \cup_{f\circ\varphi} e^\lambda.$$

*Proof.* Since $f : X \to Y$ is a homotopy equivalence, there exist a homotopy inverse $g : Y \to X$ to $f$ and $h_t : X \to X$ a homotopy such that $h_0 = g \circ f$ and $h_1 = id_X$. Let $H : [0,1] \times \partial(e^\lambda) \to X$ defined by $H(t,x) = h_t(\varphi(x))$. Then we have $H(0,x) = g \circ f \circ \varphi(x)$ and $H(1,x) = \varphi(x)$. Thus $g \circ f \circ \varphi$ and $\varphi$ are homotopic maps from $\partial(e^\lambda)$ to $X$. By the Lemma 2.4.1, there exists a homotopy equivalence

$$k : X \cup_{g\circ f\circ\varphi} e^\lambda \to X \cup_\varphi e^\lambda.$$

Define the following two maps $F : X \cup_\varphi e^\lambda \to Y \cup_{f\circ\varphi} e^\lambda$ and $G : Y \cup_{f\circ\varphi} e^\lambda \to X \cup_{g\circ f\circ\varphi} e^\lambda$ as follows

$$F(x) = \begin{cases} f(x) & \text{if } x \in X \\ x & \text{if } x \in e^\lambda \end{cases}$$

and

$$G(y) = \begin{cases} g(y) & \text{if } y \in Y \\ y & \text{if } y \in e^\lambda. \end{cases}$$

We will first prove that $F$ has a left homotopy inverse $k \circ G$. That is, the composition $k \circ G \circ F : X \cup_\varphi e^\lambda \to X \cup_\varphi e^\lambda$ is homotopic to the identity map. From the definition of $k$, $F$ and $G$, we note that

$$k \circ G \circ F(x) = \begin{cases} g \circ f(x) & \text{if } x \in X \\ 2ru & \text{if } x = ru,\ u \in \partial(e^\lambda),\ 0 \le r \le \frac{1}{2} \\ h_{2-2r} \circ \varphi(u) & \text{if } x = ru,\ u \in \partial(e^\lambda),\ \frac{1}{2} \le r \le 1 \end{cases}$$

is a continuous map. Define a family of maps $q_t : X \cup_\varphi e^\lambda \to X \cup_\varphi e^\lambda$ by

$$q_t(x) = \begin{cases} h_t(x) & \text{if } x \in X \\ \dfrac{2}{t+1} ru & \text{if } x = ru,\ u \in \partial(e^\lambda),\ 0 \le r \le \frac{t+1}{2} \\ h_{2-2r+t} \circ \varphi(u) & \text{if } x = ru,\ u \in \partial(e^\lambda),\ \frac{t+1}{2} \le r \le 1. \end{cases}$$

We then see that

$$q_0(x) = \begin{cases} h_0(x) = g \circ f(x) & \text{if } x \in X \\ 2ru & \text{if } x = ru,\ u \in \partial(e^\lambda),\ 0 \leq r \leq \frac{1}{2} \\ h_{2-2r} \circ \varphi(u) & \text{if } x = ru,\ u \in \partial(e^\lambda),\ \frac{1}{2} \leq r \leq 1, \end{cases}$$

and

$$q_1(x) = \begin{cases} h_1(x) = x & \text{if } x \in X \\ ru & \text{if } x = ru,\ u \in \partial(e^\lambda),\ 0 \leq r \leq 1 \\ h_1 \circ \varphi(u) = \varphi(u) & \text{if } x = u,\ u \in \partial(e^\lambda). \end{cases}$$

Since $q_0 = k \circ G \circ F$ and $q_1 = id$, the composition $k \circ G \circ F$ is homotopic to the identity map and hence $F$ has $k \circ G$ as a left homotopy inverse.

Similarly, $G$ has a left homotopy inverse, since $\phi = f \circ \varphi : \partial(e^\lambda) \to Y$ is an attaching map and $g : Y \to X$ is a homotopy equivalence, so $G : Y \cup_\phi e^\lambda \to X \cup_{g \circ \phi} e^\lambda$ has a left homotopy inverse.

**Claim 2.4.4:** *If a map $F$ has a left and a right homotopy inverse $L$ and $R$ respectively, then $F$ is a homotopy equivalence, and $L$ (or $R$) is a 2-sided homotopy inverse.*

*Proof.* Since $L$ and $R$ are left and right homotopy inverses to $F$, we have the relations $LF \simeq \text{id}$ and $FR \simeq \text{id}$. This implies that

$$L \simeq L(FR) = (LF)R \simeq R.$$

Hence

$$FL \simeq FR \simeq \text{id (or } RF \simeq LF \simeq \text{id)}$$

which proves that $L$ (or $R$) is a 2-sided homotopy inverse. □

To prove the Lemma 2.4.2, it only remains to prove that $F$ has a right homotopy inverse. By the Claim 2.4.4, we obtain the following:

- $k \circ (G \circ F) \simeq \text{id}$ implies that $(G \circ F) \circ k \simeq \text{id}$ since $k$ is known to have a left homotopy inverse (by Lemma 2.4.1).
- $G \circ (F \circ k) = (G \circ F) \circ k \simeq \text{id}$ implies that $(F \circ k) \circ G \simeq \text{id}$ since $G$ is known to have a left homotopy inverse.
- $F \circ (k \circ G) = (F \circ k) \circ G \simeq \text{id}$ implies that $F$ has $k \circ G$ as a right homotopy inverse.

Therefore, $F$ is a homotopy equivalence. This completes the proof of Lemma 2.4.2. □

*Proof.* **(of Theorem 2.4.3)** Let $a \in \mathbb{R}$ and $p_{ik_i}$ be critical points belonging to $f^{-1}(c_i)$ with index $\lambda_{ik_i}$. If $f^{-1}(a) = \emptyset$, then $M^a = \emptyset$ and so we have nothing to do.

If $f^{-1}(a) \neq \emptyset$, then $M^a \neq \emptyset$.

**Base case:** We may assume that $c_1 < a < c_2$. Since $M^a$ is compact, $f$ has a global minimum value $c_1 \in \mathbb{R}$ (i.e, $c_1 \leq f(p)$ for all $p \in M$). According to the Theorem 2.4.1,

$M^{c_1+\epsilon}$ is homotopy equivalent to $M^a$ for some small $\epsilon > 0$. Since the critical points belonging to $f^{-1}(c_1)$ have index 0, by Proposition 2.4.1, $M^{c_1+\epsilon}$ has the homotopy type of a disjoint union of 0 cells. Therefore, $M^a$ has the homotopy type of a $CW$-complex.

**Induction hypothesis:** Suppose that $a \neq c_1,\ c_2,\ c_3,\ \cdots$ such that $M^a$ is homotopy equivalent to a $CW$-complex $K$ via $g$. Let $c = c_{j_0}$ be the smallest critical value of $f$ bigger than $a$. According to the Theorem 2.4.1 and Proposition 2.4.1, for some small $\epsilon > 0$ we have that $M^{c-\epsilon}$ is homotopy equivalent to $M^a$ via $h$ and that $M^{c+\epsilon}$ has the homotopy type of $M^{c-\epsilon} \cup_{\varphi_{j_0 1}} e^{\lambda_{j_0 1}} \cup_{\varphi_{j_0 2}} \cdots \cup_{\varphi_{j_0 k_{j_0}}} e^{\lambda_{j_0 k_{j_0}}}$ for some attaching maps $\varphi_{j_0 1}, \cdots, \varphi_{j_0 k_{j_0}}$. Then, by Lemma 2.4.2 we see that

$$M^{c-\epsilon} \cup_{\varphi_{j_0 1}} e^{\lambda_{j_0 1}} \cup_{\varphi_{j_0 2}} \cdots \cup_{\varphi_{j_0 k_{j_0}}} e^{\lambda_{j_0 k_{j_0}}} \simeq M^a \cup_{h\circ\varphi_{j_0 1}} e^{\lambda_{j_0 1}} \cup_{h\circ\varphi_{j_0 2}} \cdots \cup_{h\circ\varphi_{j_0 k_{j_0}}} e^{\lambda_{j_0 k_{j_0}}}.$$

Since $M^a$ is homotopy equivalent to $K$ via $g$, Lemma 2.4.2 shows that

$$M^a \cup_{h\circ\varphi_{j_0 1}} e^{\lambda_{j_0 1}} \cup_{h\circ\varphi_{j_0 2}} \cdots \cup_{h\circ\varphi_{j_0 k_{j_0}}} e^{\lambda_{j_0 k_{j_0}}} \simeq K \cup_{g\circ h\circ\varphi_{j_0 1}} e^{\lambda_{j_0 1}} \cup_{g\circ h\circ\varphi_{j_0 2}} \cdots \cup_{g\circ h\circ\varphi_{j_0 k_{j_0}}} e^{\lambda_{j_0 k_{j_0}}}$$

By cellular approximation, for each $r$, $1 \leq r \leq k_{j_0}$, the map $g \circ h \circ \varphi_{j_0 r}$ is homotopic to a cellular map $\psi_{j_0 r} : \partial(e^{\lambda_{j_0 r}}) \to K^{(\lambda_{j_0 r}-1)}$, where $K^{(\lambda_{j_0 r}-1)}$ is the $(\lambda_{j_0 r} - 1)$-skeleton of $K$. Applying lemma 2.4.1 shows that

$$K \cup_{g\circ h\circ\varphi_{j_0 1}} e^{\lambda_{j_0 1}} \cup_{g\circ h\circ\varphi_{j_0 2}} \cdots \cup_{g\circ h\circ\varphi_{j_0 k_{j_0}}} e^{\lambda_{j_0 k_{j_0}}} \simeq K \cup_{\psi_{j_0 1}} e^{\lambda_{j_0 1}} \cup_{\psi_{j_0 2}} \cdots \cup_{\psi_{j_0 k_{j_0}}} e^{\lambda_{j_0 k_{j_0}}}.$$

Hence $K \cup_{\psi_{j_0 1}} e^{\lambda_{j_0 1}} \cup_{\psi_{j_0 2}} \cdots \cup_{\psi_{j_0 k_{j_0}}} e^{\lambda_{j_0 k_{j_0}}}$ is a $CW$-complex since the attaching maps are cellular. Therefore, we conclude that $M^{c+\epsilon}$ has the homotopy type of a $CW$-complex.

By induction, if $\widetilde{c}$ is the smallest critical value of $c_j$'s such that $c_j > c$, then $M^{\widetilde{a}}$ has the homotopy type of a $CW$-complex for every $\widetilde{a} \in (c, \widetilde{c})$.

Finally, since $M$ is compact, the Morse function $f$ has a finite number of critical points (see Corollary 2.2.1) and a finite number of critical values. Thus the inductive step above completes the proof for all of $M$. $\square$

## 2.5 The Morse Inequalities

In this section we will see a series of inequalities proved by **Marston Morse** which give bounds on the Betti numbers of a smooth manifold $M$. More precisely, the Morse inequalities establish a relationship between the number of critical points of index $\lambda$ of a real valued Morse function on $M$ and the $\lambda$-th Betti number on $M$.

Let us denote a tuple of topological spaces such that $X_n \supset X_{n-1} \supset \cdots \supset X_0$ by $(X_n, X_{n-1}, \cdots, X_0)$. In particular, if the tuple consists of two spaces or three spaces, then it is called a pair or triple respectively.

**Definition 2.5.1:** *Let $S$ be a function from a pair of spaces to the integers. We say that $S$ is* ***subadditive*** *if for all triples $(X, Y, Z)$ the inequality $S(X, Z) \leq S(X, Y) + S(Y, Z)$ holds. If equality holds, then $S$ is called* ***additive****.*

For any pair of spaces $(X, Y)$ and a given field $\mathbb{F}$ as coefficient of the $\lambda$-th relative homology group $H_\lambda(X, Y)$, we denote by

$$b_\lambda(X, Y) = \text{ rank over } \mathbb{F} \text{ of } H_\lambda(X, Y, \mathbb{F})$$

the $\lambda$-th Betti number of $(X, Y)$ and by

$$\chi(X, Y) = \sum(-1)^\lambda b_\lambda(X, Y)$$

the Euler characteristic of $(X, Y)$.

Given a pair $(X, \emptyset)$, we will write $S(X) := S(X, \emptyset)$.

Given a triple $(X, Y, Z)$, we can construct the following long exact sequence of relative homology

$$\cdots \xrightarrow{h_{\lambda+1}} H_\lambda(Y, Z) \xrightarrow{f_\lambda} H_\lambda(X, Z) \xrightarrow{g_\lambda} H_\lambda(X, Y) \xrightarrow{h_\lambda} H_{\lambda-1}(Y, Z) \xrightarrow{f_{\lambda-1}} \cdots \tag{2.5.1}$$

**Lemma 2.5.1:** *$b_\lambda$ is subadditive and $\chi$ is additive.*

*Proof.* Form (2.5.1), we can construct short exact sequences as follows:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \mathrm{Ker} f_\lambda & & \mathrm{Ker} g_\lambda & & \mathrm{Ker} h_\lambda & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 \xrightarrow{h_{n+1}} \cdots \xrightarrow{h_{\lambda+1}} & & H_\lambda(Y, Z) & \xrightarrow{f_\lambda} & H_\lambda(X, Z) & \xrightarrow{g_\lambda} & H_\lambda(X, Y) & \xrightarrow{h_\lambda} & H_{\lambda-1}(Y, Z) \xrightarrow{f_{\lambda-1}} \cdots \xrightarrow{h_0} 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \mathrm{Im} f_\lambda & & \mathrm{Im} g_\lambda & & \mathrm{Im} h_\lambda & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & \\
 & & (1_\lambda) & & (2_\lambda) & & (3_\lambda) & &
\end{array}$$

By the short exact sequence $(2_\lambda)$ above, we have

$$\begin{aligned}
b_\lambda(X, Z) &= \text{rank}(H_\lambda(X, Z)) \\
&= \text{rank}(Kerg_\lambda) + \text{rank}(Img_\lambda) \\
&= \text{rank}(Imf_\lambda) + \text{rank}(Kerh_\lambda) \\
&\leq \text{rank}(Kerf_\lambda) + \text{rank}(Imf_\lambda) + \text{rank}(Kerh_\lambda) + \text{rank}(Imh_\lambda) \\
&= \text{rank}(H_\lambda(Y, Z)) + \text{rank}(H_\lambda(X, Y)) \\
&= b_\lambda(X, Y) + b_\lambda(Y, Z),
\end{aligned}$$

which shows that $b_\lambda$ is subadditive.

To see that $\chi$ is additive, we first note from the short exact sequence $(2_\lambda)$ above that

$$b_\lambda(X, Z) = \text{rank}(Imf_\lambda) + \text{rank}(Img_\lambda). \tag{2.5.2}$$

From the short exact sequences $(1_\lambda)$ and $(3_\lambda)$, similar reasoning leads to the following results

$$b_\lambda(Y,Z) = \mathrm{rank}(Imh_{\lambda+1}) + \mathrm{rank}(Imf_\lambda), \tag{2.5.3}$$

and

$$b_\lambda(X,Y) = \mathrm{rank}(Img_\lambda) + \mathrm{rank}(Imh_\lambda). \tag{2.5.4}$$

Therefore, by putting (2.5.2), (2.5.3) and (2.5.4) together, gives

$$b_\lambda(Y,Z) - b_\lambda(X,Z) + b_\lambda(X,Y) = \mathrm{rank}(Imh_{\lambda+1}) + \mathrm{rank}(Imh_\lambda). \tag{2.5.5}$$

Multiplying (2.5.5) by $(-1)^\lambda$ and summing over $\lambda$ we then see that

$$\sum_{\lambda=0}^{n}(-1)^\lambda\Big(b_\lambda(Y,Z) - b_\lambda(X,Z) + b_\lambda(X,Y)\Big)= (-1)^n\mathrm{rank}(Imh_{n+1}) + \mathrm{rank}(Imh_0). \tag{2.5.6}$$

Since we have $\mathrm{rank}(Imh_{n+1}) = \mathrm{rank}(Imh_0) = 0$, (2.5.6) shows that

$$\chi(Y,Z) - \chi(X,Z) + \chi(X,Y) = 0.$$

□

**Lemma 2.5.2:** *If $S$ is subadditive and we have a tuple of spaces $(X_n, X_{n-1}, \cdots, X_0)$, then $S(X_n, X_0) \leq \sum_{i=1}^{n} S(X_i, X_{i-1})$. If $S$ is additive then equality holds.*

*Proof.* We will prove the lemma by induction on $n$.

**Base case:** If $n = 2$, then $S(X_2, X_0) \leq S(X_2, X_1) + S(X_1, X_0)$ since $S$ is subadditive.

**Induction hypothesis:** We suppose that the inequality is true for $n-1$, that is,

$$S(X_{n-1}, X_0) \leq \sum_{i=1}^{n-1} S(X_i, X_{i-1}).$$

Since $S$ is subadditive, we have $S(X_n, X_0) \leq S(X_n, X_{n-1}) + S(X_{n-1}, X_0)$. By hypothesis, we then have $S(X_n, X_0) \leq S(X_n, X_{n-1}) + \sum_{i=1}^{n-1} S(X_i, X_{i-1}) = \sum_{i=1}^{n} S(X_i, X_{i-1})$. Therefore it is true for $n$.

A similar proof shows that $S(X_n, X_0) = \sum_{i=1}^{n} S(X_i, X_{i-1})$ if $S$ is additive. □

**Theorem 2.5.1:** ***(Weak Morse Inequalities)*** *Let $M$ be a compact smooth manifold and $f : M \to \mathbb{R}$ be a Morse function on $M$. We denote the number of critical points of $f$ of index $\lambda$ by $\mu_\lambda$. Then we have*

$$b_\lambda(M) \leq \mu_\lambda, \tag{2.5.7}$$

*and*

$$\chi(M) = \sum(-1)^\lambda \mu_\lambda. \tag{2.5.8}$$

Before proving this theorem, let us recall the following theorem (see theorem 2.20 of [5]).

**Theorem 2.5.2:** ***(Excision)*** *Let $A, Z \subset X$ be topological spaces such that the closure of $Z$ is contained in the interior of $A$. Then the inclusion $(X \setminus Z, A \setminus Z) \hookrightarrow (X, A)$ induces isomorphisms $H_r(X \setminus Z, A \setminus Z) \to H_r(X, A)$ for all $r$. Equivalently, if we have a subspaces $A, B$ whose interior cover $X$, then the inclusion $(B, A \cap B) \hookrightarrow (X, A)$ induces isomorphisms $H_r(B, A \cap B) \to H_r(X, A)$ for all $r$.*

*Proof.* **(of Theorem 2.5.1)** Since $f$ is a Morse function and $M$ is compact, by Corollary 2.2.1 $f$ has a finite number of critical points and each critical point is isolated. Let $\{p_1, p_2, \cdots, p_n\}$ be the set of critical points of $f$ with indices $\lambda_1, \lambda_2, \cdots, \lambda_n$ respectively. For simplicity, assume that $f(p_i) \neq f(p_j)$ for $i \neq j$. There exists $a_i$ with $a_i < a_{i+1}$ for all $i \in \{0, 1, 2, \cdots, n\}$ such that $M^{a_0} = \emptyset, M^{a_n} = M$, and $M^{a_i}$ contains only the critical point $p_i$ of $f$. That is, $p_j$ the only critical point of $f$ with index $\lambda_j$ in $M^{a_j} \setminus M^{a_{j-1}}$ for each $j \in \{1, 2, \cdots, n\}$. By the Theorem 2.4.2, we then have $M^{a_j}$ has the homotopy type of $M^{a_{j-1}} \cup e^{\lambda_j}$, and hence, by the Theorem 2.5.2,

$$\begin{aligned}
H_r(M^{a_j}, M^{a_{j-1}}, \mathbb{F}) &\simeq H_r(M^{a_{j-1}} \cup e^{\lambda_j}, M^{a_{j-1}}, \mathbb{F}) \\
&\cong H_r(e^{\lambda_j}, \partial(e^{\lambda_j}), \mathbb{F}) \\
&\cong H_{r-1}(\partial(e^{\lambda_j}), \mathbb{F}) && \text{(by the exact sequence of a pair)} \\
&\cong H_{r-1}(S^{\lambda_j - 1}, \mathbb{F}) \\
&\cong \begin{cases} \mathbb{F} & \text{if } r = \lambda_j \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

This shows that

$$b_r(M^{a_j}, M^{a_{j-1}}, \mathbb{F}) = \begin{cases} 1 & \text{if } r = \lambda_j \\ 0 & \text{otherwise.} \end{cases}$$

Since $b_\lambda$ is subadditive and we have a tuple of spaces $(M^{a_n}, M^{a_{n-1}}, \cdots, M^{a_0})$, Lemma 2.5.2 gives us that

$$\begin{aligned}
b_\lambda(M) &= b_\lambda(M^{a_n}, M^{a_0}) \\
&\leq \sum_{i=1}^{n} b_\lambda(M^{a_i}, M^{a_{i-1}}) \\
&= \mu_\lambda
\end{aligned}$$

since

$$b_\lambda(M^{a_i}, M^{a_{i-1}}) = \begin{cases} 1 & \text{if } \lambda = \lambda_i \\ 0 & \text{otherwise.} \end{cases}$$

This proves inequality (2.5.7).

We prove the last part of this theorem by using Lemma 2.5.2,

$$\begin{aligned}
\chi(M) &= \chi(M^{a_n}, M^{a_0}) \\
&= \sum_{i=1}^{n} \chi(M^{a_i}, M^{a_{i-1}}) \\
&= \sum_{i=1}^{n} \sum (-1)^{\lambda} b_{\lambda}(M^{a_i}, M^{a_{i-1}}) \\
&= \sum (-1)^{\lambda} \Big( \sum_{i=1}^{n} b_{\lambda}(M^{a_i}, M^{a_{i-1}}) \Big) \\
&= \sum (-1)^{\lambda} \mu_{\lambda}
\end{aligned}$$

□

**Observation 2.5.1:** *If $\mu_\lambda = 0$, then $b_\lambda = 0$.*

**Lemma 2.5.3:** *The function $S_\lambda$ defined by $S_\lambda(X, Y) = \sum_{i=0}^{\lambda} (-1)^i b_{\lambda-i}(X, Y)$ is subadditive.*

*Proof.* Note that (2.5.5) can be expressed as

$$\text{rank}(Imh_{\lambda+1}) = b_\lambda(Y, Z) - b_\lambda(X, Z) + b_\lambda(X, Y) - \text{rank}(Imh_\lambda). \tag{2.5.9}$$

Since $\text{rank}(Imh_{\lambda+1}) \geq 0$ and $\text{rank}(Imh_0) = 0$, (2.5.9) tells us that

$$\sum_{i=0}^{\lambda} (-1)^i \Big( b_{\lambda-i}(Y, Z) - b_{\lambda-i}(X, Z) + b_{\lambda-i}(X, Y) \Big) \geq 0. \tag{2.5.10}$$

This means that

$$S_\lambda(Y, Z) - S_\lambda(X, Z) + S_\lambda(X, Y) \geq 0$$

which implies that $S_\lambda$ is subadditive. □

**Theorem 2.5.3:** ***(Strong Morse Inequalities)*** *Let $M$ be a compact smooth manifold and $f : M \to \mathbb{R}$ be a Morse function on $M$. We denote the number of critical points of $f$ of index $\lambda$ by $\mu_\lambda$. Then the inequality*

$$\sum_{i=0}^{\lambda} (-1)^i b_{\lambda-i}(M) \leq \sum_{i=0}^{\lambda} (-1)^i \mu_{\lambda-i} \tag{2.5.11}$$

*holds for every $\lambda \in \{0, 1, \cdots, n\}$.*

*Proof.* Since we have a tuple of spaces $(M = M^{a_n}, M^{a_{n-1}}, \cdots, M^{a_0} = \emptyset)$ and $S_\lambda$ is subadditive, by Lemma 2.5.2

$$S_\lambda(M, \emptyset) = S_\lambda(M) \leq \sum_{j=1}^{n} S_\lambda(M^{a_j}, M^{a_{j-1}}).$$

Hence, applying Lemma 2.5.3 gives

$$\begin{aligned}\sum_{i=0}^{\lambda}(-1)^i b_{\lambda-i}(M) &\leq \sum_{`j=1}^{n}\sum_{i=0}^{\lambda}(-1)^i b_{\lambda-i}(M^{a_j}, M^{a_{j-1}}) \\ &= \sum_{i=0}^{\lambda}(-1)^i\Big(\sum_{j=1}^{n} b_{\lambda-i}(M^{a_j}, M^{a_{j-1}})\Big) \\ &= \sum_{i=0}^{\lambda}(-1)^i \mu_{\lambda-i}\end{aligned}$$

since

$$b_{\lambda-i}(M^{a_j}, M^{a_{j-1}}) = \begin{cases} 1 & \text{if } \lambda - i = \lambda_j \\ 0 & \text{otherwise.}\end{cases}$$

□

To see that these inequalities are definitely stronger than the previous ones, we consider the following cases of (2.5.11):

$$\sum_{i=0}^{\lambda}(-1)^i b_{\lambda-i}(M) \leq \sum_{i=0}^{\lambda}(-1)^i \mu_{\lambda-i}, \tag{2.5.12}$$

$$\sum_{i=0}^{\lambda-1}(-1)^i b_{\lambda-1-i}(M) \leq \sum_{i=0}^{\lambda-1}(-1)^i \mu_{\lambda-1-i} \tag{2.5.13}$$

By adding the inequalities (2.5.12) and (2.5.13), we get (2.5.7). If $\mu_\lambda = 0$, then inequality (2.5.12) together with Observation 2.5.1 imply

$$\sum_{i=0}^{\lambda-1}(-1)^i b_{\lambda-1-i}(M) \geq \sum_{i=0}^{\lambda-1}(-1)^i \mu_{\lambda-1-i}, \tag{2.5.14}$$

and so, by (2.5.13) and (2.5.14), we have the equality

$$b_{\lambda-1}(M) - b_{\lambda-2}(M) + \cdots + (-1)^{\lambda-1} b_0(M) = \mu_{\lambda-1} - \mu_{\lambda-2} + \cdots + (-1)^{\lambda-1}\mu_0, \tag{2.5.15}$$

or equivalently,

$$b_0(M) - b_1(M) + \cdots + (-1)^{\lambda-1} b_{\lambda-1}(M) = \mu_0 - \mu_1 + \cdots + (-1)^{\lambda-1}\mu_{\lambda-1}. \tag{2.5.16}$$

Since $\mu_\lambda = 0$ for every $\lambda \geq n+1$, if $\lambda \geq n+1$, then (2.5.16) is exactly the same as (2.5.8).

**Corollary 2.5.1:** *If $\mu_{\lambda+1} = \mu_{\lambda-1} = 0$, then $b_\lambda = \mu_\lambda$ and $b_{\lambda+1} = b_{\lambda-1} = 0$.*

*Proof.* If $\mu_{\lambda+1} = \mu_{\lambda-1} = 0$, then Observation 2.5.1 gives that

$$b_{\lambda+1} = b_{\lambda-1} = 0. \tag{2.5.17}$$

A similar calculation as that case of (2.5.15) leads to the following results

$$b_\lambda(M) - \cancel{b_{\lambda-1}}(M) + \cdots + (-1)^\lambda b_0(M) = \mu_\lambda - \cancel{\mu_{\lambda-1}} + \cdots + (-1)^\lambda \mu_0, \tag{2.5.18}$$

and

$$b_{\lambda-2}(M) - b_{\lambda-3}(M) + \cdots + (-1)^{\lambda-2} b_0(M) = \mu_{\lambda-2} - \mu_{\lambda-3} + \cdots + (-1)^{\lambda-2} \mu_0. \tag{2.5.19}$$

By subtracting the (2.5.19) from (2.5.18), we obtain $b_\lambda = \mu_\lambda$. $\square$

# Chapter 3

# Simple applications of Morse theory

In this chapter, we will give some simple applications of theorems of the previous chapter.

## 3.1 Examples

**Example 3.1.1:** ***($n$-sphere $S^n$)*** *As in Example 1.1.3, the height function $f$ from $S^n$ to $\mathbb{R}$ has only two non-degenerate critical points, one of index $0$ and one of index $n$.*

Hence, Theorem 2.4.3 implies that $S^n$ has the homotopy type of a $CW$-complex of the form $e^0 \cup e^n$. So, the chain complex of $S^n$ is of the form

$$\begin{array}{ccccccc} 0 \to & C_n(S^n) & \to 0 \to \cdots & \to 0 \to & C_0(S^n) & \to 0 \\ & \| & & & \| & \\ & \mathbb{Z}.e^n & & & \mathbb{Z}.e^0 & \end{array} \tag{3.1.1}$$

From (3.1.1), we see that the boundary homomorphisms are $\partial_r = 0$ for all $r$. Therefore, the homotopy groups of $S^n$ are

$$H_r(S^n, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } r = 0, n \\ 0 & \text{otherwise.} \end{cases}$$

**Example 3.1.2:** ***(Complex projective space $\mathbb{C}P^n$)*** *From Example 2.1.2, we know that $p_0, p_1, \cdots, p_n$ are the only critical points of $f$, and that the index of $p_j$ is equal to twice the number of $k$ with $c_k < c_j$. Hence, we will get every even index between $0$ and $2n$ exactly once.*

Applying Theorem 2.4.3 gives us that $\mathbb{C}P^n$ has the homotopy type of a $CW$-complex of the form $e^0 \cup e^2 \cup \cdots \cup e^{2n}$. This shows that the chain complex of $\mathbb{C}P^n$ is of the form

$$\begin{array}{ccccccccccc} 0 \to & C_{2n}(\mathbb{C}P^n) & \to 0 \to & C_{2(n-1)}(\mathbb{C}P^n) & \to \cdots \to & C_2(\mathbb{C}P^n) & \to 0 \to & C_0(\mathbb{C}P^n) & \to 0 \\ & \parallel & & \parallel & & \parallel & & \parallel & \\ & \mathbb{Z}.e^{2n} & & \mathbb{Z}.e^{2(n-1)} & & \mathbb{Z}.e^2 & & \mathbb{Z}.e^0 & \end{array} \tag{3.1.2}$$

(3.1.2) tells us that the boundary homomorphisms are $\partial_r = 0$ for all $r$. Therefore, the homotopy groups of $\mathbb{C}P^n$ are given by:

$$H_r(\mathbb{C}P^n, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } r = 0, 2, \cdots, 2n \\ 0 & \text{otherwise} \end{cases}.$$

**Remark 3.1.1:** *We can use Corollary 2.5.1 to find the homology groups of spaces above without using Theorem 2.4.3. For the first example, since $\mu_{n+1} = 0$ and $\mu_{n-1} = 0$, then $b_n = \mu_n = 1$. This implies that $H_n(S^n, \mathbb{Z}) = \mathbb{Z}$. Similarly, $H_0(S^n, \mathbb{Z}) = \mathbb{Z}$ since $b_0 = \mu_0 = 1$.*

*For the second example, we will have $b_0 = b_2 = \cdots = b_{2n} = 1$ since $\mu_{k-1} = 0$ and $\mu_{k+1} = 0$ for each $k = 0, 1, \cdots, 2n$. Therefore, $H_k(\mathbb{C}P^n, \mathbb{Z}) = \mathbb{Z}$ for all $k = 0, 2, \cdots, 2n$.*

## 3.2 Reeb's theorem

**Theorem 3.2.1:** *Let $f : M \to \mathbb{R}$ be a Morse function on a compact smooth manifold $M$ of dimension $n$ with exactly two critical points. Then $M$ is homeomorphic to $S^n$.*

*Proof.* Let $p$ and $q$ be the critical points of $f$. We observe that $p$ and $q$ must be the minimum and maximum points of $f$ since $M$ is compact. We suppose that $f$ takes minimum and maximum values at $p$ and $q$ respectively. According to Lemma 2.2.1, it is easy to show that the index of $p$ is 0. Indeed, if the index of $p$ is $\lambda \neq 0$, then there exist a suitable local coordinate system $X : V \subset \mathbb{R}^n \to U_p$ in a neighborhood $U_p$ of $p$ with $0 \in V$ and $X(0) = p$ such that

$$f \circ X = f(p) - \sum_{i=1}^{\lambda} x_i^2 + \sum_{i=\lambda+1}^{n} x_i^2 \tag{3.2.1}$$

holds throughout $V$. In particular, we have $(\delta, 0, \cdots, 0) \in V$ for some $\delta > 0$ and so $f \circ X(\delta, 0, \cdots, 0) = f(p) - \delta < f(p)$ which is contradiction since $f$ takes the minimum value at $p$.

Similarly, the index of $q$ is $n$ because $f$ takes maximum value at this point.

Without loss of generality, we assume that $f(p) = 0$ and $f(q) = 1$. Therefore, $f$ can be expressed in terms of the coordinate systems $(x_1, \cdots, x_n)$ in a neighborhood $U_p$ of $p$ and $(y_1, \cdots, y_n)$ in a neighborhood $U_q$ of $q$ as the following form:

$$f = \begin{cases} x_1^2 + x_2^2 + \cdots + x_n^2 \\ 1 - y_1^2 - y_2^2 - \cdots - y_n^2. \end{cases} \tag{3.2.2}$$

Choose a small positive number $\epsilon$ such that $0 \le \sum_{i=1}^{n} x_i^2 \le \epsilon$ and $1-\epsilon \le 1 - \sum_{i=1}^{n} y_i^2 \le 1$, so that the sets $f^{-1}([0,\epsilon])$ and $f^{-1}([1-\epsilon,1])$ are diffeomorphic to closed $n$-disks $D_p^n$ and $D_q^n$ respectively. Moreover, since $f^{-1}([\epsilon,1-\epsilon])$ is a closed subset of the compact set $M$ and contains no critical point of $f$, Theorem 2.4.1 tells us that $f^{-1}([0,\epsilon]) = M^\epsilon$ is diffeomorphic to $f^{-1}([0,1-\epsilon]) = M^{1-\epsilon}$. Hence, $M = f^{-1}([0,1-\epsilon]) \cup f^{-1}([1-\epsilon,1])$ is diffeomorphic to $D_p^n \cup_{S^{n-1}} D_q^n$ which is the union of two closed $n$-disks glued along their boundary.

To show that $D_p^n \cup_{S^{n-1}} D_q^n$ is homeomorphic to $S^n$, we will use the following lemmas, which we state here without proof.

**Lemma 3.2.1:** ***(The Universal Properties of the Quotient Topology)*** *Let $p : X \twoheadrightarrow Y$ be a quotient map and let $Z$ be a topological space. Given any continuous function $f : X \to Z$ with the property that $f(x_1) = f(x_2)$ whenever $p(x_1) = p(x_2)$, then there is a unique continuous function $\widetilde{f} : Y \to Z$ so that $\widetilde{f}p = f$.*

$$\begin{tikzcd} X \arrow[d, two heads, "p"] \arrow[dr, "f"] & \\ Y \arrow[r, dashed, "\exists! \widetilde{f}"] & Z \end{tikzcd}$$

**Lemma 3.2.2:** *Let $h : X \to Y$ be a continuous bijective function. If $X$ is a compact space and $Y$ is a Hausdorff space, then $h$ is homeomorphism.*

Consider a map $f : D_p^n \cup D_q^n \to S^n$ defined by

$$f(x) = \begin{cases} f_u(x) & \text{if } x \in D_p^n \\ f_l(x) & \text{if } x \in D_q^n, \end{cases}$$

where $f_u(x) = (x, \sqrt{1-||x||^2})$ and $f_l(x) = (x, -\sqrt{1-||x||^2})$ are homeomorphism from the standard unit disk to the upper and lower hemispheres respectively. Then $f$ is continuous since $f_u$ and $f_l$ are continuous. Moreover, since $S^n = S_u^n \cup S_l^n$, $f$ is surjective. Note that $D_p^n \cup_{S^{n-1}} D_q^n$ is the quotient of $D_p^n \cup D_q^n$ by the relation " $\sim$ " that identifies those points in $D_p^n$ and in $D_q^n$ that lie in the intersection $D_p^n \cap D_q^n = S^{n-1}$. Since $f_u$ and $f_l$ are injective and if $x_1 \in D_p^n$ and $x_2 \in D_q^n$, then

$$\begin{aligned} f(x_1) = f(x_2) &\Longleftrightarrow f_u(x_1) = f_l(x_2) \\ &\Longleftrightarrow \left(x_1, \sqrt{1-||x_1||^2}\right) = \left(x_2, -\sqrt{1-||x_2||^2}\right) \\ &\Longleftrightarrow x_1 = x_2 \text{ and } ||x_1|| = ||x_2|| = 1 \\ &\Longleftrightarrow x_1 \sim x_2 \end{aligned}$$

By Lemma 3.2.1, $f$ induces a continuous map $\widetilde{f} : D_p^n \cup_{S^{n-1}} D_q^n \to S^n$, which is bijective since $f$ is surjective and $f(x_1) = f(x_2)$ implies that $x_1 \sim x_2$.

Since $D_p^n$ and $D_q^n$ are closed and bounded subsets of $\mathbb{R}^n$, they are compact. Thus, the finite union $D_p^n \cup D_q^n$ is compact, and so is its quotient $D_p^n \cup_{S^{n-1}} D_q^n$. Since $S^n$ is a metric space, it is a Hausdorff space. Therefore, $\widetilde{f}$ is a homeomorphism from $D_p^n \cup_{S^{n-1}} D_q^n$ to $S^n$, by Lemma 3.2.2. □

**Remark 3.2.1:**

1. *If $n \leq 6$, then $M$ is diffeomorphic to $S^n$ and if $n \geq 7$, then there exists $M$ such that homeomorphic to $S^n$, but it is not diffeomorphic to $S^n$. Such manifolds called* ***exotic spheres*** *(see [6], [8]).*
2. *If $f$ is smooth and its critical points are degenerate, then the theorem remains true (see [9] or Theorem $1'$ in Chapter $6$ of [11]).*

## 3.3 Morse Functions on Knots

**Definition 3.3.1:** *A* ***knot*** *is a smooth embedding of the circle ($M = S^1$) into the oriented real Euclidean 3-dimensional space $\boldsymbol{E} = \mathbb{R}^3$, with inner product $\langle \cdot, \cdot \rangle$ .*

Let $\phi : S^1 \hookrightarrow \mathbf{E}$ be a smooth embedding as in the definition. We denote by $K = \phi(S^1)$ the image of this embedding which is a compact subset of $\mathbf{E}$. Indeed, we will prove that $K$ is bounded and closed. Define

$$\psi : S^1 \to \mathbb{R} \text{ by } \psi(x) = ||\phi(x)||.$$

It is clear that $\psi$ is continuous on $S^1$. Since $S^1$ is a compact metric space, $\psi$ attains its maximum and minimum values on $S^1$. Therefore, there exists $M > 0$ such that

$$0 \leq \psi(x) \leq M, \ \forall x \in S^1.$$

Equivalently,

$$0 \leq ||\phi(x)|| \leq M, \ \forall x \in S^1.$$

Now, suppose that $x^*$ is an accumulation point of $K = \phi(S^1)$. There exists a sequence $\{y_i\}$ in $K$ such that

$$\lim_{i\to\infty} y_i = x^*.$$

Since $\phi$ is an embedding, $\phi$ is injective. Thus there exists a unique $z_i \in S^1$ such that $\phi(z_i) = y_i$, for every $i$. Since $S^1$ is compact and $\{z_i\}$ is a sequence in $S^1$, then there exists a sub-sequence $\{z_{i_j}\}$ such that

$$\lim_{j\to\infty} z_{i_j} = z^* \in S^1,$$

and

$$x^* = \lim_{j\to\infty} y_{i_j} = \lim_{j\to\infty} \phi(z_{i_j}) = \phi(\lim_{j\to\infty} z_{i_j}) = \phi(z^*) \in K.$$

Therefore, $K$ is a compact subset of $\mathbf{E}$.

Let $\mathbf{S}$ be the unit sphere in $\mathbf{E}$. Then, for each $v \in \mathbf{S}$, it determines a linear map

$$\begin{array}{rcl} L_v : \mathbf{E} & \to & \mathbb{R} \\ x & \mapsto & \langle v, x \rangle . \end{array}$$

This function can be restricted to $K \subset \mathbf{E}$ to give a Morse function for almost all $v$, and can be viewed as a height function

$$h_v := L_v|_K : K \to \mathbb{R}$$

(see Corollary 2.3.1). Let $\mu_K(v)$ be the number of critical points of $h_v$. We define $\mu_K(v) = 0$ if $h_v$ is not a Morse function, and if $h_v$ is a Morse function on $K$, Note that $\mu_K(v) \geq 2$ since a Morse function on a compact set has at least two critical points.

By the coarea formula (see Theorem 1.1.2), we have the following.

**Theorem 3.3.1:** *Let $g : \boldsymbol{S} \to \mathbb{Z}$ be the function defined by $g(v) = \mu_K(v)$. Then*

*(1) $g$ is measurable.*

*(2) the average size of $g$ is given by*

$$\overline{\mu_K} = \frac{1}{area(\boldsymbol{S})} \int_{\boldsymbol{S}} \mu_K(v) dA(v) = \frac{1}{4\pi} \int_{\boldsymbol{S}} \mu_K(v) dA(v),$$

*where $dA$ denotes the Euclidean area element on $\boldsymbol{S}$.*

Consider the smooth embedding $\phi : S^1 \to \mathbf{E}$ as a simple closed smooth curve

$$\phi : [0, 2\pi] \to \mathbf{E}.$$

Then

$$\frac{d\phi(t)}{dt} = \phi'(t) \neq 0$$

since its derivative is injective. Let $L$ be the length of $K = \phi([0, 2\pi])$. Thus, we can obtain a curve

$$\psi : [0, L] \to \mathbf{E}$$

parametrized by arc length which has the same image set as $\phi$. Indeed, we define

$$\begin{array}{rcl} s : [0, 2\pi] & \to & [0, L] \\ t & \mapsto & s(t) = \displaystyle\int_0^t |\phi'(u)| du. \end{array}$$

Since $\dfrac{ds}{dt} = |\phi'(t)| > 0$, the function $s = s(t)$ has a smooth inverse $t = t(s)$ with

$$\frac{dt}{ds} = \frac{1}{|\phi'(t)|} > 0.$$

We set

$$\psi = \phi \circ t = \phi(t) : [0, L] \to \mathbf{E}.$$

Hence $\psi([0, L]) = \phi([0, 2\pi]) = K$ and

$$|\frac{d\psi}{ds}| = |\phi'(t).\frac{dt}{ds}| = |\phi'(t).\frac{1}{\phi'(t)}| = 1.$$

If $x \in K$, then $x = \phi(s_x)$ for some $s_x \in [0, L]$ and $|\phi'(s_x)| = 1$. Let

$$T(s_x) = \phi'(s_x),$$

the unit vector tangent to $K$ at $x$. Define

$$\mathbf{S(K)} = \{(x, v) \in K \times \mathbf{S} : v \perp T(s_x)\},$$

the unit sphere bundle associated to the normal bundle of $K$ in $\mathbf{E}$. Thus, there are natural projections

$$\begin{aligned} \lambda : K \times \mathbf{S} &\to K, \\ \rho : K \times \mathbf{S} &\to \mathbf{S}. \end{aligned}$$

The restriction of these projections to $\mathbf{S(K)}$ give smooth maps

$$\begin{aligned} \lambda_K : \mathbf{S(K)} &\to K \\ \rho_K : \mathbf{S(K)} &\to \mathbf{S}. \end{aligned}$$

**Lemma 3.3.1:** *The vector $v \in \boldsymbol{S}$ is a regular value of the map $\rho_K : \boldsymbol{S(K)} \to \boldsymbol{S}$ if and only if $h_v : K \to \mathbb{R}$ is a Morse function. Moreover,*

$$\mu_K(v) = N_{\rho_K}(v), \ \forall v \in \boldsymbol{S}. \tag{3.3.1}$$

*Proof.* Consider the map

$$\begin{aligned} g : [0, L] &\xrightarrow{\phi} K \xrightarrow{h_v} \mathbb{R}, \\ s &\mapsto h_v(\phi(s)) = \langle v, \phi(s) \rangle . \end{aligned}$$

The differential of $g$ at $s_y$ is given

$$\frac{dg}{ds}\Big|_{s=s_y} = \frac{d}{ds} \langle v, \phi(s) \rangle \Big|_{s=s_y} = \langle v, \phi'(s_y) \rangle .$$

This shows that $\phi(s_y)$ is a critical point of $h_v$ if and only if $\phi'(s_y) = T(s_y) \perp v$. Since

$$\frac{d^2 g}{ds}\Big|_{s=s_y} = \frac{d}{ds} \langle v, \phi'(s) \rangle \Big|_{s=s_y} = \langle v, \phi''(s_y) \rangle = \kappa(s_y) \langle v, N(s_y) \rangle ,$$

then

$$h_v \ \text{ is a Morse function if and only if } T(s_y) \perp v \ \text{ and } \kappa(s_y) \langle v, N(s_y) \rangle \neq 0. \tag{3.3.2}$$

Secondly, define

$$\begin{aligned} \alpha : \mathbb{R}/L\mathbb{Z} \times \mathbb{R}/2\pi\mathbb{Z} &\to \mathbf{S(K)} \\ (s,\theta) &\longmapsto \big(\phi(s), \cos(\theta)N(s) + \sin(\theta)B(s)\big), \end{aligned}$$

where

$$N(s) = \frac{\phi''(s)}{||\phi''(s)||}$$

and

$$B(s) = T(s) \times N(s)$$

are the normal and binormal unit vectors respectively. It is clear that $\alpha$ is well defined and smooth. Since

$$\frac{d\alpha}{ds} = \big(\phi'(s), \cos(\theta)N'(s) + \sin(\theta)B'(s)\big)$$

and

$$\frac{d\alpha}{d\theta} = \big(0, -\sin(\theta)N(s) + \cos(\theta)B(s)\big)$$

are linearly independent for every $(s,\theta) \in \mathbb{R}/L\mathbb{Z} \times \mathbb{R}/2\pi\mathbb{Z}$, the map $\alpha$ has a smooth inverse. Moreover, we observe that $\alpha$ is a diffeomorphism.

Let $v \in \mathbf{S}$ and

$$B = \rho_K^{-1}(v) = \{(x,v) \in \mathbf{S(K)} \ : v \perp T(s_x)\}.$$

We assume that $B \neq \emptyset$. Suppose that $z = (y,v) \in B$. We can then express $z$ and $v$ as follows:

$$z = \big(\phi(s_y), \cos(\theta)N(s_y) + \sin(\theta)B(s_y)\big),$$

and

$$v = \cos(\theta)N(s_y) + \sin(\theta)B(s_y)$$

for some $(s_y,\theta) \in \mathbb{R}/L\mathbb{Z} \times \mathbb{R}/2\pi\mathbb{Z}$. We have

$$\rho_K : \mathbf{S(K)} \to \mathbf{S}$$

is the restriction of

$$H : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3,\ (u,v) \longmapsto (0_3, I_3)\begin{pmatrix} u \\ v \end{pmatrix} = v.$$

Thus,

$$(d\rho_K)_z : T_z\mathbf{S(K)} \subset \mathbb{R}^3 \times \mathbb{R}^3 \to T_v\mathbf{S} \subset \mathbb{R}^3$$

is the restriction of

$$(dH)_z : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3,\ (u,v) \longmapsto (0_3, I_3)\begin{pmatrix} u \\ v \end{pmatrix} = v.$$

Then

$$\begin{aligned}(d\rho_K)_z\Big(\frac{d\alpha}{ds}\Big) &= (dH)_z\Big|_{T_z\mathbf{S(K)}}\begin{pmatrix}\phi'(s_y)\\ \cos(\theta)N'(s_y)+\sin(\theta)B'(s_y)\end{pmatrix}\\ &= \cos(\theta)N'(s_y)+\sin(\theta)B'(s_y)\\ &= -\kappa(s_y)\cos(\theta)T(s_y)-\tau(s_y)\sin(\theta)N(s_y)+\tau(s_y)\cos(\theta)B(s_y),\end{aligned}$$

and

$$\begin{aligned}(d\rho_K)_z\Big(\frac{d\alpha}{d\theta}\Big) &= (dH)_z\Big|_{T_z\mathbf{S(K)}}\begin{pmatrix}0\\ -\sin(\theta)N(s_y)+\cos(\theta)B(s_y)\end{pmatrix}\\ &= -\sin(\theta)N(s_y)+\cos(\theta)B(s_y).\end{aligned}$$

Therefore, $v \in \mathbf{S}$ is a regular value of $\rho_K$ if and only if $(d\rho_K)_z\Big(\frac{d\alpha}{ds}\Big)$ and $(d\rho_K)_z\Big(\frac{d\alpha}{d\theta}\Big)$ are linearly independent. Equivalently,

$$v \in \mathbf{S} \text{ is a regular value of } \rho_K \text{ if and only if } \kappa(s_y)\cos(\theta) \neq 0 \text{ with } (y,v)\in B. \tag{3.3.3}$$

Note that

$$\kappa(s_y)\cos(\theta) = \kappa(s_y)\left\langle \cos(\theta)N(s_y)+\sin(\theta)B(s_y), N(s_y)\right\rangle = \kappa(s_y)\left\langle v, N(s_y)\right\rangle$$

for $(y,v)\in B$. This means that

$$h_v \text{ is a Morse function by (3.3.2).}$$

To prove the second assertion, we will show that for every $v \in \mathbf{S}$, an element of the set of critical points of $h_v$ produces only an element of $\rho_K^{-1}(v)$ and vice versa. If $\phi(s),\ s \in \mathbb{R}/L\mathbb{Z}$ is a critical point of $h_v$, then $\langle \phi'(s), v\rangle = 0$ and there exist $a, b$ such that $v = aN(s) + bB(s)$ with $||v|| = 1$. Since $||v|| = 1$, there exists $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ which satisfies $v = \cos(\theta)N(s) + \sin(\theta)B(s)$. Thus, there is $(s,\theta) \in \mathbb{R}/L\mathbb{Z} \times \mathbb{R}/2\pi\mathbb{Z}$ which produces a unique element of $\rho_K^{-1}(v)$ via the above diffeomorphism map $\alpha$. If $z = (y,v) \in B = \rho_K^{-1}(v)$, then $v \perp T(s_y)$ and there exist $(s_y,\theta)$ such that $v = \cos(\theta)N(s_y)+\sin(\theta)B(s_y)$ and $\kappa(s_y)\cos(\theta) \neq 0$ (since $v$ is a regular value of $\rho_K$). Thus,

$$\kappa(s_y)\left\langle v, N(s_y)\right\rangle = \kappa(s_y)\cos(\theta) \neq 0$$

and

$$\langle v, \phi'(s_y)\rangle = \langle v, T(s_y)\rangle = 0,$$

i.e. $\phi(s_y)$ is a critical point of $h_v$. Therefore,

$$\mu_K(v) = N_{\rho_K}(v),\ \forall v \in \mathbf{S}.$$

□

By Theorem 3.3.1, for every $v \in \mathbf{S}$, we have that $\mu_K(v)$ is measurable, which together with identity (3.3.1), implies $N_{\rho_K}(v)$ is measurable and

$$\overline{\mu}_K = \frac{1}{4\pi} \int_{\mathbf{S}} N_{\rho_K}(v) dA_{g_S}(v), \tag{3.3.4}$$

where $dA_{g_S}$ denotes the area element on $\mathbf{S}$ with the induced metric $g_s = \langle \cdot, \cdot \rangle$ from the usual inner product on $E$.

On the other hand, we will use the Theorem 1.1.2 for the map

$$\rho_K : (\mathbf{S(K)}, g_K) \to (\mathbf{S}, g_s),$$

where $g_K$ denotes the metric on $\mathbf{S(K)}$ defined by $g_K = ds^2 + d\theta^2$ from the diffeomorphism $\alpha$. We will compute the Jacobian $|J_K|$ of $\rho_K$. Let

$$\begin{array}{rcl} \Phi := \rho_K \circ \alpha : \mathbb{R}/L\mathbb{Z} \times \mathbb{R}/2\pi\mathbb{Z} & \rightarrow & \mathbf{S} \\ (s, \theta) & \longmapsto & \cos(\theta)N(s) + \sin(\theta)B(s). \end{array}$$

Then

$$\begin{aligned} \frac{d\Phi}{ds} &= D\rho_K(\alpha(s,\theta)).\frac{d\alpha}{ds} \\ &= \cos(\theta)N'(s) + \sin(\theta)B'(s) \\ &= -\kappa(s)\cos(\theta)T(s) - \tau(s)\sin(\theta)N(s) + \tau(s)\cos(\theta)B(s) \end{aligned}$$

and

$$\begin{array}{rcl} \dfrac{d\Phi}{d\theta} & = & D\rho_K(\alpha(s,\theta)).\dfrac{d\alpha}{d\theta} \\ & = & -\sin(\theta)N(s) + \cos(\theta)B(s) \end{array}$$

form the Jacobian $J_K$ as follows:

$$|J_K|^2 = \det \begin{pmatrix} \left\langle \dfrac{d\Phi}{ds}, \dfrac{d\Phi}{ds} \right\rangle_{g_S} & \left\langle \dfrac{d\Phi}{ds}, \dfrac{d\Phi}{d\theta} \right\rangle_{g_S} \\ \left\langle \dfrac{d\Phi}{d\theta}, \dfrac{d\Phi}{ds} \right\rangle_{g_S} & \left\langle \dfrac{d\Phi}{d\theta}, \dfrac{d\Phi}{d\theta} \right\rangle_{g_S} \end{pmatrix} = \det \begin{pmatrix} \kappa^2(s)\cos^2(\theta) + \tau^2(s) & \tau(s) \\ \tau(s) & 1 \end{pmatrix}.$$

Therefore, the Jacobian of $\rho_K$ is $|J_K| = |\kappa(s)\cos(\theta)|$ and we can now apply Theorem 1.1.2

$$\begin{aligned} \int_{\mathbf{S}} N_{\rho_K}(v) dA_{g_S}(v) &= \int_{\mathbf{S(K)}} |J_k| dA_{g_K}(x, v) \\ &= \int_0^L \int_0^{2\pi} |\kappa(s)\cos(\theta)| d\theta ds \\ &= \Big( \int_0^{\frac{\pi}{2}} \cos(\theta) d\theta - \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos(\theta) d\theta + \int_{\frac{3\pi}{2}}^{2\pi} \cos(\theta) d\theta \Big) \int_0^L |\kappa(s)| ds \\ &= 4 \int_0^L |\kappa(s)| ds \\ &= 4T_K, \end{aligned}$$

where

$$T_K = \int_0^L |\kappa(s)| ds$$

is called the total curvature of the knot $K$.

By (3.3.4), we conclude

$$\overline{\mu}_K = \frac{1}{4\pi}\int_{\mathbf{S}} \mu_K(v) dA_{gs}(v) = \frac{1}{4\pi}\int_{\mathbf{S}} N_{\rho_K}(v) dA_{gs}(v) = \frac{1}{\pi} T_K. \tag{3.3.5}$$

**Remark 3.3.1:** *$T_K$ measures how "**twisted**" is the curve $K$. That is, large $T_K$ means that $K$ is very twisted. Therefore, (3.3.5) shows that if $K$ is very twisted, then the height function $h_v$ will have lots of critical points on $K$ (since $T_K$ is large when $\mu_K(v)$ is large).*

In [7], the number

$$c_K = \frac{1}{2}\overline{\mu}_K \tag{3.3.6}$$

was called **the crookedness** of the knot $K$. We observe from (3.3.5) and (3.3.6) that

$$c_K = \frac{1}{4\pi}\int_{\mathbf{S}} \frac{1}{2}\mu_K(v) dA_{gs}(v) = \frac{1}{2\pi} T_K. \tag{3.3.7}$$

Moreover, any Morse function $h$ on a circle has an even number of critical points, half of which are local minima. In order to see this, consider the composition

$$g : [0, L] \subset \mathbb{R} \xrightarrow{\psi} K \xrightarrow{h} \mathbb{R}.$$

The function $g$ has a finite number of non-degenerate critical points only. The values of $g$ on these points must alternate between local minima and maxima (by Rolle's theorem), which implies that there must be the same number of local minima as that of local maxima. We then conclude that $\frac{1}{2}\mu_K(v)$ is the number of local minima of the Morse function $h_v$.

**Corollary 3.3.1:** *For any knot $K \hookrightarrow \boldsymbol{E}$, we have $T_K \geq 2\pi$.*

*Proof.* Since every Morse function on $K$ has at least two critical points, we have $\frac{1}{2}\mu_K \geq 1$ and, by (3.3.7),

$$\frac{1}{2\pi} T_K = c_K = \frac{1}{4\pi}\int_{\mathbf{S}} \frac{1}{2}\mu_K(v) dA_{gs}(v) \geq \frac{1}{4\pi}\int_{\mathbf{S}} dA_{gs}(v) = 1.$$

That is, $T_K \geq 2\pi$. $\square$

**Corollary 3.3.2:** *If $K$ is a planar convex curve, then $T_K = 2\pi$.*

*Proof.* Note that

$$\begin{array}{rcl} h_v : K \subset \mathbb{R}^3 & \to & \mathbb{R} \\ x & \longmapsto & \langle v, x \rangle \end{array}$$

is the restriction of a linear continuous function. We will now prove that if $K$ is planar and convex, then any local minimum of $h_v$ must be an absolute minimum.

Suppose $h_v$ has two local minima at $x_1, x_2 \in K$. If $h_v(x_1) > h_v(x_2)$, by continuity of $h_v$ on $K$, there exists $x_3 \in K - \{x_1, x_2\}$ such that

$$h_v(x_3) = h_v(x_1).$$

The straight line segment between $x_1$ and $x_3$ is totally contained in $K$ and for any point on that segment

$$\begin{aligned} \langle v, tx_1 + (1-t)x_3 \rangle &= t \langle v, x_1 \rangle + (1-t) \langle v, x_3 \rangle \\ &= th_v(x_1) + (1-t)h_v(x_3) \\ &= th_v(x_1) + (1-t)h_v(x_1) \\ &= h_v(x_1), \end{aligned}$$

where $t \in [0,1]$. There exist points on $K$ arbitrarily close to $x_1$ and they have to be on one side of such a line. Since $x_1$ is a local minimum, such points must be on the side where $\langle v, \cdot \rangle$ is greater that $h_v(x_1)$, but this means that the line segment above is not contained in $K$, which is a contradiction. Therefore, there is only an absolute minimum of $h_v$. Thus, (3.3.7) gives

$$\frac{1}{2\pi} T_K = c_K = \frac{1}{4\pi} \int_{\mathbf{S}} \frac{1}{2} \mu_K(v) dA_{gs}(v) = \frac{1}{4\pi} \int_{\mathbf{S}} dA_{gs}(v) = 1.$$

That is, $T_K = 2\pi$. □

**Corollary 3.3.3:** *If $T_K < 4\pi$, then $K$ is not knotted.*

*Proof.* If $T_K < 4\pi$ and $\mu_K \geq 4$, then

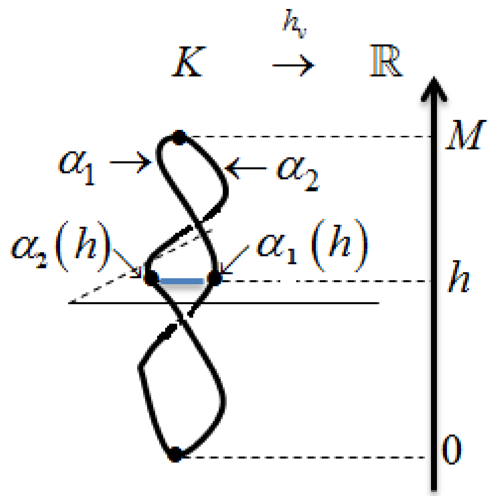
$$T_K = \pi \overline{\mu}_K = \frac{1}{4} \int_{\mathbf{S}} \mu_K(v) dA_{gs}(v) \geq \frac{1}{4} \int_{\mathbf{S}} 4 dA_{gs}(v) = 4\pi,$$

which contradicts to the hypothesis. Thus, there exists $v \in \mathbf{S}$ such that $\mu_K(v) < 4$ and $h_v$ is a Morse function. This proves that $\mu_K(v) = 2$ so that $h_v$ has only two critical points on $K$.

Without loss of generality, by means of a rotation and a translation, we can assume that $v = e_3 = (0,0,1)$, and that 0 and $M$ are the global minimum and maximum values of $h_v$ respectively. Let

$$\alpha_1, \alpha_2 : [0,M] \to \mathbb{R}^2, \; h \longmapsto \alpha_1(h), \alpha_2(h),$$

with $\alpha_1(0) = \alpha_2(0)$ and $\alpha_1(M) = \alpha_2(M)$. Next, we observe that for every $h \in [0,M]$ the intersection of the hyperplane at height $h$ with the knot $K$ consists precisely of two points $\alpha_1(h)$ and $\alpha_2(h)$ (as in the figure on the right).




Let $C_h = \{t\alpha_1(h) + (1-t)\alpha_2(h) : \; 0 \le t \le 1\}$. We claim that the set

$$C := \bigcup_{h \in [0,M]} C_h$$

is a closed disk. Consider the homotopy map

$$\begin{array}{rcl} F : [0,1] \times C & \to & C \\ (s, t\alpha_1(h) + (1-t)\alpha_2(h)) & \mapsto & \big((1-s)t + s\big)\alpha_1(h) + (1-s)(1-t)\alpha_2(h). \end{array}$$

We have

$$\begin{array}{rcl} F(0, t\alpha_1(h) + (1-t)\alpha_2(h)) & = & t\alpha_1(h) + (1-t)\alpha_2(h) \\ F(1, t\alpha_1(h) + (1-t)\alpha_2(h)) & = & \alpha_1(h). \end{array}$$

which means that when $s = 0$ it gives the identity map on $C$, and when $s = 1$ it maps everything to the contractible curve described by $\alpha_1$. $\square$

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