CIMAT

Centro de Investigación en Matemáticas, A.C.

CATEGORIFICATION AND COMBINATORICS OF SCHUBERT POLYNOMIALS

T H E S I S

In the degree of

Master of Science

with specialty in

Basic Mathematics

By

Juan Fernando Valdés Cruz

Advisers:

Dr. José Antonio de la Peña

and

Dr. Aaron D. Lauda

Guanajuato, Gto., June 2, 2017

"...Y un pájaro cantó, delgada flecha.
Pecho de plata herido vibró el cielo,
se movieron las hojas,
las yerbas despertaron...
Y sentí que la muerte era una flecha
que no se sabe quién dispara
y en un abrir los ojos nos morimos."

— Octavio Paz, 'el pájaro'.

Contents

Acknowledgements

This work was written after I finished the coursework in my master studies. I want to thank Aaron Lauda for proposing a problem of relating the graphical method of Billey, Stanley, and Bergeron, to the diagrammatics for the categorified $\mathfrak{sl}_2$.

I want to thank José Antonio de la Peña for travel support during my master course and CONACYT and CIMAT for the scholarship I had during most of my coursework. I am very grateful to the CIMAT community, where this work was written, and I thank my professors for all I learned for them.

The new results in chapter three were written, for the most part, from January 2016 to March of the same year. I studied the formula for Schubert polynomials using RC graphs during October and November the previous year. However, chapter one and two were written in February and March 2017. I want to thank Aaron Lauda for his guidance, which lead me to the final version of this work. I want to thank Christof Geiss and Luis Ñuñez for carefully reading this document and improving its overall quality. I want to thank Stephanie Dunbar for helping me proofread the document. I want to thank Nantel Bergeron for supplying the LaTeX template for drawing Pipe Dreams and pointing out where to find the proof of the diagrammatic formula. I want to thank Christof Geiss for streaming the Representation Theory Seminar in the Fall semester of 2015 and Stephen Sontz for a course in Noncommutative Geometry given in CIMAT during Spring 2016, and also thank both of them and Nicolle Sandoval, at the moment Lauda's student, for helpful discussions.

Finally, I want to thank my parents for their financial support in the extra time I spent in Mexico.

June 2017, Guanajuato, Mexico

Introduction

This document discusses Schubert polynomials and how they relate to other parts of mathematics. These other topics are noncommutative algebra, quantum groups, and combinatorics. There are alternative definitions of them. They are also related to intersection problems in the Grassmanian from the Algebraic Geometry viewpoint, but the perspective we take is combinatorial. This is possible because they have non negative coefficients and non negative structure constants.

The first definition is given through the action of the nilCoxeter algebra in the polynomial ring $\mathbb{Z}[x_1, \ldots, x_a]$. The nilCoxeter algebra is generated by differentials ∂_i that satisfy some relations similar to the symmetric group S_a. It is generated by $a-1$ elements and is a subalgebra of the nilHecke algebra $\mathcal{NH}_a$, which consists of $\mathbb{Z}\langle x_1, \ldots, x_a, \partial_1, \ldots, \partial_{a-1}\rangle$ and some relations. Khovanov and Lauda use diagrammatics for this ring. This algebra has an action in the polynomial ring $\mathbb{Z}[x_1, \ldots, x_a]$ so that there is a representation of $\mathcal{NH}_a$ in the endomorphism ring of the polynomial ring. With this action the usual definition of Schubert polynomials is given. The nilCoxeter algebra is similar to the symmetric group $\mathcal{NH}_a$ with the exception that its generators are nilpotent, instead of being their own inverses, as adjacent transpositions are in the symmetric group. The adjacent transposition is represented diagrammatically by a crossing. Dots will stand for variables. The 0-Hecke algebra generated by $\bar{\partial}_i$ has the relation $\bar{\partial}_i^2 = \bar{\partial}_i$ is also related. Each 0-Hecke element encodes certain information that relates the Schubert polynomial to the permutation it corresponds to, and the generators of the 0-Hecke algebra consist of a dot followed by a crossing. If $\bar{\pi}$ is a 0-Hecke element then there is a mapping that forgets, or untangles, the crossings so that $u(\bar{\pi})$ represents a monomial m. Forgetting the dots in $\bar{\pi}$ we get a diagram $\pi \in \mathcal{NH}_a$ that also represents a reduced wiring diagram for π. The leading monomial of s_π is $u(\bar{\pi})$.

There is another way of interpreting the wiring diagram π such that we get $u(\bar{\pi})$ from certain isotopy of the diagram. This approach is more rigid and precedes ours but has some advantages. It is very combinatorial, allowing us to read reduced words from the diagram. The set of RC graphs of π was introduced by Billey, Bergeron and Stanley as a diagrammatic way to realize a formula for calculating Schubert polynomials that depends on a subfamily of the reduced words for π that have compatible sequences. With RC graphs Monk's rule can be proved and it gives a computational tool and easier understanding of the formula of Stanley for these polynomials. Reinterpreting their diagrammatics does not intend to replace RC graphs but to provide other means of using them while making them consistent with the diagrammatic approach for quantum groups that Khovanov and Lauda developed. The diagrammatics use are inspired in diagrams for n-categories and by previous work of Khovanov and Frenkel. Diagrammatics have helped study the canonical basis of tensor products of $\mathfrak{sl}_2$ representations since early work of Khovanov. There are relations to many areas, from TQFTs and knot invariants to problems involving the Symmetric Group.

The document is organized in the following way. Chapter one consists in a review of the main properties of the nilHecke ring and its action on the polynomial ring. We introduce the diagrammatics of Khovanov and Lauda for the nilHecke ring. Here we define Schubert polynomials and provide their first application. Chapter two aims to give a clear idea of how the nilHecke algebra is related to $\mathbf{U}_q(\mathfrak{sl}_2)^+$ and its role in the categorified theory. We introduce the quantum group

$\mathbf{U}_q(\mathfrak{sl}_2)$ and give some background on the Grothendieck group of an algebra A, denoted $K_0(A)$, and calculate $K_0(\mathcal{NH})$ where $\mathcal{NH} = \bigoplus_{a\geq 0}\mathcal{NH}_a$. We explain how the generators of $\mathbf{U}_q(\mathfrak{sl}_2)^+$ are elements of $K_0(\mathcal{NH})$. In order to do this we need to use Schubert polynomials and it is one of the applications used in the categorified $\mathbf{U}_q(\mathfrak{sl}_2)$. The categorification theorem is more complex, more computations are needed to establish the bialgebra structure of $K_0(\mathcal{NH})$. Chapter three consists in presenting the diagrammatic version for computing Schubert polynomials through RC graphs, also called pipe dreams, and we reinterpret pipe dreams as nilHecke elements. We define a set of diagrams, which we name the abacus of π. We compute it in an algorithmic fashion as in the RC graph method. Using these diagrams a special type of Schubert polynomials is found, related to inclusions of the symmetric group, which we call Hanoi Towers. Afterwards we try to adapt an insertion in order to mimic Monk's rule proof as given by Billey and Bergeron. However, there is not an obvious way of realizing an analog of a Schensted insertion and it is more natural to knit the monomials with the mapping that assigns 0-Hecke elements. Our method does not even immediately prove Monk's rule. At least, it allows us to introduce a bound for the structure constants of Schubert polynomials through a diagrammatic method. The bound we give is very natural, similar to the idea that $\{s_\pi\}_{\pi\in S^\infty}$ is Gröbner basis for $\mathbb{Z}[x_1, x_2, \dots]$.

Chapter 1

The nilHecke algebra

The nilHecke ring is the ring that contains the nilCoxeter algebra and the polynomial ring $\mathbb{Z}[x_1, \dots, x_n]$. The nilHecke algebra appears in different parts of mathematics. In the categorified quantum $\mathfrak{sl}_2$ the nilHecke algebra represents the positive part of the quantum group. The nilCoxeter ring, the subring generated by the differential operators, has an action on the polynomial ring, and hence, the whole nilHecke ring acts on $\mathbb{Z}[x_1, \dots, x_n]$. This action later helps us define Schubert polynomials. In this chapter we introduce the nilHecke ring, its properties and its diagrammatics. We review the diagrammatics for the symmetric group, which help us introduce diagrammatics for the nilCoxeter ring and the nilHecke ring.

1.1 The symmetric group

Definition 1.1.1. The **symmetric group** S_a is the group of bijections of the set $\{1, \dots, a\}$. An element of S_a is called a permutation.

We recall that S_a can be generated by transpositions. An **adjacent transposition** exchanges two adjacent numbers. We write σ_i for the adjacent permutation $(i, i+1)$. We have the following lemma.

Lemma 1.1.2. *Any permutation in S_a is a product of transpositions. Even more, it is the product of adjacent transpositions σ_i for $i = 1, \dots, a-1$.*

Proof. Observe that every permutation is the product of disjoint cycles. To find a cycle take $i \in \{1, \dots, a\}$ and consider de sequence $\{\pi(i), \pi(i)^2, \dots\}$ which is finite because $\pi^n(i)$ cannot take infinite different values.

We proceed by induction on the length of a cycle. Suppose that we have a cycle $(i_1, \dots, i_n)$. A lenght one cycle is a transposition. By induction, assume that if an n cycle is given it is a product of $n-1$ transpositions. Given a $n+1$ cycle, take the first two elements, (i_1, i_2) and then replace the cycle $(i_1, \dots, i_n, i_{n+1})$ by $(i_1, i_3, \dots, i_n)(i_1, i_2)$. The induction is then complete.

Finally, note that every transposition can be written as a product of adjacent transpositions. To move i_1 to i_2, where $i_1 < i_2$, just move i_1 up to i_2 using adjacent transpositions. $\square$

We can also represent S_a diagrammatically. We represent adjacent transpositions by a crossing of adjacent strings. The following is taken from Allen Knutson notes on Schubert polynomials [13].

Let a string be a curve joining two given different points in $\mathbb{R}^3$. Considered a strings in space aligned in order, so that the strings can be described by paths that begin and end in two fixed hyperplanes. The Artin braid group B_a is the group of strings with crossings whose composition is given by concatenation of strings, without allowing self-crossings, and whose generators are adjacent string crossings.

Definition 1.1.3. The Artin Braid Group B_a is the fundamental group of the configuration space of a points X_a. That is $B_a = \pi_1(X_a)$.

Lemma 1.1.4. *The canonical presentation of* B_a *is* $B_a = \langle b_1, \dots, b_a | b_i b_j = b_j b_i (i \neq j) \qquad b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1} \rangle$.

Let us describe B_a's relations. If crossings do not involve consecutive strings then they are independent, so they commute. This is the first relation. We do not allow self-crossings, which is the Redemeister 1 move. The braid move, the Redemeister 3 move, is the another relation. It matters if one crosses two strings by b_i or b_i^{-1}, so the first string in the crossing is underneath or above the second string. If one makes a loop then this loop goes around the second string and is not isotopic to the identity braid. But if one makes the inverse crossing then one can pull the string back, making a Redemiester 2 move, which represents the relation $b_i b_i^{-1}$. We have inverses for generators, then also for every other element. The identity braid is a braid with no crossings.

Lemma 1.1.5. *There is a nontrivial homomorphism* $\phi : B_a \to S_a$ *where the new relation is* $\sigma_i^2 = 1$ *for adjacent transpositions. As a consequence* S_a *can be represented by planar string diagrams.*

Proof. Let b be a braid. Label the starting points of the strings with the set {1,. . . ,a}. Define a permutation by the action of the braid in these points, taking the starting point of the strings to the permuted endpoints. We claim this defines a group homomorphism ϕ. The identity braid does not permute the endpoints. The composition of braids gives composition of permutations. Then ϕ is a homomorphism and by the first homomorphism theorem, $S_a \cong B_a / \ker \phi$.

We claim that $\ker \phi = \langle \sigma_i^2 | \sigma_i$ is a generator of $B_a \rangle$. Given a generator σ_i of the braid group, then $\sigma_i^2 = 1$ gives $\sigma_i = \sigma_i^{-1}$. This means braiding below the next string or on top of the next string is the same, it just matters that the strings cross. Thus, the diagram is planar.

We prove that the kernel is generated by the elements σ_i^2. Let N be the normal subgroup generated by elements σ_i^2. Any permutation σ_i^2 has the same starting points as endpoints, then $\sigma_i^2 \in \ker \phi$. Conjugations the form $\sigma_{i_1} \dots \sigma_{i_s} \sigma_i^2 \sigma_{i_s} \dots \sigma_{i_1}$ are also in $\ker \phi$, and products of these elements also. So $N \subset \ker \phi$. If $b \in \ker \phi$ then it has the same endpoints. For each string crossing any other string, it has to cross back, so $b \in N$ and $\ker \phi \subset N$.

It follows that a presentation for S_a is

$$\langle \sigma_1, \dots, \sigma_{a-1} | \sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i-j| > 1,\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1},\ \sigma_i^2 = 1 \rangle.$$ □

The kernel of this homomorphism is the subgroup of B_a called pure braids. We are motivated to give the following definition.

Definition 1.1.6. A **wiring diagram** of a permutation π is a diagram that represents a permutation connecting a labeling of opposite sides numbered 1 to a, and for which a wire or string starting at i is connected to $\pi(i)$.

We map out a diagram to an expression for a permutation by reading the crossings. Set out a starting position for the strings, say at the bottom, so we find the image of π in the top. A self intersection is redundant so we do not take those into account. A crossing of two strings is an adjacent transposition, so the expression is given by the multiplication of adjacent transpositions in the order they appear.

Lemma 1.1.7 (Reduced expression in diagrammatics)**.** *An expression for* $\pi \in S_a$ *is reduced if no strings cross twice in its wiring diagram.*

Proof. Suppose a string crosses twice. Taking cases, suppose that the first crossings are consecutive crossings following the path given by the string. This implies the expression contains σ^2 for an adjacent transposition and is not reduced. It could be the expression involved is more complicated. Still two crossings can be eliminated and the diagram still has the same connectivity, which means that it represents the same permutation. Then the expression of the permutation was not reduced. $\square$

In a wiring diagram two crossings indicate the expression is not reduced. We do not use these diagrams but only reduced ones. We introduce some important definitions.

Definition 1.1.8. (Sign and inversions) An **inversion** in π is a pair (i, j) with $i < j$ such that $\pi(i) > \pi(j)$. We define the **sign** of a permutation by $(-1)^{\#inversions}$. The **length of a permutation** π is the minimal number of transpositions needed to write π as a product of adjacent transpositions. Denote the length of π by $l(\pi)$.

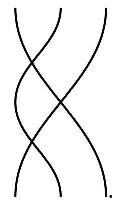
Proposition 1.1.9. *There is a group homomorphism $\phi : S_a \to \{-1, 1\}$ such that $\phi(\sigma_i) = -1$ for each adjacent transposition. This homomorphism satisfies $\phi(\pi)(-1)^{l(\pi)}$ and $\phi(\pi) = (-1)^{\#inversions}$.*

Proof. Let π be a permutation. Let $\pi(1, \dots, a) = (\pi(1), \dots, \pi(a))$ be a permutation. Let $i < j$ be a pair such that $i, j \in \{1, \dots, a\}$ and $\pi(i) < \pi(j)$. Let w be a reduced expression for π and consider the wiring diagram of w. Observe that string i has to cross string j in order that $\pi(i) < \pi(j)$. However, it cannot cross string j back otherwise the strings cross twice. For each inversion there is a crossing of strings.

Now define $\phi : S_a \to \mathbb{Z}_3^*$ the multiplicative group isomorphic to the multiplicative group $\{-1, 1\}$ such that $\phi(\sigma_i) = -1$. Then $\phi(\sigma_i^2) = 1$ so it is a group homomorphism. This homomorphism counts the parity of crossings in w, so $\phi(\pi) = (-1)^{l(\pi)}$. The other equality comes from the observation in the previous paragraph. $\square$

We can choose to draw a permutation's wiring diagram in two ways. We can set the starting positions $[1, \dots, a]$ at the bottom of the diagram and have $[\pi(1), \dots, \pi(a)]$ and the top of the diagram. This will be the notation used later for RC graphs, though the bottom side is drawn at the left making a tile such that the upper part of the antidiagonal is occupied by the drawing. In the nilHecke algebra, the convention used by Ellis, Lauda and Khovanov is that the algebra acts on the polynomial ring fed on the top. So a permutation will take the upper positions of the strings to the image in the bottom. These two conventions are related by an algebra homomorphism we define in the next section.

Example 1.1.10. *A wiring diagram for the permutation* (13). *This permutation is the one with the longest expression in S_3, and it is a transposition (not adjacent). It's length is 3 and in the diagram we can see 3 crossings on it. Each string crosses every other string. The permutation with the longest expression in S_a is the order reversing permutation for* $1, \dots, a$.



1.2 The nilHecke algebra

Let us introduce the algebra that motivates this work. The nilHecke algebra over $\mathbb{Z}$ represents the positive part of the categorified quantum group $U_q(\mathfrak{sl}_2)$ and can be defined diagrammatically.

Definition 1.2.1. (The nilHecke ring $\mathcal{NH}_a$ in a variables). Let $a \in \mathbb{N}$ and consider the freely generated algebra over $\mathbb{Z}$ with generators $\{x_1, \ldots, x_a, \partial_1, \ldots, \partial_{a-1}\}$ and relations:

$$\begin{aligned} x_i x_j &= x_j x_i \qquad \text{for all } i, j \in \{1, \ldots, a\}; && (1.2.1)\\ \partial_i \partial_j &= \partial_j \partial_i \qquad \text{for all } i, j \in \{1, \ldots, a-1\} \text{ and } |i-j| > 1; && (1.2.2)\\ \partial_k^2 &= 0 \qquad \text{for all } k \in \{1, \ldots, a-1\}; && (1.2.3)\\ \partial_i \partial_{i+1} \partial_i &= \partial_{i+1} \partial_i \partial_{i+1} \qquad \text{for all } i, j \in \{1, \ldots, a-2\}; && (1.2.4)\\ x_k \partial_k - \partial_k x_{k+1} &= \partial_k x_k - x_{k+1} \partial_k = 1 \qquad \text{for all } k \in \{1, \ldots, a-1\}. && (1.2.5) \end{aligned}$$

The subring generated by only the differentials $\{\partial_i\}_{i=1}^{a-1}$ is called the nilCoxeter ring.

- We observe that the commutative polynomial ring $\mathbb{Z}[x_1, \ldots, x_a]$ is a subring of $\mathcal{NH}_a$. For simplicity write $\mathbb{Z}[\bar{x}_a]$ for the polynomial ring in a variables $x_1, \ldots, x_a$.
- Write $Z(\mathcal{NH}_a) = \mathbb{Z}[\bar{x}_a]^{S_a}$, where the the superscript S_a for the invariant ring fixed by the action of the symmetric group S_a on $\mathbb{Z}[\bar{x}_a]$. We show later the invariant ring $Z(\mathcal{NH}_a) = \mathbb{Z}[x_1, \ldots, x_a]^{S_a}$ is the center of $\mathcal{NH}_a$.
- The algebra $\mathcal{NH}_a$ is noncommutative and finitely generated. There is a natural inclusion $\mathcal{NH}_a \hookrightarrow \mathcal{NH}_b$ if $b > a$. In particular $\mathcal{NH}_a \hookrightarrow \mathcal{NH}_{a+1}$.
- Observe ∂_i is nilpotent. Thus the algebra $\mathcal{NH}_a$ is not a domain, and the nilCoexter subring is not a domain either.
- The nilHecke algebra is the nilHecke ring taken over a field, which we do in Chapter 2 by extension of scalars to $\mathbb{Q}$.

The next equation is an extension of 1.2.5.

Proposition 1.2.2. *The following equation holds:*

$$\partial_i x_i^m - x_{i+1}^m \partial_i = x_i^m \partial_i - \partial_i x_{i+1}^m = \sum_{l_1 + l_2 = m-1} x_i^{l_1} x_{i+1}^{l_2}. \qquad (1.2.6)$$

Proof. We prove only $\partial_i x_i^m - x_{i+1}^m \partial_i = \sum_{l_1+l_2=m-1} x_i^{l_1} x_{i+1}^{l_2}$ other computation is similar. By induction on m, start with $x_i \partial_i - \partial_i x_{i+1} = 1$, and for the case $m = 2$

$$x_i(x_i \partial_i - \partial_i x_{i+1}) + (x_i \partial_i - \partial_i x_{i+1}) x_{i+1}.$$

On the other side of the equation we get

$$x_i(1) + (1) x_{i+1} = x_i + x_{i+1}.$$

The induction step follows from a similar calculation. Using strong induction we replace the relation for $m-1$ and $m-2$. From $x_i^m \partial_i - \partial_i x_{i+1}^m = \sum_{l_1+l_2=m-1} x_i^{l_1} x_{i+1}^{l_2}$ get, on the left hand side of 1.2,

$$\begin{aligned} & x_i(x_i^m \partial_i - \partial_i x_{i+1}^m + (x_i^m \partial_i - \partial_i x_{i+1}^m) x_{i+1} \\ = \;& x_i^{m+1} \partial_i - x_i(x_i^{m-1} \partial_i - \partial_i x_{i+1}^{m-1}) x_{i+1} - \partial_i x_{i+1}^{m+1} \\ = \;& x_i^{m+1} \partial_i - \partial_i x_{i+1}^{m+1} + x_i \left(\sum_{l_1+l_2=m-2} x_i^{l_1} x_{i+1}^{l_2} \right) x_{i+1} \end{aligned}$$

and on the right hand side

$$x_i \left(\sum_{l_1+l_2=m-1} x_i^{l_1} x_{i+1}^{l_2} \right) + \left(\sum_{l_1+l_2=m-1} x_i^{l_1} x_{i+1}^{l_2} \right) x_{i+1}.$$

Observe that

$$x_i \left(\sum_{l_1+l_2=m-1} x_i^{l_1} x_{i+1}^{l_2} \right) + \left(\sum_{l_1+l_2=m-1} x_i^{l_1} x_{i+1}^{l_2} \right) x_{i+1} + x_i \left(\sum_{l_1+l_2=m-2} x_i^{l_1} x_{i+1}^{l_2} \right) x_{i+1} = \sum_{l_1+l_2=m} x_i^{l_1} x_{i+1}^{l_2}$$

so that 1.2 proves the equation. □

Let $w = s_{i_1} \dots s_{i_k}$ be a reduced expression for $w \in S_a$. Write $\partial_w := \partial_{i_1} \dots \partial_{i_k}$. We now show that ∂_w is independent of the reduced expression for w. That is, if w is another reduced expression we still get the same element ∂_w. We have two relations for the set $\{\partial_i\}$. They are

$$\partial_i \partial_{i+1} \partial_i = \partial_{i+1} \partial_i \partial_{i+1} \qquad \partial_i \partial_j = \partial_j \partial_i \text{ if } |i-j| > 1. \tag{1.2.7}$$

Remember a reduced expression of a permutation never has a transposition two consecutive times in its expression, so we should never find ∂_i^2 in a reduced expression. Two expressions for a permutation are equivalent if each can be obtained from the other by a finite sequence of moves involving the previous relations. The relations for the symmetric group besides s_i being idempotent are

$$s_i s_{i+1} s_i = s_{i+1} s_i s_i \qquad s_i s_j = s_j s_i,$$

the same conditions as for the nilCoxeter algebra generators. As a consequence the expression

$$\partial_w = \partial_{s_1} \dots \partial_{s_k} \text{ for } w = s_{i_1} \dots s_{i_k}$$

is well defined. If we have that ∂_w is not zero then w is a reduced expression for the corresponding permutation. To see this observe that the two crossings Lemma 1.1.7 and the relation $\partial_i^2 = 0$ imply a non reduced expression gives a zero nilCoxeter element. This leads to the following.

Lemma 1.2.3. *Reduced expressions for the same permutation represent the same nilCoxeter element. In particular the nilCoxeter subring of $\mathcal{NH}_a$ is spanned by $\{\partial_\pi : \pi \in S_a\}$.*

Lemma 1.2.4. *The symmetric group S_a acts on $\mathbb{Z}[x_1, \dots, x_a]$.*

Proof. The natural action of $\pi \in S_a$ is given by $\pi f(x_1, \dots, x_a) = f(x_{\pi(1)}, \dots, x_{\pi(a)})$. □

Lemma 1.2.5. *There is a subring Λ_a of $\mathbb{Z}[\bar{x}_a]$ that consists of polynomials that satisfy $s_i(f) = f$ for all $i \in \{1, \dots, a-1\}$.*

Proof. Let f and g be polynomials such that $s_i(f) = f$ and $s_i(g) = g$ for all i. Let α be a scalar. For any i we have that

$$\begin{aligned} s_i(f+g) &= s_i(f) + s_i(g) = f + g \\ s_i(fg) &= s_i(f) s_i(g) = fg, \\ s_i(\alpha f) &= \alpha s_i(f) = \alpha f, \\ s_i(1) &= 1. \quad \square \end{aligned}$$

Definition 1.2.6. A polynomial f is symmetric in x_i and x_{i+1} if $s_i(f) = f$. Define the ring of symmetric polynomials in a variables as the ring of invariant polynomials under the S_a action. We denote the symmetric polynomials as Λ_a. In the literature sometimes $\mathbb{Z}[x_1, \ldots, x_a]^{S_a}$ is used.

We introduce skew derivations [21] in relation to the nilHecke action on $\mathbb{Z}[\bar{x}_a]$. It turns out that ∂_i is a σ_i-derivation of the polynomial ring through the action we consider.

Definition 1.2.7. Let $EndA$ denote the endomorphisms of A over R, where A is an algebra over a ring R. Let $\sigma \in EndA$ be given. A skew derivation of A is an R-linear map δ that satisfies

$$\delta(ab) = \delta(a)b + \sigma(a)\delta(b).$$

We say that δ is a σ-derivation and the set of all σ-derivations is $Der_\alpha A$.

If $\sigma \in EndA$ then $\sigma - 1 \in Der_\sigma A$ so it is not empty, as seen in the following example.

Example 1.2.8. *Assume that δ is a σ derivation. We have that $\sigma - 1$ and $\sigma\delta\sigma^{-1}$ are σ derivations. We know $\sigma - 1$ is R-linear, and see that*

$$\begin{aligned} (\sigma-1)(ab) &= \sigma(ab) - ab; \\ (\sigma-1)(a)b + \sigma(a)(\sigma-1)(b) &= \sigma(a)b - ab + \sigma(a)\sigma(b) - \sigma(a)b \\ &= \sigma(ab) - ab. \end{aligned}$$

Now we prove $\sigma\delta\sigma^{-1}$ is a derivation. That it is R-linear is immediate. Note that

$$\begin{aligned} \sigma\delta\sigma^{-1}(ab) &= \sigma\delta\sigma^{-1}(a)\sigma^{-1}(b) \\ &= \sigma\left(\delta(\sigma^{-1}(a))\sigma^{-1}(b) + a\delta(\sigma^{-1}(b))\right) \\ &= (\sigma\delta\sigma^{-1}(a))\sigma\sigma^{-1}(b) + \sigma(a)(\sigma\delta\sigma^{-1}(b)). \end{aligned}$$

Now suppose that A is commutative. For $a \in A$ if $a \neq \sigma(a)$ we have

$$\delta(a^n) = (a^{n-1} + a^{n-2} + \cdots + \sigma(a)^{n-1})\delta(a) = \left(\frac{a^n - \sigma(a^n)}{a - \sigma(a)}\right)\delta(a) \tag{1.2.8}$$

If f is a function on A,

$$\delta(f) = \frac{f - \sigma(f)}{a - \sigma(a)}\delta(a).$$

Theorem 1.2.9. *The nilHecke ring $\mathcal{NH}_a$ acts on $\mathbb{Z}[\bar{x}_a]$. Let f be a polynomial. This action is defined by $x_i(f) = x_i f$, multiplication, and ∂_i by the divided difference operator*

$$\partial_i(f) = \frac{f - s_i(f)}{x_i - x_{i+1}}.$$

We need to show that the action of ∂_i satisfies the nilCoxeter relations, which we leave for Lemma 1.2.14 and Corollary 1.2.15. In particular we prove that $\partial_i(f)$ is a polynomial. In the following pages we prove that $\mathcal{NH}_a$ acts on the polynomial ring. If $\mathcal{NH}_a$ acts on $\mathbb{Z}[\bar{x}_a]$ then by restriction to the nilCoxeter subring of $\mathcal{NH}_a$ generated by the differentials the following corollary is obtained.

Corollary 1.2.10. *The nilCoxeter ring acts on $\mathbb{Z}[\bar{x}_a]$.*

Example 1.2.11. *We calculate the action on the variables.*

$$\partial_i(x_i) = \frac{x_i - x_{i+1}}{x_i - x_{i+1}} = 1 \quad (1.2.9)$$

$$\partial_i(x_{i+1}) = \frac{x_{i+1} - x_i}{x_i - x_{i+1}} = -1 \quad (1.2.10)$$

$$\partial_i(x_j) = \frac{x_j - x_j}{x_i - x_{i+1}} = 0. \quad (1.2.11)$$

Lemma 1.2.12. *Let $f \in \mathbb{Z}[\bar{x}_a]$. Then*

1. *If f is symmetric in x_i and x_{i+1} then $\partial_i(f) = 0$.*
2. *The image of ∂_i consists of symmetric polynomials in x_i and x_{i+1}.*

Proof. 1. If f is symmetric we get

$$\partial_i(f) = \frac{f - s_i(f)}{x_i - x_{i+1}} = \frac{f - f}{x_i - x_{i+1}} = 0.$$

2. Suppose that $\partial_i(f) \neq 0$. We prove $\partial_i(f)$ is symmetric. This means $s_i(\partial_i(f)) = \partial_i(f)$.

$$s_i(\partial_i(f)) = s_i\left(\frac{f - s_i(f)}{x_i - x_{i+1}}\right) = \frac{s_i(f - s_i(f)}{s_i(x_i - x_{i-1})} = -\frac{s_i(f) - f}{x_i - x_{i+1}} = \frac{f - s_i(f)}{x_i - x_{i+1}} = \partial_i(f).$$

Therefore, $\partial_i(f)$ is symmetric for x_i and x_{i+1}. □

In particular if $f \in \mathbb{Z}$ (or $f \in k$ for the nilHecke algebra) we have that $\partial_i(f) = 0$. Transpositions s_i that are adjacent also generate S_a. If $\partial_i(f) = 0$ for $1 \leq i < a$ then $f = s_i(f)$ so $f \in \Lambda_a$, because it is invariant under the action of a generating set of S_a.

Lemma 1.2.13. *The kernel of ∂_i are symmetric polynomials and the image of ∂_i are symmetric polynomials. That is, the ring $\Lambda_a = \cap_{i=1}^{a-1} \ker \partial_i = \cap Im\partial_i$. As a consequence the nilHecke relation $\partial_i^2(f) = 0$ is satisfied.*

Proof. Let f be a symmetric polynomial in all the variables. Then $s_i(f) = f$ for any s_i, $i = 1, \ldots, a-1$. Thus $\partial_i(f) = 0$ for all i, so $f \in \ker \partial_i$ for all i. Given $i \in \{1, \ldots, a-1\}$, let $g = x_i f$. We calculate $\partial_i(g) = \partial_i(f) = (x_i f - s_i(x_i f))/(x_i - x_{i+1}) = (x_i f - x_{i+1} f)/(x_i - x_{i+1}) = f$. Then $f \in Im\partial_i$, for each $i \in \{1, \ldots, a-1\}$. This gives two inclusions.

Given f such that $f = \partial_i(g_i)$ for some polynomial g_i is a symmetric polynomial in x_i and x_{i+1}. If $f \in \cap Im\partial_i$ then $f = \partial_i(g_i)$ for some g_i for each $i \in \{1, \ldots, a-1\}$. This implies f is symmetric in all the variables, thus $f \in \Lambda_a$. Let $\partial_i(f) = 0$. This implies $f = s_i(f)$ so f is symmetric in x_i and x_{i+1}. In the same fashion if $\partial_i(f) = 0$ for each i, it is symmetric in all the variables, so that $f \in \cap Im\partial_i$.

We already know that $\partial_i(f)$ is symmetric so $\partial_i(\partial_i(f)) = 0$. □

We derive our main tool for computations of $\partial_i(f)$, we prove the Liebniz rule. Then we verify that the action we defined satisfies the nilHecke relations involving differentials.

Lemma 1.2.14. *(Twisted Liebniz rule and nilHecke relations) Let f be a polynomial. Then*

1. *if $\partial_i(f) = 0$, $\partial(fg) = f\partial(g)$.*
2. *$\partial_i(fg) = \partial_i(f)g + s_i(f)\partial_i(g)$.*

3. $\partial_i\partial_{i+1}\partial_i(f) = \partial_{i+1}\partial_i\partial_{i+1}(f)$.

4. $\partial_i\partial_j(f) = \partial_j\partial_i(f)$ *for* $|i-j|>1$.

5. $(\partial_i x_i - x_{i+1}\partial_i)(f) = f$ *and* $(x_i\partial_i - \partial_i x_{i+1})(f) = f$.

Proof. Let us prove 2 first. Take the right hand side,

$$\begin{aligned}
\partial_i(f)g + s_i(f)\partial_i(g) &= \frac{f-s_i(f)}{x_i-x_{i+1}}g + s_i(f)\frac{g-s_i(g)}{x_i-x_{i+1}} && (1.2.12)\\
&= \frac{fg - s_i(fg) + s_i(f)g - s_i(f)s_i(g)}{x_i-x_{i+1}} && (1.2.13)\\
&= \frac{fg - s_i(f)s_i(g)}{x_i-x_{i+1}} && (1.2.14)\\
&= \frac{fg - s_i(fg)}{x_i-x_{i+1}} = \partial_i(fg). && (1.2.15)
\end{aligned}$$

Observe that $s_i(f)s_i(g) = s_i(fg)$ because s_i simply exchanges x_i and x_{i+1}, so the product of the polynomials with exchanged variables is the same as exchanging variables in the product. Now, if $\partial_i(f) = 0$ so f is symmetric in x_i and x_{i+1}, the formula reduces to

$$\partial_i(fg) = \partial(f)g - s_i(f)\partial_i(g) = f\partial_i(g)$$

using that $s_i(f) = f$. This proves 1.

We prove the nilHecke relations, which are 3-5. If $|i-j|>1$

$$\begin{aligned}
\partial_i\partial_j(f) &= \partial_i\frac{f-s_i(f)}{x_i-x_{i+1}} && (1.2.16)\\
&= \frac{f-s_i(f)}{(x_i-xi+1)(x_j-x_{j+1})} + s_j\frac{f-s_i(f)}{(x_i-x_{i+1})(x_j-x_{j+1})} && (1.2.17)\\
&= \frac{f-s_i(f)-s_j(f)+s_js_i(f)}{(x_i-x_{i+1})(x_j-x_{j+1})} && (1.2.18)\\
&= \frac{f-s_i(f)-s_j(f)+s_js_i(f)}{(x_i-x_{i+1})(x_j-x_{j+1})} = \partial_j\partial_i(f). && (1.2.19)
\end{aligned}$$

Rememeber that $\partial_i(f)$ is symmetric so $s_i(\partial_i(f)) = \partial_i(f)$. Get that

$$
\begin{aligned}
\partial_i\partial_{i+1}\partial_i(f) &= \partial_i\partial_{i+1}\left(\frac{f-s_i(f)}{x_i-x_{i+1}}\right)\\
&= \partial_i\left(\frac{f(x_i-x_{i+2})-x_i(s_i(f)+s_{i+1}(f))+x_{i+2}s_i(f)+x_{i+1}s_{i+1}(f)+(x_i-x_{i+1})s_{i+1}s_i(f)}{(x_i-x_{i+1})(x_i-x_{i+2})(x_{i+1}-x_{i+2}}\right)\\
&= \frac{f(x_i-x_{i+2})-x_i(s_i(f)+s_{i+1}(f))+x_{i+2}s_i(f)+x_{i+1}s_{i+1}(f)+(x_i-x_{i+1})s_{i+1}s_i(f)}{(x_i-x_{i+1})^2(x_i-x_{i+2})(x_{i+1}-x_{i+2})}\\
&+ \frac{f(x_i-x_{i+1})-x_is_i(f)(x_i-x_{i+1})-x_is_{i+1}(f)(x_i-x_{i+1})+s_{i+1}s_i(f)(x_i-x_{i+1})}{(x_i-x_{i+1})^2(x_i-x_{i+2})(x_{i+1}-x_{i+2})}\\
&+ \frac{s_is_{i+1}(f)(x_i-x_{i+1})-s_is_{i+1}s_i(f)(x_i-x_{i+1})}{(x_i-x_{i+1})^2(x_i-x_{i+2})(x_{i+1}-x_{i+2})}\\
&= \frac{f-s_i(f)-s_{i+1}(f)+s_is_{i+1}(f)+s_{i+1}s_i(f)-s_is_{i+1}s_i(f)}{(x_i-x_{i+1})(x_i-x_{i+2})(x_{i+1}-x_{i+2})}\\
&= \frac{f-s_i(f)-s_{i+1}(f)+s_is_{i+1}(f)+s_{i+1}s_i(f)-s_{i+1}s_is_{i+1}(f)}{(x_i-x_{i+1})(x_i-x_{i+2})(x_{i+1}-x_{i+2})}\\
&= \partial_{i+1}\partial_i\partial_{i+1}(f).
\end{aligned}
$$

In the last step we use the braiding relation for s_i. At last,

$$
\begin{aligned}
(\partial_ix_i-x_{i+1}\partial_i)(f) &= \frac{x_if-s_i(x_if)}{x-x_{i+1}}-x_{i+1}\frac{f-s_i(f)}{x-x_{i+1}} && (1.2.20)\\
&= \frac{x_if-x_{i+1}s_i(f)-x_{i+1}f+x_{i+1}s_i(f)}{x_i-x_{i+1}} && (1.2.21)\\
&= \frac{f(x_i-x_{i+1})}{x_i-x_{i+1}}=f && (1.2.22)
\end{aligned}
$$

In a similar way one can compute $x_i\partial_i-\partial_ix_{i+1}=1$. Thus, the proof is complete. □

Corollary 1.2.15. *For any polynomial f, $\partial_i(f)$ is a polynomial.*

Proof. Write $f\in\mathbb{Z}[\bar{x}_a]$ as a sum of monomials. As the operators ∂_i are linear, we can apply them to monomials. The operator applied to each monomial renders a polynomial using the Liebniz rule we just proved, and the result follows.

In particular, ∂_π for $\pi\in S_a$ applied to a polynomial renders a polynomial. □

The divided difference operator's action satisfies the nilHecke relations. Thus, we have finally proved theorem 1.2.9 that $\mathcal{NH}_a$ acts on the polynomial ring $\mathbb{Z}[\bar{x}_a]$. We return in section 1.5 to the action of $\mathcal{NH}_a$ in order to define Schubert polynomials. We wish to give the ring $\mathcal{NH}_a$ a $\mathbb{Z}$-grading. s

Definition 1.2.16 (Additive Grading)**.** Let A be an algebra and G a commutative group. A **grading** on A by G is a decomposition of A,

$$A=\bigoplus_{g\in G}A_g$$

of $G-$components such that $A_gA_h\subseteq A_{g+h}$. A component is called an homogeneous component, and its elements homogeneous elements. A graded homomorphism of G-graded algebras $f:A\to B$

satisfies that $f(A_g) \subset B_g$.[1] A module M over a G-graded algebra is a G-graded module if $A_g M_{g'} \subset M_{gg'}$. A morphism of graded modules is a mapping $f : N \to M$ such that $f(N_g) \subset M_g$.

Given a grading there is an implicit degree function associated to it. We define $\deg(f) = g$ if and only if $f \in A_g$. Consider the additive group $\mathbb{Z}$ and A and algebra over a field. This way, given $n \in \mathbb{Z}$, the homogeneous components A_n are vector spaces and $\deg a = n$ for $a \in A_n$. We consider $\mathbb{Z}$ additive gradings int the present work. Graded homomorphisms are more restrictive that homomorphisms and many known algebras can be graded by the integers. In occasion we would have that a $\mathbb{Z}$-graded ring A has nonezero homogeneous components only for $n \geq 0$. In that case we say that A is graded by the semigroup of positive integers.

Example 1.2.17. *The polynomial ring $\mathbb{Z}[\bar{x}_a]$ admits a $\mathbb{Z}$ grading. It is graded by the semiring of positive integers. Let $\deg(x_i) = 1$ for $i = 1, \ldots, a$. If f is a polynomial let $\deg(f)$ be the maximum degree of its monomials. Then $\deg(fg) = \deg(f) + \deg(g)$ defines a grading, and $\deg$ is called the total degree.*

The ring $\mathbb{Z}[\bar{x}_a]$ admits other gradings, for example multi-indices. In such case we consider the separate the degree of each variable, and set an order in the a-tuples using the lexicographic order. These gradings are important in division algorithms.

Definition 1.2.18. An action of an algebra A in another algebra B is a **graded action** if its action defines a graded endomorphism of B. Then the assignment from A to $EndB$ is a graded homomorphism. We say that the action defines a graded representation. The grading A and B have are considered in the equation $\deg_B(a(b)) = \deg_A(a) + \deg_B(b)$.

Theorem 1.2.19. *The algebra $\mathcal{NH}_a$ is a graded algebra. The grading is given by $\deg(x_i) = 2$ and $\deg(\partial_i) = -2$ for all i. Furthermore, if the polynomial ring $\mathbb{Z}[\bar{x}_a]$ is graded with $\deg(x_i) = 2$ for all i, then the action of $\mathcal{NH}_a$ on polynomials defines a graded representation.*

Proof. We show that the grading is well defined, that the relations preserve the grading. We also show that the action of the nilHecke algebra in the polynomial ring is degree preserving.

First we check that $\mathcal{NH}_a$ is a graded algebra. In particular we check equations involving differentials. The commutation relation gives the same degree in both sides. The braiding relation for differentials is also preserved. We verify that $\partial_i x_i - x_{i+1}\partial_i$ has degree zero, which is forced by $\partial_i x_i - x_{i+1}\partial_i = 1$. Calculate $\deg(\partial_i x_i) - \deg(x_{i+1}\partial_i) = (2-2) + (2-2) = 0$, so the sum of two elements in the zero degree is zero as we expected. The equation $x_i\partial_i - \partial_i x_{i+1} = 1$ is seen to be preserved in a similar fashion. Then deg is a grading in $\mathcal{NH}_a$.

We prove the second part of the theorem, that the representation is graded. The action on polynomials in $\mathbb{Z}[\bar{x}_a]$ is degree preserving by setting $\deg(x_i) = 2$ for all $i = 1, \ldots, a$. This means the following equation holds for a polynomial p and $h \in \mathcal{NH}_a$,

$$\deg(hp) = \deg(h) + \deg(p).$$

Notice that it is enough to check the action on generators of $\mathcal{NH}_a$. For the action of the variables x_i it is clear this works. For the action differentials ∂_i by the previous equations we have

$$\begin{aligned} \partial_i x_i &= 1 && (1.2.23)\\ \partial_i x_{i+1} &= -1 && (1.2.24)\\ \partial_i x_j &= 0 \text{ if } j \neq i, i+1 && (1.2.25)\\ & && (1.2.26) \end{aligned}$$

[1] A graded homomorphism is of degree s if $f(A_g) \subset B_{g+s}$. We use only degree zero homomorphisms in this document, and we are not interested in other gradings at the moment, such as gradings by noncommutative groups.

with degrees

$$\begin{aligned}
\deg(\partial_i(x_i)) &= \deg(\partial_i) + \deg(x_i) = -2 + 2 = 0 && (1.2.27)\\
\deg(\partial_i(x_{i+1}) &= \deg \partial_i + \deg x_i = -2 + 2 = 0 && (1.2.28)\\
\deg(\partial_i x_j) &= \deg \partial_i + \deg x_j && (1.2.29)\\
\deg(1) &= \deg(-1) = \deg(0) = 0. && (1.2.30)
\end{aligned}$$

Thus, the degree is well defined and the degree is preserved in the action of $\mathcal{NH}_a$. □

Remark 1.2.20. The reason a $2\mathbb{Z}$ grading is used is that the nilHecke algebra has an interpretation in terms of complex cohomology of flag varieties. The algebra $\mathcal{NH}_a$ is one example of a family of algebras called KLR algebras associated to semisimple Lie algebras. The nilHecke algebra corresponds to the simplest Lie algebra $\mathfrak{sl}_2$, as seen in Chapter 2. For other Lie algebras the full $\mathbb{Z}$-grading appears.

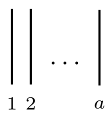
Example 1.2.21. *(Action of $\mathcal{NH}_a$) We have that:*

$$\begin{aligned}
\partial_1(x_1^2 x_2) &= x_1 x_2 \partial_1(x_1) = x_1 x_2 && (1.2.31)\\
\partial_1(x_1^2) &= \frac{x_1^2 - x_2^2}{x_1 - x_2} = \frac{(x_1 - x_2)(x_1 + x_2)}{x_1 - x_2} = x_1 + x_2. && (1.2.32)
\end{aligned}$$

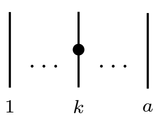
1.3 Diagrammatics for the nilHecke algebra

The diagrammatic approach to the nilHecke algebra comes from the work of Lauda in the categorification of the quantum algebra $U_q(\mathfrak{sl}_2)$. Graphical calculus related to Quantum Groups and $\mathfrak{sl}_2$ are traced back to Kauffman and Penrose, later Frenkel and Khovanov.

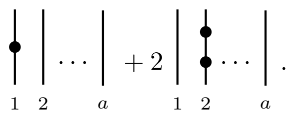
First, we describe the diagrammatics for the polynomial ring $k[\bar{x}_a]$. Take a vertical ordered strands representing each variable. For each power of x_i, $i \in \{1, \ldots, a\}$, place a dot into the strand. In this way, the unit element 1 is encoded by

$1 \ :=$  (1.3.1)

where we have a strings. Now we encode the a variables x_i, $1 \leq i \leq a$ by

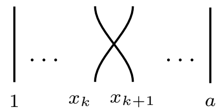
$x_k \ :=$ 

Take linear combinations of diagrams to form, for example $x_1 + 2x_2^2$, in the following way

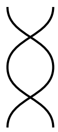


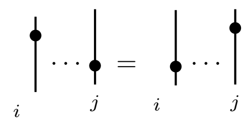
Multiplication of polynomials is done by stacking up diagrams. Multiplying the last equation by x_1 slides one dot in the first string in both of the diagrams above as $x_1(x_1 + 2x_2^2) = x_1^2 + 2x_1 x_2^2$. Instead of placing several dots on one string we can write a number m at the side of the dot to signal m dots. In the polynomial ring we have the commutativity relation $x_i x_j = x_j x_i$, so the order, or height of the dot, is not important. This relation is an isotopy of the diagram. But in case of $\mathcal{NH}_a$, the algebra is noncommutative. Following the convention of Lauda and Khovanov, we stack up diagrams from top to bottom, and read the expression from left to right.

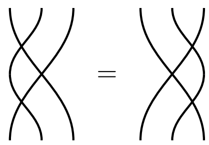
The differentials ∂_k in $\mathcal{NH}_a$ are encoded by

$\partial_k :=$  .

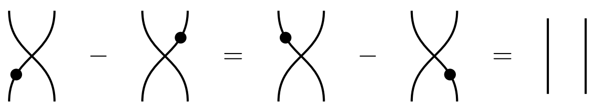
We may omit the extra vertical lines and represent the relations locally around the affected strands. The relations for our generators, drawing only affected strands, are depicted as follows:

 $= 0$ $\qquad$ $\partial_k^2 = 0;$

 $\qquad$ $x_i x_j = x_j x_i;$

 $\qquad$ $\partial_i \partial_{i+1} \partial_i = \partial_{i+1} \partial_i \partial_{i+1}$

and



which stands for

$$\partial_k x_k - x_{k+1}\partial_k = x_k \partial_k - \partial_k x_{k+1} = 1;$$

where $1 \leq i, j, k \leq a$ and we omit strands that are not altered in the relations for brevity. The other relations are

$$\partial_i \partial_j = \partial_j \partial_i$$

for $|i - j| > 1$, which is depicted as two crossings on separate strands that commute, and other relations which are also given by planar isotopy of the diagrams.

1.4 The 0-Hecke Algebra

Definition 1.4.1. The 0-Hecke subring is the subring of $\mathcal{NH}_a$ generated by the elements $\bar{\partial}_i := x_i \partial_i$, for $i = 1, \ldots, a - 1$.

Remark 1.4.2. Diagrammatically the 0-Hecke subring is generated by elements that consist of a dot followed by a crossing. We note the similarity with the nilCoxeter ring. The nilCoxeter subring of $\mathcal{NH}_a$ is generated diagrammatically by crossings. Even more, a wiring diagram that represents a reduced expression is a nonzero nilCoxeter element.

Theorem 1.4.3. *The 0-Hecke algebra generators satisfy the following relations*

$$\bar{\partial}_i^2 = \bar{\partial}_i; \tag{1.4.1}$$
$$\bar{\partial}_i\bar{\partial}_{i+1}\bar{\partial}_i = \bar{\partial}_{i+1}\bar{\partial}_i\bar{\partial}_{i+1}; \tag{1.4.2}$$
$$\bar{\partial}_i\bar{\partial}_j = \bar{\partial}_j\bar{\partial}_i \text{ if } |i-j|>1. \tag{1.4.3}$$

Proof. That $\bar{\partial}_i\bar{\partial}_j = \bar{\partial}_j\bar{\partial}_i$ for $|i-j|>1$ follows from an isotopy of the diagram.

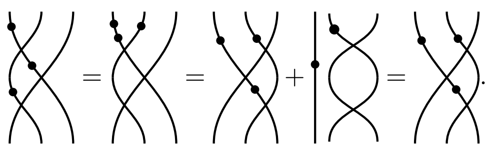
Let us prove the generators are idempotent. For this we use the nilHecke relation $x_k\partial_k = 1 + \partial_k x_{k+1}$, $x_{k+1}\partial_k = \partial_k x_k - 1$, and $\partial_k^2 = 0$ in the following way:

$$\begin{aligned}
\bar{\partial}_k^2 &= (x_k\partial_k)^2 = (1+\partial_k x_{k+1})(1+\partial_k x_{k+1}) && (1.4.4)\\
&= 1 + \partial_k x_{k+1} + \partial x_{k+1} + \partial_k(x_{k+1}\partial_k)x_{k+1} && (1.4.5)\\
&= 1 + \partial_k x_{k+1} + \partial x_{k+1} + \partial_k(\partial_k x_k - 1)x_{k+1} && (1.4.6)\\
&= 1 + \partial_k x_{k+1} + \partial_k x_{k+1} + \partial_k^2 x_k - \partial_k x_{k+1} && (1.4.7)\\
&= 1 + \partial_k x_{k+1} = \bar{\partial}_k. && (1.4.8)
\end{aligned}$$

Finally, we calculate the braiding relation,

$$\begin{aligned}
\bar{\partial}_i\bar{\partial}_{i+1}\bar{\partial}_i &= x_i\partial_i x_{i+1}\partial_{i+1}x_i\partial_i && (1.4.9)\\
&= x_i\partial_i(x_i x_{i+1})\partial_{i+1}\partial_i && (1.4.10)\\
&= x_i^2 x_{i+1}\partial_i\partial_{i+1}\partial_i && (1.4.11)\\
&= x_i^2 x_{i+1}\partial_{i+1}\partial_i\partial_{i+1} && (1.4.12)\\
&= x_{i+1}\partial_{i+1}x_i^2\partial_i\partial_{i+1} && (1.4.13)\\
&= x_{i+1}\partial_{i+1}x_i(x_i\partial_i)\partial_{i+1} && (1.4.14)\\
&= x_{i+1}\partial_{i+1}x_i(1+\partial_i x_{i+1})\partial_{i+1} && (1.4.15)\\
&= x_{i+1}\partial_{i+1}x_i\partial_{i+1} + x_{i+1}\partial_{i+1}x_i\partial_i x_{i+1}\partial_{i+1} && (1.4.16)\\
&= x_{i+1}x_i\partial_{i+1}^2 + \bar{\partial}_{i+1}\bar{\partial}_i\bar{\partial}_{i+1} && (1.4.17)\\
&= \bar{\partial}_{i+1}\bar{\partial}_i\bar{\partial}_{i+1}. \quad \square && (1.4.18)
\end{aligned}$$

Example 1.4.4. *Diagrammatics can work better. We see how to prove the braiding relation again:*



(1.4.19)

Example 1.4.5. *The action of 0-Hecke elements preserves the relation $\bar{\partial}_i^2 = \bar{\partial}_i$. By direct calculation,*

$$\begin{aligned}
\bar{\partial}_i\bar{\partial}_i(f) &= \bar{\partial}_i\left(x_i\partial_i(f)\right) && (1.4.20)\\
&= x_i\left(\frac{x_i\partial_i(f) - x_{i+1}\partial_i(f)}{x_i - x_{i+1}}\right) && (1.4.21)\\
&= x_i\partial_i(f) && (1.4.22)\\
&= \bar{\partial}_i(f). && (1.4.23)
\end{aligned}$$

Lemma 1.4.6. *Let π be a nilCoxeter element. We can assign π a 0-Hecke element $\bar{\pi}$ by setting $\partial_i \mapsto \bar{\partial}_i$. The number of dots in $\bar{\pi}$ is the length of the permutation that it represents.*

Remark 1.4.7. This is not an algebra map. If we are given two permutations π and π' such that $l(\pi)+l(\pi')$ is bigger than the length for the longest permutation in S_a we have that $\partial_\pi \partial_{\pi'} = 0$, but the composition of the 0-Hecke images in nonzero because $\bar{\partial}_i$ is idempotent.

For the 0-Hecke elements we face an issue with notation. The RC graphs we use later have the image on the top, but the convention of Ellis, Lauda and Khovanov is to stack diagrams from top to bottom, as in [5, 16, 17]. We handle these notations through the following trick. Let ψ be the nilHecke automorphism given by reflecting the diagram vertically, and let σ be the nilHecke anti-homomorphism given by reflecting the diagram in a horizontal line. Both maps are defined easily in terms of diagrammatics. The change of notation between the RC graph convention and the nilHecke convention is given by σ.

Let $\bar{\pi}$ be a 0-Hecke element. Then there is an underlying wiring diagram of a permutation obtained by forgetting the dots in the diagram. Observe that $w(\bar{\pi}) = 0$ if the number of dots in $\bar{\pi}$ is greater than $l(w_0)$, the permutation with longest length w_0. In any other case, this gives us a reduced wiring diagram of some permutation. Write that as $w(\bar{\pi}) = \pi$. In section 3.3 we define a similar map u that untangles the diagram.

Lemma 1.4.8. *The map w is extended to the nilHecke algebra as a forgetful map that omits the dots of diagrams. The image of w is the nilCoxeter algebra.*

Remark 1.4.9. The map w is not an algebra map. However, it is diagrammatic.

Theorem 1.4.10. *Given a permutation with and a reduced wiring diagram for π, the diagram is a nilCoxeter element, for which we write $\pi \in \mathcal{NH}_a$. To π there corresponds a unique 0-Hecke element such that $w(\bar{\pi}) = \pi$.*

There is an odd theory for the nilHecke ring introduced by Khovanov, Lauda, and Ellis [5]. The theory changes by defining an analog of the nilHecke ring that acts on the anticommuting polynomial ring $\mathbb{Z}\langle \bar{x}_a \rangle / \langle x_i x_j = -x_j x_i \rangle$ for $i \neq j$, and $i = 1, \dots, a$. The odd nilHecke algebra relations change by some signs, for example the one in the quotient for the skew ring and $\partial_i \partial_j = -\partial_j \partial_i$, for $i \neq j$. The 0-Hecke algebra is a subalgebra of the odd nilHecke algebra also. In this odd theory the 0-Hecke algebra is important because these elements commute with the elements in the algebra, instead of anticommuting as most elements do.

1.5 Schubert Polynomials

In this section we define Schubert polynomials using the nilCoxeter action and prove some important properties.

Definition 1.5.1. Fix a positive integer a. Let w_0 be the permutation with the longest length in S_a and let $\underline{x^\delta} = x_1^a x_2^{a-1} \dots x_a$, so $\underline{x^\delta} \in \mathbb{Z}[\bar{x}_a]$. Define the Schubert polynomial s_π by

$$s_\pi = \partial_{\pi^{-1} w_0}(\underline{x^\delta}).$$

The following lemma is very useful in our study.

Lemma 1.5.2. *The action of divided difference operators on Schubert polynomials is given by*

$$\partial_w s_\pi = \begin{cases} s_{\pi w^{-1}} & \text{if } l(\pi w^{-1}) = l(\pi) - l(w) \\ 0 & \text{otherwise} \end{cases}. \tag{1.5.1}$$

Proof. By definition $s_\pi = \partial_{\pi^{-1}w_0}(\underline{x}^\delta)$. Given $u, v \in S_a$ it is immediate that

$$\partial_u \partial_v = \begin{cases} \partial_{uv} & \text{if } l(uv) = l(u) + l(v); \\ 0 & \text{otherwise.} \end{cases}$$

This way, if $l(\pi w^{-1}) = l(\pi) - l(w)$,

$$\partial_w(s_\pi) = \partial_w \partial_{\pi^{-1}w_0}(\underline{x}^\delta) = \partial_{w\pi^{-1}w_0}(\underline{x}^\delta) = \partial_{\pi w^{-1-1}w_0}(\underline{x}^\delta) = s_{\pi w^{-1}}.$$

In particular $\partial_\pi(s_\pi) = 1$ for any $\pi \in S_a$. □

Theorem 1.5.3. *The set of Schubert polynomials $\{s_\pi : \pi \in S_a\}$ is linearly independent over $\mathbb{Z}$.*

Proof. Let $\sum_\pi c_\pi s_\pi = 0$ where $c_\pi \in \mathbb{Z}$ and finitely many terms are different from zero. We proceed by induction on Coxeter length. A direct calculation then goes as follows. As the sum is finite then we can assume all permutations belong to some group S_a and that all coefficients for elements of length greater than $l(\pi)$ are zero. We apply the endomorphism ∂_w for some $w \in S_a$, to get

$$\partial_w \left(\sum_\pi c_\pi s_\pi \right) = \sum_\pi c_\pi \partial_w(s_\pi) = 0.$$

Then $\partial_w(s_\pi) = 1$ if and only if $w = \pi$. Therefore $c_w = 0$. We can do this for any permutation in the sum. □

We leave the proof of the following lemma for the end of Chapter 3.

Lemma 1.5.4. *Any monomial in $H_a := \{x_1^{\alpha_1} \dots x_a^{\alpha_a} | \alpha_i \leq a - i, \ \ \forall i$ such that $1 \leq i \leq a\}$ can be written as a linear combination $\sum_{\pi \in S_a} c_\pi s_\pi$ such that $c_\pi \in \mathbb{Z}$ for all $\pi \in S_a$.*

If $\underline{x}^\alpha \in \mathcal{H}_a$ say that $\alpha \subseteq \delta$ or $\alpha \leq \delta$, which means $\alpha_i \leq a_i$ as $\delta_i = a - i$. We have that $|\mathcal{H}| = a!$ using the multiplication principle. There is one Schubert polynomial for each $\pi \in S_a$, so there are $a!$ different ones. The cardinalities of both sets coincide.

Lemma 1.5.5. *[19] Let $\delta = (a-1, a-2, \dots, 1)$ for $a \in \mathbb{N}$. The polynomial ring $\mathbb{Z}[\bar{x}_a] \cong \sum_{\alpha \subset \delta} \Lambda_a \underline{x}^\alpha$ where $\alpha \in \mathbb{N}^{a-1}$, and the abelian group $\mathcal{H}_a$ generated by $\{\underline{x}^\alpha : \alpha \subset \delta\}$ has rank $a!$.*

Theorem 1.5.6. *The set of Schubert polynomials $\mathcal{S} = \{s_\pi : \pi \in S_a\}$ is a basis for the free module $\mathbb{Z}[\bar{x}_a]$ over Λ_a. That is, $\mathbb{Z}[\bar{x}_a]$ is a free module over Λ_a of rank $a!$ with basis the set of Schubert polynomials.*

Proof. We sketch the proof as it can be found in Manivel's book [19, 2.5.5]. The idea of the proof is a change of basis. The polynomial ring can be written in terms of a more classical basis, $\mathcal{H}_a := \{x_1^{\alpha_1} \dots x_a^{\alpha_a} | \alpha_i \leq a - i \forall 1 \leq i \leq a\}$ has rank $a!$. This basis of monomials can be written terms of Schubert polynomials from lemma 1.5.4 with integral coefficients. We have $a!$ Schubert polynomials and $|\mathcal{H}_a| = a!$ also. [2] The change of basis from the Schubert basis to the monomial basis is immediate. The bases have the same cardinalities so the homogeneous components of $\Lambda_a \otimes \mathcal{H}_a$ and $\Lambda_a \otimes \{s_\pi\}_{\pi \in S_n}$ have all the same dimension. □

[2] Schubert polynomials for S_a and the abelian group $\mathcal{H}_a$ have the same graded ranks. Lauda in [16,17] gives another proof when he compares graded ranks to ensure the mapping is an isomorphism. By tensoring by Λ_a we get again the same graded rank, so there should be an isomorphism of graded rings. The graded rank is an analog for groups of a Hilbert series. This idea appears in several parts of mathematics where we have graded rings. For example, one is cohomology rings, Euler characteristics, the categorified Jones Polynomial, and the Hilbert polynomial in algebraic geometry.

Corollary 1.5.7. *The matrix ring $Mat(a!, \Lambda_a)$ is isomorphic to $End_{\Lambda_a}(\mathbb{Z}[\bar{x}_a])$ through the Schubert polynomial basis.*

We have the following formulas presented in the notes of Allen Knutson on Schubert Polynomials and Pipe Dreams [13]. Recall $i < a$ is a descent if $\pi(i) < \pi(i+1)$.

Theorem 1.5.8. *(Lascoux's transition formula) Let π be different of 1 and i a descent of π. Let $j := \max\{i' : \pi(i') < \pi(i)\}$ so $j \geq i+1$. Let $\pi' = \pi(ij)$. Then*

$$s_\pi = x_i s_{\pi'} + \sum_{a<i, \pi'(ai) \geq_B \pi'} s_{\pi'(ai)}.$$

This last result is useful because it expresses Schubert polynomials in terms of other previous Schubert polynomials in a non negative linear combination. Then Schubert polynomials are seen, inductively, to have non negative coefficients.

Corollary 1.5.9. *Schubert polynomials have non negative coefficients.*

1.6 The nilHecke algebra as a matrix ring.

We give some applications of Schubert polynomials and describe $\mathcal{NH}_a$ through its natural representation in the endomorphism ring $End_{\Lambda_a}(\mathbb{Z}[\bar{x}_a])$. Proofs of these propositions can be found in Lauda's papers [16, 17] and for the odd case by Ellis, Khovanov and Lauda in [5].

Lemma 1.6.1. *In the nilHecke algebra $x_i x_{i+1} \partial_i = \partial_i x_i x_{i+1}$.*

Proof. By the employment of the nilHecke relation twice,

$$\partial_i x_i x_{i+1} = x_{i+1} + x_{i+1}\partial_i x_{i+1} == x_{i+1} + x_i x_{i+1}\partial_i - x_{i+1} = x_i x_{i+1}\partial_i. \quad \square \tag{1.6.1}$$

Lemma 1.6.2. *Let $f \in \Lambda_a$ and $g \in \mathbb{Z}[\bar{x}_a]$. For any reduced expression w of a permutation $\pi \in S_a$ we have that $\partial_w(fg) = f\partial_w g$. The converse holds, f commutes with ∂_w for reduced expressions of any permutation $\pi \in S_a$ only if $f \in \Lambda_a$. In particular f commutes with ∂_i if and only if f is symmetric in x_i and x_{i+1}.*

Proof. It is enough to consider generators of the nilCoxeter algebra. Suppose that $f \in \Lambda_a$ so $\partial_i(f) = 0$ and $s_i(f) = f$. Using the twisted Liebniz rule we get

$$\partial_i(fg) = \partial_i(f)g + s_i(f)\partial_i(g) = f\partial_i(g).$$

On the other hand, we can also calculate explicitly that

$$\partial_i(fg) = \frac{fg - s_i(fg)}{x_i - x_{i+1}} = \frac{fg - fs_i(g)}{x_i - x_{i+1}} = f\partial_i(g).$$

Now, we prove the converse. We show first that if a polynomial commutes with the action of ∂_i then it is symmetric in x_i and x_{i+1}. Let g be a polynomial, and assume $\partial_i(fg) = f\partial_i(g)$. Calculate

$$\partial_i(fg) = \frac{fg - s_i(fg)}{x_i - x_{i+1}}; \qquad f\frac{g - s_i(g)}{x_i - x_{i+1}} = f\partial_i(g),$$

which combined give $fg - s_i(fg) = f(g - s_i(g))$. This implies that $fs_i(g) = s_i(fg)$, so that f is invariant under s_i. Therefore it is symmetric in x_i and x_{i+1}. If this happens for every ∂_i, $i = 1, \dots, a$, then $f \in \Lambda_a$. $\square$

Theorem 1.6.3. *(Basis for $\mathcal{NH}_a$) Consider the set $\mathcal{B} = \{\underline{x}^A\partial_\pi\}_{\pi\in S_a, A\in\mathbb{N}^a}$ in $End(\mathbb{Z}[\bar{x}_a])$. Elements in $\mathcal{B}$ are linearly independent in the natural representation.*

Proof. We proceed again by induction on length. For $l(\pi) = 0$, from lemma 1.5.2 we get that $\underline{x}^A\partial_\pi 1 = \underline{x}^A$ if $\pi = 1$ and zero otherwise. Suppose that it is true for all π' such that $l(\pi) > l(\pi')$. Notice that, if $l(\pi') < l(\pi)$,

$$\underline{x}^A\partial_\pi s_{\pi'} = 0$$

and if $l(\pi) = l(\pi')$,

$$\underline{x}^A\partial_\pi s_{\pi'} = \begin{cases} \underline{x}^A & \pi = \pi' \\ 0 & otherwise. \end{cases}$$

We claim that if $\underline{x}^A\partial_{\pi''}$ is a linear combination of other elements in the representation, then it is a linear combination of previous elements. If it is a linear combination of elements of bigger length then its action on $s_{\pi''}$ is zero, from the previous equation, which is not true. But it contradicts the induction hypothesis. Then they are linearly independent. $\square$

Theorem 1.6.4. *The nilHecke algebra $\mathcal{NH}_a$ is isomorphic to $Mat(a!, \Lambda_a)$. In particular, the regular representation of $\mathcal{NH}_a$ is faithful.*

Proof. We give an isomorphism to the matrix ring $Mat(a!, \Lambda_a)$ of matrices of size $a!$ over the symmetric polynomials Λ_a. Remember $a!$ is the cardinality of the Schubert basis for the polynomial ring as a free module over Λ_a.

Let A be the polynomial ring in a variables. Let $\mathcal{NH}_a$ act on $A = \bigoplus_{\pi\in S_a}\Lambda_a s_\pi$ through ϕ. This way $\phi : \mathcal{NH}_a \to End(\Lambda_a)$ is a representation of $\mathcal{NH}_a$ and we prove it is an isomorphism. For a $\mathcal{NH}_a$ element $f = \sum_{\pi\in S_a} f_\pi\partial_\pi$ we assign the natural element in the endomorphism ring of A that corresponds to f. We can write that as $\phi(f) = f(-)$, where $f(-) \in EndA$. Let $p \in A$ be a polynomial. We want to see that the representation ϕ is faithful, that is, $\phi(f)(p) = 0$ only if f is zero. Assume $\phi(f)(p) = 0$. But now choose $\pi' \in S_a$ of minimal length in the expression of f. Then $\phi(f)(s_{\pi'}) = 0$ and also

$$\partial_\pi s_{\pi'} = \begin{cases} s_{\pi'\pi^{-1}} & \text{if } l(\pi'\pi) = l(\pi') - l(\pi) \\ 0 & \text{otherwise.} \end{cases}$$

That implies the only contribution comes from $\pi \in S_a$ such that $l(\pi) = l(\pi')$. From $l(\pi'\pi^{-1}) = l(\pi') - l(\pi) = 0$, get that $\pi = \pi'$ and $f_\pi = 0$. By induction on Coxeter length we get the result.

The surjectivity of ϕ comes from making sure that the elementary matrices, which span the matrix ring, are in the image. Order the Schubert basis using the length of permutations, so that the matrix acts on a column vector $f_1 s_1 + \cdots + f_\pi s_\pi + \cdots + f_{w_0} s_{w_0}$. From the formula get

$$\phi(s_\pi\partial_{w_0})(s_{\pi'} = \begin{cases} s_\pi & \pi' = w_0 \\ 0 & \text{otherwise.} \end{cases}$$

Observe that the last equation, using the ordered basis, translates to

$$\phi(s_\pi\partial_{w_0}) \mapsto \begin{pmatrix} 0 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 1 \\ \vdots & \ddots & \vdots & 0 \\ 0 & \dots & 0 & 0 \end{pmatrix}$$

with the entry one in the position (π, w_0). This gives the elements in the last column. If w has length $l(w_0) - 1$, so $w = \sigma_i w_0$, for a transposition σ_i, our formula says

$$\phi(s_\pi \partial_w)(s_{\pi'}) = \begin{cases} s_{\sigma_i} & \text{if } \pi' = w_0 \\ s_\pi & \text{if } \pi' = w \\ 0 & \text{otherwise} \end{cases}.$$

For the case $\pi' = w_0$ the computation is

$$s_\pi \partial_w (s_{w_0}) = \partial_{\pi^{-1} w_0 w}(\underline{x}^\delta) = \partial_{w_0^{-1} w_0 w}(\underline{x}^\delta) = \partial_w(\underline{x}^\delta) = \partial_{\sigma_i \sigma_i^{-1} w}(\underline{x}^\delta) = \partial_{\sigma_i w_0}(\underline{x}^\delta) = s_{\sigma_i}.$$

This matrix is the sum of two elementary matrices, and the one corresponding to $\pi' = w_0$ is in the last column which we computed before. Subtracting the elementary matrix we had before we get the elementary matrix with one on the entry (π, w). We can proceed by induction and the computation is similar to the one we just did. Therefore, we obtain the desired isomorphism. $\square$

Corollary 1.6.5. *From the last theorem we derive that:*

1. *The nilHecke algebra can be seen as a free module over Λ_a.*
2. *The center of the nilHecke algebra is $\mathbb{Z}[\bar{x}_a]^{S_a} = \Lambda_a$.*

Proof. For the first part observe that $Mat(\Lambda_a, a!)$ is a Λ_a module.

We calculate the center of the matrix ring. Remember that if $z \in Z(Mat(n, R))$, the center of $Mat(n, R)$, then z is a multiple of I. It should be clear z has to be sum of the diagonal elements, which are idempotents in $Mat(n, R)$. Write e_i for $i = 1, \ldots, n$ for the orthogonal basic idempotents in $Mat(n, R)$, i.e. the diagonal elements in the basis. Let $A \in Mat(n, R)$. Then Ae_i is the $i-$th column of A, and $e_i A$ is the i-th row of A. Let $z = \sum_i c_i e_i$ with $c_i \in R$ then $c_i \neq 0$. Otherwise there is a matrix A such that $Az \neq zA$ from the last calculation. If $Az = zA$ then non diagonal elements are nonzero and coincide, so $c_i = c_j$ for all $i, j \in \{1, \ldots, n\}$. The identity I_n generates the center, so $RI_n \cong R$ is the center of the matrix ring. Thus, the center of $\mathcal{NH}_a$ is isomorphic to Λ_a. $\square$

Chapter 2

Categorification

In this section we review how the nilHecke algebra categorifies $\mathbf{U}_q(\mathfrak{sl}_2)^+$. The idea of categorification goes back to Igor Frenkel, who conjectured higher categorical quantum groups existed. This was possible from the existence of the algebra ${}_{\mathcal{A}}\dot{\mathbf{U}}$, the idempotented integral version of the quantum enveloping algebra introduced by Lusztig. We need some preliminaries in categorification, and to describe the quantum group $\mathbf{U}_q(\mathfrak{sl}_2)$. After describing $\mathbf{U}_q(\mathfrak{sl}_2)$ we make the modification that gives the integral version. The integral version of the enveloping algebra is due to Kostant, and is generalized to quantum groups by Lusztig. In the work of Khovanov and Lauda the algebra categorified is Lusztig's integral idempotented version $\dot{\mathbf{U}}$ of $\mathbf{U}_q(\mathfrak{sl}_2)$, the algebra that has a canonical basis that was categorified diagrammatically, where the nilHecke algebra plays a part. It turns out that $\dot{\mathbf{U}}^+ = \mathbf{U}_q(\mathfrak{sl}_2)^+$ and we skip $\dot{\mathbf{U}}$ and give an easier introduction. We focus on showing an application of Schubert polynomials, they help to categorify the generators of $\mathbf{U}_q(\mathfrak{sl}_2)^+$. The algebras that categorify the positive parts of the deformed enveloping algebras are called KLR algebras, studied in Khovanov and Lauda's work in [12, 16, 17] and by Rouquier in *2-Kac Moody algebras* [23]. The nilHecke algebra is the KLR algebra associated to $\mathfrak{sl}_2$.

Remark 2.0.1. The ring of polynomials $\mathbb{Z}[\bar{x}_a]$ seen as a Λ_a module is a free module with basis the set of Schubert polynomials $\{s_\pi : \pi \in S_a\}$. As it is free we can always make an extension of scalars $- \otimes_{\mathbb{Z}} \mathbb{Q}$ and consider the nilHecke algebra over $\mathbb{Q}$. We will deal with the nilHecke algebra and not the nilHecke ring in the present chapter in order to apply the categorification theorems of Section 2.2. The nilHecke algebra acts over $\mathbb{Q}[\bar{x}_a]$ and the results in the previous chapter still hold. In Chapter 3 we work again over $\mathbb{Z}$.

2.1 The quantum group $\mathbf{U}_q(\mathfrak{sl}_2)$

Here we review some important facts of the algebra $\mathfrak{sl}_2$ and the quantum group $\mathbf{U}_q(\mathfrak{sl}_2)$. Most of this section's material can be found in Kassel's book on Quantum Groups. For this reason we omit most of the proofs. We show that $\mathbf{U}_q(\mathfrak{sl}_2)$ is a Hopf algebra, so we can call it a quantum group, in the sense of Drinfeld. We also review how representations behave.

Definition 2.1.1. The Lie algebra $\mathfrak{sl}_2$ over any field of characteristic different from 2 is the Lie algebra of traceless square matrices of size 2. The Lie bracket is defined as $[a, b] = ab - ba$.

Lemma 2.1.2. *The algebra $\mathfrak{sl}_2$ is generated by the matrices*

$$e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \qquad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

with product $[-,-]$ *and with relations*

$$[e,f]=h \qquad [h,e]=2e \qquad [h,f]=-2f.$$

We study the representations of $\mathfrak{sl}_2$ over $\mathbb{C}$. This algebra is nonassociative and the operation $[a,b]=ab-ba$ is called a Lie bracket. In this case this is a simple Lie algebra of dimension 3 and it is very important in the classification of finite dimensional Lie algebras. It turns out that the representation categories of $\mathfrak{sl}_2$ and the enveloping algebra $U(\mathfrak{sl}_2)$ are the same, though we do not give a proof of this. With this excuse, we are concerned with finite dimensional representations of $\mathfrak{sl}_2$. Call v the **highest weight** vector of an $\mathfrak{sl}_2$ representation if $ev=0$ and if it has the highest eigenvalue for H. An irreducible $\mathfrak{sl}_2$ representation is a vector space V in which h is diagonalizable and it has a decomposition into the eigenvectors of h such that it has no non trivial subrepresentation. From the $\mathfrak{sl}_2$ relations it can be proved that if v' has eigenvalue λ then ev' has eigenvalue $\lambda+2$ and fv' has eigenvalue $\lambda-2$. Induction on these elements give a description of the irreducible representations V_n with highest weight $\lambda=n$.

Theorem 2.1.3 (Kassel [10])**.** *Let* v_k *be as we just defined, then*

$$hv_k=(n-2k)v_k, \qquad ev_k=(n-k+1)v_{k-1}, \qquad fv_k=(k+1)v_{k+1}.$$

We do not study $\mathfrak{sl}_2$ as it is nonassociative, but prefer the enveloping algebra $U(\mathfrak{sl}_2)$. This algebra is the associative algebra generated by elements $\{E,F,H\}$ with the $\mathfrak{sl}_2$ relations

$$[E,F]=H, \qquad [H,E]=2E \qquad [H,F]=-2F.$$

It has the universal property that it factors any Lie homomorphsim from $\mathfrak{sl}_2$ to any associative algebra with unit. The enveloping algebra has a Hopf structure. This algebra can be 'deformed' to give another algebra $\mathbf{U}_q(\mathfrak{sl}_2)$ which is the one of interest for us. Before doing that we introduce an algebra presented by Kostant in *Groups over Z* [15, p. 485]. As stated there, *if we select the basis of the irreducible representation* V *in a way that* $ev_i=\pm iv_{1+i}$ *then the* $\mathbb{Z}$ *span of the basis* $\{v_i\}_i$ *is stable under the divided powers* $E^m/m!$ *and* $F^n/n!$ *for* $m,n\in\mathbb{Z}$*. The algebra generated by the divided powers and* h *is called the integral version of* $U(\mathfrak{sl}_2)$ *denoted* ${}_{\mathbb{Z}}U(\mathfrak{sl}_2)$*.*

Now, we introduce the quantum case. Let q be a nonzero element in the field. In our case it will be a generic element different from the roots of unity[1]. We introduce the quantum version of $\mathfrak{sl}_2$. Define the quantum factorial

$$[n]=\frac{q^n-q^{-n}}{q-q^{-1}}=q^{n-1}+q^{n-3}+\cdots+q^{-n+3}+q^{-n+1}$$

with analog definitions of factorials and binomials as follows

$$\begin{aligned} [-n] &= -[n] && (2.1.1)\\ [m+n] &= q^n[m]+q^{-m}[n] && (2.1.2)\\ [k]! &= [1][2]\cdots[k] && (2.1.3)\\ \begin{bmatrix} n\\ k\end{bmatrix} &:= \frac{[n]!}{[k]![n-k]!} && (2.1.4)\\ [n] &= q^{-(n-1)}(n)_{q^2}. && (2.1.5)\\ & && (2.1.6)\end{aligned}$$

[1]We do not deal with that case in this document.

Definition 2.1.4. Let $q \in \mathbb{Q}^*$ such that q is not a root of unity. Let $\mathbf{U}_q(\mathfrak{sl}_2)$ be the algebra freely generated over $\mathbb{Q}(q)$ the field of rational functions over $\mathbb{Q}$, by four variables E, F, K, K^{-1} that satisfy the following relations

$$\begin{aligned} KK^{-1} &= K^{-1}K = 1 && (2.1.7)\\ KEK^{-1} &= q^2E, && (2.1.8)\\ KFK^{-1} &= q^{-2}F && (2.1.9)\\ [E, F] &= \frac{K - K^{-1}}{q - q^{-1}}. && (2.1.10) \end{aligned}$$

Theorem 2.1.5. *(PBW theorem for $\mathbf{U}_q(\mathfrak{sl}_2)$) The algebra $\mathbf{U}_q(\mathfrak{sl}_2)$ is Noetherian and has no nonzero divisors. The set $\{E^iF^jK^l\}_{i,j\in\mathbb{N},l\in\mathbb{Z}}$ is a basis of $\mathbf{U}_q(\mathfrak{sl}_2)$.*

The proof of the theorem rests in a noncommutative tool called an Ore extension. A proof of the previous theorem can be found in the book by Kassel [10]. The Ore extension is used to get the analog version of Hilbert Basis Theorem in the noncommutative setting. If R is Noetherian the Ore extension is unique and Noetherian too. To define an Ore extension one needs an $\alpha-$derivation. This derivation is a twisted type of derivation, just as the action $\partial_i(fg) = \partial_i(f)g + s_i(f)\partial_i(g)$ is twisted by the automorphism s_i.

One expects to get back $U(\mathfrak{sl}_2)$ by setting $q = 1$ but it is not achieved form this presentation of $\mathbf{U}_q(\mathfrak{sl}_2)$, it has to be modified. The presentation we use is the usual one. The representation theory of $\mathbf{U}_q(\mathfrak{sl}_2)$ is similar to the one of $\mathfrak{sl}_2$. The highest eigenvalue and eigenvector for eigenspaces of H in the representation is called the highest weight and highest weight vector. This determines the irreducible representations also.

Theorem 2.1.6 (Finite dimensional representations of $\mathbf{U}_q(\mathfrak{sl}_2)$)**.** *Let V_n be the n+1 dimensional representation of $\mathbf{U}_q(\mathfrak{sl}_2)$ with basis $\{v_m\}$, $m \leq |n|$, $m = 2 \ mod 2$, such that*

$$\begin{aligned} Kv_m &= q^m v_m; && (2.1.11)\\ Ev_m &= \left[\frac{n+m}{2} + 1\right] v_{m+2}; && (2.1.12)\\ Fv_m &= \left[\frac{n-m}{2} + 1\right] v_{m-2}. && (2.1.13) \end{aligned}$$

These are all the irreducible finite dimensional representations of $\mathbf{U}_q(\mathfrak{sl}_2)$.

Representations of $U(\mathfrak{sl}_2)$ and $\mathbf{U}_q(\mathfrak{sl}_2)$ are very similar. Let n and an irreducible representation of $U(\mathfrak{sl}_2)$ of dimension n be given. The matrix for E in this representation has the numbers $n - 1, \ldots, 1$ in the diagonal just above the main diagonal. For $\mathbf{U}_q(\mathfrak{sl}_2)$ the matrix has the quantum numbers $[n - 1], \ldots, [1]$. Our objective now is to endow $\mathbf{U}_q(\mathfrak{sl}_2)$ a Hopf algebra structure.

Definition 2.1.7. A coalgebra is an algebraic structure with a counit and comultiplication. These are maps that satisfy dual diagrams to the unit and multiplication diagrams. Cocommutativity is defined in the same fashion. A **bialgebra** is an algebra with a coalgebra and an algebra structure that are compatible and the counit and unit maps are algebra maps.

If A is an algebra then let the map $\mu : A \otimes A \to A$ multiply the entries. Let $*$ represent the convolution of A maps, so that $f*g : A^{op}\otimes A^{op} \to A\otimes A$ is given by $f\otimes g$. Let ν denote the unit map $\nu : k \to A$. Recall that $\Delta : A \to A \otimes A$. An antipode $S : A \to A^{op}$ in A is an antihomomorphism such that the following equations hold:

$$\mu \circ (I * S) \circ \Delta = \mu \circ (S * I) \circ \Delta = \nu \circ \varepsilon.$$

A **Hopf algebra** is a bialgebra with an antipode.

Example 2.1.8. *Let G be a finite group and and consider the group algebra kG. Define the coalgebra structure by $\Delta(x) = x \otimes x$ and $\varepsilon(x) = 1$. Let $S(x) = x^{-1}$. Extend these maps linearly to give a Hopf structure for kG. Elements in a Hopf algebra that satisfy $\Delta(x) = x \otimes x$ are called grouplike elements.*

We define a comultiplication Δ, antipode S, and counit ε for $\mathbf{U}_q(\mathfrak{sl}_2)$. The map Δ and ε are defined as

$$\begin{aligned}
\Delta(E) &= 1 \otimes E + E \otimes K; && (2.1.14)\\
\Delta(F) &= K^{-1} \otimes F + F \otimes 1 && (2.1.15)\\
\Delta(K) &= K \otimes K && (2.1.16)\\
\Delta(K^{-1}) &= K^{-1} \otimes K^{-1} && (2.1.17)\\
\varepsilon(E) &= \varepsilon(F) = 0; && (2.1.18)\\
\varepsilon(K) &= \varepsilon(K^{-1}) = 1 && (2.1.19)
\end{aligned}$$

and

$$S(E) = -EK^{-1}, \qquad S(F) = -KF, \qquad S(K) = K^{-1}, \qquad S(K^{-1}) = K. \tag{2.1.20}$$

Theorem 2.1.9. *The algebra $\mathbf{U}_q(\mathfrak{sl}_2)$ has a Hopf algebra structure.*

Proof. We use the counit, antipode and comultiplication as defined above on generators. The theorem is broken into the following steps, where we check that the maps satisfy the desired relations on the algebra's generators.

(1) The map Δ is an algebra map.

$$\begin{aligned}
\Delta(K)\Delta(K^{-1}) &= (K \otimes K)(K^{-1} \otimes K^{-1}) = 1 \otimes 1 \cong 1 && (2.1.21)\\
\Delta(K^{-1})\Delta(K) &= (K^{-1} \otimes K^{-1})(K \otimes K) = 1 \otimes 1 \cong 1 && (2.1.22)\\
\Delta(K)\Delta(E)\Delta(K^{-1}) &= (K \otimes K)(1 \otimes E + E \otimes K)(K^{-1} \otimes K^{-1}) && (2.1.23)\\
&= 1 \otimes KEK^{-1} + KEK^{-1} \otimes K && (2.1.24)\\
&= q^2(1 \otimes E + E \otimes K) && (2.1.25)\\
&= q^2\Delta(E); && (2.1.26)\\
\Delta(K)\Delta(F)\Delta(K^{-1}) &= (K \otimes K)(K^{-1} \otimes F + F \otimes 1)(K^{-1} \otimes K^{-1}) && (2.1.27)\\
&= (K^{-1} \otimes KFK^{-1} + KFK^{-1} \otimes 1) && (2.1.28)\\
&= q^{-2}(K^{-1} \otimes F + F \otimes 1) && (2.1.29)\\
&= q^{-2}\Delta(F); && (2.1.30)\\
[\Delta(E), \Delta(F)] &= (1 \otimes E + E \otimes K)(K^{-1} \otimes F + F \otimes 1) && (2.1.31)\\
&= K^{-1} \otimes EF + F \otimes E + EK^{-1} \otimes KF + EF \otimes K && (2.1.32)\\
&- K^{-1} \otimes FE - K^{-1}E \otimes FK - F \otimes E - FE \otimes K && (2.1.33)\\
&= K^{-1} \otimes EF - K^{-1} \otimes FE + EF \otimes K - FE \otimes K && (2.1.34)\\
&= K^{-1} \otimes [E, F] + [E, F] \otimes K && (2.1.35)\\
&= \left(\frac{1}{q - q^{-1}}\right)\big(K^{-1} \otimes (K - K^{-1}) + (K - K^{-1}) \otimes K\big) && (2.1.36)\\
&= \frac{\Delta(K) - \Delta(K^{-1})}{q - q^{-1}}. && (2.1.37)
\end{aligned}$$

In the last equation we use that $K^{-1}E \otimes FK = q^2EK^{-1} \otimes q^{-2}KF = EK^{-1} \otimes KF$.

(2) The map Δ is coassociative.

$$\begin{aligned}
(\Delta \otimes Id)\Delta(E) &= (\Delta \otimes Id)(1 \otimes E + E \otimes K) && (2.1.38)\\
&= 1 \otimes 1 \otimes E + 1 \otimes E \otimes K + E \otimes K \otimes K && (2.1.39)\\
(Id \otimes \Delta)\Delta(E) &= (Id \otimes \Delta)(1 \otimes E + E \otimes K) && (2.1.40)\\
&= 1 \otimes 1 \otimes E + 1 \otimes E \otimes K + E \otimes K \otimes K && (2.1.41)\\
(\Delta \otimes Id)\Delta(F) &= (\Delta \otimes Id)(K^{-1} \otimes F + F \otimes 1) && (2.1.42)\\
&= K^{-1} \otimes K^{-1} \otimes F + K^{-1} \otimes 1 + F \otimes 1 \otimes 1 && (2.1.43)\\
(Id \otimes \Delta)\Delta(F) &= (Id \otimes \Delta)(K^{-1} \otimes F + F \otimes 1) && (2.1.44)\\
&= K^{-1} \otimes K^{-1} \otimes F + K^{-1} \otimes 1 + F \otimes \Delta(1) && (2.1.45)\\
(\Delta \otimes Id)\Delta(K) &= K \otimes K \otimes K = (Id \otimes \Delta)\Delta(K) && (2.1.46)\\
(\Delta \otimes Id)\Delta(K^{-1}) &= K^{-1} \otimes K^{-1} \otimes K^{-1} = (Id \otimes \Delta)\Delta(K^{-1}) && (2.1.47)
\end{aligned}$$

(3) The map ε is an algebra map.

$$\begin{aligned}
\varepsilon(K)\varepsilon(K^{-1}) = 1 = \varepsilon(K^{-1})\varepsilon(K) && (2.1.48)\\
\varepsilon(K)\varepsilon(E)\varepsilon(K^{-1}) = 0 = q^2 \times 0 = q^2\varepsilon(E) && (2.1.49)\\
\varepsilon(K)\varepsilon(F)\varepsilon(K^{-1}) = 0 = q^{-2} \times 0 = q^{-2}\varepsilon(F) && (2.1.50)\\
[\varepsilon(E), \varepsilon(F)] = 0; && (2.1.51)\\
\frac{\varepsilon(K) - \varepsilon(K^{-1})}{q - q^{-1}} = \frac{1-1}{q-q^{-1}} = 0. && (2.1.52)
\end{aligned}$$

(4) ε satisfies the counit axiom. Let η be the map that sends $A \mapsto k \otimes A$. The counit axiom states that

$$(\varepsilon \otimes Id) \circ \Delta = \eta = (Id \otimes \varepsilon) \circ \Delta.$$

We just check that the left hand side equals η. We have that

$$\begin{aligned}
(\varepsilon \otimes 1)\Delta(K) &= (\varepsilon \otimes 1)(K \otimes K) = 1 \otimes K; && (2.1.53)\\
(\varepsilon \otimes 1)\Delta(K^{-1}) &= (\varepsilon \otimes 1)(K^{-1} \otimes K^{-1}) = 1 \otimes K^{-1}; && (2.1.54)\\
(\varepsilon \otimes 1)\Delta(E) &= (\varepsilon \otimes 1)(1 \otimes E + E \otimes K) && (2.1.55)\\
&= \varepsilon(1) \otimes E + \varepsilon(E) \otimes K = 1 \otimes E + 0 = 1 \otimes E; && (2.1.56)\\
(\varepsilon \otimes 1)\Delta(F) &= (\varepsilon \otimes 1)(K^{-1} \otimes F + F \otimes 1) && (2.1.57)\\
&= \varepsilon(K^{-1}) \otimes F + \varepsilon(F) \otimes 1 = 1 \otimes F + 0 = 1 \otimes F. && (2.1.58)
\end{aligned}$$

(5) The map $S : \mathbf{U}_q(\mathfrak{sl}_2) \to \mathbf{U}_q(\mathfrak{sl}_2)^{op}$ is an algebra map.

$$\begin{aligned}
S(K)S(K^{-1}) &= K^{-1}K = 1 = KK^{-1} = S(K^{-1})S(K); && (2.1.59)\\
S(K^{-1})S(E)S(K) &= -K(EK^{-1})K^{-1} = -q^2EK^{-1} = q^2S(E); && (2.1.60)\\
S(K^{-1})S(F)S(K) &= K(-KF)K^{-1} = -K(q^{-2}F) = -q^{-2}KF = q^{-2}S(F); && (2.1.61)\\
[S(F), S(E) &= KFEK^{-1} - EK^{-1}KF = [F, E] && (2.1.62)\\
&= \frac{K^{-1} - K}{q - q^{-1}} = \frac{S(K) - S(K^{-1})}{q - q^{-1}}. && (2.1.63)
\end{aligned}$$

(6) The map S is an antipode. We prove that the antipode equation holds on generators.

$$\begin{aligned}
\mu \circ (S * 1) \circ \Delta(K) &= \mu(K^{-1} \otimes K) = K^{-1}K = 1 = \varepsilon(K); && (2.1.64)\\
\mu \circ (1 * S) \circ \Delta(K^{-1}) &= \mu(K^{-1} \otimes K) = K^{-1}K = 1 = \varepsilon(K^{-1}); && (2.1.65)\\
\mu \circ (S * 1) \circ \Delta(E) &= \mu(1 \otimes S(E) + E \otimes S(K)) && (2.1.66)\\
&= \mu(1 \otimes -EK^{-1} + E \otimes K^{-1}) = -EK^{-1} + EK^{-1} && (2.1.67)\\
&= 0 = \varepsilon(E); && (2.1.68)\\
\mu \circ (S * 1) \circ \Delta(F) &= \mu(K^{-1} \otimes S(F) + F \otimes S(1)) && (2.1.69)\\
&= \mu(K^{-1} \otimes -KF + F \otimes 1) = K^{-1}(-KF) + F && (2.1.70)\\
&= F - F = 0 = \varepsilon(F). \quad \square && (2.1.71)
\end{aligned}$$

Theorem 2.1.10. *We have that $S^2(u) = KuK^{-1}$ for any $u \in \mathbf{U}_q(\mathfrak{sl}_2)$. This means that S^2 is not the identity, but it is an inner automorphism.*

That $S^2 \neq Id$ implies that $\mathbf{U}_q(\mathfrak{sl}_2)$ is neither commutative or cocommutative. The generators K and K^{-1} are grouplike elements. Remember the tensor product of representations carries an action defined by $a(v \otimes w) = (av \otimes aw)$. There is the important formula for tensor product of representations

$$V_n \otimes V_m \cong V_{n+m} \oplus V_{n+m-2} \oplus \cdots \oplus V_{n-m}$$

which is proved verifying that $V_n \otimes V_m$ contains a highest weight vector of weight q^{n+m-2p} for $0 \leq p \leq m$.

2.2 Categorification and K_0 of a ring

It might be unclear what we mean by categorification. Khovanov's article *Linearization and categorification* [11] is a short and simple introduction. To categorify is to do the inverse problem of decategorifying. A decategorification is a functor from an $n-$category $\mathcal{C}$ to a $n-1$ category $\mathcal{C}'$. It is natural to understand some algebras as categories, for example, path algebras, in the sense of Grabriel. Given a category $\mathcal{C}$ a functor $\mathcal{C} \to R$, where R is a ring or algebra is a decategorification functor. To categorify is to construct a category $\mathcal{C}$ and a functor with this property. What we discuss comes from [3, 22]. We are interested in the split Grothendieck group of a monoidal category. For the definiion of a monoidal category we refer the reader to Etingof's book [7].

Definition 2.2.1. A monoidal category $\mathcal{C}$ categorifies an algebra B if $K_0(\mathcal{C}) = B$.

In the following we introduce the Grothendieck group of the module category of a ring.

Definition 2.2.2. Let $\mathcal{C}$ be an abelian category $\mathcal{C}$. The Grothendieck group $K_0(\mathcal{C})$ is the quotient of the free abelian group generated by isoclasses of objects of $\mathcal{C}$ by split exact sequences in $\mathcal{C}$. That is, take $Fr(\mathcal{C}) = \{[X] | X \in \mathcal{C}\}$ with quotient induced by the relations $[X] = [Y] + [Z]$ if there is a **split exact sequence** $0 \to Y \to X \to Z \to 0$ in $\mathcal{C}$.

Remark 2.2.3. Equivalent categories give rise to the same Grothendieck groups. In particular Morita equivalent rings will have isomorphic Grothendieck groups.

Theorem 2.2.4. *If the additive category $\mathcal{C}$ is an abelian monoidal category the Grothendieck group $K_0(\mathcal{C})$ has a ring structure with multiplication induced from the tensor product in $\mathcal{C}$.*

We wish to calculate $K_0(\mathcal{NH}_a)$ and define an algebra $\mathcal{NH}$ such that $K_0(\mathcal{NH}) =_{\mathcal{A}} \mathbf{U}_q(\mathfrak{sl}_2)^+$. Not every module category is monoidal. In the case of $\mathcal{NH}$ we do not have a bialgebra, but the induction and restriction functors for modules from the tower of algebras that we define in Section 2.3 give $K_0(\mathcal{NH})$ a Hopf algebra structure. That $K_0(\mathcal{NH})$ is a bialgebra is a result of Khovanov-Lauda, which also holds for towers of finite dimensional algebras, a notable theorem of Nantel Bergeron [2]. We start making our way to present Grothendieck groups of rings.

Definition 2.2.5. A **projective module** is a summand of a free module.

Lemma 2.2.6. *An exact sequence ending in a projective R module splits. In particular an exact sequence of projective modules splits.*

Definition 2.2.7. A module M in mod$A$ is called flat if $- \otimes_A M$ is an exact functor.

Remark 2.2.8. A projective module is flat.

Definition 2.2.9. Given an algebra A the Grothendieck group of A, $K_0(A)$, is the Grothendieck group of the category of finitely generated projective left A modules.

Remark 2.2.10. We wish to make theorem 2.2.4 more clear in the case $\mathcal{C}$ is the module category of a ring. If we consider a bialgebra A the module category is a monoidal category where the coproduct makes the tensor product of modules M and N, $M \otimes N$, also an A module through $a \cdot (m \otimes n) = \Delta(a)(m \otimes n)$ for $m \otimes n \in M \otimes N$ and $a \in A$. Associativity for product in the ring $K_0(A)$ follows from associativity in the tensor product. The tensor product of projective modules is projective: If M is a summand of M' and N is a summand of N' where M' and N' are free, $M \otimes N$ is a summand of $M' \otimes N'$, which is free. Therefore $M \otimes N$ is projective. The ring has an identity $[A]$, as $A \otimes_A M \cong M \cong M \otimes_A A$. The zero of $K_0(A)$ is the zero module. We check the distributivity also holds. Given a direct sum $M \oplus N$ and an A module S we can tensor by S and the relation $(M \oplus N) \otimes_A S \cong (M \otimes_A S) \oplus (N \otimes_A S)$ is an isomorfism. To prove the distributivity we can look at the split exact sequence $0 \to M \to M \oplus N \to N \to 0$ and tensor getting $0 \to M \otimes S \to (M \otimes S) \oplus (N \otimes S) \to N \otimes S \to 0$ what we want. The module S is projective, so it is flat and the sequence is exact.

If A is commutative $M \otimes N$ is an A module and there is an isomorphism $M \otimes N \cong N \otimes M$.

Corollary 2.2.11. *If A is a commutative ring then $K_0(A)$ is a commutative algebra.*

Let us categorify $\mathbb{Z}$. Given the ring of integers $\mathbb{Z}$ we wish to find a category such that $K_0(\mathcal{C}) = \mathbb{Z}$. The algebra or category $\mathcal{C}$ of this example must be simple. Picture the additive semigroup of integers. We want it to have a generator corresponding to 1 and a semigroup of elements generated by it, additively, such that by adding two copies we get the direct sum of them. The category $\mathcal{C}$ is a category of modules. An indecomposable in the category can generate the Grothendick group and this indecomposable has to be unique. In the case of vector spaces over a field k of characteristic zero we have an indecomposable one dimensional vector space that factors any othe vector space. We also have a zero vector space which is neutral element for addition. Let us choose $\mathcal{C}$ as the category of finite dimensional vector spaces over the field k. Call this category $Vect_k$. The group induced from that semigroup is the additive group of integers $\mathbb{Z}$. A decategorification functor from $Vect_k$ to $\mathbb{Z}$ is given by dimension. The ring $\mathbb{Z}$ has also a product. In the category $Vect_k$ we have a product given by the tensor product. It satisfies that $\dim(V \otimes W) = \dim V \times \dim W$. The one dimensional generator $[k]$ satisfies $[k] \otimes W \cong W \otimes [k] \cong W$ has the same dimension. We have explained the following example.

Example 2.2.12. *Let $Vect_k$ be the category of finite dimensional vector spaces over an arbitrary field of characteristic zero. Then $K_0(Vect_k) \cong \mathbb{Z}$, where the product in the Grothendieck ring is the tensor product of modules. There is decategorification functor* $\dim : Vect_k \to \mathbb{Z}$ *that sends any vector space V to its corresponding class* $\dim(V) \in \mathbb{Z}$. *In this case $K_0(Vect_k)$ is a ring.*

We want to calculate two examples of Grothendieck groups, for commutative local rings and for matrix rings. The following proof requires a common tool in ring theory. The main idea of the proof is that the rank, or dimension, is well defined modulo the radical and can be lifted up. It is important to bear in mind the characterizations of the radical.

Definition 2.2.13. The **Jacobson radical** of a not necessarily commutative ring R is the intersection of all maximal left ideals. We write $\mathrm{rad}A$ for the Jacobson radical.

Lemma 2.2.14. *The Jacobson radical equals both of the following sets.*

1. *The intersection of maximal right ideals,*
2. *and $\{x \in A : \forall a \in A, 1 - ax$ has a left inverse$\}$.*

Lemma 2.2.15. *(Nakayama) Let A be a ring and M a finitely generated A-module with $rad(AM) = M$. Then $M = 0$.*

Corollary 2.2.16. *Let M be a finitely generated module over A. A set $\{x_i\}_{i=1}^m$ generate M if and only if their images generate $M/(radA)M$.*

Lemma 2.2.17. *Let A be a local commutative ring. Then $K_0(A) = \mathbb{Z}$.*

Remark 2.2.18. The last result can be made more general. Namely, if A is a local noncommutative ring then $K_0(A) = \mathbb{Z}$. Given a module M by the radical of the A-module M we can lift the basis used here to give a rank function. Through Nakayama's lemma the result is obtained. Even more, if A is commutative and local by Kaplansky's theorem a projective module is free.

The function dim is a decategorification functor. It is well defined for isoclasses of finitely generated projective modules. In particular it gives an isomorphism of $K_0(A)$ and $\mathbb{Z}$ by passing to the quotient $\dim : K_0(A) \to \mathbb{Z}$.

We review the theorem of Morita to give more background on the importance of K_0 as an algebraic invariant. The best way to state the theorem of Morita is as it appears in [24]. The other way is to show that the progenerator gives rise to a Morita context, which through some bimodules give a pair of functors that define an equivalence.

Definition 2.2.19. A generator G for a module category $\mathcal{C}$ is a module such that any module $M \in \mathcal{C}$ we have that M is the image of G^J for some index set J. A progenerator is a finitely generated projective generator.

Theorem 2.2.20 (Morita)**.** *Let A and A' be rings. Then A is Morita equivalent to A' if there is a progenerator P of A' such that $A \cong End_{A'}(P)$.*

Lemma 2.2.21. *A full matrix ring of size n over a ring A is Morita equivalent to A.*

Proof. We show how to apply the theorem of Morita as we stated it. Let $Mat(n, A)$ be the matrix ring of $n \times n$ matrices over A. The category we consider is of finitely generated left modules. Let c be a column vector. As an A module c is projective, because it is a summand of A^{n+1}, which is free. We claim it is a progenerator. The modules over the matrix rings consist of sums of columns, which can be of different sizes, of some A modules. The module c is finitely generated as c is the sum of n times A. Observe that A itself is the image of c and is a generator for R-mod. The column modules are built from finite sums of projective A modules and thus any finitely generated module is the image of c^J for some index J.[2]

Now observe that the matrix ring is the endomorphism ring of c over A, then $Mat(n, A) = End_A(c)$. By Morita's theorem their module categories are equivalent. □

[2] In the category of A modules G is a generator if there is a surjective map $G \to A$.

The progenerator of the category is the column vector, which is also the module that generates the Grothendieck group. Using Morita's theorem may seem obscure, so we focus on a direct calculation given by Rosenberg in [22] for the next lemma.

Lemma 2.2.22. *Let $Mat(A, n)$ be the matrix ring of $n \times n$ matrices over A a commutative ring. Then $K_0(Mat(A, n)) \cong K_0(A)$.*

Proof. We study $K_0(A)$ through idempotents. Suppose P is a projective module so that $P \oplus M = A^n$ for some $n > 0$. Let π be the projection in P so that $\pi^2 = \pi$. As A^n has a rank and π is an idempotent in $End_A(A^n)$ it is a matrix idempotent! This module is equivalent to any other P' such that its corresponding endomorphism π' is conjugate in $Mat(m, A)$ for largely enough m. One can embed the matrices by filling the rest of the entries with zeroes.

On the other hand consider $S := Mat(n, A)$ the ring of matrices and ${}_SS$ over itself as a module. The matrix ring has n left ideals which are isomorphic. These left ideals are columns in the matrix ring, and given a column c, we have that $c^n = S$ as S modules. This module is a summand of S so it is projective, and even indecomposable. Observe that c is generated by the column vector c with a one in the first entry and zero elsewhere. Informally, we claim this identifies c with A.

The matrix idempotent π corresponding to the column c inside $End(M)$ for any module M with summand c is primitive. That means $\pi = e_1 + e_2$ for other idempotents e_1 and e_2, then $e_1 = 0$ or $e_2 = 0$ as c is indecomposable. Any sum of columns is isomorphic to a sum of this module inductively, obtained from the split sequence $0 \to c \to c^2 \to c \to 0$.

Now, we look at idempotents in the matrix ring. These correspond to sums of diagonal elements $\pi_{i,i}$ for $i = 1, \dots \dim End(M)$. Observe that $End_S(M)$ has finite dimension, it is a matrix ring of matrices with size $\dim M \times \dim M$ where $M = A^n$. Under conjugation we only have one class corresponding to $\pi_{1,1}$ and this corresponds to a column c vector of S.

Let A be any ring. Consider the inclusions of matrices stacking matrix rings in the upper left corner for $Mat(n, A)$ and any n. We can take the union of them and end up with $M(A)$ the infinite ring of matrices. In this sense we take the union of invertible matrices $Gl(n, R)$ and get $Gl(A)$ the invertible matrices in $M(A)$. Idempotents in this matrices under conjugation correspond to isoclasses of indecomposable projective modules. A projective module in this presentation is the finite sum of primitive idempotents in $M(A)$, as observed in Rosbenberg's book [22].

There is the usual isomorphism $M_r(S) \cong M_{nr}(A)$ which induces isomorphisms of $M(A)$ and $M(S)$, and also for the conjugations. Then the previous description of the Grothendieck group implies that $K_0(S) \cong K_0(A)$. □

This proof gives a more tangible explanation of the Morita equivalence, where we have that the indecomposable progenerator c of S corresponds to the indecomposable progenerator R under the equivalence.

Corollary 2.2.23. *We have that $K_0(\mathcal{NH}_a) = \mathbb{Z}$.*

Proof. Simply consider the last two theorems we proved. If R is a unital commutative ring we have that $K_0(Mat(n, R))) \cong K_0(R)$. If R is a local commutative unital ring then $K_0(R) = \mathbb{Z}$ through the dimension functor. We obtained in the last chapter that $\mathcal{NH}_a \cong Mat(a!, \Lambda_a)$. Putting these together

$$K_0(\mathcal{NH}_a) \cong K_0(Mat(a!, \Lambda_a) \cong K_0(\Lambda_a) \cong \mathbb{Z}.$$ □

We introduce the Grothendieck group of a graded algebra. In order to do so we follow again the example of the category $Vect_k$ of vector spaces.

Example 2.2.24. *Define the category of* **graded vector spaces over the field** *k, denoted by $gVect_k$, in the following way. Let the objects be $\mathbb{Z}$ graded vector spaces, so that $V = \oplus_{i \in \mathbb{Z}} V_i$. A* *morphism of graded vector spaces, such as a morphism of graded algebras is given componentwise. In the case of graded modules we want the compatibility $A_g M_{g'} \subset M_{gg'}$ preserved under the morphism. We calculate $K_0(gVect_k)$. For a vector space $V \in gVect_k$ set*

$$[V] = [\oplus_{i\in\mathbb{Z}} V_i] = \sum_{i\in\mathbb{Z}} q^i [V_k] = \sum_{i\in\mathbb{Z}} q^i \dim_k V_i[k].$$

Then define $\dim_q V = \sum_{i\in\mathbb{Z}} q^i \dim_k V_i$*, to obtain* $[V] = \dim_q V[k]$*. This implies* $K_0(gVect_k) = \mathbb{Z}[q, q^{-1}]$*.*

Definition 2.2.25. Let A be a $\mathbb{Z}$ graded ring. The graded Grothendieck group of an algebra A is the Grothendieck group of the graded module category of projective modules with the additional relation

$$[x_t] = q^t[x]$$

for $x_t \in M_t$ that makes $K_0(A)$ a $\mathbb{Z}[q, q^{-1}]$ module.

In a similar form as before we have the following results.

Corollary 2.2.26. *1. If the category of modules is monoidal then the graded $K_0(R)$ is a ring.*

2. Let R be a local commutative unital graded ring. Then $K_0(R) = \mathbb{Z}[q, q^{-1}]$.

3. We still have that $K_0(Mat(n, R)) \cong K_0(R)$ for the graded version of K_0.

Example 2.2.27 (Euler characteristic)**.** *Let $\mathcal{C}^\bullet$ be the category of complexes of $gVect_k$. We define a map from complexes of graded vector spaces to $K_0(gVect_k)$ which is isomorphic to $\mathbb{Z}[q, q^{-1}]$ with the Euler characteristic $\chi(C^\bullet) = \sum_{i\in\mathbb{Z}}(-1)^i[C_i] = \sum_{i\in\mathbb{Z}}(-1)^i \dim_q(C_i)[k] \in K_0(gVect_k)$.*

To apply this theorem to $\mathcal{NH}_a$ through the isomorphism $\mathcal{NH}_a = Mat(a!, \Lambda_a)$ we need the following lemma.

Lemma 2.2.28. *The ring of symmetric functions over a field is a graded local commutative unital ring.*

Proof. Let the ground field be k. Consider $k[\bar{x}_a]$. We claim that there is a graded maximal ideal I in $k\bar{x}_a]$. The grading is given by the non negative integers. If we multiply by a polynomial f, in the extreme case the lowest degree is for $f \in k$, and for any $g \in I$, $\deg(fg) = \deg(g) + \deg(f) = \deg(g) > 0$, so $fg \in I$. Now take the quotient $\Lambda_a/I = A_0 = k$. Then I is maximal. We claim that $m := I \cap \Lambda_a$ is a maximal graded ideal in Λ_a. If it was not maximal then there is an ideal m' such that $I \subsetneq m'$. But, if $f \in m' - I$ then $f \in \langle m' \rangle - m$ in $k[\bar{x}_a]$ and if $\langle m' \rangle$ does not equal m it has to be $k[\bar{x}_a]$. Then $m' = k[\bar{x}_a] \cap \Lambda_a = \Lambda_a$ is not a maximal proper ideal. $\square$

Theorem 2.2.29. *Let $\mathcal{NH}_a$ be considered as a graded algebra. Thus $K_0(\mathcal{NH}_a) \cong \mathbb{Z}[q, q^{-1}]$.*

2.3 The categorification of $\mathbf{U}_q(\mathfrak{sl}_2)^+$.

Our final objective in the chapter is to briefly explain part of Lauda's work, showing how Schubert polynomials are related to the $\mathfrak{sl}_2$ categorification theorem. For $\mathfrak{sl}_2$ the diagrammatic category that includes the nilHecke ring as a subring is not exactly what categorifies $\mathbf{U}_q(\mathfrak{sl}_2)$. There are several technical steps, taking a subcategory of morphisms that preserve degree and a categorical construction called a Karoubi envelope. Then the Grothendieck group is computed and $K_0(\mathcal{K}(\mathcal{U}))$ gives $\dot{\mathbf{U}}$. As reviewing the entire $\mathfrak{sl}_2$ paper is too much for the present document, we are be concerned in explaining the main role of the nilHecke algebra in the $\mathfrak{sl}_2$ categorification.

We define the positive part of $\mathbf{U}_q(\mathfrak{sl}_2)$ in the following way. The algebra admits a triangular decomposition $\mathbf{U}_q(\mathfrak{sl}_2) \cong U^- \otimes U^z \otimes U^+$ where U^- is generated by F, U^z by K and K^{-1}, and U^+ by E. The positive part is U^+. Our goal is to prove the following theorem.

Theorem 2.3.1 (Khovanov, Lauda [12, 17])**.** *The nilHecke algebra $\mathcal{NH}$ categorifies the positive part of the integral idempotented version of $\mathbf{U}_q(\mathfrak{sl}_2)$.*

We need to introduce two algebras, the positive part of the integral quantum group ${}_{\mathcal{A}}\mathbf{U}_q(\mathfrak{sl}_2)^+$ and the nilHecke algebra $\mathcal{NH}$. The algebra $\mathbf{U}_q(\mathfrak{sl}_2)$ is defined by Lusztig over $\mathbb{Q}(q)$. He also introduced an analog quantum integral version, denoted by ${}_{\mathcal{A}}\mathbf{U}_q(\mathfrak{sl}_2)$, where $\mathcal{A} = \mathbb{Z}[q, q^{-1}]$. This is an integral subalgebra of $\mathbf{U}_q(\mathfrak{sl}_2)$ generated by K, K^{-1}, and the divided powers

$$E^{(n)} = \frac{E^n}{[n]!} \qquad \text{and} \qquad F^{(n)} = \frac{F^n}{[n]!}.$$

By taking account that

$$E^{(a)}E^{(b)} = \begin{bmatrix} a+b \\ a \end{bmatrix} E^{(a+b)} \qquad \text{with} \qquad \begin{bmatrix} a+b \\ a \end{bmatrix} \in \mathcal{A},$$

it is an algebra over $\mathcal{A}$, as mentioned in [18, 4.1,4.2]. The last equation is shown in Lusztig's book [18, 1.3.1.(d)]. The positive part of the integral quantum group is the $\mathcal{A}$ algebra generated by the divided powers. Thus, as a $\mathbb{Z}[q, q^{-1}]$ algebra, $\mathbf{U}_q(\mathfrak{sl}_2)^+ := \langle E^{(a)} | a \in \mathbb{N}\rangle$. This is the same as $\dot{\mathbf{U}}^+$ for $\mathfrak{sl}_2$, where $\dot{\mathbf{U}}$ is the idempotented version of $\mathbf{U}_q(\mathfrak{sl}_2)$ introduced by Lusztig.

We already know that $\mathcal{NH}_a \cong Mat(a!, \Lambda_a)$. We introduce $\mathcal{NH} = \bigoplus_{a\geq 0} \mathcal{NH}_a$.

Definition 2.3.2. A tower of algebras is a collection $\{A_n\}_{n\geq 0}$ of unital algebras over a field k with mappings $\mu_{m,n} : A_m \otimes A_n \to A_{n+m}$ that satisfy $\mu_{l+m,n} \circ (\mu_{l,m} \otimes id_n) = \mu_{l,m+n} \circ (id_l \otimes \mu_{m,n})$. We say $A = (\bigoplus_n A_n, \mu)$ is a tower of algebras.

Lemma 2.3.3. *The nilHecke algebras $\{\mathcal{NH}_a\}_{a\geq 1}$ form a tower of algebras.*

Sketch. We already mentioned before the inclusion $\mathcal{NH}_a \hookrightarrow \mathcal{NH}_{a+1}$. Let the map μ be defined by taking the juxtaposition of diagrams, multiplying coefficients. That is putting one at the side of each other. This gives the desired associative product, as explained in [12]. □

We have the following notable result of Bergeron, though it does not apply directly to the case of $\mathcal{NH}$ because $\mathcal{NH}_a$ is not finite dimensional. We use another result of Bergeron in Chapter 3.

Theorem 2.3.4 (Nantel Bergeron [2])**.** *The Grothendieck group of a tower of finite dimensional algebras has a Hopf algebra structure.*

Theorem 2.3.5 (Khovanov-Lauda [12, 17])**.** *There is a mapping $K_0(\mathcal{NH}) \to {}_{\mathcal{A}}\mathbf{U}_q(\mathfrak{sl}_2)^+$.*

Sketch. Take $\mathcal{NH} = \bigoplus_{a\geq 0} \mathcal{NH}_a$. We calculate $K_0(\mathcal{NH})$. As $\mathcal{NH}$ is a tower of algebras we know it should have a Hopf structure. A projective module for $\mathcal{NH}$ is a projective module splits in a sum of finite projective modules over $\mathcal{NH}_{a_{i_j}}$ for $j = 1, \ldots, s$. These each need to be $\mathcal{NH}_{a_{i_j}}$ finitely generated projective modules. The module category we are interested in is

$$\bigoplus_{a\geq 0} \text{Finitely generated projective modules of } \mathcal{NH}_a.$$

We already know that $K_0(\mathcal{NH}_a) \cong \mathbb{Z}[q, q^{-1}]$ for the graded version of K_0. Thus

$$K_0(\mathcal{NH}) \cong \bigoplus K_0(\mathcal{NH}_a) \cong \bigoplus_{a\geq 0} \mathbb{Z}[q, q^{-1}].$$

The product for K_0 comes from induction functors and the coproduct comes from restriction functors. They give $K_0(\mathcal{NH})$ a bialgebra structure. It is harder to define the antipode. We describe how multiplication is induced through induction functors. The key idea is that we need to multiply the regular way representation such that there is a mapping

$$\mathcal{NH}_a \otimes \mathcal{NH}_b \longrightarrow \mathcal{NH}_{a+b}.$$

Given an $\mathcal{NH}_a$ module M to multiply this module by a $\mathcal{NH}_b$ module M' we need to embed both modules into modules over $\mathcal{NH}_c$ where $c = a + b$ (at least). To embed these modules we can look at the nonunital embedding of unital modules $i : \mathcal{NH}_a \hookrightarrow \mathcal{NH}_b$ and setting $i(1) = e$ an idempotent in $\mathcal{NH}_b$ take

$$M \mapsto \mathcal{NH}_b e \otimes_{\mathcal{NH}_a} M.$$

This functor turns out to have an adjoint, the restriction functor, taking M to eM, viewed as a $\mathcal{NH}_b$ module. With the induction of two modules we can take the tensor product of the modules as $\mathcal{NH}_c$ modules for $c = a + b$.

We show how to categorify the divided powers and to define the mapping of algebras. Consider the regular representation r_a of $\mathcal{NH}_a$, which is isomorphic to $Mat(a!, \Lambda_a)$. Let the column c_a correspond to the first column of the matrix ring as a representation. As the matrix r_a has size $a! \times a!$ from the cardinality of the Schubert basis (or S_a actually) we find the relation $r_a = [a]!c_a$. Define a map from $K_0(\mathcal{NH})$ to ${}_{\mathcal{A}}\mathbf{U}_q(\mathfrak{sl}_2)^+$ as follows. Let

$$r_a \mapsto E^{(a)}; \tag{2.3.1}$$

$$c_a \mapsto E^a. \tag{2.3.2}$$

As a consequence we find the categorified relation $E^a = [a]!E^{(a)}$ for the divided powers of E. $\square$

More details are found in Lauda's work on $\mathfrak{sl}_2$ [16,17]. We leave the curious reader to investigate more about the 2-category for $\mathfrak{sl}_2$.

Remark 2.3.6. The diagrammatic version of the induction functor $\mathcal{NH}_a \hookrightarrow \mathcal{NH}_b \otimes \mathcal{NH}_a$ of a diagram in $\mathcal{NH}_a$ places b strings before the diagram. A similar idea is used to define Hanoi Towers. The result of Bergeron we mentioned also uses induction and restriction functors.

Chapter 3

Schubert polynomials

In this chapter we study the combinatorial definition of Schubert polynomials, there are alternative definitions for them. In Chapter 1 we used the formula $\mathbf{S}_w = \partial_{w^{-1}w_0}(\underline{x}^\delta)$. We work with a combinatorial definition that arises of permutation diagrams called RC graphs, or pipe dreams. We introduce a class of rings called $\mathbb{Z}_+$ rings.

Definition 3.0.1. Let A be an algebra (or ring) that is free as a $\mathbb{Z}$ module, such that there is a basis $\mathcal{B} = \{b_i\}_{i\in I}$ with nonnegative structure constants. That means $b_i b_j = \sum_{\lambda\in I} c_{i,j}^\lambda b_\lambda$ where $c_{i,j}^\lambda \geq 0$ are integers. A ring A that is freely generated over the integers with a $\mathbb{Z}_+$ basis $\mathcal{B}$ that is unital, which means $1 \in A$ is called a $\mathbb{Z}_+$ ring. If $1 \in \mathcal{B}$ we say that A is a $\mathbb{Z}_+$ unital ring.

Some $\mathbb{Z}_+$ rings are important in the theory of quantum groups. They are related to tensor categories and treated in more depth in Etingof's book [7]. The polynomial ring with the basis of Schubert polynomials is a $\mathbb{Z}_+$ unital ring.

Theorem 3.0.2. *The basis of Schubert polynomials $\mathbf{S}_\pi$ for $\pi \in S^\infty$ makes the integer polynomial ring $\mathbb{Z}[x_1, x_2, ..]$ a unital $\mathbb{Z}_+$ ring. That is, the product of Schubert polynomials is a $\mathbb{Z}$ linear combination of Schubert polynomials; the coefficients of this expansion are called structure constants which are non negative integers.*

Remark 3.0.3. The proof of the previous theorem is not combinatorial. The positivity and integrality of structure constants can be proved using algebraic geometry and gives rise to a topic called Schubert calculus. It is related to the cohomology of the Schubert variety and Poincaré Duality.

The nilHecke algebra gives another setting in which they are important.

Theorem 3.0.4. *The Schubert polynomials for S_a give an additive basis of $\mathbb{Z}[x_1, \ldots, x_a]$ as a module over the symmetric polynomials Λ_a.*

The aim of the current chapter is to carry the diagrammatics of the categorified quantum group for $\mathfrak{sl}_2$ to the combinatorial and diagrammatic presentation of Schubert polynomials, which are already present in the categorification theory. For this endeavor we turn to a diagrammatic formula of Bergeron and Billey [1] shown first in *Some combinatorial properties of Schubert polynomials* [4].

3.1 Pipe Dreams and the combinatorial construction

The following is taken from Billey, Jockusch and Stanley's paper *Some combinatorial properties of Schubert polynomials* [4]. The formula presented has a diagrammatic interpretation made possible by a result of Fomin and Kirrilov [8]. That is how RC graphs were introduced. These diagrams represent a history of the inversions of a permutation. The sketch of the following proof describes the proof that appears in [4] but the formula we use is stated in the paper on RC graphs of Billey and Bergeron [1, Theorem 2.1] and in Allen Knutson's notes on Schubert polynomials [13].

Definition 3.1.1. Let $R(\pi)$ denote the set of reduced expressions for π. Let $w \in R(\pi)$ and $w = w_1 \dots w_p$. We say that a p-tuple $\alpha = (\alpha_1, \dots, \alpha_p)$ of strictly positive integers is w-compatible if

$$\begin{aligned} \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_p; & \\ \alpha_i \leq \alpha_j; \qquad & \text{for } 1 \leq j \leq p; \\ \alpha_j < \alpha_{j+1}; \qquad & \text{if } w_j < w_{j+1}. \end{aligned}$$

Let $C(a)$ denote the set of **compatible sequences** associated to a.

Theorem 3.1.2. *Let $\pi \in S^\infty$. We have the positive formula*

$$\mathbf{S}_\pi = \sum_{w \in R(\pi),\, \alpha \in C(w)} x_{\alpha_1} \dots x_{\alpha_p}$$

with the diagrammatic version

$$\mathbf{S}_\pi = \sum_{D \in RC(\pi)} x^D$$

where $RC(\pi)$ is the set of RC graphs for π we define in this section.

Sketch of proof. The proof of Billey, Jockusch and Stanley [4, 1.1,1.2] follows this order. The expression they give is not diagrammatic.

- Define a polynomial R_π that depends on descending initial expressions for w that are $a-$compatible sequences, that is given in the formula.
- Prove the formula of Lauscoux for the polynomial R_π.
- Separate two cases for R_π. We may have that $\pi(1) \neq 1$, assuming that π is the $a-$cycle one reduces the formula to a previous case, as it found in Section 3.4 in this document. Then prove the case for $\pi(1) < \pi(2)$. A computation using the nilCoxeter algebra is required.
- Interpret R_π diagrammatically through RC graphs. Each compatible sequence can be read from one of the diagrams, which are wiring diagrams of a permutation set up in a tiling. ☐

There might be reduced expressions with no compatible sequences. Compatible sequences can be read from the RC graphs.

We describe how to make an RC graph. In the literature they are also called pipe dreams. Let a be a positive number and consider a tiling of size $a \times a$. We draw crosses in some of the boxes such that every cross is above the antidiagonal. In the rest draw elbows so that one can connect the strings to obtain the wiring diagram of a permutation. We want the wiring diagram to be reduced. We label $1, 2, \dots, a$ in the left side of the diagram and follow the string to the top of the diagram, taking i to a position $\pi(i)$. The image of π is in the top of the diagram.

Definition 3.1.3. A **reduced word compatible sequence graph**, or RC graph, for π is a reduced wiring diagram for a permutation set up in a tiling as described above.

Several of these tilings correspond to the same reduced expression and different reduced expressions of a permutation, but always to different compatible sequences. Given a reduced expression Q of π we write D_π or D_Q for an RC graph for it. We can write just D if there is no confusion. Let set of RC graphs of π be denoted as $RC(\pi)$. For brevity we write the configuration of crosses and dots instead of drawing the complete wiring diagram in the RC graph.

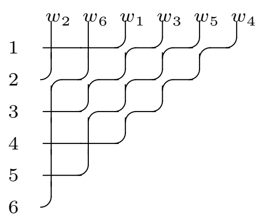
Remark 3.1.4. We describe the notations we use for permutations, as there are several ways to write them. We denote a permutation by $\underline{\pi(1)\dots\pi(n)}$, represent it in the cycle notation by

$$(a_1^1 \dots a_{n_1}^1)(a_1^2 \dots a_{n_2}^2)\dots(a_1^k \dots a_{n_k}^k)$$

in k disjoint cycles, and by $i_1 \dots i_{l(\pi)}$ for a reduced expression.

Example 3.1.5. *This example is from Billey's paper [4]. We draw an RC graph with only crosses and the full diagram. The RC graph drawn with just + and dots contains all the information.*

$$\begin{array}{cccccc} + & + & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \\ + & \cdot & \cdot & \cdot & & \\ + & + & \cdot & & & \\ + & \cdot & & & & \\ \cdot & & & & & \end{array} \tag{3.1.1}$$



(3.1.2)

We introduce an operation in $RC(\pi)$. The following properties are required:

- The operation changes the configuration of crosses.
- The permutation is preserved. If we do an operation such that a cross is moved to a new row where the row is full the permutation is changed.
- We do not create a double crossing. When this happens the expression is not reduced and the diagram is not in $RC(\pi)$.
- We do not run out of space to move the cross a slot up.

Definition 3.1.6 (Operations in RC graphs)**.** A **ladder move** in an RC graph is an operation changing the configuration of crosses in the following way

$$\begin{array}{cc} \cdot & \cdot \\ + & + \\ \vdots & \vdots \\ + & + \\ + & \cdot \end{array} \longrightarrow \begin{array}{cc} \cdot & + \\ + & + \\ \vdots & \vdots \\ + & + \\ \cdot & \cdot \end{array}$$

where the number of rows involved is arbitrary.

A **chute move** in an RC graph is an operation changing the configuration of crosses in the following way

$$\begin{array}{cccccc} \cdot & + & + & \cdots & + & + \\ \cdot & + & + & \cdots & + & \cdot \end{array} \longrightarrow \begin{array}{cccccc} \cdot & + & + & \cdots & + & \cdot \\ + & + & + & \cdots & + & \cdot \end{array}$$

where the number of columns involved is arbitrary.

Proposition 3.1.7. *[1, 3.5] Ladder and chute moves preserve the permutation associated with an RC-graph. Even more, they generate the set of RC graphs of π.*

Example 3.1.8. *In an RC graph the following moves do not preserve the permutation.*

$$\begin{array}{cc} + & \cdot \\ + & \cdot \end{array} \longrightarrow \begin{array}{cc} + & + \\ \cdot & \cdot \end{array} \qquad \begin{array}{cc} \cdot & \cdot \\ + & + \end{array} \longrightarrow \begin{array}{cc} \cdot & + \\ \cdot & + \end{array} \qquad \begin{array}{cc} + & \cdot \\ + & + \\ + & \cdot \end{array} \longrightarrow \begin{array}{cc} + & + \\ + & + \\ \cdot & \cdot \end{array}$$

Remark 3.1.9. The proof of 3.1.7 and of the example are given by inspection. This gives a hint of how RC graph moves work, which is topological in the sense that a move preserves a permutation if it locally preserves it. This lemma allows us to translate the algorithm for generating pipe dreams. Both moves represent isotopies of the wiring diagram and the other moves change the permutation or create an unreduced RC graph, which represents another permutation also.

We follow the next convention when drawing an RC graph. Write the reduced expression and the permutation in cycle notation each time we make a diagram. We label the corner in order to see how the diagram is placed. To read the action of π in the ordered tuple $(1, \ldots, a)$ then we label the column $1, \ldots a$ and the expression w_i for $i = 1, \ldots, a$ in the top of the diagram.

Example 3.1.10. *We list RC graphs for S_2 and S_3. We have first the only element in S_1.*

$$\begin{array}{cc} () & \emptyset \\ NW & 1 \\ 1 & . \end{array}$$

Then the first transposition.

$$\begin{array}{ccc} (12) & 2 & 1 \\ NW & 1 & 2 \\ 1 & + & . \\ 2 & . & \end{array}$$

We list S_3 elements remaining, which are $(23), (123), (132), (13)$.

$$\begin{array}{cccc} (23) & [2] & & \\ NW & 1 & 3 & 2 \\ 1 & . & . & . \\ 2 & + & . & \\ 3 & . & & \end{array} \qquad \begin{array}{cccc} (23) & & & \\ NW & 1 & 3 & 2 \\ 1 & . & + & . \\ 2 & . & . & \\ 3 & . & & \end{array}$$

$$\begin{array}{cccc} (13) & [121] & & \\ NW & 3 & 2 & 1 \\ 1 & + & + & . \\ 2 & + & . & \\ 3 & . & & \end{array}$$

(123)	[21]		
NW	3	1	2
1	+	+	.
2	.	.	
3	.		

(132)	[12]		
NW	2	3	1
1	+	.	.
2	+	.	
3	.		

Remark 3.1.11. Similarly, reduced expressions can be obtained by performing moves on expressions. Observe there is a distinguished diagram that has more crosses to the left.

Definition 3.1.12. We say an RC graph **flushes left** or is **flush left** if the crosses(+) are stacked in each row from the begging, read from left to right, starting at column one and on, in all of the rows of the RC graph.

Theorem 3.1.13. *Let Q be a reduced expression for π. Other reduced expressions are obtained by making the following changes in the expression:*

1. *Switching adjacent numbers ij such that $|i - j| > 1$, that is they are not consecutive.*

2. *Swiching $i(i+1)i$ for $(i+1)i(i+1)$.*

Proof. We recall that the symmetric group S_a is finitely presented by the adjacent transpositions s_i for $i = 1, \ldots, a-1$ and satisfy the relations:

$$s_i s_j = s_j s_i \text{ for } |i-j| > 1 \qquad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$$ □

To each RC graph we associate a monomial. This allows us to calculate the Schubert polynomial.

Definition 3.1.14. Let D be a reduced RC graph of a permutation $\pi \in S_a$. The monomial x^D associated to an RC graph is defined as follows. Let c_i be the number of crosses in the i-th row of the reduced RC graph D. Then $x^D := \prod_i x_i^{c_i}$ for $i \in \{1, \ldots, a\}$.

Remark 3.1.15. This is well defined for $\pi \in S^\infty$. Let $\pi \in S_k$. If m is the biggest entry permuted by π then we have that $\pi(k) = k$ for $k > m$ and this rows have no crosses on D. In this case we can take a subdiagram D' such that D' is an RC graph of a permutation in S_m and $x^{D'} = x^D$.

Example 3.1.16. *The maximal RC graph, the one with the most + is the RC graph with the permutation of longest length, which is the order reversing permutation. The order reversing permutation might be the product of disjoint cycles, for example $(14)(23)$ which has the longest length in S_4.*

Lemma 3.1.17. *The number of inversions of the permutation π corresponds to the number of + in a RC graph P_w of a reduced expression w of π. That is, the number of + in the RC graph is the length of π. This is also the degree of the monomial m_P of the RC graph, which is a homogeneous element.*

Example 3.1.18. *Multiplication table of S_4.*

length	*cyclefactorization*		
1	()		
2	(12)	(23)	(34)
	(13)	(14)	(24)
3	(123)	(132)	(124)
	(142)	(134)	(143)
	(234)	(243)	
4	(12)(34)	(13)(24)	(14)(23)
	(1234)	(1243)	(1342)
	(1324)	(1432)	(1432)

Example 3.1.19. *RC graphs for S_4. We already have those for S_3, so we only need to do 18 permutations more, and diagrams for each reduced expression for them. We start with length 1 and the longest permutation* (24)*.*

(24)				
NW	1	4	3	2
1	.	.	.	.
2	+	+	.	
3	+	.		
4	.			

(24)				
NW	1	4	3	2
1	.	+	.	.
2	+	+	.	
3	.	.		
4	.			

(24)				
NW	1	4	3	2
1	.	.	+	.
2	+	.	.	
3	+	.		
4	.			

(24)				
NW	1	4	3	2
1	.	+	+	.
2	.	.	.	
3	+	.		
4	.			

(24)				
NW	1	4	3	2
1	.	+	+	.
2	.	+	.	
3	.	.		
4	.			

(14)	32123			
NW	4	2	3	1
1	+	+	+	.
2	+	.	.	
3	+	.		
4	.			

(34)	3			
NW	1	2	4	3
1	.	.	.	.
2	.	.	.	
3	+	.		
4	.			

(34)	3			
NW	1	2	4	3
1	.	.	.	.
2	.	+	.	
3	.	.		
4	.			

(34)	3			
NW	1	2	4	3
1	.	.	+	.
2	.	.	.	
3	.	.		
4	.			

Length 2 permutations.

(234)	32			
NW	1	4	2	3
1	.	.	.	.
2	+	+	.	
3	.	.		
4	.			

(234)				
NW	1	4	2	3
1	.	.	+	.
2	+	.	.	
3	.	.		
4	.			

(243)				
NW	1	4	2	3
1	.	+	+	.
2	.	.	.	
3	.	.		
4	.			

(243)	23			
NW	1	3	4	2
1	.	.	.	.
2	+	.	.	
3	+	.		
4	.			

(243)				
NW	1	3	4	2
1	.	+	.	.
2	.	.	.	
3	+	.		
4	.			

(234)				
NW	1	4	2	3
1	.	+	.	.
2	.	+	.	
3	.	.		
4	.			

(124)	3213			
NW	4	1	3	2
1	+	+	+	.
2	.	.	.	
3	+	.		
4	.			

(124)				
NW	4	1	3	2
1	+	+	+	.
2	.	+	.	
3	.	.		
4	.			

(142)	1323			
NW	2	4	3	1
1	+	.	.	.
2	+	+	.	
3	+	.		
4	.			

(142)				
NW	2	4	3	1
1	+	.	+	.
2	+	.	.	
3	+	.		
4	.			

(134)	3212			
NW	4	2	1	3
1	+	+	+	.
2	+	.	.	
3	.	.		
4	.			

(143)	2123			
NW	3	2	4	1
1	+	+	.	.
2	+	.	.	
3	+	.		
4	.			

Length 2, 2 disjoint cycles.

(12)(34)	13			
NW	2	1	4	3
1	+	.	.	.
2	.	.	.	
3	+	.		
4	.			

(12)(34)	13			
NW	2	1	4	3
1	+	.	.	.
2	.	+	.	
3	.	.		
4	.			

(12)(34)	13			
NW	2	1	4	3
1	+	.	+	.
2	.	.	.	
3	.	.		
4	.			

(13)(24)	2132			
NW	3	4	1	2
1	+	+	.	.
2	+	+	.	
3	.	.		
4	.			

Order reversing permutation of length 6.

(14)(23)	321323			
NW	4	3	2	1
1	+	+	+	.
2	+	+	.	
3	+	.		
4	.			

Length 3, we have the 4 cycles.

(1234)	321			
NW	4	1	2	3
1	+	+	+	.
2	.	.	.	
3	.	.		
4	.			

(1243)	213			
NW	3	1	4	2
1	+	+	.	.
2	.	.	.	
3	+	.		
4	.			

(1243)	213			
NW	3	1	4	2
1	+	+	.	.
2	.	+	.	
3	.	.		
4	.			

(1324)	132			
NW	3	5	2	4
2	+	.	.	.
3	+	+	.	
4	.	.		
5	.			

(1324)	2	4	1	3
NW	1	2	3	4
1	+	.	+	.
2	+	.	.	
3	.	.		
4	.			

(1432)	123			
NW	2	3	471	
1	+	.	.	.
2	+	.	.	
3	+	.		
4	.			

And to finish, the length 5 permutations.

(1342)	32132			
NW	4	3	2	1
1	+	+	+	.
2	+	+	.	
3	.	.		
4	.			

(1423)	21323			
NW	3	4	2	1
1	+	+	.	.
2	+	+	.	
3	+	.		
4	.			

3.2 Bumping and Schur polynomials

The reason to introduce Schur polynomials is the discussion of the Schensted insertion for RC graphs in Section 3.2.1 and the material of Sections 3.3 and 3.4. Let a be a positive integer. The ways to write $n = \sum_i k_i$ for $0 < k_i \leq a$ in decreasing order are called the partitions of a number, and one of these sums is a partition of the number a. A partition has size k if it has k summands.[1] We associate polynomials to partitions. The theory of Schur polynomials and its combinatorics is beautifully presented in Fulton's *Young Tableaux, with applications to Representation Theory and Geometry* [9]. Given a partition of a number of size k we can associate a diagram to it as follows.

Definition 3.2.1. A semi standard Young tableaux (SSYT) of a partition $[\lambda_1, \dots, \lambda_a]$ in **a parts** is the set of boxes $\{(i,j) : j \leq \lambda(i)\}$ with numbers such that the labeling is weakly increasing in columns and strictly increasing in rows.

Example 3.2.2. *The SSYT for the partition* $\lambda = [5,3,1]$ *are:*

11111	11112	11122	11222	12222	12223	12233
222	222	222	223	233	233	233
3	3	3	3	3	3	3

We want to associate a monomial to the tableau τ.

Definition 3.2.3. Given a tableau τ define by $\alpha(i)$ the number of times that i appears in the tableau. In this way $\alpha(i) : \mathbb{N} \to \mathbb{N}$ is a function that is zero for $i > a$, where a is the number of rows in the tableau. We can associate a monomial to τ by $x^\tau := x_1^{\alpha(1)} x_2^{\alpha(2)} \dots x_n^{\alpha(a)}$. The Schur polynomial s_λ is defined by the formula

$$s_\lambda = \sum_\tau x^\tau$$

where the sum runs over all SSYT τ for λ.

Definition 3.2.4. Given $a > 0$, an integer, we define the **complete homogeneous polynomials** by

$$h_k(x_1, \dots, x_a) = \sum_{1 \leq l_1 \leq \dots \leq l_k \leq a} x_{l_1} \dots x_{l_k}.$$

and the **elementary symmetric polynomials** by

$$e_k(x_1, \dots, x_a) = \sum_{1 \leq l_1 < \dots < l_k \leq a} x_{l_1} \dots x_{l_k}.$$

Remark 3.2.5. Both families of polynomials are bases for the symmetric polynomial ring.

Lemma 3.2.6. *All complete homogeneous polynomials are Schur polynomials.*

Proof. The array of n 1's, 11111...1, defines a tableau that can be increassed weakly with no restriction imposed by the second row and covers all possible combinations, thus the polynomial is completely homogeneous. This is the shape $\lambda = (a)$. □

Lemma 3.2.7. *All elementary symmetric polynomials are Schur polynomials.*

[1] How many partitions are there for a given number n?

Proof. We find the appropriate tableau. Allow letters from $\{1, \dots, a\}$ and consider the shape $\lambda = (1, 1, 1, \dots, 1)$, an $m-$tuple. Then fill up the numbers strictly decreasing from the top starting with 1. We can have another strictly decreasing sequence by adding 1 to each entry until the last one is a. This gives each monomial of an elementary symmetric polynomial $e_m(x_1, \dots, x_a)$. □

Lemma 3.2.8. *The polynomial defined by taking the sum of the monomials of all tableaux for a partition λ, $s_\lambda = \sum_\tau x^\tau$ is a symmetric polynomial.*

Theorem 3.2.9. *The set of Schur polynomials in a parts is an additive basis for the symmetric polynomial ring.*

The Littlewood-Richardson rule of multiplication of Schur polynomials uses SSYT, and skew shapes λ/μ for tableaux $\mu \subset \lambda$. These skew shapes are also called skew tableaux. They encode the product of Schur polynomials. Two operations are involved. These are the Schensted insertion and the Schützenberger slide. The first is the inspiration for the RC graphs insertion for Schubert polynomials of Billy and Bergeron. The slide operation is similar to the *jeu de taquin*, the French name of a game in which one slides tiles around on a board of tiles that has only one free space. We describe the Schensted insertion, which we refer to as bumping, as Fulton does. The following is taken from Fulton's book on Young Tableaux [9].

Let x be a positive integer and T a tableau. We insert x into T getting a tableau $T \leftarrow x$. The algorithm is:

1. Bump x into the first row. For this, look for a number as large as all the entries in the first row, and add x in a new box at the end.

2. This may not happen. If not, find the left most entry in the first row that is strictly larger than x. Bump this entry with x and take the new entry instead.

3. Do this procedure in the next row with the new entry until it stops.

We claim that the new diagram is a SSYT. As in each row we place the bump where it suits the weakly increasing order, columns are weakly increasing. We move the bigger element down, so a bump is placed below the strictly bigger element making the rows strictly increasing. This defines a SSYT. The path of the bumps define a bumping route. A route R can be strictly to the left (weakly left) of a route R' if in each row which contains a box of R', R has a box which is left of (left or equal to) the box of R'. This is an issue with the insertion algorithm in RC graphs, mentioned in Billey and Bergeron's article, *RC graphs and Schubert polynomials* [1]. In the example they show the path of consecutive insertions might fail to remain weakly to the right of the first insertion, as in the case of $s_{\underline{12543}}\ s_{\underline{12453}}$.

Lemma 3.2.10. *Let consecutive row insertions $T \leftarrow x$ and $(T \leftarrow x) \leftarrow x'$ which give rise to routes R and R' be given. If $x \leq x'$ then R is strictly to the left of R' and if $X > x'$ then R' is weakly to the left of R.*

Definition 3.2.11. (Skew shape) A skew shape or diagram is obtained by removing a smaller Young diagram from a larger one. If $\lambda = (\lambda_1, \lambda_2, \dots)$ and $\mu = (\mu_1, \mu_2, \dots)$ then we have $\mu \subset \lambda$ if the diagram of μ is contained in that of λ. This means that $\mu_i \leq \lambda_i$. The resulting skew shape is denoted λ/μ, which consists of the remaining boxes.

Theorem 3.2.12. *The multiplication of Schur polynomials is determined by tableaux in the following way. Let λ and μ be partitions such that s_λ and s_μ are their Schur polynomials. Then $s_\lambda s_\mu = \sum_\pi c^\pi_{\lambda,\mu} \mathbf{S}_\pi$ where the number $c^\pi_{\lambda,\mu}$ only depends on λ, μ and π, such that $|\lambda| + |\mu| = |\pi|$ and π contains both tableaux.*

There are variants of the bumping procedure reviewed in the appendix A of Fulton's book [9]. For example, we can bump columns also. Bumpings in SSYT have inverse operations also, given some not too stringent conditions.

3.2.1 Bumping an RC graph

We bump RC graphs in Fulton's style as was done by Billey, Bergeron and Stanley. Here is an example of how to bump RC graphs of π with one cross(+), to get an RC graph of a permutation with length $l(\tau) = l(\pi) + 1$. Suppose the RC graph is flush left. Then bumping is given just by including one more cross in the end of the row.

Example 3.2.13. *The diagram* $\begin{array}{lll} + & + & . \\ + & . & \\ . & & \end{array}$ *when bumped in the second row gives* $\begin{array}{llll} + & + & . & . \\ + & + & . & \\ . & . & & \\ . & & & \end{array}$

which is a permutation in S_4 and not in S_3.

Lemma 3.2.14. *Bumping a flush left RC graph is invertible. Bumping a flush left RC graph gives a flush left RC graph.*

Bumping is more involved if the diagram is not flush left. If we insert a cross one could be lucky to obtain an RC graph. In case we do not the RC graph corresponds to an unreduced wiring diagram of a permutation of smaller length. We repair this. One can flush left the crosses in the double crossing, in an ordered fashion, until one gets a well posed RC graph. The path of the modifications is the insertion path. Insertion paths are known not to be well behaved in consecutive insertions. This gives an obstacle for the generalization of Monk's rule to arbitrary multiplications. The procedure introduced by Billey and Bergeron makes the RC graphs a monoid. It gives an elegant proof of Monk's rule [1]. This is an example similar to the one in the paper.

Example 3.2.15. *The insertion path for the insertion shown is (3,1), (1,3), (1,2).*

$$\begin{array}{lll} + & . & + \\ . & . & \\ . & & \end{array} \quad \rightarrow \quad \begin{array}{llll} + & . & + & . \\ . & . & . & \\ + & . & & \\ . & & & \end{array} \quad \rightarrow \quad \begin{array}{llll} + & + & . & . \\ . & . & . & \\ + & . & & \\ . & & & \end{array}$$

Lemma 3.2.16. *Let i be the greatest nonempty row in a flush left RC graph D. Denote by X a full column of crosses, from 1 to i, such that if we bump $D \leftarrow X$ we get another flush left RC graph D'. This operation is invertible and the family of RC graphs of the permutation that represents D is the same cardinality as the one of D'.*

Theorem 3.2.17. *The family of RC graphs is a monoid.*

The identity is the empty diagram. The multiplication is given by bumping. If we bump the RC graphs of σ with the RC graphs of π we should get the same as when bumping the RC graphs of π with those of σ. We are set to define a commutative operation $\pi * \sigma$.

3.3 Reinterpreting pipe dreams

In this section we introduce the diagrammatic method for Schubert polynomials. We make a slight modification that allows us to perceive the RC graphs as compatible with the diagrammatics of Khovanov and Lauda. We understand diagrammatically the Lehmer code, the permutation, and the largest monomial in the lexicographic order of the Schubert polynomial in a natural way. The diagram we introduce is related to the flush left RC graph that appears in the theory. Schubert polynomials are defined through the action of the nilCoxeter algebra on the ring of polynomials $\mathbb{Z}[\bar{x}_a]$. The formula is $\mathbf{S}_\pi = \partial_{\pi^{-1}w_0}(\underline{x}^\delta)$ for $\pi \in S_a$ where $\underline{x}^\delta = x_1^n x_2^{n-1} \dots x_a$. Here is the diagrammatic version of Stanley's formula of Schubert polynomials, which uses the family of RC graphs of π.

Theorem 3.3.1 ([1, 13]). *Let π be a permutation. Let P denote a reduced RC graph and π_P the permutation it represents. The Schubert polynomial for π equals*

$$\mathbf{S}_\pi := \sum_{P \in RC(\pi)} \prod_i x_i^{\#+ \text{ in } i\text{-th row of } P}.$$

Example 3.3.2 (Schubert polynomials for n=2 and s_{σ_i}). *We only have two RC graphs. One is the identity which has no crossings and $\sigma_1 = (12)$ with only a crossing in the northwest corner, the only place to put a cross. The Schubert polynomials are $\mathbf{S}_{()} = 1$ and $\mathbf{S}_{(12)} = x_1$, respectively.*

Let $\sigma_i = (i, i+1)$ the adjacent transposition. Let us prove that $\mathbf{S}_{\sigma_i} = \sum_{k=1}^{i} x_k$. The flush left RC graph is the diagram with one cross on the first column, $i-$th row. We can do ladder moves until we move the cross up to the first row. That is, we get P_i, $P_{i-1}, \dots P_1$, a sequence of diagrams where P_k is obtained by a ladder move on P_{k+1}. These RC graphs all have one cross in the i-row so their monomial is x_i. Adding up gives the sum we want. Observe that they only have one crossing and that these ladder moves preserve the permutation.

Example 3.3.3 (Schubert polynomials for n=3). *The generators of S_3 are* $(12) := 1$ *and* $(23) := 2$. *We have the following table.*

length	*permutation*	*cycle factorization*	*expression*
0	$\underline{123}$	()	$\emptyset$
1	$\underline{213}$	(12)	1
1	$\underline{132}$	(23)	2
2	$\underline{312}$	(123)	12
2	$\underline{231}$	(132)	21
3	$\underline{321}$	(13)	$121, 212$

Observe how the braid move $121 = 212$ *gives equivalent expressions. We omit the corresponding table for S_4 as it is too big. The polynomials are*

$$\mathbf{S}_{(23)} = x_1 + x_2; \tag{3.3.1}$$
$$\mathbf{S}_{(123)} = x_1^2; \tag{3.3.2}$$
$$\mathbf{S}_{(132)} = x_1 x_2; \tag{3.3.3}$$
$$\mathbf{S}_{(13)} = x_1^2 x_2. \tag{3.3.4}$$

Example 3.3.4. *(Products of $\mathbf{S}_\pi$ for $\pi \in S_3$).*

$$\begin{array}{rcl}
\mathbf{S}_{(12)}\mathbf{S}_{(23)} = & x_1(x_1+x_2) = x_1^2 + x_1x_2 & = \mathbf{S}_{(123)} + \mathbf{S}_{(132)}; \\
\mathbf{S}_{(12)}\mathbf{S}_{(123)} = & x_1x_1^2 = x_1^3 & = \mathbf{S}_{(1234)}; \\
\mathbf{S}_{(12)}\mathbf{S}_{(132)} = & x_1(x_1x_2) = x_1^2x_2 & = \mathbf{S}_{(13)}; \\
\mathbf{S}_{(12)}\mathbf{S}_{(13)} = & x_1(x_1^2x_2) = x_1^3x_2 & = \mathbf{S}_{(134)}; \\
\mathbf{S}_{(23)}\mathbf{S}_{(123)} = & (x_1+x_2)x_1^2 = x_1^3 + x_1^2x_2 & = \mathbf{S}_{(1234)} + \mathbf{S}_{(13)}; \\
\mathbf{S}_{(23)}\mathbf{S}_{(132)} = & (x_1+x_2)x_1^2 = x_1^2x_2 + x_1^3 & = \mathbf{S}_{(13)} + \mathbf{S}_{(1234)}; \\
\mathbf{S}_{(23)}\mathbf{S}_{(13)} = & (x_1+x_2)x_1^2x_2 = x_1^3x_2 + x_1^2x_2^2 & = \mathbf{S}_{(1324)} + \mathbf{S}_{(13)(24)}; \\
\mathbf{S}_{(13)}\mathbf{S}_{(123)} = & (x_1^2x_2)x_1^2 = x_1^4x_2 & = \mathbf{S}_{(1345)}; \\
\mathbf{S}_{(13)}\mathbf{S}_{(132)} = & x_1^2x_2(x_1x_2) = x_1^3x_2^2 & = \mathbf{S}_{(13)(14)}; \\
\mathbf{S}_{(13)}\mathbf{S}_{(132)} = & x_1^2x_2(x_1x_2) = x_1^3x_2^2 & = \mathbf{S}_{(13)(14)}; \\
\mathbf{S}_{(123)}\mathbf{S}_{(132)} = & x_1^2(x_1x_2) = x_1^3x_2 & = \mathbf{S}_{(134)}.
\end{array}$$

Example 3.3.5. *The list of Schubert polynomials for the missing elements in S_4 are:*

$$\begin{array}{rcl}
\mathbf{S}_{14} & = & x_1^3x_2x_3; \\
\mathbf{S}_{(24)} & = & x_1^2x_2 + x_1^2x_2 + x_1^2x_3 + x_1x_2x_3 + x_2^2; x_3 \\
\mathbf{S}_{(34)} & = & x_1 + x_2 + x_3; \\
\mathbf{S}_{(124)} & = & x_1^3x_3 + x_1^3x_2; \\
\mathbf{S}_{(142)} & = & x_2x_3(x_1x_2 + x_1^2) = x_1x_2^2x_3 + x_1^2x_2x_3; \\
\mathbf{S}_{(234)} & = & x_1^2 + x_1x_2 + 2^x; \\
\mathbf{S}_{(243)} & = & x_1x_2 + x_1x_3 + x_2x_3; \\
\mathbf{S}_{(12)(34)} & = & x_1^2 + x_1x_2 + x_1x_3; \\
\mathbf{S}_{(13)(24)} & = & x_1^2x_2^2; \\
\mathbf{S}_{(14)(23)} & = & x_1^3x_2^2x_1; \\
\mathbf{S}_{(1234)} & = & x_1x_2x_3; \\
\mathbf{S}_{(1243)} & = & x_1x_2^2x_1^2x_2; \\
\mathbf{S}_{(1342)} & = & x_1^2(x_2+x_3) = x_1^2x_2 + x_1^2x_3; \\
\mathbf{S}_{(1324)} & = & x_1x_2^2 + x_1^2x_2; \\
\mathbf{S}_{(1432)} & = & x_1^3; \\
\mathbf{S}_{(1423)} & = & x_1^3x_2^2.
\end{array}$$

Example 3.3.6 (x_i and x_1^n)**.** *We look for an expression for x_i and x_1^n*

- *Remember $\mathbf{S}_{(i,i+1)} = \sum_{k=1}^{i} x_i$. This is precisely $\mathbf{S}_{(12)}$, the RC graph with a $+$ on the northwest corner. To get x_i for $i > 1$ we need to subtract, $\mathbf{S}_i - \mathbf{S}_{i-1} = x_i$.*
- *The polynomial x_1^n corresponds to the permutation $\underline{23\ldots(n-1)n1}$.*

Definition 3.3.7 (Lehmer code [13])**.** Let $\pi \in S_a$. The Lehmer code of π is the list of a numbers $c_\pi(i) := \#\{i > j : \pi(j) < \pi(i)\}$ with $c_\pi(i) \in \{0, \ldots, a-i\}$. For example, the Lehmer code of $\underline{15423}$ is 03200.

Example 3.3.8. *The table following are the Lehmer codes for S_3.*

Permutation	$\underline{123}$	$\underline{213}$	$\underline{132}$	$\underline{312}$	$\underline{231}$	$\underline{321}$
Lehmer code	000	100	010	110	200	210

Lemma 3.3.9. *Let $\pi \in S_a$.*

1. *The Lehmer code of π always ends in zero.*
2. *A permutation is uniquely determined by its Lehmer code.*
3. *Adding a zero at the end of the Lehmer code does not change the permutation, it represents the same permutation inside S_{a+1}.*
4. *Given a Lehmer code c, and given any other code such that it equals c with more zeroes at the right, the code still represents a permutation with the same action as the original permutation.*
5. *Let c be a sequence that is eventually zero. We can choose $\pi \in S^{\infty}$ such that c is the Lehmer code of π. Actually, $\pi \in S_a$ where $i + c(i) \leq a$.*
6. *There is an equivalence relation induced by Lehmer codes on sequences that are eventually zero.*
7. *The sum $\sum_i c(i)$ equals $l(\pi)$.*

Proof. Using the notation $\underline{\pi^{-1}(1)\dots\pi^{-1}(n)}$, the last number in the permutation cannot be bigger that any other before. Thus, the Lehmer code ends in zero. We argue a permutation is uniquely determined by a Lehmer code. The permutation cannot have a wiring diagram with less than $c(i)+i$ strings. Let $a \geq i + c(i)$ for every i with $c_\pi(i) \neq 0$. We build a wiring diagram for π.

Let $c = c(1)c(2)\dots$ be the Lehmer code of π. Then $c(1)$ is the number of times string one crosses bigger strings, $c(2)$ the number times string two crosses bigger strings, and so on. To be precise we build the corresponding RC graph. A crossing of string one produces a cross in the first row, placed on the first entry. The same for the second with a slight change. If the crossing involves string one, then this crossing was already taken in account. In general in each row we add a cross left-most (in the first column) if the string i crosses some string k with $k > i$. This defines a flush left RC graph, which means it does not admit a chute move. This is the flush left RC graph of π. We prove a key statement to complete the argument.

We claim a flush left RC graph gives an reduced wiring diagram for π. We show that $l(\pi)$ equals the number of crosses in the flush left diagram we have described. Take any string i and suppose it crosses other strings. There are only two cases. If it crosses a string $j > i$ then there are crosses in row i until this happens, as it is flush left. If it crosses a string $j < i$ then it the cross was already considered in row j. The flush left RC graph of a permutation defines it completely. We can see the last part here. As each crossing is needed, the number of crosses in each row is $c(i)$ and the number of inversions is the number of crosses. Therefore $l(\pi) = \sum c(i)$.

When we add a zero at the end we see π as included in the next symmetric group of size $a+1$. As we do not permute the last letters, set $c(i) = 0$, for $i > a$. This could be carried on to identify π with the same permutation in any S_b with $b > a$. This is precisely the quotient identification for S^{∞}. So sequences that are eventually zero identify the equivalence class of π in S^{∞}. ☐

We can also exhibit the inverse of the permutation, as in terms of RC graphs it is given by the transpose of the graph. That is, take the transpose and flush it left to get the flush left RC graph of π^{-1}. This also proves that $l(\pi) = l(\pi^{-1})$. In the last lemma we can observe that the Lehmer code of π and the flush left RC graph of π are related. By the previous lemma we know that the diagram we build is flush left. Now, notice that there was no ambiguity in the process so the flush left diagram is unique. If we do not have a flush left diagram then we can move a cross down and to the left, which is the chute or ladder move, so any other RC graph of π has a monomial that is strictly smaller in the lexicographic order.

Lemma 3.3.10. *Let D be a flush left RC graph of π. The following hold.*

1. *The flush left RC graph is the only diagram of π whose crosses flush left.*
2. *An RC graph that is not flush left admits a chute or ladder move.*
3. *The flush left RC graph encodes largest monomial in the lexicographic order of $\mathbf{S}_\pi$.*

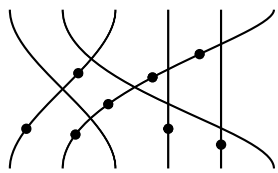
Theorem 3.0.2 is an important theorem concerning Schubert polynomials as a basis of $\mathbb{Z}[x_1, x_2, \dots]$ as a $\mathbb{Z}$ module. We show the proof given in Knutson's notes.

Proof of Theorem 3.0.2. By lemma 1.5.3 we know that Schubert polynomials are linearly independent. Let $p \in \mathbb{Z}[x_1, x_2, \dots]$ and let m be its largest monomial in the lexicographic order. We choose the permutation π such that the Lehmer code of π, c_π, gives the powers of the variables in the monomial m. Choose $\pi \in S^\infty$ such that $\deg(p) - \deg(a_\pi \mathbf{S}_\pi) < \deg(p)$, with $a_\pi \in \mathbb{Z}$, and continue the division inductively. □

We find a diagrammatic way of making the assignment $m \to c_\pi$ and $m \to \pi$ very precise. The code c_π gives the number of crosses in row i in the flush left RC graph of π as shown in Lemma 3.3.9. This is the motivation of the following definition that does not involve RC graphs but the usual string diagrams. We translate the RC graphs $RC(\pi)$ of a permutation into a family of monomials in the nilHecke algebra. We do not worry about the isotopies that the diagram has, which is rather technical in the RC graph approach. The flush left RC graph is represented by a 0-Hecke element in $\mathcal{NH}_a$.

Definition 3.3.11. An **irreducible pipe dream** π is a 0-Hecke monomial with coefficient one such that $w(\pi)$ is a nonzero nilCoxeter element.

Example 3.3.12. *The nilHecke irreducible pipe dream monomial for the permutation $\pi = \underline{361452}$.*



There is a permutation of the a strings attached to the monomial (and viceversa). Let the irreducible pipe dream be denoted by π. In the nilHecke algebra we write $\bar{\pi}$ for the 0-Hecke element but the nilHecke algebra is not discussed in this chapter. By convention we always draw a strings even if they are not permuted.

The dot configuration determined locally by $\partial_i \mapsto \bar{\partial}_i$. We define an **abacus move** which replaces chute/ladder moves. The abacus move generates a set of diagrams through which we can calculate the Schubert polynomial for π.

Lemma 3.3.13. *The irreducible pipe dream does not depend on the reduced expression of π.*

Proof. The underlying wiring diagram for the permutation in the nilHecke irreducible pipe dream of π is well defined from the flush left RC graph for π. From the wiring diagram we obtain a nilCoxeter element. The nilCoxeter generators are nilpotent. But the pipe dream represents a reduced expression, the only possible equivalent expressions are obtained through wiring diagrams by isotopies $\partial_i\partial_j = \partial_i\partial_j$ or by the braid relation. The 0-Hecke elements for these wiring diagrams are also equivalent. The dot configuration for these monomials is the same so the Lehmer code of π is read from the dots of any reduced 0-Hecke element. $\square$

We draw the irreducible pipe dream string diagram from the Lehmer code. Take the monomial x^{c_π} and to each dot we anticipate a crossing, setting $x_i \mapsto \bar{\partial}_i$, as a local relation. Then we connect the strings in a way such that we do not introduce other crossings. There are several wiring diagrams of a permutation. To draw the irreducible pipe dream in a way such that the underlying wiring diagram is the flush left RC graph wiring diagram we can do the following. Start by connecting first the string i such that $\pi(i) = 1$, then draw the string i' such that $\pi(i') = 2$, and so on. Drawing the strands in this order draws the wiring diagram of the flush left RC graph.

There is an issue in notation we mentioned in Section 1.4. The RC graphs are drawn where the initial strand configuration is in the left side and the top of the diagram has the images $\pi(i)$. With an isotopy we carry the left side to the bottom. This is not the convention for the categorified quantum group $U_q(\mathfrak{sl}_2)$. Drawing the wiring diagram upside down gives the inverse permutation. This is equivalent to the transpose RC graph representing π^{-1}. Refer to the drawing convention that has the images of the permutation on top as the RC graph convention and to the usual one the nilHecke convention. To maintain similarity with RC graphs we use the RC graph convention which is obtained through the antihomomorphism σ of Section 1.4.

We continue with our diagrammatic construction.

Definition 3.3.14. A **pipe dream** is a diagram obtained from another pipe dream by an abacus move. This makes sense as irreducible pipe dreams are already given.

Definition 3.3.15. (Untangling map) To each monomial $\tau \in \mathcal{NH}_a$ we assign a monomial $u(\tau) \in \mathcal{P}_a$ by untangling the strings. That is, the powers of x_k in the monomial are obtained by counting the dots in the $k-$th string of the diagram τ. This map can be extended linearly to $\mathcal{NH}_a$. We call $u : \mathcal{NH}_a \to \mathcal{P}_a$ the **untangling map**.

Remark 3.3.16. This is not a homomorphism of algebras. We might have two non zero nilHecke monomials such that their multiplication is zero but their untangled multiplication is clearly not zero.

Corollary 3.3.17. *There is an implicit map that assigns to a monomial a 0-Hecke element. To a Lehmer code c there corresponds a permutation π such that π is the underlying wiring diagram of the natural 0-Hecke element that corresponds to c. This element π satisfies that $u(\pi) = x^D$ for the flush left RC graph D of π.*

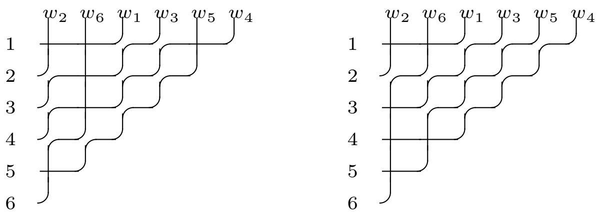
When mapping the monomial to the 0-Hecke element we **knit** the monomial m to get the irreducible pipe dream π. The underlying wiring diagram of π is the wiring diagram of the permutation. We can also say we knit the Lehmer code, by knitting the monomial $\prod_i x_i^{c_\pi(i)}$. This means we have four objects that are directly related to each other, which are: a permutation, a 0-Hecke element, a monomial, and a Schubert polynomial.

Example 3.3.18. *(Also taken form [1]) Consider the following diagram of the permutation $\underline{314652}$. The expression 521345 is obtained by reading the position of each cross starting from row one going down from right to left. If the cross is in the position (i, j) the number it represents is $j + i$. The compatible sequence 111235 is obtained by counting the number of crosses in the corresponding row.*

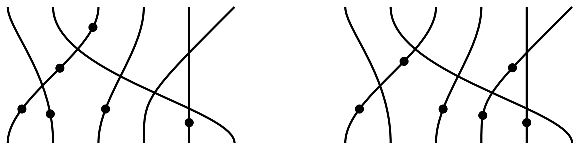
The diagram is depicted below with the flush left RC graph also.

$$\begin{array}{cccccc} + & + & \cdot & \cdot & + & \cdot \\ \cdot & + & \cdot & \cdot & \cdot & \\ \cdot & + & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & & & \\ + & \cdot & & & & \\ \cdot & & & & & \end{array} \qquad \begin{array}{cccccc} + & + & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \\ + & \cdot & \cdot & \cdot & & \\ + & + & \cdot & & & \\ + & \cdot & & & & \\ \cdot & & & & & \end{array}$$

The pipe dream and flush left pipe dream are:



The nilHecke pipe dream and irreducible pipe dream monomials are:



Our objective is to create two coordinates as in the RC graphs to mimic the ladder moves and chute moves.We already have the row coordinates of the crossings encoded in each string, so the strings need to have heights, which we refer to as levels that the dots can occupy. These play the role of columns in the RC graph. Only one dot can occupy a level in a string. A wiring diagram of a permutation in S_a has at most a levels. The $k-th$ string has $a-k+1$ levels. The levels of string i are determined by the strings crossing it. If we see a dot placed just before the k crossing on it, then the dot is in the level k. When we perform an abacus move on the irreducible pipe dream it is convenient to place below the string the levels occupied by dots. We define the abacus move in what follows. We follow the same algorithm that the one for RC graphs, but instead of crosses in the diagrams we have dots in the diagram, and the moves performed are the same. The irreducible pipe dream determines the initial dot configuration.

Definition 3.3.19. An **abacus move** is a ladder move or chute move performed on dots in the strings within the boundaries given by levels in the strings.

We mimic the restrictions for ladder/chute moves in RC graphs. Several things can obstruct an abacus move:

1. The dot reached or is in the biggest level.
2. There is a dot in the same level at its immediate left. The left-most dot in that level has to be moved first, if it is possible.
3. There is a dot in the same string at the next level.
4. There is a dot in the string at the left in the next level, we cannot move the dot to a place already taken.

Recall that in an RC graph $D(w,a)$ the expression and the compatible sequence are read from the diagram. The expression is read by the position of the crosses in the columns read in order, from top to bottom. In the pipe dream monomial we can find the expression by reading the levels in decreasing order in each string, always reading strings from left to right. The monomial of the compatible sequence is obtained by multiplying x_i once for each cross in the i-th row, in the case of RC graphs. In the pipe dream monomial is obtained by adding a power of x_i once for each dot in the i-th string. This is stated as follows.

Lemma 3.3.20. *The position of the dots in the strings of a pipe dream correspond to a compatible sequence of a reduced expression of π. The levels of dots in a pipe dream read in decreasing order from the first string to the last string correspond to the reduced expression of the pipe dream. The length of the permutation is the number of dots in the monomial.*

We recall that not every reduced expression has compatible sequences, sometimes there is no pipe dream corresponding to a reduced expression.

Lemma 3.3.21. *The reduced expression and compatible sequence corresponding to an irreducible pipe dream is the highest weighted compatible sequence among all compatible sequences of the permutation.*

Proof. By the construction we just have to argue that the wiring diagram of the irreducible pipe dream itself corresponds to the flush left RC graph of π. Consider a wiring diagram given by the RC graph as in done in Lemma 3.3.9, which is flush left. Any abacus move, as a chute/ladder move, gives a diagram with a smaller monomial. Then $u(\pi)$ is the largest in the lexicographic order as it has the biggest number of dots in the farthest most string. The monomial $u(\pi)$ is the monomial of the flush left RC graph. $\square$

Remark 3.3.22 (Topological construction of $\bar{\pi}$ from the flush left RC graph). Let D be the flush left RC graph of π. Replace locally ∂_i by $\bar{\partial}_i$, so we have the dots in the same configurations as the $+$ in the RC graph. Move with an isotopy the strings crossing in horizontal direction towards the right side of the columns in the RC graph diagram. Then we can identify the strings with rows in the RC graph and the horizontal crossings with columns, or levels in the pipe dream once the ambient space is changed. That is, once the left side of the diagram is moved downwards so that $\bar{\pi}$ is a wiring diagram as it is usually drawn.

Lemma 3.3.23. *Let D be an RC graph and D' be another RC graph obtained from a chute or ladder move on D. There is a pipe dream τ that corresponds to D or D' such that τ' corresponds to the remaining RC graph and τ' is obtained from τ by an abacus move.*

Proof. Let D be flush left. By the construction given a ladder/chute move on the RC graph can happen if and only if an abacus move is admissible. The diagram D corresponds to an irreducible pipe dream τ. Performing the same sequence of moves gives a pipe dream τ' such that $u(\tau') = x^{D'}$. This is enough to assert the lemma. $\square$

Lemma 3.3.24. *Let D be any RC graph of a given permutation. There exists a pipe dream τ obtained by abacus moves from an irreducible pipe dream π such that $u(\tau) = x^D$.*

Proof. Let D be an RC graph with expression w and compatible sequence a. It is obtained by moves from the flush left RC graph D. Then $D(w,a)$ can be assigned a pipe dream by making the same abacus moves made to get $D(w,a)$ from the flush left RC graph D. As the configuration of crosses matches the configuration of dots we have that $u(\tau) = x^D$. $\square$

Lemma 3.3.25. *The Schubert polynomial satisfies the formula $\mathbf{S}_\pi = \sum_\tau u(\tau)$ where the sum runs over pipe dreams τ of π.*

Proof. The formula of Stanley in terms of RC graphs[2] is $\mathbf{S}_\pi = \sum_D x^D$. As there is a pipe dream with $x^D = u(\tau)$ by the previous lemma, the last formula is rewritten as $\mathbf{S}_\pi = \sum_D x^D = \sum_\tau u(\tau)$ where the sum goes over pipe dreams τ of π. □

Theorem 3.3.26. *To each irreducible pipe dream there corresponds a Schubert polynomial.*

Definition 3.3.27. The set of pipe dreams of a permutation π, which includes π, obtained by abacus moves, is **the abacus of** π. This set is denoted $\mathcal{A}_\pi$ and its cardinality by $a(\pi)$.[3]

Definition 3.3.28. A permutation is **dominant** if $a(\pi) = 1$.

Lemma 3.3.29. *The Lehmer code of a dominant permutation is non increasing. A permutation with a non increasing Lehmer code is a dominant permutation.*

Proof. Let π have a Lehmer code such that $c_\pi(i) < c_\pi(i+1)$ for some i and the inequality is strict. Then an abacus move can be performed on π and $a(\pi) > 1$. Suppose that $a(\pi) > 1$. For the irreducible pipe dream to allow an abacus move on a given dot we need to have an increasing Lehmer code for an abacus move. If the move takes more space on the left the sequence has to be increasing also. As both moves need space at their left to be allowed, the Lehmer code is not non increasing. □

There is a natural operation in Lehmer codes which preserves $a(\pi)$.

Definition 3.3.30. A permutation is a **step permutation** if the Lehmer code of π is a step function such that $c_\pi(1) > 0$. In this way, for some $m \geq 1$,

$$c_\pi(i) = \begin{cases} n & i \in \{1, \dots, m\} \\ 0 & i > m. \end{cases}$$

The integer m is the size of the step permutation π and n the height of the step permutation.

Remark 3.3.31. Let π be a step permutation as defined above. The total degree $\deg(\mathbf{S}_\pi) = m$ and m is the largest variable with nonzero exponent, $\mathbf{S}_\pi$ is symmetric, and $a(\pi) = 1$.

Lemma 3.3.32. *A step permutation is a dominant permutation. Whenever there is a permutation π such that we can subtract the Lehmer code of a step permutation σ and get a new Lehmer code, that is $c_\pi - c_\sigma \geq 0$, then $\mathbf{S}_\sigma$ divides $\mathbf{S}_\pi$.*

Proof. Let $c_\lambda = c_\pi - c_\sigma$ define λ. Then $a(\lambda) = a(\pi)$, as the abaci produced have the same behavior, and we can factor the symmetric polynomial $\mathbf{S}_\sigma$ from $\mathbf{S}_\pi$ getting $\mathbf{S}_\sigma \mathbf{S}_\lambda = \mathbf{S}_\pi$. □

Another proof uses that the symmetric polynomial $\mathbf{S}_\sigma$ can be factored as $\mathcal{NH}_a$ is a Λ_a module. We can also consider an untangled abacus where we forget the original underlying wiring diagram. However, to build it we cannot leave out the levels on the pipe dream. An untangled pipe dream τ contains more information than $u(\tau)$.

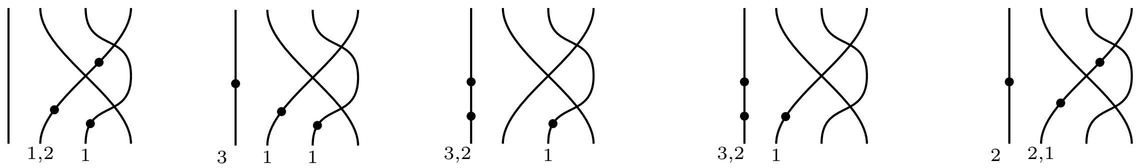
The untangling map is defined for the nilHecke algebra, but it is implicitly defined in a set of monomials if we use the convention that it simply adds them. In order to obtain a polynomial, the set has to be finite. That means, if S is a set of polynomials in the nilHecke algebra, let $i : S \to \mathcal{NH}$ be the inclusion map that adds all elements in S, define $u(S) := u(i(S))$.

Lemma 3.3.24 implies there is an algorithm to obtain the abacus. We can start with the irreducible diagram π and make a graph in the following way. Let the vertices be pipe dreams and connect them if one is obtained from the other through an abacus move. The graph is connected because the irreducible pipe dream generates them.

[2] Or pipe dreams, used with this meaning by other authors, like Knutson [13], but we want to keep the name for nilHecke elements.

[3] The abacus is strictly a set. Each monomial x^D or $u(\tau)$ appears only once in the polynomial.

Example 3.3.33. *Consider the permutation $\underline{1432}$. This is the abacus is obtained through moves making a tree as mentioned before. In this case there are two branches, where one of them has only one diagram given by an abacus braid move.*



This is the whole abacus. In the last diagram, the dot in string two cannot take position 2 in the first string. The Schubert polynomial associated is

$$\mathbf{S}_{\underline{1432}} = u(\mathcal{A}_{\underline{1432}}) = x_2^2x_3 + x_1x_2x_3 + x_1^2x_3 + x_1^2x_2 + x_1x_2^2.$$

The algorithm produces more than a tree. If we add the edges to the repeated monomials in the process we obtain a graph that can be understood as a CW complex and by adding the unreduced RC graphs in between the vertices we can paste two cells that make this 2 skeleton CW complex homeomorphic to a sphere. This apparently has some remarkable properties we skip, but to the curious reader we refer to Knutson and Miller's work [14] or to Escobar and Mézráros paper [6].

Lemma 3.3.34. *Let $\tau \in \mathcal{A}_\pi$. If $\tau(1) = 1$ if and only if $\tau = \pi$. In addition, $\tau(1) = 0$ if and only if $\tau \neq \pi$, that is, $\tau(1) = 0$ in any other case.*

Proof. It is straightforward to see $\pi(1) = 1$. Note that if τ is any other pipe dream, to some crossing there anticipates no dot. If the diagram does not feed any other dot, then the application of this crossing gives zero. Thus, $\tau(1) = 0$ in any other case. If $\tau(1)$ is not zero, every crossing was fed a dot at the left string to be crossed, so the action gives 1, and the final untangling sums 1. As every crossing was fed a dot at the left, this means $\tau = \pi$. □

3.4 Hanoi Towers

We study a special case of the Schubert polynomials here. These arise from certain inclusions of the symmetric group. Consider the inclusion $S_a \hookrightarrow S_b$ with $b > a$ acting in the last entries. For example, take $i : S_3 \hookrightarrow S_5$ into the last entries, then $i(\underline{321}) = \underline{12543}$. We speculate the Schubert polynomials of π and the inclusion $i(\pi)$ should be related.

Definition 3.4.1. Let $\pi \in S_a$ be a permutation. The inclusion of π into S_b with $b > a$ permuting the last entries $b-a, \ldots, b$ gives a permutation $h(\pi) = \sigma$. The abacus of σ is called a **Hanoi Tower**, denoted $\pi \hookrightarrow \sigma$, and the inclusion h used here is the **Hanoi inclusion**. We say that the Hanoi Tower and the inclusion are of size **a into b**.

If we choose another inclusion such that the entries permuted are not consecutive, then we change the diagrams, the length of the permutation changes, and they are not similar. However, if we use a Hanoi inclusion we do not. It is be redundant to refer to Hanoi Towers of permutations that do not fix one. Assume $\pi(1) \neq 1$ for any Hanoi tower $\pi \hookrightarrow \sigma$.

Lemma 3.4.2. *Let h be a Hanoi inclusion and π a permutation. The diagram of $h(\pi)$ is the diagram π anticipated by a number of strings; these strings are not tangled and have no dots. If a diagram is not the identity map and starts with a number of untangled strings with no dots then it is an image of a Hanoi inclusion.*

Theorem 3.4.3. *The Hanoi inclusion $S_a \hookrightarrow S_b$ for $b \geq a$ is a homomorphism of groups.*

The first example of a Hanoi Tower is the one for adjacent transpositions $\sigma_i = (i, i+1)$.

Example 3.4.4. *The Schubert polynomial of σ_i is $\mathbf{S}_{\sigma_i} = x_1 + \cdots + x_i$. The abacus of σ_i carries a dot from the $i - th$ string all the way to the first. It is a Hanoi Tower of size 2 into i.*

There are several Hanoi inclusions into a fixed S_b and each S_a can be injected to all $b > a$. For a fixed b write $h_l : S_l \hookrightarrow S_b$ for $l = 1, \ldots, b$, the set of Hanoi inclusions, where $h_b = id_{S_b}$. For a fixed a and $l > a$ write h^l for $l = 1, 2, \ldots$ for the Hanoi inclusions $h^l : S_a \hookrightarrow S_l$. Write only h when a Hanoi inclusion is fixed.

Example 3.4.5. *Consider the polynomial for the Hanoi inclusion $\sigma = (12) \hookrightarrow S_a$. We always have a monomial x_1, which is $u(\sigma)$, a monomial x_a, which is the monomial $u(h(\sigma))$ for $\sigma = (12)$ and we have the remaining set which we shall refer to as transition diagrams. The transition diagrams are $u \circ h^l(12)$ for $l = 2, 3, \ldots a - 2$.*

We calculate Hanoi towers of some elements in S_3.

<u>312</u> The polynomial of is $\mathbf{S}_{\underline{312}} = x_1x_2$. Some Hanoi Towers are

$$x_1x_2 + x_1x_3 + x_2x_3$$
$$x_1x_2 + x_1x_3 + x_2x_3 + x_2x_4 + x_3x_4 + x_1x_4 +$$
$$x_1x_2 + x_1x_3 + x_2x_3 + x_2x_4 + x_3x_4 + x_1x_4 + x_3x_5 + x_4x_5 + x_1x_5 + x_2x_5.$$

<u>231</u> The polynomial is $\mathbf{S}_{\underline{231}} = x_1^2$. Some Hanoi Towers are

$$x_1^2 + x_1x_2 + x_2^2$$
$$x_1^2 + x_1x_2 + x_2^2 + x_2x_3 + x_3^2 + x_1x_3$$
$$x_1^2 + x_1x_2 + x_2^2 + x_2x_3 + x_3^2 + x_1x_3 + x_3x_4 + x_4^2 + x_2x_4 + x_1x_4.$$

<u>321</u> The polynomial is $\mathbf{S}_{\underline{321}} = x_1^2x_2 + x_2^2x_3 + x_1x_2x_3 + x_1^2x_3 + x_1x_2^2$. Hanoi Towers rapidly seem to get more complicated!

By observing the above example we can see that the abacus of the Hanoi Tower is split into sets in the following way. We have several inclusions of the abacus in consecutive strings, but we have some other monomials. These other monomials added in the process are called transition monomials, and form the **transition set** of the Hanoi Tower, with respect to the previous Hanoi Tower. That is, each time we add another string, we include the previous Hanoi Tower, the one with one string less, in the first strings, in the last strings, and get some transition set. We have counted more monomials because the two Hanoi Towers intersect in the middle consecutive strings in a Hanoi Tower that has two strings less. Thus, we can use the inclusion-exclusion principle to count the abacus.

Theorem 3.4.6. *The cardinality of the k-th Hanoi tower H_k of a permutation σ is given by the recursive formula $|H_k| = 2|H_{k-1}| - |H_{k-2}| + |T_k|$ where T_k is the transition set of the abacus H_k.*

In a combinatorial setting it should be clear that the inclusion-exclusion principle can be used to get an expression in terms of $|H_i|$ for any $i < k$. It turns out that this expression should be also obtained from the application of the theorem to $|H_{k-1}|$ in the formula repeatedly, so we do not bother going further.

The Lehmer code that corresponds to Hanoi Towers is special as the one that corresponds to the natural inclusion of $S_a \hookrightarrow S_{a+1}$. The Lehmer code in that case that corresponds to the same permutation seen inside S_{a+1} we just define $c_\pi(a+1) = 0$. For Hanoi towers we do have to change all values, but they are defined in terms of the code c_π.

Lemma 3.4.7. *Let $\pi \in S_a$ with Lehmer code $c_\pi(i) : \{1, \ldots, a\} \to \mathbb{N}$. Let $b > a$ an $h : S_a \hookrightarrow S_b$. The Lehmer code of $h(\pi)$ satisfies the shifting rule $c_{h(\pi)}(i+(b-a)) = c_\pi(i)$. In addition $c_{h(\pi)}(i) = 0$ for $i < b - a$.*

Proof. As the Lehmer code c_π is the function that counts the dots on the strings of π, the result is immediate from observing the diagram $h(\pi)$. The new code $c_{h(\pi)}$ is

$$c_{h(\pi)}(i) := \begin{cases} 0 & i < b-a,\ i > b, \\ c_\pi(k) & k = i - (b-a). \end{cases} \tag{3.4.1}$$

□

We could have given a more constructive proof of the function $c_{h(\pi)} : \{1, \ldots, b\} \to \mathbb{N}$ using the definition but our proof is simpler. The Hanoi Tower can be seen then as an operation in Lehmer codes also. It can be interesting to study abaci of Hanoi Towers of dominant permutations but we restrict ourselves now to step permutations hoping for an easy description of their Hanoi Towers. Remember a step permutation is dominant. Then the following is true.

Lemma 3.4.8. *A step permutation π is always dominant and $a(\pi) = 1$.*

Example 3.4.9. *Let π be a step permutation with a 1 size step Lehmer code. Then π is a $a-$cycle for some $a > 0$. If $a(\pi) = 1$ and $u(\pi) = x_1^a$ then π is the reversed $a-$cycle, and is a step permutation.*

Lemma 3.4.10. *A Hanoi Tower $\pi \hookrightarrow \sigma$ is the Hanoi Tower of a step permutation π if and only if σ exchanges two consecutive sets of strings which may differ in cardinality.*

Proof. Let c_π be the Lehmer code of a step permutation. We know then that $c_\pi(i) = s$ for $i = 1, \ldots, a$, some $a > 0$. This means that s strings cross the first string, and the second string also, and so on. The diagram of π is such that two adjacent groups of strings are exchanged without braiding themselves. A Hanoi tower of such permutation looks the same but has some strings in the beginning that are untangled. Then, any Hanoi tower of π exchanges consecutive sets of strings. □

We consider the next special case to calculate $a(\pi \hookrightarrow \sigma)$,

$$c_n(i) := \begin{cases} 1 & 1 \leq i \leq n \\ 0 & \text{otherwise,} \end{cases} \tag{3.4.2}$$

and

$$c^n(i) := \begin{cases} n & i = 1 \\ 0 & \text{otherwise .} \end{cases} \tag{3.4.3}$$

Lemma 3.4.11. *Let $k \geq 2$. The abacus of c^k satisfies*

$$a(h^m(c^k)) = \sum_{j=0}^{m-1} a(h^j(c^{k-1})).$$

Proof. Let $n > 0$ be big enough so that RC graph of $h^m(c^k)$ can be drawn in the $n \times n$ tiling. By fixing the first dot we look at the square $[1, \ldots, n] \times [2, \ldots, n]$ which fixes the first column of the RC graph. In the 0-Hecke element it fixes the first string level. Then, we move $k-1$ dots just as in $h^m(c^{k-1})$, fixing the last dot. Now move the fixed dot up and to the left making an abacus move, and fix it again. We move $k-1$ dots just as in $h^{m-1}(c^{k-1})$. If we continue this process we count smaller Hanoi Towers until we consider c^{k-1}, and we have all the summands. □

Corollary 3.4.12. *The same formula holds for c_k.*

Proof. We consider a mirror argument. Let $\Delta = \{(n, n) : n \in \mathbb{N}'\}$ where $\mathbb{N}' = \mathbb{N} - \{0\}$. Denote by R the operation of reflecting the pipe dream's dots along this diagonal. We have that $Rc^k = c_k$ and that $Rc_k = c^k$. Observe chute and ladder moves are inverses of each other if there are zero rows and zero columns involved. Given a pipe dream of c^k and a chute move performed on it reflecting both pipe dreams we get a pipe dream of c_k and a ladder move performed on it. Thus, $R\mathcal{A}_{h^j(c^k)} = \mathcal{A}_{h^j(c_k)}$, and in particular $a(h^j(c^k)) = a(h^j(c_k))$. $\square$

The previous result may help calculating $a(\pi \hookrightarrow \sigma)$ in more generality.

Definition 3.4.13. The **Hanoi sequence of** π is the sequence $(a(h^k(\pi))_k$.

Example 3.4.14. *We have that $a(h^m(c^2(i))) = \frac{m(m+1)}{2}$ as a particular case of 3.4.11. For each Hanoi Tower $\pi \hookrightarrow \sigma$ the Hanoi sequence of σ is a subsequence of the Hanoi sequence of π.*

3.5 Special cases of products

In this section we discuss some products of Schubert polynomials. We will prove a new rule that shifts the structure constants in one case, and bound the constants in all cases through the diagrammatic method we presented. Monk's rule is a well known result and the one below is mentioned in Billey and Bergeron's paper of RC graphs. Monk's rule is proved by bumping RC graphs. Both results are found there and we just write them in our new terms.

Theorem 3.5.1. *(Disjoint permutations) Let $\sigma \in S_a$ and $\lambda \in S_b$ with $b > a$ such that λ arises from a Hanoi tower $\omega \hookrightarrow \lambda$ such that λ fixes $\{1, \ldots, a\}$. If γ is the permutation that corresponds to the juxtaposition of irreducible pipe dreams σ and ω in this order, then*

$$\mathbf{S}_\sigma \mathbf{S}_\lambda = \mathbf{S}_\gamma.$$

This rule was proved Monk in [20], and simultaneously by Chevalley.

Theorem 3.5.2 (Monk's rule)**.** *Let σ be a Hanoi tower of* (12)*, and π be any permutation. Let σ_i exchange i and $i+1$. Then*

$$\mathbf{S}_\pi \mathbf{S}_{\sigma_i} = \sum_{\lambda = \pi\sigma_{ab}} \mathbf{S}_\lambda$$

where $l(\lambda) = l(\pi) + 1$ and k runs over transpositions σ_{ab} that exchange a and b, with $a \leq i < b$, $\lambda_a < \lambda_b$ and there is no k with $a < k < b$ such that $\lambda_a < \lambda_k < \lambda_b$ (these are covers in the Bruhat order).

We can approach this result in another way. A way of generalizing Monk's rule is to consider the Hanoi towers of permutations in S_3, which we already studied before. The permutation (13) might be a bit complicated but the other Hanoi towers for S_3 are simpler.

In the following we should assume that the action of π and σ do not fix one. We recall the case of the step permutations which we proved before in 3.3.32.

Theorem 3.5.3. *Let π be a step permutation and σ any permutation. If λ is the permutation whose Lehmer code satisfies $c_\lambda = c_\pi + c_\sigma$ then $\mathbf{S}_\pi \mathbf{S}_\sigma = \mathbf{S}_\lambda$.*

Conjecture 3.5.4. Let $\pi, \sigma \in S_a$. Suppose we know the coefficients for the expression $\mathbf{S}_\pi \mathbf{S}_\sigma = \sum_\lambda c^\lambda_{\sigma,\pi} \mathbf{S}_\lambda$, where $c^\lambda_{\sigma,\pi} \in \mathbb{N}$. Given a Hanoi inclusion the coefficients $c^{h(\lambda)}_{h(\pi),h(\sigma)}$ of $\mathbf{S}_{h(\pi)} \mathbf{S}_{h(\sigma)}$ could be predicted also. There is a partition of the set of coefficients where some are determined by $c^\lambda_{\pi,\sigma}$ and some are transition coefficients. We are using the same inclusion for both permutations. The product for the Hanoi tower h^k is the k-th Hanoi product. We refer to this statement as the **Hanoi rule for structure constants**.

We sketch an algorithm for calculating Hanoi products. Shift the coefficients as indicated by the Hanoi inclusion h. That is, we take $c_{h(\pi),h(\sigma)}^{h(\lambda} = c_{\pi,\sigma}^{\lambda}$ for all coefficients in the original product. Then repair the additional constants identifying the sequences of powers of the product monomials that are not weakly decreasing, after removing the known constants. The new biggest leading monomials may be chosen from this set by inspection. The last constants should be abaci of one element.

Now, we proceed to define a set of (irreducible) pipe dreams that are obtained from the knitting of the monomials of the product. Knit the Lehmer codes of the monomials to get the set of pipe dreams. Note that for each diagram we have two options, the diagram represents a plus one added to the coefficients of the expansion or signals that the monomial is obtained by untangling a pipe dream of another permutation that is counted in the expansion. Informally, say that is counted in a constant. We formalize these ideas.

Definition 3.5.5. The family of irreducible pipe dreams knitted from the product of $\mathbf{S}_\lambda \mathbf{S}_\pi$, taking as a disjoint union to count repetitions, is called the Lehmer set of π and σ. Denote it by $L(\pi, \sigma)$ or $\pi * \sigma$. The number of occurrences of λ in $\pi * \sigma$ is called the **Lehmer number** of λ, denoted $L_{\pi,\sigma}^{\lambda}$ or just L_λ to avoid redundancy.

Example 3.5.6. *The case for $a(\pi)a(\sigma) = 1$ should give the appropriate constant. Suppose that $a(\pi) = a(\sigma) = 1$ (that they are dominant permutations). Observe that $\pi * \sigma = \{\lambda\}$ where $u(\lambda) = u(\pi)u(\sigma)$. Then $L_\lambda = 1 = c^\lambda$ and the Lehmer set predicts the constant in the trivial case. Our statement, the product of polynomials of dominant permutations is the polynomial of the dominant permutation knitted from the unique monomial in the Lehmer set. Even more, the Lehmer code, with domain $\mathbb{N}$, of the knitted permutation satisfies $c_\lambda(i) = c_\pi(i) + c_\sigma(i)$ for all i.*

This multiplication is similar to an insertion. The multiplication of the ring $\mathbb{Z}[\bar{x}_a]$ in diagrammatics slides new dots to the monomials. But this is the multiplication of the polynomial ring and is commutative. The Lehmer set satisfies $L(\pi, \sigma) = L(\sigma, \pi)$ which was not clear when bumping RC graphs. In the case of SSYT bumping them makes the tableaux a monoid that sometimes has inverses. We do not worry about each bumping being 'good' in the sense of SSYT, where paths are well behaved, which RC graph bumping needs. The previous example motivates the following.

Definition 3.5.7. The product induced by the addition of Lehmer codes of permutations in S^∞ is called the Lehmer product, denoted $*_L$, of π and σ, so that $\pi *_L \sigma = \lambda$ where λ is the permutation knitted from the addition of Lehmer codes.

These definitions can be made more general. The advantage of the case of Schubert polynomials is that the Lehmer map is diagrammatic. Let the **Lehmer map** be an assignment of basis elements of a polynomial ring to monomials, which is well defined whenever we can divide. The diagram knitted from the monomial is the image of the Lehmer map of Schubert polynomials. For noncommutative rings in good cases we have left and right division, or just one of them. We can repair this by setting a right Lehmer map and a left Lehmer map. The Lehmer map would have the property that it assigns the monomial the basis element which has this monomial as a largest monomial, in the order used for division, which is required. If this Lehmer map is defined it also defines a Lehmer set and Lehmer numbers for a product (of positive linear combinations). It even defines a Lehmer product for the polynomials (not for permutations) where the leading monomial of the permutation is chosen and uses the Lehmer map (in permutations $\mathbf{S}_\pi *_L \mathbf{S}_\sigma = \mathbf{S}_{\pi *_L \sigma}$ is the same product).

Example 3.5.8. *Consider $\mathbb{Z}[x_1, \ldots, x_a]$ with the monomial basis. The Lehmer map is the identity on the set of monomials and can be extended linearly. The Lehmer constants match the constants in the expansion for products of positive linear combinations of the basis elements.*

Definition 3.5.9. Let R be a Z_+ ring that admits one sided division, not necessarily commutative and not necesarilly with 1. Let $\mathcal{B} = \{b_\lambda\}$ a basis with non negative structure constants. Let the **Lehmer map** $L : \mathcal{B} \times \mathcal{B} \to \mathcal{B} \times \mathbb{N}$ be a map that assigns the set of basis elements whose leading monomial appears in the expression of $b'b''$ for $b', b'' \in \mathcal{B}$ and the number of times such monomial appears. The number of L_λ in the pair (b_λ, L_λ) assigned to b' and b'' is called the **Lehmer number** of b' and b'' and we say it counts the number of occurrences of b_λ in the **Lehmer set** $L(b', b'')$. The Lehmer set $L(b', b'')$ is the subset $L(\mathcal{B} \times \mathcal{B})$ of $\mathcal{B} \times \mathbb{N}$.

Corollary 3.5.10. *If R admits a Lehmer map then the Lehmer numbers are non negative and greater than or equal to the structure constants for R.*

We attempt to provide an application of the Lehmer set. Consider the case we have a finite dimensional $\mathbb{Z}_+$ ring, such that the following theorem holds. A proof of this theorem and more on $\mathbb{Z}_+$ rings is found in Etingof's book [7]. In this case the multiplication by an element is coded into an integer matrix satisfying the Frobenius-Perron theorem and we can define a Frobenius dimension. The corollary is a simple implication of the theorem.

Theorem 3.5.11. *Let R be a $\mathbb{Z}_+$ ring of finite dimension. Define the group homomorphism $FDim : \mathcal{B} \to \mathcal{C}$ such that $FDim(x)$ is the biggest eigenvalue in the multiplication matrix associated to multiplication by x by the left. The norm, or spectral radius, of the matrix is positive. This mapping is extended to the whole ring R by additivity.*

Corollary 3.5.12. *Let R be a ring that admits a Lehmer map. In the conditions of the previous theorem, fix an element $x \in \mathcal{B}$ and let l_λ be the Lehmer number of a basis element $b_\lambda \in \mathcal{B}$ for multiplication by x. Then if $l_\lambda > c \cdot FDim(x)$ for some $c \in \mathbb{N}$, there is a another nonzero positive constant different from l_λ in the product xb_λ.*

It is possible to choose c as the maximum between the dimension of the ring R over $\mathbb{Z}$. Then $\dim_{\mathbb{Z}} R$ accounts for the change of basis to eigenvectors in the theorem. We return to our study of Schubert polynomials.

Example 3.5.13. *Division by Schubert polynomials can be done by hand. This is not in essence combinatorial except by the knitting of monomials. Consider a product $\mathbf{S}_\pi\mathbf{S}_\sigma$. Find the biggest monomial in the lexicographic order. Knit the monomial and get $\mathbf{S}_\lambda$. Now subtract, $\mathbf{S}_1 := \mathbf{S}_\pi\mathbf{S}_\sigma - \mathbf{S}_\lambda$, and repeat the process. In this division we have to calculate $\mathbf{S}_\lambda$, subtract it, and inspect again $\mathbf{S}_k$ looking for the monomial to knit.*

Theorem 3.5.14. *Let π and σ be given, and denote $c^\lambda_{\pi,\sigma}$ the Schubert structure constant of $\mathbf{S}_\sigma\mathbf{S}_\pi$. Then $L_\lambda \geq c^\lambda$. Even more, $L_\lambda = c^\lambda + n_\lambda$, where n_λ is the number of occurences of the monomial $u(\lambda)$ in the product $\mathbf{S}_\pi\mathbf{S}_\sigma$ on other constants. Then $|\pi * \sigma| = a(\pi)a(\sigma)$.*

Proof. It follows from Theorem 3.0.2 that searching for the largest degree we find a polynomial to divide by. The Lehmer code of the polynomial does not only determine it, but any polynomial with that monomial has a bigger order monomial. We are left with the unique choice of the permutation knitted by this code. Each Schubert constant has to be knitted thus is found in the Lehmer set. The Lehmer number does not always match the constant, but exceeds it. The number by which it exceeds counts the number of occurrences the monomial is found in other abaci. □

Corollary 3.5.15. *The equation $a(\pi)a(\sigma) - \sum_\lambda n(\lambda) = \sum_\lambda c^\lambda$ holds.*

Proof. The number of coefficients in the product is $a(\pi)a(\sigma)$ and it equals $\sum_\lambda L_\lambda$. Then

$$a(\pi)a(\sigma) - \sum_\lambda n(\lambda) = \sum_\lambda L_\lambda - \sum_\lambda n(\lambda) = c^\lambda.$$ □

Corollary 3.5.16. *Given a known constant c^λ, the number of $\lambda \in L(\sigma, \pi)$ with nonzero constants is at least $n(\lambda)$.*

Corollary 3.5.17. *If $L_\lambda = 1$ for all λ then no structure constant is greater than one. The biggest structure constant cannot be greater than the $L'_\lambda s$. Let $L := \max L_\lambda$ so that $\max c^\lambda \leq L$. There always exists a constant $L_\lambda = 1$.*

Proof. The result follows from the theorem except the last conclusion. Each polynomial has a largest monomial in the lexicographic order whose powers are the Lehmer code of the permutation. The multiplication of these largest monomials is a largest monomial in the product, with coefficient one, so $L_\lambda = 1$. As it is the largest the monomial, it is not untangled from the abacus of another permutation. Then there is a constant $c^\lambda = 1$ where λ has Lehmer code $u(\pi)u(\sigma)$ and $L_\lambda = c^\lambda$. $\square$

We can restate the last conclusion by saying that $\pi *_L \sigma \in L(\pi, \sigma)$ and $c^{\pi *_L \sigma} = 1$. As Schubert polynomials are a basis over $\mathbb{Z}$ for the ring $Z[x_1, x_2, \ldots]$ we know the coefficients are determined uniquely.

Definition 3.5.18. A subcover $\mathcal{B}$ of $\pi * \sigma$ is a family of abaci for $\lambda \in \pi * \sigma$ such that the number of occurrences of λ is less than L_λ. Define $D := u(\pi * \sigma) - \sum b_\lambda u(\lambda)$. Denote by $b_\lambda \leq L_\lambda$ the number of occurrences of λ in $\mathcal{B}$. Define the cardinality of a subcover to be $\sum_\lambda b_\lambda a(\lambda)$. Say that the subcover is **non intersecting** if $D \geq 0$.

We are looking for the maximum subcover properly contained in $\pi * \sigma$. If the contention is not proper then $D < 0$. In the Littlewood-Richardson rule the constants count some number of skew shapes and in the present case, the constants could be related to $a(-)$ and the Bruhat order. The uniqueness of the coefficients and the non intersecting property whose requirement implies $D \geq 0$ gives the following theorem. There is not much added meaning because we still check that $D \geq 0$.

Theorem 3.5.19. *The structure constants determine uniquely the nonintersecting subcover $\mathcal{B}$ which equals $\sigma * \pi$ in cardinality and $D = 0$. We solve the equation $\sum b_\lambda a(\lambda) = |\pi * \sigma|$ and then $b_\lambda = c^\lambda_{\pi,\sigma}$.*

Remark 3.5.20. We do not know if the solution $\sum b_\lambda a(\lambda) = |\pi * \sigma|$ is unique.

Conjecture 3.5.21. There is a way of weakening the condition of the theorem to subcovers such that $D < 0$. If we do not need to check $D \geq 0$ for subcovers, there might be another condition missing that assures the solution to $\sum b_\lambda a(\lambda) = |\pi * \sigma|$ is unique for covers such that $b_\lambda \leq L_\lambda$.

Even if this is true we still have to look for appropriate methods of calculating $c^\lambda_{\pi,\sigma}$. For example, one could hope that there is a relation of the abacus, Lehmer codes, and the nilHecke action on the product that gives the value of the constants. We hope there exists a closed formula for the number $a(\pi)$ for any permutation. There is a special case where the Lehmer set comes out naturally.

Corollary 3.5.22. *Suppose that $a(\lambda) = 1$ for all $\lambda \in L(\pi, \sigma)$. Then every $\lambda \in L(\pi, \sigma)$ is a dominant permutation and $c^\lambda = 1 = L_\lambda$ for all λ.*

With this new terminology we state a lemma towards the Hanoi rule. For the proof of the lemma just observe that the shifted monomials $u(h(\tau))u(h(\tau')) = u(h(\lambda))$ are now in the product of the Hanoi Towers of σ and π.

Lemma 3.5.23. *Let λ be a permutation such that $\mathbf{S}_\lambda$ has a nonzero coefficient in the expression for $\mathbf{S}_\pi \mathbf{S}_\sigma$. Then λ is in the Lehmer set $\pi * \sigma$ and $h(\lambda) \in h(\pi) * h(\sigma)$.*

We can do a bit more, but in a more traditional way. We set to prove that each $h(\lambda)$ increments by one a constant if λ did before. Assume for the following that the diagrams of π and σ have no unpermuted strings in the beginning (they do not fix one).

Of the Hanoi rule. Picture the coefficients as boxes with a counter. There are finite boxes as indicated by the knitting of the Lehmer set of $\mathbf{S}_\pi \mathbf{S}_\sigma$. For each time $u(\lambda)$ appears in the sum the box with label λ goes up by one. There is an order for these counter increments imposed by the division algorithm. The Lehmer set of the product of the Hanoi towers have the shifted boxes $h(\lambda) \in L(h(\pi), h(\sigma))$. This same increment of counters appears in the shifted counters for the Hanoi Towers in the same order. For each time $u \circ h(\lambda)$ appears in the sum the box with label $h(\lambda)$ goes up by one, so the structure constants are shifted. Only one inconvenient can arise which we are set to rule out. Any other monomial in the Hanoi Towers cannot have be larger in the lexicographic order than $u \circ h(\lambda)$. The other monomials come from the same abacus but with monomials shifted to a lower degree, so these cannot be bigger. Then the other monomials that have bigger variables are transition monomials. Any transition monomial always has a nonzero exponent for a smaller variable than the smallest variable with power bigger than zero in $u(\lambda)$. Then the transition monomial is smaller in the lexicographic order and $h(\lambda)$ is chosen also to increment the counters in the division algorithm. As this happens for each λ, by the division algorithm $c^{h(\lambda)}_{h(\pi),h(\sigma)} = c^{\lambda}_{\pi,\sigma}$. $\square$

The rest of the coefficients are called transition coefficients. The Hanoi rule cannot be used repeatedly for each inclusion generating new coefficients inductively, for the new transition monomials in the Hanoi towers are bigger than these inductive transition coefficients. Unless we prove that the products of these new transition monomials are transition monomials of the shifted constants, they interrupt the order of the division and we cannot iterate the rule by adding one string and computing the constants, and adding the next string and shifting all the previous constants. This might be proved for special permutations, such as dominant permutations or for adjacent transpositions (as the transition set is empty). Transition monomials are yet to be determined.

At a first glance there are two things that can account for the constants. First, the cardinalities $a(\lambda)$ for $\lambda \in L(\sigma, \pi)$, and the Bruhat order or weak Bruhat order for the Lehmer set. An algorithm could be:

- Given $\sigma, \pi \in S^\infty$ build $L(\sigma, \pi)$.
- For $\lambda \in L(\sigma, \pi)$ calculate $a(\lambda)$.
- For $\lambda \in L(\sigma, \pi)$ identify if $\pi \leq_B \lambda$ or $\sigma \leq_B \lambda$ and call this subset of $L(\sigma, \pi)$ the Bruhat set; denote the Bruhat set by $B(\sigma, \pi)$. Order $B(\sigma, \pi)$ using the lexicographic order $\leq_L$.
- Starting from the biggest element, add the numbers $a(\lambda)$ for $\lambda \in B(\sigma, \pi)$ in such way that $\sum_\lambda a(\lambda) = |\pi * \sigma|$ and that bigger monomials are not left out; so they are chosen by descending the chain given by $\leq_L$ in $B(\sigma, \pi)$.

Definition 3.5.24. Define the Lehmer set of π as the set of permutations $L(\pi)$ knitted from the monomials of $\mathbf{S}_\pi$. It is understood that $L(\pi)$ also represents a polynomial given by the untangling map; the polynomial is $L(\pi) := u(L(\pi)) = u(\mathcal{A}_\pi) = \mathbf{S}_\pi$.

We give another method through brute knitting. We find the expansion of any monomial in the Schubert basis. If we know how to expand through knitting any monomial into the Schubert basis we can compute the constants by adding all the expansions of the monomials $u(\lambda) \in L(\sigma, \pi)$. In this case we do not need to study the product $L(\sigma, \pi)$ but instead we should understand $L(\pi)$. The idea is as follows. Take the monomial m and we have no choice but to knit m so we get $L(\pi_1)$ and a Schubert polynomial $\mathbf{S}_\pi$. If this is not a dominant permutation we have $a(\pi_1) > 1$, so we need to add correction terms. We subtract then $\mathbf{S}_{\pi_i}$ for $\pi_i \neq \pi_1$ in $L(\pi_1)$ and get $a(\pi_1) - 1$ polynomials. Repeat the procedure recursively. Our claim is that the algorithm stops. Each time we add correction terms we knit polynomials such that in the step i we have $\deg L(\pi_k) < \deg L(\pi_i)$ for $k > i$. The set

$$\{\pi : l(\pi) = \pi_1 \text{ and } \deg(\mathbf{S}_\pi) \leq_L \deg(\mathbf{S}_{\pi_1})\}$$

is finite. The permutations with the lowest monomials in the lexicographic order in the set are dominant. When we decrease order we make our way to dominant permutations. Thus, the algorithm stops. Let

$$S(\pi) = \mathbf{S}_\pi - \sum_{\lambda \in L(\pi); \lambda \neq \pi} \mathbf{S}_\lambda$$

and the recursion is given by, if $u(\pi) = m$ and $r(\pi) = \{\lambda \in L(\pi) : a(\lambda) > 1,\ \lambda \neq \pi\}$,

$$R(\pi) = S(\pi) - \sum_{\lambda \in r(\pi)} R(\lambda),$$

so that $m = R(\pi_1)$.

We give an example for a product $p := x_1x_2x_3(x_1+x_2+x_3+x_4) = x_1^2x_2x_3 + x_1x_2^2x_3 + x_1x_2x_3^2 + x_1x_2x_3x_4$. By Monk's rule we know that knitting the Lehmer codes 1111 and 112 give the two permutations λ_1 and λ_2 in the expansion with coefficient one. We hope that the recursion is not over complicated. To simplify the notation write, for example, $L(111) = u(\mathcal{A}_{\pi(111)})$ where $\pi(111)$ is the permutation that is knitted from the Lehmer code 111. Get

$$\begin{aligned}
x_1x_2x_3x_4 &= \mathbf{S}_{\lambda_1}; && (3.5.1)\\
x_1x_2x_3^2 &= \mathbf{S}_{\lambda_2} - L(121) - L(211) + L(211) = \mathbf{S}_{\lambda_2} - L(121); && (3.5.2)\\
x_1x_2^2x_3 &= L(121) - L(211); && (3.5.3)\\
x_1^2x_2x_3 &= L(211); && (3.5.4)\\
p: &= \mathbf{S}_{\lambda_1} + \mathbf{S}_{\lambda_2} - L(121) + L(121) - L(211) - L(211) = \mathbf{S}_{\lambda_1} + \mathbf{S}_{\lambda_2}. && (3.5.5)
\end{aligned}$$

With our efforts we are able to finally prove Lemma 1.5.4 of Chapter 1.

Lemma 3.5.25. *Let x^α be a monomial in $\mathbb{Z}[\bar{x}_a]$ such that $\alpha \subset \delta$ where δ is the Lehmer code of w_0, the powers of $\underline{x}^\delta$. Then x^α is a linear combination of Schubert polynomials $x^\alpha = \sum_{\pi \in S_a} c_\pi \mathbf{S}_\pi$ such that $\pi \in S_a$.*

Proof. Using the brute knitting formula of monomials for x^α, every Schubert polynomial in the sum has lexicographic degree less than or equal to x^α. Write π for the permutation with $u(\bar{\pi}) = x^\alpha$. For any $D \in RC(\pi)$ we have that D is a configuration of $l(\pi)$ crosses that sit in the upper antidiagonal matrix with dots and crosses. If α' is the sequence of powers of the monomial x^D then $\alpha' \subset \delta$. Thus, every monomial $x^{\alpha'}$ we knit in the recursion satisfies $\alpha' \subset \delta$ and knits the permutation π' with $u(\pi') = x^D$ where $\pi' \in S_a$. □

Bibliography

[1] Nantel Bergeron and Sara Billey. RC-graphs and Schubert polynomials. *Experiment. Math.*, 2(4):257–269, 1993.

[2] Nantel Bergeron and Huilan Li. Algebraic structures on Grothendieck groups of a tower of algebras. *J. Algebra*, 321(8):2068–2084, 2009.

[3] A. J. Berrick and M. E. Keating. *Categories and modules with K-theory in view*, volume 67 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2000.

[4] Sara C. Billey, William Jockusch, and Richard P. Stanley. Some combinatorial properties of Schubert polynomials. *J. Algebraic Combin.*, 2(4):345–374, 1993.

[5] Alexander P. Ellis, Mikhail Khovanov, and Aaron D. Lauda. The odd nilHecke algebra and its diagrammatics. *Int. Math. Res. Not. IMRN*, (4):991–1062, 2014.

[6] Laura Escobar and Karola Mészáros. Subword complexes via triangulations of polytopes. Available online in *https://arxiv.org/abs/1502.03997v2*, 2015.

[7] Pavel Etingof, Shlomo Gelaki, Dmitri Nikshych, and Victor Ostrik. *Tensor categories*, volume 205 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2015.

[8] Sergey Fomin and Anatol N. Kirillov. The Yang-Baxter equation, symmetric functions, and Schubert polynomials. *Discrete Math.*, 153(1-3):123–143, 1996.

[9] William Fulton. *Young tableaux*, volume 35 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1997. With applications to representation theory and geometry.

[10] Christian Kassel. *Quantum groups*, volume 155 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.

[11] Mikhail Khovanov. Linearization and categorification. *Port. Math.*, 73(4):319–335, 2016.

[12] Mikhail Khovanov and Aaron D. Lauda. A diagrammatic approach to categorification of quantum groups. I. *Represent. Theory*, 13:309–347, 2009.

[13] Allen. Knutson. Schubert polynomials and symmetric functions. *Notes for the Lisbon combinatorics summer school 2012*, Available online in *http://www.math.cornell.edu/ allenk/schubnotes.pdf*.

[14] Allen Knutson and Ezra Miller. Gröbner geometry of Schubert polynomials. *Ann. of Math. (2)*, 161(3):1245–1318, 2005.

[15] Bertram Kostant. *Groups over Z.* Amer. Math. Soc., Providence, R.I., 1966.

[16] Aaron Lauda. Introduction to diagrammatic algebra and the categorified quantum $\mathfrak{sl}_2$. Available online in *https://arxiv.org/abs/1106.2128*.

[17] Aaron D. Lauda. A categorification of quantum sl(2). *Adv. Math.*, 225(6):3327–3424, 2010.

[18] G. Lusztig. Quantum deformations of certain simple modules over enveloping algebras. *Adv. in Math.*, 70(2):237–249, 1988.

[19] L Manivel. *Symmetric functions, Schubert polynomials and degeneracy loci.* American Mathematical Society, volume 6 of smf/ams texts and monographs edition, 2001.

[20] D. Monk. The geometry of flag manifolds. *Proc. London Math. Soc. (3)*, 9:253–286, 1959.

[21] S. Montgomery and S. Paul Smith. Skew derivations and $U_q(\mathrm{sl}(2))$. *Israel J. Math.*, 72(1-2):158–166, 1990. Hopf algebras.

[22] Jonathan Rosenberg. *Algebraic K -Theory and its applications.* Springer Graduate Texts in Mathematics, no 147 edition, 1994.

[23] Raphael Rouquier. 2-kac-moody algebras. 2008.

[24] Louis H. Rowen. *Ring theory.* Academic Press, Inc., Boston, MA, student edition, 1991.