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A Lamperti type representation of real-valued self-similar Markov processes and Lévy processes conditioned to avoid zero

T H E S I S

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Contents

Introducción

Esta tesis comprende dos capítulos. El Capítulo 1 es el artículo aceptado el mes de junio 2012 para publicación en la revista Bernoulli "The Lamperti representation of real-valued self-similar Markov processes". El Capítulo 2 corresponde al artículo en preparación "On Lévy processes conditioned to avoid zero". El punto en común de ambos capítulos son los procesos de Lévy a valores reales. A continuación daremos una breve descripción sobre los procesos de Lévy y la relación que éstos guardan con cada capítulo. También daremos una descripción general del contenido de los Capítulos 1 y 2.

Procesos de Lévy

Los procesos de Lévy a valores reales son procesos estocásticos con incrementos independientes y estacionarios. En el Capítulo 1, tenemos que todos los procesos de Markov autosimilares positivos se pueden expresar como la exponencial de un proceso de Lévy cambiado de tiempo por la inversa de su funcional exponencial. Generalizaremos esta propiedad al caso en el cual el proceso de Markov autosimilar toma valores en los reales. En el Capítulo 2, la regularidad por si misma del punto cero para los procesos de Lévy implica la existencia de una densidad continua del q-resolvente (ver [5]). Bajo el supuesto adicional de que el proceso de Lévy no es un proceso de Poisson compuesto, se puede hallar una función invariante para el semigrupo del proceso matado en su primer tiempo de llegada a cero. La función invariante se obtiene como un límite de una sucesión de funciones determinada por la densidad del q-resolvente. Con ayuda de la h transformada de Doob, construimos la ley de una nueva clase de procesos de Markov.

En los dos capítulos, la tripleta (a, σ, π), que caracteriza a los procesos de Lévy, es fundamental en las pruebas, fórmulas y ejemplos. Recordamos la definición de la tripleta de un proceso de Lévy. Si ξ es un proceso de Lévy con ley $\mathbb{P}$, entonces para cualquier $t > 0$, ξ_t es una variable aleatoria infinitamente divisible y su transformada de Fourier admite la fórmula de Lévy-Khintchine, i.e., existe una función $\psi : \mathbb{R} \to \mathbb{C}$ tal que

$$\mathbb{E}(e^{i\lambda\xi_t}) = e^{-t\psi(\lambda)}, \quad \lambda \in \mathbb{R},$$

con ψ dada por

$$\psi(\lambda) = ia\lambda + \frac{\sigma^2}{2}\lambda^2 + \int_{\mathbb{R}} (1 - e^{i\lambda x} + i\lambda x \mathbf{1}_{\{|x|<1\}})\pi(dx), \quad \lambda \in \mathbb{R}, \tag{1}$$

donde $a \in \mathbb{R}$, $\sigma \geq 0$ y π es una medida definida en $\mathbb{R} \setminus \{0\}$ que cumple $\int_{\mathbb{R}} (1 \wedge x^2)\pi(dx) < \infty$. La constante a es conocida como deriva, σ es el coeficiente Gaussiano y π es llamada una medida de Lévy. Para nuestra comodidad al momento de hacer cálculos, escogemos el exponente característico de manera diferente en cada capítulo. Por ejemplo, en el Capítulo 2, tomaremos ψ como en (1), mientras que en el Capítulo 1, ψ sastisface $\mathbb{E}(e^{i\lambda\xi_t}) = e^{t\psi(\lambda)}$, $t > 0$, $\lambda \in \mathbb{R}$, con

$$\psi(\lambda) = ia\lambda - \frac{\sigma^2}{2}\lambda^2 + \int_{\mathbb{R}} (e^{i\lambda x} - 1 - i\lambda x \mathbf{1}_{\{|x|<1\}})\pi(dx), \quad \lambda \in \mathbb{R}.$$

Un ejemplo importante de un proceso de Lévy, que estará presente en toda esta tesis, es el proceso α-estable. Para $\alpha = 2$, el proceso ξ es el bien conocido y estudiado movimiento Browniano. En el caso $\alpha \in (0, 2)$, el proceso α no tiene coeficiente Gaussiano y la medida de Lévy tiene densidad ν con respecto a la medida de Lebesgue, la cual está dada por

$$\nu(y) = c^+ y^{-\alpha-1}\mathbf{1}_{\{y>0\}} + c^-|y|^{-\alpha-1}\mathbf{1}_{\{y<0\}},$$

con c^+ and c^- dos constantes no negativas tales que $c^+ + c^- > 0$. Además, se puede mostrar que ψ se escribe de la siguiente manera,

$$\psi(\lambda) = c|\lambda|^\alpha(1 - i\beta \operatorname{sgn}(\lambda)\tan(\alpha\pi/2)), \quad \lambda \in \mathbb{R},$$

donde

$$c = -\frac{(c^+ + c^-)\Gamma(2-\alpha)}{\alpha(\alpha-1)}\cos(\alpha\pi/2), \quad \beta = \frac{c^+ - c^-}{c^+ + c^-}.$$

Para mayor detalle sobre la teoría de los procesos de Lévy, sugerimos los libros [5, 35, 45].

Capítulo 1. La representación de Lamperti de procesos de Markov autosimilares a valores reales

En este capítulo hacemos contribuciones a la teoría de los procesos de Markov autosimilares a valores reales. Específicamente, obtenemos una representación tipo Lamperti para los procesos de Markov autosimilares a valores reales matados en su primer tiempo de llegada a cero, esto es, representamos procesos de Markov autosimilares a valores reales como procesos invariantes multiplicativos cambiados de tiempo.

Representación de Lamperti

Sea E el conjunto $[0, \infty)$ o $\mathbb{R}^n$. Una familia càdlàg de procesos de Markov fuerte con espacio de estados E, $\{X^{(x)} = (X, \mathbb{P}_x), x \in E\}$, es llamada autosimilar de índice $\alpha > 0$ si para toda $c > 0$, la ley de $(cX_{c^{-\alpha}t}, t \geq 0)$ bajo $\mathbb{P}_x$, es la misma que la ley de $(X_t, t \geq 0)$ bajo $\mathbb{P}_{cx}$, para toda x. En el caso cuando el proceso toma valores positivos fue estudiado por Lamperti in 1972. En su artículo, él probó varias propiedades sobre esta clase de procesos. El resultado el cual es de nuestro interés es conocido como la representación de Lamperti. La representación de Lamperti, algunas veces llamado transformación de Lamperti, establece que cualquier proceso de Markov autosimilar positivo matado en su primer tiempo de llegada a cero, se puede representar como la exponencial de un proceso de Lévy cambiado de tiempo por la inversa de su funcional exponencial. Formalmente, si X es un proceso de Markov autosimilar positivo de índice $\alpha > 0$, entonces el proceso $(\xi_t, t \geq 0)$ definido por

$$\exp\{\xi_t\} = x^{-1} X_{\nu(t)}, \quad t \geq 0,$$

donde

$$\nu(t) = \inf\left\{ s > 0 : \int_0^s (X_u)^{-\alpha} du > t \right\},$$

con la convención usual $\inf\{\emptyset\} = +\infty$, es un $\mathbf{P}$-proceso de Lévy. Aquí, $\mathbf{P} = \mathbb{P}_1$.

Nuestro objetivo en esta parte de la tesis es generalizar el resultado anterior al caso en el cual el proceso tiene como espacio de estados la línea real. Para este fin, seguimos algunas ideas en [22] para poder caracterizar a los procesos subyacentes en tal representación. Los procesos resultantes son los llamados procesos invariantes multiplicativos que satisfacen la propiedad de Feller, los cuales aparecen en [34] como procesos de Markov autosimilares con valores en $\mathbb{R}^n$ cambiados de tiempo. En el caso $n = 1$, a este tipo de procesos los llamaremos procesos de Lamperti-Kiu. Formalmente, un proceso de Lamperti-Kiu, $Y = (Y_t, t \geq 0)$, es un proceso càdlàg tomando valores en $\mathbb{R}^* := \mathbb{R} \setminus \{0\}$, satisfaciendo la propiedad de Feller y que además cumple lo siguiente

$$\{(aY_t, t \geq 0), \mathbb{P}_x\} \stackrel{\mathcal{L}}{=} \left\{(\mathrm{sgn}(a) Y_t, t \geq 0), \mathbb{P}_{|a|x}\right\}, \tag{2}$$

para toda $x, a \neq 0$. En esta tesis también damos una representación de los procesos de Lamperti-Kiu como la exponencial de una cierta clase de procesos. Al hacer esto, se completa el trabajo de Kiu [34].

Dos resultados principales son establecidos en este capítulo. Teorema 1 (abajo) establece que, dependiendo del signo del proceso, el comportamiento de los procesos de Markov autosimilares entre cambios de signo es como un proceso de Markov autosimilar positivo (o negativo). Escribimos esto formalmente, sea H_n el n-ésimo cambio de signo del proceso X:

$$H_0 = 0, \quad H_n = \inf\{t > H_{n-1} : X_t X_{t-} < 0\}, \quad n \geq 1.$$

Suponga que $\mathbb{P}_x(H_1 < \infty) = 1$, para toda $x \in \mathbb{R}^*$. Defina,

$$\mathcal{X}_t^{(n)} = \frac{X_{H_n + |X_{H_n}|^\alpha t}}{|X_{H_n}|}, \quad 0 \leq t < |X_{H_n}|^{-\alpha}(H_{n+1} - H_n), \tag{3}$$

y

$$J_n = \frac{X_{H_{n+1}}}{X_{H_{n+1}-}}, \quad n \geq 0. \tag{4}$$

Entonces, $(\mathcal{X}^{(n)}, n \geq 0)$ y $(J_n, n \geq 0)$ satisfacen lo siguiente.

Teorema 1. *Sea $X^{(x)} = (X, \mathbb{P}_x)_{x \in \mathbb{R}^*}$ una familia de procesos de Markov autosimilares a valores reales de índice $\alpha > 0$, tal que $\mathbb{P}_x(H_1 < \infty) = 1$, para toda $x \in \mathbb{R}^*$. Entonces*

(i) Las trayectorias entre cambios de signo, $(\mathcal{X}^{(n)}, n \geq 0)$, definidos en (3), son independientes bajo $\mathbb{P}_x$, para $x \in \mathbb{R}^*$. Además, para toda $n \geq 0$,

$$\left\{ \left(\mathcal{X}_t^{(n)}, 0 \leq t < |X_{H_n}|^{-\alpha}(H_{n+1} - H_n) \right), \mathbb{P}_x \right\} \stackrel{\mathcal{L}}{=} \left\{ (X_t, 0 \leq t < H_1), \mathbb{P}_{sgn(x)(-1)^n} \right\}.$$

De aquí, son procesos de Lévy matados en un tiempo exponential cambiados de tiempo.

(ii) *Las variables aleatorias $J_n, n \geq 0$, definidas en (4), son independientes bajo $\mathbb{P}_x$, para $x \in \mathbb{R}^*$ y para $n \geq 0$, se satisface la identidad*

$$\{J_n, \mathbb{P}_x\} \stackrel{\mathcal{L}}{=} \left\{J_0, \mathbb{P}_{sgn(x)(-1)^n}\right\}.$$

(iii) *Para cada $n \geq 0$, el proceso $\mathcal{X}^{(n)}$ y la variable aleatoria J_n son independientes, bajo $\mathbb{P}_x$, para $x \in \mathbb{R}^*$.*

El Teorema 1 implica que seis son los objetos aleatorios que definen al proceso subyacente en la representación de Lamperti, esto es, tenemos $(\xi^+, \xi^-, \zeta^+, \zeta^-, U^+, U^-)$, donde ξ^+, ξ^- son dos procesos de Lévy, ζ^+, ζ^- son dos variables aleatorias exponenciales y U^+, U^- son dos variables aleatorias que toman valores en los reales, todos independientes. Con esto en mente, construimos el proceso estocástico siguiente.

Sean ξ^+, ξ^- procesos de Lévy a valores reales; ζ^+, ζ^- variables aleatorias exponenciales con parámetros q^+, q^-, respectivamente, y U^+, U^- variables aleatorias con valores en los reales. Sean $(\xi^{+,k}, k \geq 0)$, $(\xi^{-,k}, k \geq 0)$, $(\zeta^{+,k}, k \geq 0)$, $(\zeta^{-,k}, k \geq 0)$, $(U^{+,k}, k \geq 0)$, $(U^{-,k}, k \geq 0)$ sucesiones independientes de variables aleatorias i.i.d. tales que

$$\xi^{+,0} \stackrel{\text{Law}}{=} \xi^+, \quad \xi^{-,0} \stackrel{\text{Law}}{=} \xi^-, \quad \zeta^{+,0} \stackrel{\text{Law}}{=} \zeta^+, \quad \zeta^{-,0} \stackrel{\text{Law}}{=} \zeta^-, \quad U^{+,0} \stackrel{\text{Law}}{=} U^+, \quad U^{-,0} \stackrel{\text{Law}}{=} U^-.$$

Para cada $x \in \mathbb{R}^*$ fijo, consideramos la sucesión $((\xi^{(x,k)}, \zeta^{(x,k)}, U^{(x,k)}), k \geq 0)$, donde para $k \geq 0$,

$$(\xi^{(x,k)}, \zeta^{(x,k)}, U^{(x,k)}) = \begin{cases} (\xi^{+,k}, \zeta^{+,k}, U^{+,k}), & \text{if } \operatorname{sgn}(x)(-1)^k = 1, \\ \\ (\xi^{-,k}, \zeta^{-,k}, U^{-,k}), & \text{if } \operatorname{sgn}(x)(-1)^k = -1. \end{cases}$$

Sea $(T_n^{(x)}, \geq 0)$ la sucesión definida por

$$T_0^{(x)} = 0, \quad T_n^{(x)} = \sum_{k=0}^{n-1} \zeta^{(x,k)}, \quad n \geq 1,$$

y $(N_t^{(x)}, t \geq 0)$ un proceso de renovación alternante:

$$N_t^{(x)} = \max\{n \geq 0 : T_n^{(x)} \leq t\}, \quad t \geq 0.$$

Escribimos

$$\sigma_t^{(x)} = t - T_{N_t^{(x)}}^{(x)}, \quad \xi_{\sigma_t}^{(x)} = \xi_{\sigma_t^{(x)}}^{(x,N_t^{(x)})}, \quad \xi_\zeta^{(x,k)} = \xi_{\zeta^{(x,k)}}^{(x,k)}.$$

Sea $(Y_t^{(x)}, t \geq 0)$ el proceso definido por

$$Y_t^{(x)} = x \exp\{\mathcal{E}_t^{(x)}\}, \quad t \geq 0, \tag{5}$$

donde

$$\mathcal{E}_t^{(x)} = \xi_{\sigma_t}^{(x)} + \sum_{k=0}^{N_t^{(x)}-1} \left(\xi_\zeta^{(x,k)} + U^{(x,k)}\right) + i\pi N_t^{(x)}, \quad t \geq 0.$$

El Teorema 2 establece que cualquier proceso de Lamperti-Kiu se puede obtener como en (5).

Teorema 2. *Sea $Y^{(x)}$ el proceso definido en (5). Entonces,*

(i) el proceso $Y^{(x)}$ es Felleriano en $\mathbb{R}^*$ y satisface (2). Además, para cualquier tiempo de paro finito $\mathbf{T}$:

$$((Y^{(x)}_{\mathbf{T}})^{-1}Y^{(x)}_{\mathbf{T}+s}, s \geq 0) \stackrel{\mathcal{L}}{=} (\exp\{\widetilde{\mathcal{E}}_s^{(sgn(Y^{(x)}_{\mathbf{T}}))}\}, s \geq 0),$$

donde $\widetilde{\mathcal{E}}^{(\cdot)}$ es una copia de $\mathcal{E}^{(\cdot)}$, independiente de $(\mathcal{E}^{(\cdot)}_u, 0 \leq u \leq \mathbf{T})$.

(ii) Sea $(X^{(x)})_{x\in\mathbb{R}^*} = (X, \mathbb{P}_x)_{x\in\mathbb{R}^*}$ una familia de procesos de Markov autosimilares a valores reales de índice $\alpha > 0$, tal que $\mathbb{P}_x(H_1 < \infty) = 1$, para toda $x \in \mathbb{R}^*$. Para cada $x \in \mathbb{R}^*$ defina el proceso $\mathcal{Y}^{(x)}$ por

$$\mathcal{Y}^{(x)}_t = X^{(x)}_{\nu^{(x)}(t)}, \quad t \geq 0,$$

donde

$$\nu^{(x)}(t) = \inf\left\{s > 0 : \int_0^s |X^{(x)}_u|^{-\alpha} du > t\right\}.$$

Entonces $\mathcal{Y}^{(x)}$ se puede descomponer como en (5). Más aun, cada proceso de Lamperti-Kiu se puede construir como en (5).

(iii) Recíprocamente, sea $(Y^{(x)})_{x\in\mathbb{R}^*}$ una familia de procesos definidos como en (5) y considere los procesos $(X^{(x)})_{x\in\mathbb{R}^*}$ dados por

$$X^{(x)}_t = Y^{(x)}_{\tau(t|x|^{-\alpha})}, \quad t \geq 0,$$

donde

$$\tau(t) = \inf\left\{s > 0 : \int_0^s |\exp\{\alpha\mathcal{E}^{(x)}_u\}| du > t\right\}, \quad t < T,$$

para algún $\alpha > 0$. Entones $(X^{(x)})_{x\in\mathbb{R}^*}$ es una familia de procesos de Markov autosimilares a valores reales de índice $\alpha > 0$.

El Teorema 2 también nos permite calcular el generador infinitesimal de los procesos de Lamperti-Kiu, esto se consigue en la sección 1.2.2 de esta tesis. Con ayuda del generador infinitesimal de los procesos de Lamperti-Kiu y del teorema de Volkonski, podemos dar dos ejemplos en donde las características de los procesos subyacentes se pueden calcular en forma explícita. El primero de ellos es el bien conocido proceso α-estable y el segundo es el proceso α-estable condicionado a evitar cero, el cual es un ejemplo particular de los procesos de Lévy condicionados a evitar cero, construidos en la segunda parte de esta tesis.

Capítulo 2. Procesos de Lévy condicionados a evitar cero

Una de las aportaciones de este capítulo consiste en construir la ley de los procesos de Lévy condicionados a evitar cero. Para esta construcción, seguimos algunas ideas de los artículos [17], [48], siendo que en éste último el caso simétrico ha sido estudiado. La herramienta matemática usada para esta construcción es la técnica de la h-transformada. El procedimiento es como sigue, consideramos un proceso de Lévy cumpliendo dos hipótesis: el punto cero es regular por si mismo y el proceso de Lévy no es un proceso de Poisson compuesto, encontramos una función invariante para el semigrupo del proceso de Lévy matado en su primer tiempo de llegada a cero. Con ayuda de esta función, la h-transformada y la medida de excursiones fuera del cero, generamos una nueva familia de medidas de probabilidad. A continuación damos algunos detalles matemáticos y una breve discusión sobre el procedimiento empleado.

Para enunciar los resultados principales, introducimos algo de notación. Sea $\mathcal{D}[0,\infty)$ el espacio de trayectorias càdlàg $\omega : [0,\infty) \to \mathbb{R} \cup \{\Delta\}$ con tiempo de vida $\zeta(\omega) = \inf\{s : \omega_s = \Delta\}$, donde Δ es un punto cementerio. Sean X el proceso coordenadas y $\mathcal{F}_t = (X_s, s \leq t)$, la filtración natural de X. Sea $(\mathbb{P}_x, x \geq 0)$ una familia de medidas tal que $(X, \mathbb{P}_x)$ es un proceso de Lévy iniciando en x, escribimos $\mathbb{P} = \mathbb{P}_0$. Denotemos por n la medida de excursiones fuera del cero. Sea T_0 el primer tiempo de llegada a cero para X, esto es, $T_0 = \inf\{s > 0 : X_s = 0\}$, con $\inf\{\emptyset\} = \infty$. La familia $(P_t^0, t \geq 0)$ es llamada semigrupo del proceso de Lévy matado en T_0.

Es bien conocido que bajo la hipótesis de que el punto cero sea regular en si mismo y suponiendo que $(X, \mathbb{P})$ no es un proceso Poisson compuesto, es posible asegurar la existencia de una densidad continua para el q-resolvente. Denotemos por u_q la densidad del q-resolvente y sea $(h_q, q > 0)$ la sucesión de funciones dada por

$$h_q(x) = u_q(0) - u_q(-x) = [n(\zeta > \mathbf{e}_q)]^{-1} \mathbb{P}_x(T_0 > \mathbf{e}_q), \quad q > 0, \quad x \in \mathbb{R}. \tag{6}$$

Tenemos que para cualquier $q > 0$, la función h_q es una función excesiva para el semigrupo P_t^0. Definimos la función h por

$$h(x) = \lim_{q \to 0} h_q(x), \quad x \in \mathbb{R}.$$

La función h está bien definida y es una función P_t^0-invariante tal como lo establece el teorema siguiente.

Teorema 3. *La función $h(x)$ está bien definida y es invariante con respecto al semigrupo del proceso de Lévy matado en T_0, i.e.,*

$$P_t^0 h(x) = h(x), \quad t > 0, x \in \mathbb{R}.$$

Además,

$$n(h(X_t), t < \zeta) = 1, \quad \forall\, t > 0.$$

El Teorema 3 nos permite definir una nueva familia de medidas sobre el conjunto $\mathcal{H}_0 = \mathcal{H} \cup \{0\}$, donde $\mathcal{H} = \{x \in \mathbb{R} : h(x) > 0\}$. Sea $(\mathbb{P}_x^{\updownarrow}, x \in \mathcal{H}_0)$ la única familia de medidas tales que para $x \in \mathcal{H}_0$,

$$\mathbb{P}_x^{\updownarrow}(\Lambda) = \begin{cases} \dfrac{1}{h(x)} \mathbb{E}_x^0(\mathbf{1}_\Lambda h(X_t)), & x \in \mathcal{H}, \\ \\ n(\mathbf{1}_\Lambda h(X_t) \mathbf{1}_{\{t<\zeta\}}), & x = 0, \end{cases}$$

para toda $\Lambda \in \mathcal{F}_t$, para toda $t \geq 0$. De esta manera, tenemos:

Teorema 4. *Las medidas $(\mathbb{P}_x^{\updownarrow})_{x \in \mathcal{H}_0}$ es una familia de medidas Markoviana tales que*

(i) $\mathbb{P}_x^{\updownarrow}(X_0 = x) = 1$, $\forall\, x \in \mathcal{H}_0$.

(ii) $\mathbb{P}_x^{\updownarrow}(T_0 = \infty) = 1$, $\forall\, x \in \mathcal{H}_0$.

con semigrupo:

$$P_t^{\updownarrow}(x, dy) := \frac{h(y)}{h(x)} P_t^0(x, dy), \quad x \in \mathcal{H}, \quad t \geq 0,$$

y ley de entrada bajo $\mathbb{P}_0^{\updownarrow}$ *dada por*

$$\mathbb{P}_0^{\updownarrow}(X_t \in dy) = n(h(y)\mathbf{1}_{\{X_t \in dy\}}\mathbf{1}_{\{t<\zeta\}}).$$

Ya que $\mathbb{P}_x^{\updownarrow}(T_0 = \infty) = 1$, para toda $x \in \mathcal{H}_0$, entonces llamamos a $(X, \mathbb{P}_x^{\updownarrow})_{x \in \mathcal{H}_0}$ procesos de Lévy condicionados a evitar cero. Además, como se mencionó anteriormente, h_q es una función P_t^0-excesiva, entonces por (6), para $x \in \mathcal{H}$, es posible construir un proceso Lévy condicionado a evitar cero hasta un tiempo exponencial con parámetro $q > 0$. Si tomamos límite cuando $q \to 0$, como el teorema siguiente muestra, podemos obtener la ley $\mathbb{P}_x^{\updownarrow}$. Esta es otra razón por la cual usamos el nombre procesos de Lévy condicionados a evitar cero.

Teorema 5. *Sea* $\mathbf{e}_q$ *una variable aleatoria exponencial con parámetro* $q > 0$*, independiente de* $(X, \mathbb{P})$*. Entonces, para cualquier* $x \in \mathcal{H}$ *y cualquier* $(\mathcal{F}_t)_{t\geq 0}$*-tiempo de paro* T*,*

$$\lim_{q\to 0} \mathbb{P}_x(\Lambda, T < \mathbf{e}_q \mid T_0 > \mathbf{e}_q) = \mathbb{P}_x^{\updownarrow}(\Lambda), \quad \forall\, \Lambda \in \mathcal{F}_T.$$

En el caso α-estable, la función h se puede calcular explícitamente, esto es,

$$h(x) = \begin{cases} |x|, & \alpha = 2, \\ \\ K(\alpha)(1 - \beta \operatorname{sgn}(x))|x|^{\alpha-1}, & \alpha \in (1, 2), \end{cases}$$

donde

$$K(\alpha) = \frac{\Gamma(2-\alpha)\sin(\alpha\pi/2)}{c\pi(\alpha-1)(1+\beta^2\tan^2(\alpha\pi/2))}$$

y

$$c = -\frac{(c^+ + c^-)\Gamma(2-\alpha)}{\alpha(\alpha-1)}\cos(\alpha\pi/2), \quad \beta = \frac{c^+ - c^-}{c^+ + c^-}.$$

El proceso resultante es el proceso α-estable condicionado a evitar cero, el cual aparece en el Capítulo 1 como ejemplo de un proceso de Markov autosimilar a valores reales.

Finalmente, si X es un proceso de Lévy espectralmente negativo satisfaciendo las condiciones mencionadas al principio, entonces

$$h(x) = \begin{cases} \dfrac{1}{\Psi'(\Phi(0)+)}(1 - e^{\Phi(0)x}) + W(x), & \text{si } \lim_{t\to\infty} X_t = -\infty, \\ \dfrac{-x}{\Psi''(0+)} + W(x), & \text{si } \limsup_{t\to\infty} X_t = -\liminf_{t\to\infty} X_t = \infty, \\ W(x), & \text{si } \lim_{t\to\infty} X_t = \infty, \end{cases}$$

donde Ψ es el exponente de Laplace del proceso $(X, \mathbb{P})$, $\Phi(q)$ es la raiz más grande de la ecuación $\Psi(\lambda) = q$ y W es la función 0-escala del proceso $(X, \mathbb{P})$.

Organización de la tesis

La tesis está organizado como sigue. El Capítulo 1 está divido en cuatro secciones. La Sección 1.1 es sobre algunos hechos de los procesos de Markov autosimilares y la representación de Lamperti. La Sección 1.2.1 está dedicada a algunos resultados preliminares acerca de los procesos de Markov autosimilares a valores reales. En la Sección 1.2.2 construimos el proceso subyacente en la representación de Lamperti y establecemos el resultado de que todo proceso de Lamperti-Kiu se puede escribir de esta forma. También se muestra la representación de Lamperti y se calcula el generador infinitesimal de los procesos de Lamperti-Kiu en esta sección. La Sección 1.3 está dedicada a probar los resultados principales. En la Sección 1.4 proporcionamos dos ejemplos en donde es posible calcular explícitamente las características de los procesos de Lamperti-Kiu: el proceso α-estable y el proceso α-estable condicionado a evitar cero. El Capítulo 2 se encuentra dividido en cinco secciones. La Sección 2.1 es una breve introducción sobre algunos la construcción de los procesos de Lévy condidicionados a evitar cero en el caso simétrico y los procesos de Lévy condicionados a permanecer positivos. Se introduce notación en la Sección 2.2.1. Los resultados principales son enunciados en la Sección 2.2.2 y sus pruebas se encuentran en la Sección 2.3. Algunas propiedades de la sucesión de funciones que define a h y de h misma, se encuentran en la Secciones 2.3.2 y 2.3.3. En la Sección 2.3.4 se introduce una función auxiliar y se demuestran algunas propiedades de ésta. La Sección 2.4 está dedicada exclusivamente a probar los resultados principales. Finalmente, en la Sección 2.5 estudiamos los casos α-estable y espectralmente negativo.

Introduction

This thesis is divided into two chapters. Chapter 1 is the paper accepted for publication in the Bernoulli journal, "The Lamperti representation of real-valued self-similar Markov processes". Chapter 2 corresponds to a paper still in process, "On Lévy processes conditioned to avoid zero". The common point of these two chapters is real-valued Lévy processes. We will give a brief summary on real-valued Lévy processes and their relation to these in each chapter. Also, we will give a general description of the content of Chapters 1 and 2.

Lévy processes

The real-valued Lévy processes are càdlàg stochastic processes having independent and stationary increments. In Chapter 1, we have that all positive self-similar Markov processes can be expressed as the exponential of Lévy processes time changed by the inverse of their exponential functional. We generalize this property to the real-valued case, that is, we obtain a similar representation in the case when the self-similar Markov process is taking values in the real line. In Chapter 2, the regularity of the point zero for a Lévy process implies the existence of a continuous density for the q-resolvent kernel (see p.g. [5]). Under the additional assumption that the Lévy process is not a compound Poisson process, we find an invariant function for the semigroup of the killed process at its first hitting time. The invariant function is obtained as a limit of a sequence of functions determined by the continuous density of the q-resolvent kernel. With help of the h-Doob transformation, we construct the law of a new kind of Markov processes.

In both chapters, the triple (a, σ, π), which characterize the Lévy processes, is fundamental in proofs, formulas and examples. We recall the definition of the triple of a Lévy process. If ξ is a Lévy process with law $\mathbb{P}$, then for any $t > 0$, ξ_t is an infinite divisible random variable and its Fourier transform admits the Lévy-Khintchine decomposition, i.e., there exists a function $\psi : \mathbb{R} \to \mathbb{C}$ such that

$$\mathbb{E}(e^{i\lambda\xi_t}) = e^{-t\psi(\lambda)}, \quad \lambda \in \mathbb{R},$$

with ψ given by

$$\psi(\lambda) = ia\lambda + \frac{\sigma^2}{2}\lambda^2 + \int_{\mathbb{R}} (1 - e^{i\lambda x} + i\lambda x \mathbf{1}_{\{|x|<1\}})\pi(dx), \quad \lambda \in \mathbb{R}, \tag{1}$$

where $a \in \mathbb{R}$, $\sigma \geq 0$ and π is a measure defined on $\mathbb{R} \setminus \{0\}$ satisfying $\int_{\mathbb{R}}(1 \wedge x^2)\pi(dx) < \infty$. The constant a is known as drift, σ is the Gaussian coefficient and π is called a Lévy measure.

For computational convenience, the characteristic exponent is chosen in different ways in each chapter. For instance, in Chapter 2 we will take ψ as in (1), while in Chapter 1, ψ satisfies $\mathbb{E}(e^{i\lambda\xi_t}) = e^{t\psi(\lambda)}$, $t > 0$, $\lambda \in \mathbb{R}$ with

$$\psi(\lambda) = ia\lambda - \frac{\sigma^2}{2}\lambda^2 + \int_{\mathbb{R}} (e^{i\lambda x} - 1 - i\lambda x \mathbf{1}_{\{|x|<1\}})\pi(dx), \quad \lambda \in \mathbb{R}.$$

An important example of a Lévy process, which will be presented in this thesis, is the so called α-stable process. For $\alpha = 2$, the process ξ is the well known and studied Brownian motion. In the case $\alpha \in (0, 2)$, the α-stable process has no Gaussian coefficient and the Lévy measure has a density ν with respect to Lebesgue measure given by

$$\nu(y) = c^+ y^{-\alpha-1}\mathbf{1}_{\{y>0\}} + c^-|y|^{-\alpha-1}\mathbf{1}_{\{y<0\}},$$

with c^+ and c^- being two nonnegative constants such that $c^+ + c^- > 0$. Furthermore, it is verified that ψ can be expressed as

$$\psi(\lambda) = c|\lambda|^\alpha(1 - i\beta\mathrm{sgn}(\lambda)\tan(\alpha\pi/2)), \quad \lambda \in \mathbb{R},$$

where

$$c = -\frac{(c^+ + c^-)\Gamma(2-\alpha)}{\alpha(\alpha-1)}\cos(\alpha\pi/2), \quad \beta = \frac{c^+ - c^-}{c^+ + c^-}.$$

For a detailed account on the theory of Lévy processes, see [5, 35, 45].

Chapter 1. The Lamperti representation of real-valued self-similar Markov processes

In this chapter we make contributions to the theory of real-valued self-similar Markov processes. To be specific, we obtain a Lamperti type representation for real-valued self-similar Markov processes killed at their first hitting time of zero, that is, we represent real-valued self-similar Markov processes as time changed multiplicative invariant processes.

Lamperti representation

Let E be $[0, \infty)$ or $\mathbb{R}^n$. A càdlàg strong Markov family $\{X^{(x)} = (X, \mathbb{P}_x), x \in E\}$ is called a E-valued self-similar Markov process of index $\alpha > 0$ if for all $c > 0$, the law of $(cX_{c^{-\alpha}t}, t \geq 0)$ under $\mathbb{P}_x$, is the same as $(X_t, t \geq 0)$ under $\mathbb{P}_{cx}$, for all x. The case when the self-similar Markov process is taking values in the positive half-line was first studied by Lamperti in 1972. In his paper, he proved several interesting properties for this particular class of self-similar Markov processes, but the result that interests us is the well known Lamperti representation. The Lamperti representation, sometimes called Lamperti transformation, establishes that any positive self-similar Markov process killed the first time that it reaches the point zero, can be represented as the exponential of Lévy processes time changed by the inverse of their exponential functional. Formally, if X is a positive self-similar Markov process of index $\alpha > 0$, then the process $(\xi_t, t \geq 0)$ defined by

$$\exp\{\xi_t\} = x^{-1} X_{\nu(t)}, \quad t \geq 0,$$

where

$$\nu(t) = \inf\left\{ s > 0 : \int_0^s (X_u)^{-\alpha} du > t \right\},$$

with the usual convention $\inf\{\emptyset\} = +\infty$, is a $\mathbf{P}$-Lévy process. Here, $\mathbf{P} = \mathbb{P}_1$.

Our aim in this part of the thesis is to generalize the latter result to the case when the process has the real line as state space. To get our aim, we follow some ideas in [22] in order to characterize the underlying processes in this representation. The resulting underlying processes are the so called Feller multiplicative invariant processes, which appearing in [34] as time changed $\mathbb{R}^n$-valued self-similar Markov processes. In the case $n = 1$, we will call them Lamperti-Kiu processes. Formally, a Lamperti-Kiu process $Y = (Y_t, t \geq 0)$, is a càdlàg process taking values in $\mathbb{R}^* := \mathbb{R} \setminus \{0\}$, having the Feller property and satisfying

$$\{(aY_t, t \geq 0), \mathbb{P}_x\} \stackrel{\mathcal{L}}{=} \left\{(\mathrm{sgn}(a) Y_t, t \geq 0), \mathbb{P}_{|a|x}\right\}, \tag{2}$$

for all $x, a \neq 0$. We also give a representation of the Lamperti-Kiu process as the exponential of certain processes. Doing this, we also complete Kiu's work ([34]).

Two main results are established in this chapter. Theorem 1 (below) establishes that, depending on the sign of the process, the behaviour of real-valued self-similar Markov processes between time sign changes is as a positive (or negative) self-similar Markov process. We state this formally, let H_n be the n-th change of sign of the process X:

$$H_0 = 0, \quad H_n = \inf\{t > H_{n-1} : X_t X_{t-} < 0\}, \quad n \geq 1.$$

Assume that $\mathbb{P}_x(H_1 < \infty) = 1$, for all $x \in \mathbb{R}^*$. Define,

$$\mathcal{X}_t^{(n)} = \frac{X_{H_n + |X_{H_n}|^\alpha t}}{|X_{H_n}|}, \quad 0 \leq t < |X_{H_n}|^{-\alpha}(H_{n+1} - H_n), \tag{3}$$

and

$$J_n = \frac{X_{H_{n+1}}}{X_{H_{n+1}-}}, \quad n \geq 0. \tag{4}$$

Then, $(\mathcal{X}^{(n)}, n \geq 0)$ and $(J_n, n \geq 0)$ satisfy the following.

Theorem 1. *Let $X^{(x)} = (X, \mathbb{P}_x)_{x \in \mathbb{R}^*}$ be a family of real-valued self-similar Markov processes of index $\alpha > 0$, such that $\mathbb{P}_x(H_1 < \infty) = 1$, for all $x \in \mathbb{R}^*$. Then*

(i) The paths between sign changes, $(\mathcal{X}^{(n)}, n \geq 0)$, as defined in (3), are independent under $\mathbb{P}_x$, for $x \in \mathbb{R}^*$. Furthermore, for all $n \geq 0$,

$$\left\{ \left(\mathcal{X}_t^{(n)}, 0 \leq t < |X_{H_n}|^{-\alpha}(H_{n+1} - H_n) \right), \mathbb{P}_x \right\} \stackrel{\mathcal{L}}{=} \left\{ (X_t, 0 \leq t < H_1), \mathbb{P}_{sgn(x)(-1)^n} \right\}.$$

Hence, they are time changed Lévy processes killed at an exponential time.

(ii) The random variables $J_n, n \geq 0$, as defined in (4), are independent under $\mathbb{P}_x$, for $x \in \mathbb{R}^*$ and for $n \geq 0$, the identity

$$\left\{J_n, \mathbb{P}_x\right\} \stackrel{\mathcal{L}}{=} \left\{J_0, \mathbb{P}_{sgn(x)(-1)^n}\right\},$$

holds.

(iii) For every $n \geq 0$, the process $\mathcal{X}^{(n)}$ and the random variable J_n are independent, under $\mathbb{P}_x$, for $x \in \mathbb{R}^*$.

Theorem 1 implies that six random objects define the underlying processes in the Lamperti representation, that is, we have $(\xi^+, \xi^-, \zeta^+, \zeta^-, U^+, U^-)$, where ξ^+, ξ^- are two Lévy processes, ζ^+, ζ^- are two exponential random variables and U^+, U^- are two real random variables, all independent. With this in mind, the following process is constructed.

Let ξ^+, ξ^- be real valued Lévy processes; ζ^+, ζ^- be exponential random variables with parameters q^+, q^-, respectively, and U^+, U^- be real valued random variables. Let $(\xi^{+,k}, k \geq 0)$, $(\xi^{-,k}, k \geq 0)$, $(\zeta^{+,k}, k \geq 0)$, $(\zeta^{-,k}, k \geq 0)$, $(U^{+,k}, k \geq 0)$, $(U^{-,k}, k \geq 0)$ be independent sequences of i.i.d. random variables such that

$$\xi^{+,0} \stackrel{\text{Law}}{=} \xi^+, \quad \xi^{-,0} \stackrel{\text{Law}}{=} \xi^-, \quad \zeta^{+,0} \stackrel{\text{Law}}{=} \zeta^+, \quad \zeta^{-,0} \stackrel{\text{Law}}{=} \zeta^-, \quad U^{+,0} \stackrel{\text{Law}}{=} U^+, \quad U^{-,0} \stackrel{\text{Law}}{=} U^-.$$

For every $x \in \mathbb{R}^*$ fixed, we consider the sequence $((\xi^{(x,k)}, \zeta^{(x,k)}, U^{(x,k)}), k \geq 0)$, where for $k \geq 0$,

$$(\xi^{(x,k)}, \zeta^{(x,k)}, U^{(x,k)}) = \begin{cases} (\xi^{+,k}, \zeta^{+,k}, U^{+,k}), & \text{if } \operatorname{sgn}(x)(-1)^k = 1, \\ \\ (\xi^{-,k}, \zeta^{-,k}, U^{-,k}), & \text{if } \operatorname{sgn}(x)(-1)^k = -1. \end{cases}$$

Let $(T_n^{(x)}, n \geq 0)$ be the sequence defined by

$$T_0^{(x)} = 0, \quad T_n^{(x)} = \sum_{k=0}^{n-1} \zeta^{(x,k)}, \quad n \geq 1,$$

and $(N_t^{(x)}, t \geq 0)$ be the alternating renewal type process:

$$N_t^{(x)} = \max\{n \geq 0 : T_n^{(x)} \leq t\}, \quad t \geq 0.$$

Write

$$\sigma_t^{(x)} = t - T_{N_t^{(x)}}^{(x)}, \quad \xi_{\sigma_t}^{(x)} = \xi_{\sigma_t^{(x)}}^{(x, N_t^{(x)})}, \quad \xi_\zeta^{(x,k)} = \xi_{\zeta^{(x,k)}}^{(x,k)}.$$

Let $(Y_t^{(x)}, t \geq 0)$ be given by

$$Y_t^{(x)} = x \exp\{\mathcal{E}_t^{(x)}\}, \quad t \geq 0, \tag{5}$$

where

$$\mathcal{E}_t^{(x)} = \xi_{\sigma_t}^{(x)} + \sum_{k=0}^{N_t^{(x)}-1} \left(\xi_\zeta^{(x,k)} + U^{(x,k)}\right) + i\pi N_t^{(x)}, \quad t \geq 0.$$

Theorem 2 establishes that any Lamperti-Kiu process can be written as (5).

Theorem 2. *Let $Y^{(x)}$ be the process defined in (5). Then,*

(i) the process $Y^{(x)}$ is Fellerian in $\mathbb{R}^*$ and satisfies (2). Furthermore, for any finite stopping time $\mathbf{T}$:

$$((Y^{(x)}_{\mathbf{T}})^{-1}Y^{(x)}_{\mathbf{T}+s}, s \geq 0) \stackrel{\mathcal{L}}{=} (\exp\{\widetilde{\mathcal{E}}^{(sgn(Y^{(x)}_{\mathbf{T}}))}_s\}, s \geq 0),$$

where $\widetilde{\mathcal{E}}^{(\cdot)}$ is a copy of $\mathcal{E}^{(\cdot)}$ which is independent of $(\mathcal{E}^{(\cdot)}_u, 0 \leq u \leq \mathbf{T})$.

(ii) Let $(X^{(x)})_{x\in\mathbb{R}^*} = (X, \mathbb{P}_x)_{x\in\mathbb{R}^*}$ be a family of real-valued self-similar Markov processes of index $\alpha > 0$ such that $\mathbb{P}_x(H_1 < \infty) = 1$, for all $x \in \mathbb{R}^*$. For every $x \in \mathbb{R}^*$ define the process $\mathcal{Y}^{(x)}$ by

$$\mathcal{Y}^{(x)}_t = X^{(x)}_{\nu^{(x)}(t)}, \quad t \geq 0,$$

where

$$\nu^{(x)}(t) = \inf\left\{s > 0 : \int_0^s |X^{(x)}_u|^{-\alpha} du > t\right\}.$$

Then $\mathcal{Y}^{(x)}$ may be decomposed as in (5). Moreover, every Lamperti-Kiu process can be constructed as explained in (5).

(iii) Conversely, let $(Y^{(x)})_{x\in\mathbb{R}^*}$ be a family of processes as constructed in (5) and consider the processes $(X^{(x)})_{x\in\mathbb{R}^*}$ given by

$$X^{(x)}_t = Y^{(x)}_{\tau(t|x|^{-\alpha})}, \quad t \geq 0,$$

where

$$\tau(t) = \inf\left\{s > 0 : \int_0^s |\exp\{\alpha\mathcal{E}^{(x)}_u\}| du > t\right\}, \quad t < T,$$

for some $\alpha > 0$. Then $(X^{(x)})_{x\in\mathbb{R}^*}$ is a family of real-valued self-similar Markov processes of index $\alpha > 0$.

Theorem 2 also allows us to compute the infinitesimal generator of the Lamperti-Kiu processes, this is accomplished in Section 1.2.2. With help of the infinitesimal generator of Lamperti-Kiu processes and Volkonski's theorem we provide two examples where the characteristics of the underlying processes can be computed explicitly. The first one is the well known α-stable process and the second is the α-stable process conditioned to avoid zero, which is a particular example of a Lévy process conditioned to avoid zero constructed in the second part of this thesis.

Chapter 2. The Lévy processes conditioned to avoid zero

One of contributions in this chapter consists of constructing the law of Lévy processes conditioned to avoid zero. For this construction, we follow some ideas from the papers [17] and [48], in the latter, the symmetric case has been studied. The mathematical tool to perform this construction is the h-path transformation technique. The procedure is as follows, we consider a Lévy process satisfying two hypotheses: the point zero is regular for itself and the Lévy process is not a compound Poisson process. We find an invariant function for the semigroup of the Lévy process killed at its first hitting time of zero. With help of this function, the h-path transformation and the excursion measure away from zero, we generate a new family of probability measures. Here are some mathematical details and a brief discussion on the procedure.

In order to state the main results, we introduce some notation. Let $\mathcal{D}[0,\infty)$ be the space of càdlàg paths $\omega : [0,\infty) \to \mathbb{R} \cup \{\Delta\}$ with lifetime $\zeta(\omega) = \inf\{s : \omega_s = \Delta\}$, where Δ is a cemetery point. Let X be the coordinate process and $\mathcal{F}_t = (X_s, s \leq t)$. Let $(\mathbb{P}_x, x \geq 0)$ be a family of measures such that $(X, \mathbb{P}_x)$ is a Lévy process with starting point x, we set $\mathbb{P} = \mathbb{P}_0$. Denote by n the excursion measure away from zero. Let T_0 be the first hitting time of zero for X, that is, $T_0 = \inf\{s > 0 : X_s = 0\}$, with $\inf\{\emptyset\} = \infty$. The family $(P_t^0, t \geq 0)$ is called semigroup of the Lévy process killed at T_0.

It is well known that under the hypothesis of regularity for itself for the point zero and assuming that $(X, \mathbb{P})$ is not a compound Poisson process, we always can ensure the existence of continuous density for the q-resolvent. Denote by u_q the continuous density of the q-resolvent and let $(h_q, q > 0)$ be the sequence of functions given by

$$h_q(x) = u_q(0) - u_q(-x) = [n(\zeta > \mathbf{e}_q)]^{-1}\mathbb{P}_x(T_0 > \mathbf{e}_q), \quad q > 0, \quad x \in \mathbb{R}. \tag{6}$$

We have that for any $q > 0$, the function h_q is an excessive function for the semigroup P_t^0. We define the function h by

$$h(x) = \lim_{q\to 0} h_q(x), \quad x \in \mathbb{R}.$$

The function h is a well defined function and is P_t^0-invariant as the following theorem establishes.

Theorem 3. *The function $h(x)$ is well defined and is invariant with respect to the semigroup of the killed process, i.e.,*

$$P_t^0 h(x) = h(x), \quad t > 0, x \in \mathbb{R}.$$

Furthermore,

$$n(h(X_t), t < \zeta) = 1, \quad \forall\, t > 0.$$

Theorem 3 allows us to define a new family of measures on the set $\mathcal{H}_0 = \mathcal{H} \cup \{0\}$, where $\mathcal{H} = \{x \in \mathbb{R} : h(x) > 0\}$. Let $(\mathbb{P}_x^{\updownarrow}, x \in \mathcal{H}_0)$ be the unique family of measures such that for $x \in \mathcal{H}_0$,

$$\mathbb{P}_x^{\updownarrow}(\Lambda) = \begin{cases} \dfrac{1}{h(x)}\mathbb{E}_x^0(\mathbf{1}_\Lambda h(X_t)), & x \in \mathcal{H}, \\ \\ n(\mathbf{1}_\Lambda h(X_t)\mathbf{1}_{\{t<\zeta\}}), & x = 0. \end{cases}$$

for all $\Lambda \in \mathcal{F}_t$, for all $t \geq 0$. We have the following Theorem.

Theorem 4. *The measures $(\mathbb{P}_x^{\updownarrow})_{x\in\mathcal{H}_0}$ is a Markovian family of measures such that*

(i) $\mathbb{P}_x^{\updownarrow}(X_0 = x) = 1$, $\forall\, x \in \mathcal{H}_0$.

(ii) $\mathbb{P}_x^{\updownarrow}(T_0 = \infty) = 1$, $\forall\, x \in \mathcal{H}_0$.

with semigroup:

$$P_t^{\updownarrow}(x,dy) := \frac{h(y)}{h(x)} P_t^0(x,dy), \quad x \in \mathcal{H}, \quad t \geq 0,$$

and entrance law under $\mathbb{P}_0^{\updownarrow}$ *given by*

$$\mathbb{P}_0^{\updownarrow}(X_t \in dy) = n(h(y)\mathbf{1}_{\{X_t \in dy\}}\mathbf{1}_{\{t<\zeta\}}).$$

Since $\mathbb{P}_x^{\updownarrow}(T_0 = \infty) = 1$, for all $x \in \mathcal{H}_0$, we call $(X, \mathbb{P}_x^{\updownarrow})_{x\in\mathcal{H}_0}$ Lévy processes conditioned to avoid zero. Furthermore, as mentioned above, h_q is a P_t^0-excessive function, then by (6), for $x \in \mathcal{H}$, it is possible to construct a Lévy process conditioned to avoid zero up to an exponential random time with parameter $q > 0$. If we take the limit as $q \to 0$, as the following theorem shows, we can obtain the law $\mathbb{P}_x^{\updownarrow}$. This is another reason whereby we use the name Lévy processes conditioned to avoid zero.

Theorem 5. *Let* $\mathbf{e}_q$ *be an exponential time with parameter* $q > 0$ *independent of* $(X, \mathbb{P})$*. Then for any* $x \in \mathcal{H}$*, and any* $(\mathcal{F}_t)_{t\geq 0}$*- stopping time* T*,*

$$\lim_{q\to 0} \mathbb{P}_x(\Lambda, T < \mathbf{e}_q \mid T_0 > \mathbf{e}_q) = \mathbb{P}_x^{\updownarrow}(\Lambda), \quad \forall \Lambda \in \mathcal{F}_T.$$

In the α-stable case, the function h can be computed explicitly, namely,

$$h(x) = \begin{cases} |x|, & \alpha = 2, \\ \\ K(\alpha)(1 - \beta \mathrm{sgn}(x))|x|^{\alpha-1}, & \alpha \in (1,2), \end{cases}$$

where

$$K(\alpha) = \frac{\Gamma(2-\alpha)\sin(\alpha\pi/2)}{c\pi(\alpha-1)(1+\beta^2\tan^2(\alpha\pi/2))}$$

and

$$c = -\frac{(c^+ + c^-)\Gamma(2-\alpha)}{\alpha(\alpha-1)}\cos(\alpha\pi/2), \quad \beta = \frac{c^+ - c^-}{c^+ + c^-}.$$

The resulting process is the α-stable process conditioned to avoid zero, which appears in Chapter 1 as an example of a real-valued self-similar Markov process.

Finally, if X is a spectrally negative Lévy process satisfying the conditions aforementioned, then

$$h(x) = \begin{cases} \frac{1}{\Psi'(\Phi(0)+)}(1 - e^{\Phi(0)x}) + W(x), & \text{if } \lim_{t\to\infty} X_t = -\infty, \\ \frac{-x}{\Psi''(0+)} + W(x), & \text{if } \limsup_{t\to\infty} X_t = -\liminf_{t\to\infty} X_t = \infty, \\ W(x), & \text{if } \lim_{t\to\infty} X_t = \infty, \end{cases}$$

where Ψ is the Laplace exponent of the process $(X, \mathbb{P})$, $\Phi(q)$ is the largest root of the equation $\Psi(\lambda) = q$, and W is the 0-scale function of the process $(X, \mathbb{P})$.

Organization of the thesis

The thesis is organized as follows. Chapter 1 is divided into four sections. Section 1.1 is on some facts of the self-similar Markov processes and Lamperti's representation. Section 1.2.1 is devoted to some preliminary results about real-valued self-similar Markov processes. In Section 1.2.2, we construct the underlying process in Lamperti's representation and establish the result that all Lamperti-Kiu processes can be written this way. Lamperti's representation is given and the infinitesimal generator of Lamperti-Kiu processes is computed in this section. Section 1.3 is devoted to prove the main results. In Section 1.4, we provide two examples where it is possible to compute explicitly the characteristics of the Lamperti-Kiu process: the α-stable process and the α-stable process conditioned to avoid zero. Chapter 2 is also divided into four sections. Section 2.1 is on some facts of the construction of Lévy processes conditioned to avoid zero in the symmetric case and Lévy processes conditioned to stay positive. Notation is introduced in Section 2.2.1. The main results are stated in Section 2.2.2 and their proofs are found in Section 2.3. Some properties of the sequence of functions which defines h, and h itself, are given in Section 2.3.2 and 2.3.3. In Section 2.3.4, an auxiliary function is introduced and some of its properties are shown. The Section 2.4 is exclusively dedicated to prove the main results. Finally, in Section 2.5 we study the cases: α-stable and spectrally negative Lévy processes.

Chapter 1

The Lamperti representation of real-valued self-similar Markov processes

1.1 Introduction

Semi-stable processes were introduced by Lamperti in [36] as those processes satisfying a scaling property. Nowadays this kind of processes are known as self-similar processes. Formally, a càdlàg stochastic process $X = (X_t, t \geq 0)$, with $X_0 = 0$, and Euclidean state space E, is self-similar of order $\alpha > 0$, if for every $a > 0$, the processes $(X_{at}, t \geq 0)$ and $(a^\alpha X_t, t \geq 0)$, have the same law. Lamperti proved that the class of self-similar processes is formed by those stochastic processes that can be obtained as the weak limit of sequences of stochastic processes that have been subject to an infinite sequence of dilations of scale of time and space. More formally, the main result of Lamperti in [36] can be stated as follows: let $(\widetilde{X}_t, t \geq 0)$ be a stochastic process defined in some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with values in E. Assume that there exists a positive real function $f(\eta) \nearrow \infty$ such that the process $(\widetilde{X}_t^\eta, t \geq 0)$ defined by

$$\widetilde{X}_t^\eta = \frac{\widetilde{X}_{\eta t}}{f(\eta)}, \quad t \geq 0,$$

converges to a non-degenerated process X in the sense of finite-dimensional distributions. Then, X is a self-similar process of order α and $f(\eta) = \eta^\alpha L(\eta)$, for some $\alpha > 0$, where L is a slowly varying function. The converse is also true, every self-similar process can be obtained in such a way.

If X is a Markov process with stationary transition function $P_t(x, A)$, then the self-similarity property written in terms of its transition function takes the form

$$P_{at}(x, A) = P_t(a^{-1/\alpha}x, a^{-1/\alpha}A), \tag{1.1}$$

for all $a > 0$, $t \geq 0$, $x \in E$, and all measurable sets A. We will assume that X is a strong Markov process and refer to it as a *self-similar Markov process of index* $\alpha > 0$.

From now on, Ω denotes the space of càdlàg paths, X the coordinates process and $(\mathcal{F}_t, t \geq 0)$ its natural filtration, i.e, $\mathcal{F}_t = \sigma(X_s, s \leq t)$.

There are many other ways than (1.1) to define self-similar Markov processes. The definition used in this thesis is the following.

Definition 1.1. *Let E be $[0,\infty)$ or $\mathbb{R}^n$. We will say that $\{X^{(x)} = (X, \mathbb{P}_x), x \in E\}$ is a family of E-valued self-similar Markov processes with index $\alpha > 0$ if it is a càdlàg strong Markov family with state space E, and that satisfies that for every $c > 0$,*

$$\{(cX_{c^{-\alpha}t}, t \geq 0), \mathbb{P}_x\} \stackrel{\mathcal{L}}{=} \{(X_t, t \geq 0), \mathbb{P}_{cx}\}, \quad \forall\, x \in E.$$

The case $E = [0,\infty)$ was first investigated by Lamperti in [37] and has further been the object of many studies, see for instance [6]-[14] and the reference therein. Here we summarize some of his main results. Let T be the first hitting time of zero for X, i.e.,

$$T = \inf\{t > 0 : X_t = 0\},$$

with $\inf\{\emptyset\} = \infty$. Then, for any starting point $x > 0$, one and only one of the following cases holds:

C.1 $T = \infty$, a.s.

C.2 $T < \infty$, $X_{T-} = 0$, a.s.

C.3 $T < \infty$, $X_{T-} > 0$, a.s.

We refer to *C.1* as the class of processes that never reach zero, processes in the class *C.2* hit zero continuously, and those in the class *C.3* reach zero by a jump. In particular, if T is finite, then the process reaches zero continuously or by a jump. Another important result in [37] is the representation of positive self-similar Markov processes as the exponential of Lévy processes time changed by the inverse of their exponential functional. This representation is known as the Lamperti representation and its extension to real-valued processes is one of the main motivations of this thesis. Formally, the Lamperti representation can be stated as follows. Assume that the process X is absorbed at 0. Let $(\xi_t, t \geq 0)$ be the process defined by

$$\exp\{\xi_t\} = x^{-1} X_{\nu(t)}, \quad t \geq 0,$$

where

$$\nu(t) = \inf\left\{s > 0 : \int_0^s (X_u)^{-\alpha} du > t\right\},$$

with the usual convention $\inf\{\emptyset\} = +\infty$. Then, under $\mathbb{P}_x$, ξ is a Lévy process. Furthermore, ξ satisfies either (i) $\limsup_{t\to\infty} \xi_t = \infty$ a.s., (ii) $\lim_{t\to\infty} \xi_t = -\infty$ a.s. or (iii) ξ is a Lévy process killed at an independent exponential time $\zeta < \infty$ a.s., depending on whether X is in the class *C.1*, *C.2* or *C.3*, respectively. Note that since an exponential random variable with parameter q is infinite if only if $q = 0$, then we can always consider the process ξ as a Lévy process killed at an independent exponential time ζ with parameter $q \geq 0$. Conversely, let $(\xi, \mathbf{P})$ be a

Lévy process killed at an exponential random time ζ with parameter $q \geq 0$ and cemetery point $\{-\infty\}$. Let $\alpha > 0$ and for $x > 0$, define the process $X^{(x)}$ by

$$X_t^{(x)} = x \exp\{\xi_{\tau(tx^{-\alpha})}\}, \quad t \geq 0,$$

where

$$\tau(t) = \inf\left\{u > 0 : \int_0^u \exp\{\alpha\xi_s\}ds > t\right\}.$$

Then, $(X^{(x)})_{x>0}$ is a positive self-similar Markov process of index $\alpha > 0$ which is absorbed at 0. Furthermore, the latter classification depending on the asymptotic behaviour of ξ holds. An important relation between T and the exponential functional of the Lévy process ξ is $(T, \mathbb{P}_x) \stackrel{\mathcal{L}}{=} (x^\alpha \int_0^\zeta \exp\{\alpha\xi_s\}ds, \mathbf{P})$. Further details on this topic can be found in [37, 6].

In [34] the case of $\mathbb{R}^n$-valued self-similar Markov processes was studied. The main result in [34] asserts that, if X killed at T is a Feller self-similar Markov process, then the process Y defined by

$$Y_t = X_{\nu(t)}, \quad t \geq 0,$$

where

$$\nu(t) = \inf\left\{s > 0 : \int_0^s |X_u|^{-\alpha}du > t\right\},$$

is a Feller multiplicative invariant process, i.e., Y is a Feller process with semigroup Q_t satisfying

$$Q_t(x, A) = Q_t(ax, aA), \tag{1.2}$$

for all $x \neq 0$, a, t positive and $A \in \mathcal{B}(\mathbb{R}^n \setminus \{0\})$. Another way to write (1.2) is

$$Q_t(x, a^{-1}A) = Q_t(|a|x, \operatorname{sgn}(a)A),$$

for all t positive, $x, a \neq 0$ and $A \in \mathcal{B}(\mathbb{R}^n \setminus \{0\})$. This property may also be written in terms of the process Y as follows:

$$\{(aY_t, t \geq 0), \mathbb{P}_x\} \stackrel{\mathcal{L}}{=} \left\{(\operatorname{sgn}(a)Y_t, t \geq 0), \mathbb{P}_{|a|x}\right\}, \tag{1.3}$$

for all $x, a \neq 0$. In [34], the converse of this result has not been proved but using (1.3), it is easy to verify that it actually holds. Indeed, let Y be a strong Markov process taking values in $\mathbb{R}^n \setminus \{0\}$ and satisfying (1.3). Let $\alpha > 0$ and define the process X by

$$X_t = Y_{\varphi(t)}, \quad t \geq 0,$$

where

$$\varphi(t) = \inf\left\{s > 0 : \int_0^s |Y_u|^\alpha du > t\right\},$$

with $\inf\{\emptyset\} = \infty$. Since the strong Markov process is preserved under time changes by additive functionals, then X is a strong Markov process. Note that

$$\varphi(c^{-\alpha}t)(Y) = \inf\left\{s > 0 : \int_0^s |cY_u|^\alpha > t\right\} = \varphi(t)(cY).$$

Then, using the latter identity and (1.3) with $a = c$, we have

$$\{(cX_{c^{-\alpha}t}, t \geq 0), \mathbb{P}_x\} = \{(cY_{\varphi(c^{-\alpha}t)}, t \geq 0), \mathbb{P}_x\} \stackrel{\mathcal{L}}{=} \{(Y_{\varphi(t)}, t \geq 0), \mathbb{P}_{cx}\} = \{(X_t, t \geq 0), \mathbb{P}_{cx}\}.$$

This proves the self-similarity property of X. Then X is a $\mathbb{R}^n$-valued self-similar Markov process of index $\alpha > 0$ which is killed at T. It is important to mention that no explicit form of Y has been given in [34]. Giving a construction of Feller multiplicative invariant processes taking values in $\mathbb{R}^* := \mathbb{R} \setminus \{0\}$, that we will call Lamperti-Kiu processes, is another main motivation of this thesis.

Definition 1.2. *Let $Y = (Y_t, t \geq 0)$ be a càdlàg process. We say that Y is a Lamperti-Kiu process if it takes values in $\mathbb{R}^*$, has the Feller property and (1.3) is satisfied.*

A subclass of Lamperti-Kiu processes has been studied by Chybiryakov in [22] who gave the following definition. Let Y be a $\mathbb{R}^*$-valued càdlàg process defined on some probability space $(\Omega, \mathcal{F}, \mathbf{P})$ such that $Y_0 = 1$. It is said that Y is a multiplicative Lévy process if for any $s, t > 0$, $Y_t^{-1} Y_{t+s}$ is independent of $\mathcal{G}_t = \sigma(Y_u, u \leq t)$ and the law of $Y_t^{-1} Y_{t+s}$ does not depend on t. It can be shown that if Y is a multiplicative Lévy process, then Y is Markovian and its semigroup satisfies (1.2). Furthermore, there exist a Lévy process ξ, a Poisson process N and a sequence $U = (U_k, k \geq 0)$ of i.i.d. random variables, all independent, such that

$$Y_t = \exp\left\{\xi_t + \sum_{k=1}^{N_t} U_k + i\pi N_t\right\}, \quad t \geq 0. \tag{1.4}$$

The converse is also true, i.e., if ξ is a Lévy process, N a Poisson process and $U = (U_k, k \geq 0)$ a sequence of i.i.d. random variables, ξ, N and U being independent, then Y defined by (1.4) is a multiplicative Lévy process. It is easy to see that a multiplicative Lévy process is a symmetric Lamperti-Kiu process.

The reason in [22] to study the class of multiplicative Lévy processes was to establish a Lamperti type representation for real valued processes that fulfill the scaling property given in the following definition. A strong Markov family $\{X^{(x)} = (X, \mathbb{P}_x), x \in \mathbb{R}^*\}$ with state space $\mathbb{R}^*$, is self-similar of index $\alpha > 0$ in the sense of [22], if for all $c \neq 0$,

$$\left\{(cX_{|c|^{-\alpha}t}, t \geq 0), \mathbb{P}_x\right\} \stackrel{\mathcal{L}}{=} \{(X_t, t \geq 0), \mathbb{P}_{cx}\}, \tag{1.5}$$

for all $x \in \mathbb{R}^*$. The Lamperti type representation given in [22] establishes that for such a self-similar process $X^{(x)}$, the process Y, defined by

$$Y_t = x^{-1} X^{(x)}_{\nu^{(x)}(t)}, \quad t \geq 0,$$

where

$$\nu^{(x)}(t) = \inf\left\{s > 0 : \int_0^s |X_u^{(x)}|^{-\alpha} du > t\right\}, \quad t \geq 0,$$

with $\inf\{\emptyset\} = \infty$, is a multiplicative Lévy process. Conversely, let Y be a multiplicative Lévy process, and

$$\mathcal{E}_t = \xi_t + \sum_{k=1}^{N_t} U_k + i\pi N_t, \quad t \geq 0,$$

where ξ, N and $(U_k, k \geq 0)$ are as in (1.4), so that $Y_t = \exp\{\mathcal{E}_t\}$, $t \geq 0$. For $x \in \mathbb{R}^*$, define $X^{(x)}$ by

$$X_t^{(x)} = xY_{\tau(tx^{-\alpha})}, \quad t \geq 0,$$

where

$$\tau(t) = \inf\left\{u > 0 : \int_0^u |\exp\{\alpha\mathcal{E}_u\}| du > t\right\}, \quad t \geq 0,$$

with $\inf\{\emptyset\} = \infty$. Then $X^{(x)}$ is a $\mathbb{R}^*$-valued self-similar Markov process in the sense of [22], which is recalled in (1.5).

It is important to observe that if we take $c = -1$ in (1.5), it is seen that the process $X^{(x)}$ is necessarily a symmetric process and as a consequence Y is also symmetric. In this work we establish the analogous description for non-symmetric real valued self-similar Markov processes.

1.2 Preliminaries and main results

1.2.1 Real-valued self-similar Markov processes and description of Lamperti-Kiu processes

In this section, we will prove some additional properties of real-valued self-similar Markov processes, in order to characterize them as time changed Lamperti-Kiu processes.

Let X be a real-valued self-similar Markov process. Let H_n be the n-th change of sign of the process X, i.e.,

$$H_0 = 0, \quad H_n = \inf\{t > H_{n-1} : X_t X_{t-} < 0\}, \quad n \geq 1.$$

Note that

$$\begin{aligned} H_1(X) &= \inf\{t > 0 : X_t X_{t-} < 0\} \\ &= |x|^\alpha \inf\left\{|x|^{-\alpha}t > 0 : \left(|x|^{-1}X_{|x|^\alpha |x|^{-\alpha}t}\right)\left(|x|^{-1}X_{|x|^\alpha(|x|^{-\alpha}t)-}\right) < 0\right\} \\ &= |x|^\alpha H_1\left(|x|^{-1}X_{|x|^\alpha \cdot}\right). \end{aligned} \tag{1.6}$$

Hence, by the self-similarity property, for $x \in \mathbb{R}^*$, it holds that $\mathbb{P}_x(H_1 < \infty) = \mathbb{P}_{\mathrm{sgn}(x)}(H_1 < \infty)$. Furthermore, proceeding as in the proof of Lemma 2.5 in [37], it is verified that for each $x \in \mathbb{R}^*$, either $\mathbb{P}_x(H_1 < \infty) = 1$ or $\mathbb{P}_x(H_1 < \infty) = 0$. The latter and former facts allow us to conclude that there are four mutually exclusive cases, namely,

C.1 $\mathbb{P}_x(H_1 < \infty) = 1$, $\forall\, x > 0$ and $\mathbb{P}_x(H_1 = \infty) = 1$, $\forall\, x < 0$;

C.2 $\mathbb{P}_x(H_1 < \infty) = 1$, $\forall\, x < 0$ and $\mathbb{P}_x(H_1 = \infty) = 1$, $\forall\, x > 0$;

C.3 $\mathbb{P}_x(H_1 = \infty) = 1$, $\forall\, x \in \mathbb{R}^*$;

C.4 $\mathbb{P}_x(H_1 < \infty) = 1$, $\forall\, x \in \mathbb{R}^*$.

In the case **C.1**, if the process X starts at a negative point, then $\{(-X_t \mathbf{1}_{\{t<T\}}, t \geq 0), \mathbb{P}_x\}_{x<0}$ behaves as a positive self-similar Markov process, which have already been characterized by Lamperti. Now, if the process starts at a positive point, it can be deduced from Lamperti's representation (further details are given in the forthcoming Theorem 1.5 *(i)*) that the process X behaves as a time changed Lévy process until it changes of sign, and when this occurs, by the strong Markov property, its behaviour is that of X issued from a negative point. The case **C.2** is similar to the first one. For the case **C.3**, depending on the starting point, X or $-X$ is a positive self-similar Markov process, again we fall in a known case. In summary, the Lamperti representation for the cases **C.1**-**C.3** can be obtained from the Theorem 1.5 *(i)* and the Lamperti representation for the positive self-similar Markov processes. Thus, we are only interested in the case **C.4**, where the process X a.s. has at least two changes of sign (and by the strong Markov property infinitely many changes of sign). For this case, we have:

Proposition 1.3. *If $\mathbb{P}_x(H_1 < \infty) = 1$, for all $x \in \mathbb{R}^*$, then the sequence of stopping times $(H_n, n \geq 0)$ converges to the first hitting time of zero T, $\mathbb{P}_x$-a.s., for all $x \in \mathbb{R}^*$.*

The proof of this result will be given in Section 1.3. We can see that under the condition of Proposition 1.3, if X is killed at T, then X has an infinite number of changes of sign before it dies. Moreover, if T is finite, then X reaches zero at time T continuously from the left.

The result in Proposition 1.3 is well known in the case where X is an α-stable process and X is not a subordinator. In that case, if $\alpha \in (0, 1]$, $T = \infty$ a.s., while if $\alpha \in (1, 2]$, with probability one, $T < \infty$ and X makes infinitely many jumps before reaching zero. This process and its Lamperti representation will be studied in section 1.4.1.

In the cases **C.1**-**C.3**, we can not ensure the veracity of Proposition 1.3. The following example illustrates the latter.

Example 1.4. Let $(X, \mathbb{P}_x)_{x\in\mathbb{R}}$ be a real-valued self-similar Markov process with index $\alpha > 0$. Assume that condition **C.4** is satisfied. Consider the stopping time

$$S = \inf\{t > 0 : X_{t-} < 0, X_t > 0\}$$

and the process $\widetilde{X}$ defined by

$$\widetilde{X}_t = X_t \mathbf{1}_{\{t<S\}}, \quad t \geq 0.$$

Then the process $(\widetilde{X}, \mathbb{P}_x)_{x\in\mathbb{R}}$ is a real-valued self-similar Markov process with the same index as X. The behaviour of the first change of sign $\widetilde{H}_1$ is as in the case **C.1**, the other changes of sign satisfy $\widetilde{H}_n = \infty$, for $n \geq 2$ and the lifetime of $\widetilde{X}$, $\widetilde{T} = S$, is finite. Hence the conclusion of the Proposition 1.3 does not hold.

Hereafter we assume that $\mathbb{P}_x(H_1 < \infty) = 1$, for all $x \in \mathbb{R}^*$. Then, for every $n \geq 0$, the process $(\mathcal{X}_t^{(n)}, t \geq 0)$ given by

$$\mathcal{X}_t^{(n)} = \frac{X_{H_n + |X_{H_n}|^\alpha t}}{|X_{H_n}|}, \quad 0 \leq t < |X_{H_n}|^{-\alpha}(H_{n+1} - H_n), \tag{1.7}$$

is well defined. We call the random variable X_{H_n} an overshoot or undershoot when $X_{H_n-} < 0$ and $X_{H_n} > 0$ or $X_{H_n-} > 0$ and $X_{H_n} < 0$, respectively. The random variable X_{H_n-} is called the jump height before crossing of the x-axis. The case $X_{H_n-} < 0$ means that the change of sign at time H_n is from a negative to a positive value. Now, we define the sequence of random variables $(J_n, n \geq 0)$ given by the quotient

$$J_n = \frac{X_{H_{n+1}}}{X_{H_{n+1}-}}, \quad n \geq 0. \tag{1.8}$$

These random objects satisfy the following properties.

Theorem 1.5. *Let $\{X^{(x)} = (X, \mathbb{P}_x), x \in \mathbb{R}^*\}$ be a family of real-valued self-similar Markov processes of index $\alpha > 0$, such that $\mathbb{P}_x(H_1 < \infty) = 1$, for all $x \in \mathbb{R}^*$. Then*

(i) The paths between sign changes, $(\mathcal{X}^{(n)}, n \geq 0)$, as defined in (1.7), are independent under $\mathbb{P}_x$, for $x \in \mathbb{R}^*$. Furthermore, for all $n \geq 0$,

$$\left\{\left(\mathcal{X}_t^{(n)}, 0 \leq t < |X_{H_n}|^{-\alpha}(H_{n+1} - H_n)\right), \mathbb{P}_x\right\} \stackrel{\mathcal{L}}{=} \left\{(X_t, 0 \leq t < H_1), \mathbb{P}_{sgn(x)(-1)^n}\right\}. \tag{1.9}$$

Hence, they are time changed Lévy processes killed at an exponential time.

(ii) The random variables $J_n, n \geq 0$, as defined in (1.8), are independent under $\mathbb{P}_x$, for $x \in \mathbb{R}^*$ and for $n \geq 0$, the identity

$$\{J_n, \mathbb{P}_x\} \stackrel{\mathcal{L}}{=} \left\{J_0, \mathbb{P}_{sgn(x)(-1)^n}\right\}, \tag{1.10}$$

holds.

(iii) For every $n \geq 0$, the process $\mathcal{X}^{(n)}$ and the random variable J_n are independent, under $\mathbb{P}_x$, for $x \in \mathbb{R}^*$.

From (1.9) we can see that only two independent Lévy processes killed at an exponential time are involved in the Lamperti representation. In the same way, from (1.10), only two independent real random variables represent the quotient between overshoots (undershoots) and jump height before crossing of the x-axis. Furthermore, by *(iii)* all these random objects are independent. The latter theorem is at the heart of our motivation to construct the Lamperti-Kiu processes in the next section.

1.2.2 Construction of Lamperti-Kiu processes

In this section we give a generalization of time changed exponentials of Lévy processes as well as of the processes which are defined in (1.4). We will see that all Lamperti-Kiu processes can be constructed as this generalization of (1.4).

Let ξ^+, ξ^- be real valued Lévy processes; ζ^+, ζ^- exponential random variables with parameters q^+, q^-, respectively, and U^+, U^- real valued random variables. Let $(\xi^{+,k}, k \geq 0)$, $(\xi^{-,k}, k \geq 0)$, $(\zeta^{+,k}, k \geq 0)$, $(\zeta^{-,k}, k \geq 0)$, $(U^{+,k}, k \geq 0)$, $(U^{-,k}, k \geq 0)$ be independent sequences of i.i.d. random variables such that

$$\xi^{+,0} \stackrel{\text{Law}}{=} \xi^+, \quad \xi^{-,0} \stackrel{\text{Law}}{=} \xi^-, \quad \zeta^{+,0} \stackrel{\text{Law}}{=} \zeta^+, \quad \zeta^{-,0} \stackrel{\text{Law}}{=} \zeta^-, \quad U^{+,0} \stackrel{\text{Law}}{=} U^+, \quad U^{-,0} \stackrel{\text{Law}}{=} U^-.$$

For every $x \in \mathbb{R}^*$ fixed, we consider the sequence $((\xi^{(x,k)}, \zeta^{(x,k)}, U^{(x,k)}), k \geq 0)$, where for $k \geq 0$,

$$(\xi^{(x,k)}, \zeta^{(x,k)}, U^{(x,k)}) = \begin{cases} (\xi^{+,k}, \zeta^{+,k}, U^{+,k}), & \text{if } \operatorname{sgn}(x)(-1)^k = 1, \\ (\xi^{-,k}, \zeta^{-,k}, U^{-,k}), & \text{if } \operatorname{sgn}(x)(-1)^k = -1. \end{cases}$$

Let $(T_n^{(x)}, n \geq 0)$ be the sequence defined by

$$T_0^{(x)} = 0, \quad T_n^{(x)} = \sum_{k=0}^{n-1} \zeta^{(x,k)}, \quad n \geq 1,$$

and $(N_t^{(x)}, t \geq 0)$ be the alternating renewal type process:

$$N_t^{(x)} = \max\{n \geq 0 : T_n^{(x)} \leq t\}, \quad t \geq 0.$$

For notational convenience we write

$$\sigma_t^{(x)} = t - T_{N_t^{(x)}}^{(x)}, \quad \xi_{\sigma_t}^{(x)} = \xi_{\sigma_t^{(x)}}^{(x, N_t^{(x)})}, \quad \xi_\zeta^{(x,k)} = \xi_{\zeta^{(x,k)}}^{(x,k)}.$$

Finally, we define the process $Y^{(x)} = (Y_t^{(x)}, t \geq 0)$ by

$$Y_t^{(x)} = x \exp\{\mathcal{E}_t^{(x)}\}, \quad t \geq 0, \tag{1.11}$$

where

$$\mathcal{E}_t^{(x)} = \xi_{\sigma_t}^{(x)} + \sum_{k=0}^{N_t^{(x)}-1} \left(\xi_\zeta^{(x,k)} + U^{(x,k)}\right) + i\pi N_t^{(x)}, \quad t \geq 0.$$

Remark 1.6. Observe that the process $Y^{(x)}$ is a generalization of multiplicative Lévy processes. For, take $(\xi^+, U^+, \zeta^+) \stackrel{\mathcal{L}}{=} (\xi^-, U^-, \zeta^-)$ it is seen that $Y^{(x)}$ is a multiplicative Lévy process, as it has been defined in [22]. Moreover, if $q^+ = 0$ and $q^- > 0$, then for $x > 0$, $Y^{(x)}$ does not jump to the negative axis and $Y^{(x)}$ is the exponential of a Lévy process, which appears in the Lamperti representation for positive self-similar Markov processes.

The following theorem is the main result of this chapter. The first part states that $Y^{(x)}$ is a Lamperti-Kiu process, the second and third parts are the generalization of the Lamperti representation.

Theorem 1.7. *Let $Y^{(x)}$ be the process defined in (1.11). Then,*

(i) the process $Y^{(x)}$ is Fellerian in $\mathbb{R}^*$ and satisfies (1.3). Furthermore, for any finite stopping time $\mathbf{T}$:

$$((Y_{\mathbf{T}}^{(x)})^{-1} Y_{\mathbf{T}+s}^{(x)}, s \geq 0) \stackrel{\mathcal{L}}{=} (\exp\{\widetilde{\mathcal{E}}_s^{(sgn(Y_{\mathbf{T}}^{(x)}))}\}, s \geq 0),$$

where $\widetilde{\mathcal{E}}^{(\cdot)}$ is a copy of $\mathcal{E}^{(\cdot)}$ which is independent of $(\mathcal{E}_u^{(\cdot)}, 0 \leq u \leq \mathbf{T})$.

(ii) *Let $\{X^{(x)} = (X, \mathbb{P}_x), x \in \mathbb{R}^*\}$ be a family of real-valued self-similar Markov processes of index $\alpha > 0$ such that $\mathbb{P}_x(H_1 < \infty) = 1$, for all $x \in \mathbb{R}^*$. For every $x \in \mathbb{R}^*$ define the process $\mathcal{Y}^{(x)}$ by*

$$\mathcal{Y}_t^{(x)} = X_{\nu^{(x)}(t)}^{(x)}, \quad t \geq 0,$$

where

$$\nu^{(x)}(t) = \inf\left\{s > 0 : \int_0^s |X_u^{(x)}|^{-\alpha} du > t\right\}.$$

Then $\mathcal{Y}^{(x)}$ may be decomposed as in (1.11). Moreover, every Lamperti-Kiu process can be constructed as explained in (1.11).

(iii) *Conversely, let $(Y^{(x)})_{x\in\mathbb{R}^*}$ be a family of processes as constructed in (1.11) and consider the processes $(X^{(x)})_{x\in\mathbb{R}^*}$ given by*

$$X_t^{(x)} = Y_{\tau(t|x|^{-\alpha})}^{(x)}, \quad t \geq 0,$$

where

$$\tau(t) = \inf\left\{s > 0 : \int_0^s |\exp\{\alpha \mathcal{E}_u^{(x)}\}| du > t\right\}, \quad t < T,$$

for some $\alpha > 0$. Then $(X^{(x)})_{x\in\mathbb{R}^*}$ is a family of real-valued self-similar Markov processes of index $\alpha > 0$.

From now on, we denote a Lamperti-Kiu process by Y. Now, we obtain an expression for the infinitesimal generator of Y, that will be used in the examples.

Proposition 1.8. *Let $\mathcal{K}$ be the infinitesimal generator of Y. Let $\mathcal{A}^+$, $\mathcal{A}^-$ be the infinitesimal generators of ξ^+, ξ^-, respectively. Let f be a bounded continuous function such that $f(0) = 0$ and $(f \circ \exp) \in \mathcal{D}_{\mathcal{A}^+}$ and $(f \circ -\exp) \in \mathcal{D}_{\mathcal{A}^-}$. Then, for every $x \in \mathbb{R}^*$,*

$$\mathcal{K}f(x) = \mathcal{A}^{sgn(x)}(f \circ sgn(x)\exp)(\log|x|) + q^{sgn(x)}\left(\mathbf{E}[f(-x\exp\{U^{sgn(x)}\}) - f(x)]\right). \tag{1.12}$$

With the help of the latter proposition we can give the infinitesimal generator of Y in terms of the parameters of the Lévy processes ξ^+ and ξ^- as follows. Recall that the characteristic exponent of the Lévy process $\xi^\pm$ can be written as

$$\psi^\pm(\lambda) = a^\pm i\lambda - \frac{[\sigma^\pm]^2}{2}\lambda^2 + \int_{\mathbb{R}} [e^{i\lambda y} - 1 - i\lambda l(y)]\pi^\pm(dy), \quad \lambda \in \mathbb{R},$$

where $a^\pm \in \mathbb{R}$, $\sigma > 0$, $l(\cdot)$ is a fixed continuous bounded function such that $l(y) \sim y$ as $y \to 0$ and $\pi^\pm$ is the Lévy measure of the process $\xi^\pm$, which satisfies $\pi^\pm(\{0\}) = 0$ and $\int_{\mathbb{R}}(1\wedge x^2)\pi^\pm(dx) < \infty$. Furthermore, the choice of the function l is arbitrary and the coefficient $a^\pm$ is the only one which depends on this choice (see remark 8.4 in [45]). Later in the examples we will choose conveniently this function. Hence, the infinitesimal generator of the Lévy process $\xi^\pm$ can be expressed as

$$\mathcal{A}^\pm f(x) = a^\pm f'(x) + \frac{[\sigma^\pm]^2}{2} f''(x) + \int_{\mathbb{R}} [f(x+y) - f(x) - f'(x)l(y)]\pi^\pm(dy), \quad f \in \mathcal{D}_{\mathcal{A}^\pm}.$$

Then using the expression of $\mathcal{A}^{\pm}$ and (1.12), we find for $x \in \mathbb{R}^*$,

$$\begin{aligned}\mathcal{K}f(x) &= b^{\mathrm{sgn}(x)}xf'(x) + \frac{[\sigma^{\mathrm{sgn}(x)}]^2}{2}x^2 f''(x) + \int_{\mathbb{R}^+}[f(xu) - f(x) - xf'(x)l(\log u)]\Theta^{\mathrm{sgn}(x)}(du) \\ &\quad + q^{\mathrm{sgn}(x)}\{\mathbf{E}[f(-x\exp\{U^{\mathrm{sgn}(x)}\}) - f(x)]\}, \qquad (1.13)\end{aligned}$$

where $b^{\mathrm{sgn}(x)} = a^{\mathrm{sgn}(x)} + [\sigma^{\mathrm{sgn}(x)}]^2/2$, $\Theta^{\mathrm{sgn}(x)}(du) = \pi^{\mathrm{sgn}(x)}(du) \circ \log u$. Hence, by Volkonskii's theorem, the generator $\widetilde{\mathcal{K}}$ of the time changed process Y_τ is given by $\widetilde{\mathcal{K}}f(x) = |x|^{-\alpha}\mathcal{K}f(x)$, for $x \in \mathbb{R}^*$. Hence, knowing that the infinitesimal generator of Y is given by (1.13) it is possible to identify the infinitesimal generator of the self-similar Markov process X and conversely.

1.3 Proofs

Proof of Proposition 1.3. The strong Markov property implies $\mathbb{P}_x(H_n < \infty, \forall\, n \geq 0) = 1$. Thus, $(H_n, n \geq 0)$ is a strictly increasing sequence of stopping times satisfying $H_n \leq T$, for all $n \geq 0$. Let H be the limit of this sequence, then $H \leq T$. If $H = \infty$, then clearly $T = \infty$ and $H = T$. On the other hand, if $H < \infty$, then on the set $\{H < T\}$, it is possible to define the process $X^H = (X_{H+t}\mathbf{1}_{\{t<T-H\}}, t \geq 0)$. This process has no change of sign, and by the strong Markov property, for all $y \in \mathbb{R}^*$, conditionally on $X_H = y$, X^H has the same distribution as X under $\mathbb{P}_y$. This contradicts the fact that X has at least one change of sign. Therefore, $H = T$, a.s. □

Proof of Theorem 1.5. For $t \geq 0$, we denote by $\theta_t : \Omega \to \Omega$ the shift operator, i.e., for $\omega \in \Omega$, $\theta_t\omega(s) = \omega(t+s)$, $s \geq 0$.

(i) Let F be a bounded and measurable functional. From (1.6) and the self-similarity property, it follows

$$\mathbb{E}_x\left[F\left(\frac{X_{|X_0|^\alpha t}}{|X_0|}, 0 \leq t < |X_0|^{-\alpha}H_1\right)\right] = \mathbb{E}_{\mathrm{sgn}(x)}\left[F\left(X_t, 0 \leq t < H_1\right)\right],$$

for $x \in \mathbb{R}^*$. Moreover, $\mathrm{sgn}(X_{H_n}) = \mathrm{sgn}(x)(-1)^n$, $\mathbb{P}_x$-a.s. These two facts and the strong Markov property are sufficient to complete the proof. Indeed, for $\mathcal{X}^{(0)}, \ldots, \mathcal{X}^{(n)}$ as defined in (1.7) and for all $F_0, \ldots, F_n$ bounded and measurable functionals, we have

$$\begin{aligned}\mathbb{E}_x\left[\prod_{k=0}^{n} F_k\left(\mathcal{X}^{(k)}\right)\right] &= \mathbb{E}_x\left[\prod_{k=0}^{n-1} F_k\left(\mathcal{X}^{(k)}\right)\mathbb{E}_{X_{H_n}}\left[F_n\left(\frac{X_{|X_0|^\alpha t}}{|X_0|}, 0 \leq t < |X_0|^{-\alpha}H_1\right)\right]\right] \\ &= \mathbb{E}_x\left[\prod_{k=0}^{n-1} F_k\left(\mathcal{X}^{(k)}\right)\mathbb{E}_{\mathrm{sgn}(x)(-1)^n}\left[F_n\left(X_t, 0 \leq t < H_1\right)\right]\right] \\ &= \mathbb{E}_x\left[\prod_{k=0}^{n-1} F_k\left(\mathcal{X}^{(k)}\right)\right]\mathbb{E}_{\mathrm{sgn}(x)(-1)^n}\left[F_n\left(X_t, 0 \leq t < H_1\right)\right],\end{aligned}$$

where the strong Markov and self-similarity properties were used to obtain the first and second equality, respectively. Now, taking $F_0 = \ldots = F_{n-1} \equiv 1$, we have

$$\mathbb{E}_x\left[F_n\left(\mathcal{X}_t^{(n)}, 0 \leq t < |X_{H_n}|^{-\alpha}(H_{n+1} - H_n)\right)\right] = \mathbb{E}_{\mathrm{sgn}(x)(-1)^n}\left[F_n\left(X_t, 0 \leq t < H_1\right)\right].$$

This proves (1.9). In addition

$$\mathbb{E}_x\left[\prod_{k=0}^{n} F_k\left(\mathcal{X}^{(k)}\right)\right] = \mathbb{E}_x\left[\prod_{k=0}^{n-1} F_k\left(\mathcal{X}^{(k)}\right)\right]\mathbb{E}_x\left[F_n\left(\mathcal{X}^{(n)}\right)\right].$$

This proves the independence in the sequence $\{(\mathcal{X}_t^{(n)}, 0 \leq t < |X_{H_n}|^{-\alpha}(H_{n+1} - H_n)), n \geq 0\}$ under $\mathbb{P}_x$.

(ii) From (1.6) and the self-similarity property, we derive that

$$\mathbb{E}_x\left[f\left(\frac{X_{H_1}}{X_{H_1-}}\right)\right] = \mathbb{E}_{\mathrm{sgn}(x)}\left[f\left(\frac{X_{H_1}}{X_{H_1-}}\right)\right], \tag{1.14}$$

for all $x \in \mathbb{R}^*$, and f bounded Borel function. Now, let $f_0, \ldots, f_n$ bounded Borel functions. Proceeding as in *(i)*, using (1.14) and the strong Markov property, we obtain

$$\mathbb{E}_x\left[\prod_{k=0}^{n} f_k\left(\frac{X_{H_{k+1}}}{X_{H_{k+1}-}}\right)\right] = \mathbb{E}_x\left[\prod_{k=0}^{n-1} f_k\left(\frac{X_{H_{k+1}}}{X_{H_{k+1}-}}\right)\right]\mathbb{E}_{\mathrm{sgn}(x)(-1)^n}\left[f_n\left(\frac{X_{H_1}}{X_{H_1-}}\right)\right].$$

The conclusion follows as in *(i)*.

(iii) By the strong Markov property, *(i)* and *(ii)*, it is sufficient to prove the case $n = 0$. For $k \geq 1$, let $f : \mathbb{R}^{*k} \to \mathbb{R}$, $g : \mathbb{R}^* \to \mathbb{R}$ be two Borel functions, and $0 < s_1 < \ldots < s_k$. We note the following identity $\frac{X_{H_1}}{X_{H_1-}} \circ \theta_{s_k} = \frac{X_{H_1}}{X_{H_1-}}$, on $\{s_k < H_1\}$. Hence, by the Markov property and (1.14), we have

$$\begin{aligned}
\mathbb{E}_x\left[f(X_{s_1}, \ldots, X_{s_k})g\left(\frac{X_{H_1}}{X_{H_1-}}\right); s_k < H_1\right] &= \mathbb{E}_x\left[f(X_{s_1}, \ldots, X_{s_k})\mathbb{E}_{X_{s_k}}\left[g\left(\frac{X_{H_1}}{X_{H_1-}}\right)\right]; s_k < H_1\right] \\
&= \mathbb{E}_x\left[f(X_{s_1}, \ldots, X_{s_k}); s_k < H_1\right]\mathbb{E}_{\mathrm{sgn}(x)}\left[g\left(\frac{X_{H_1}}{X_{H_1-}}\right)\right] \\
&= \mathbb{E}_x\left[f(X_{s_1}, \ldots, X_{s_k}); s_k < H_1\right]\mathbb{E}_x\left[g\left(\frac{X_{H_1}}{X_{H_1-}}\right)\right].
\end{aligned}$$

This ends the proof. □

In order to prove Theorem 1.7, we first prove the following lemma. This lemma is a consequence of the lack-of-memory property of the exponential distribution and the properties of the random objects which define $Y^{(x)}$. Before we state it, we define the following process. For $x \in \mathbb{R}^*$, let $Z^{(x)}$ be the sign process of $Y^{(x)}$, i.e., $Z_t^{(x)} = \mathrm{sgn}(Y_t^{(x)})$, $t \geq 0$. Note that $Z^{(x)}$ is a continuous time Markov chain with state space $\{-1, 1\}$, starting point $\mathrm{sgn}(x)$ and transition semigroup $e^{t\mathbf{Q}}$, where

$$\mathbf{Q} = \begin{pmatrix} -q^- & q^- \\ q^+ & -q^+ \end{pmatrix}.$$

Furthermore, since the law of $Z^{(x)}$ is determined by $\mathbf{Q}$ (hence by ζ^+, ζ^-), then the process $Z^{(x)}$ is independent of $((\xi^{(x,k)}, U^{(x,k)}), k \geq 0)$.

Lemma 1.9. *Let n, m be positive integers and s, t be positive real numbers. We have the following properties*

(a) *Conditionally on* $T_n^{(x)} \le t < T_{n+1}^{(x)}$, *the random variable* $T_{n+1}^{(x)} - t$ *has an exponential distribution with parameter* $q^{(x,n)}$, *where* $q^{(x,n)}$ *equals* q^+ *if* $sgn(x)(-1)^n = 1$ *and* q^- *otherwise. Furthermore,*

$$\xi_{\zeta}^{(x,n)} - \xi_{t-T_n^{(x)}}^{(x,n)} \stackrel{\mathcal{L}}{=} \widetilde{\xi}_{\widetilde{\zeta}}^{(Z_t^{(x)},0)},$$

where $(\widetilde{\xi}^{(\cdot,0)}, \widetilde{\zeta}^{(\cdot,0)})$ *are independent of* $(\xi^{(\cdot,k)}, \zeta^{(\cdot,k)}, 0 \le k < n)$ *and with the same distribution as* $(\xi^{(\cdot,0)}, \zeta^{(\cdot,0)})$.

(b) *Conditionally on* $T_n^{(x)} \le t < T_{n+1}^{(x)}$, $T_{n+m}^{(x)} \le t + s < T_{n+m+1}^{(x)}$ *the distribution of* $\xi_{t+s-T_{n+m}^{(x)}}^{(x,n+m)}$ *is the same as the distribution of* $\widetilde{\xi}_{s-\widetilde{T}_m^{(Z_t^{(x)})}}^{(Z_t^{(x)},m)}$ *conditionally on* $\widetilde{T}_m^{(Z_t^{(x)})} \le s < \widetilde{T}_{m+1}^{(Z_t^{(x)})}$, *i.e.,*

$$\mathbf{P}(\xi_{t+s-T_{n+m}^{(x)}}^{(x,n+m)} \in dz \mid T_{n+m}^{(x)} \le t+s < T_{n+m+1}^{(x)}, T_n^{(x)} \le t < T_{n+1}^{(x)})$$
$$= \mathbf{P}(\widetilde{\xi}_{s-\widetilde{T}_m^{(Z_t^{(x)})}}^{(Z_t^{(x)},m)} \in dz \mid \widetilde{T}_m^{(Z_t^{(x)})} \le s < \widetilde{T}_{m+1}^{(Z_t^{(x)})}),$$

where $(\widetilde{\xi}^{(\cdot,m)}, \widetilde{T}_m^{(\cdot)})$ *are independent of* $(\xi^{(\cdot,k)}, T_k^{(\cdot)}, 0 \le k \le n)$ *with the same distribution as* $(\xi^{(\cdot,m)}, T_m^{(\cdot)})$.

Proof of Lemma 1.9. The first part of (a) follows from the lack-of-memory property of the exponential distribution. Now, by construction, $(\xi^{(x,n)}, \zeta^{(x,n)}, n \ge 0)$ is a sequence of independent random objects which depends on x only through its sign and $T_{n+m}^{(x)} = T_n^{(x)} + \sum_{k=0}^{m-1} \zeta^{(x,n+k)}$. Hence, it is always possible to take $(\widetilde{\xi}^{(\cdot,0)}, \widetilde{\zeta}^{(\cdot,0)})$ and $(\widetilde{\xi}^{(\cdot,m)}, \widetilde{T}_m^{(\cdot)})$ with the properties described in (a) and (b), respectively. Thus, it only remains to prove the equality in distribution in (a) and (b).

Denote by $f_{T_n^{(x)}}$ the density of the random variable $T_n^{(x)}$. Simple computations lead to

$$\mathbf{P}(\xi_{\zeta}^{(x,n)} - \xi_{t-T_n^{(x)}}^{(x,n)} \in dz, T_n^{(x)} \le t < T_{n+1}^{(x)}) = \int_0^t \int_{t-u}^{\infty} \mathbf{P}(\xi_{r-(t-u)}^{(x,n)} \in dz) q^{(x,n)} e^{-q^{(x,n)} r} dr f_{T_n^{(x)}}(u) du$$
$$= \mathbf{P}(\xi_{\zeta}^{(x,n)} \in dz) \mathbf{P}(T_n^{(x)} \le t < T_{n+1}^{(x)}),$$

where the independence and stationarity of the increments of the Lévy process $\xi^{(x,n)}$ have been used in the first equality and we made the change of variables $v = r - (t-u)$ to obtain the second. Hence, the equality in law of (a) is obtained.

By (a) we have that for all $m \ge 0$, conditionally on $T_n^{(x)} \le t < T_{n+1}^{(x)}$, the random variable $T_{n+m}^{(x)} - t$ has the same distribution as $T_m^{(Z_t^{(x)})}$ and it is independent of $(T_k^{(x)}, 0 \le k \le n)$. Hence

$$\mathbf{P}(\xi_{t+s-T_{n+m}^{(x)}}^{(x,n+m)} \in dz, T_{n+m}^{(x)} \le t+s < T_{n+m+1}^{(x)} \mid T_n^{(x)} \le t < T_{n+1}^{(x)})$$
$$= \mathbf{P}(\xi_{s-\widetilde{T}_m^{(Z_t^{(x)})}}^{(Z_t^{(x)},m)} \in dz, \widetilde{T}_m^{(Z_t^{(x)})} \le s < \widetilde{T}_{m+1}^{(Z_t^{(x)})})$$

and

$$\mathbf{P}(T_{n+m}^{(x)} \le t+s < T_{n+m+1}^{(x)} \mid T_n^{(x)} \le t < T_{n+1}^{(x)}) = \mathbf{P}(\widetilde{T}_m^{(Z_t^{(x)})} \le s < \widetilde{T}_{m+1}^{(Z_t^{(x)})}).$$

Therefore

$$\begin{aligned}&\mathbf{P}(\xi^{(x,n+m)}_{t+s-T^{(x)}_{n+m}} \in dz \mid T^{(x)}_{n+m} \le t+s < T^{(x)}_{n+m+1}, T^{(x)}_n \le t < T^{(x)}_{n+1})\\ &= \mathbf{P}(\xi^{(Z^{(x)}_t,m)}_{s-\widetilde{T}^{(Z^{(x)}_t)}_m} \in dz \mid \widetilde{T}^{(Z^{(x)}_t)}_m \le s < \widetilde{T}^{(Z^{(x)}_t)}_{m+1}).\end{aligned}$$

This finishes the proof. $\square$

Proof of Theorem 1.7. (i) First we prove that $Y^{(x)}$ satisfies the property (1.3). We note that the process $\mathcal{E}^{(\cdot)}$ depends on x only through its sign, then clearly for all $a \in \mathbb{R}^*$, $\mathcal{E}^{(|a|x)} \stackrel{\mathcal{L}}{=} \mathcal{E}^{(x)}$. Hence, we have

$$\begin{aligned}(\mathrm{sgn}(a)Y^{(|a|x)}_t, t\ge 0) &= (\mathrm{sgn}(a)|a|x\exp\{\mathcal{E}^{(|a|x)}_t\}, t\ge 0)\\ &\stackrel{\mathcal{L}}{=} (ax\exp\{\mathcal{E}^{(x)}_t\}, t\ge 0)\\ &= (aY^{(x)}_t, t\ge 0).\end{aligned}$$

Therefore, the process $Y^{(x)}$ satisfies the property (1.3).

Let $s,t\ge 0$, then by Lemma 1.9, conditionally on $T^{(x)}_n \le t < T^{(x)}_{n+1}$, $T^{(x)}_{n+m} \le t+s < T^{(x)}_{n+m+1}$, we have

$$\begin{aligned}\frac{Y^{(x)}_{t+s}}{Y^{(x)}_t} &= \exp\left\{\xi^{(x,n+m)}_{t+s-T^{(x)}_{n+m}} + \sum_{k=1}^{m-1}\left(\xi^{(x,n+k)}_\zeta + U^{(x,n+k)}\right) + \xi^{(x,n)}_\zeta - \xi^{(x,n)}_{t-T^{(x)}_n} + U^{(x,n)} + i\pi m\right\}\\ &\stackrel{\mathcal{L}}{=} \exp\left\{\widetilde{\xi}^{(Z^{(x)}_t,m)}_{s-\widetilde{T}^{(Z^{(x)}_t)}_m} + \sum_{k=0}^{m-1}\left(\widetilde{\xi}^{(Z^{(x)}_t,k)}_{\widetilde{\zeta}} + \widetilde{U}^{(Z^{(x)}_t,k)}\right) + i\pi m\right\}.\end{aligned}$$

Hence, for $s,t\ge 0$,

$$\frac{Y^{(x)}_{t+s}}{Y^{(x)}_t} \stackrel{\mathcal{L}}{=} \exp\{\widetilde{\mathcal{E}}^{(Z^{(x)}_t)}_s\}, \tag{1.15}$$

where $\widetilde{\mathcal{E}}^{(\cdot)}$ is a copy of $\mathcal{E}^{(\cdot)}$ which is independent of $(\mathcal{E}^{(\cdot)}_u, 0\le u\le t)$. Thus, $Y^{(x)}$ has the Markov property. Furthermore

$$(Y^{(x)}_{t+s}, s\ge 0) \stackrel{\mathcal{L}}{=} (\widetilde{Y}^{(Y^{(x)}_t)}_s, s\ge 0),$$

where $\widetilde{Y}^{(\cdot)}$ is a copy of $Y^{(\cdot)}$ which is independent of $(Y^{(\cdot)}_u, 0\le u\le t)$. This also ensures that all processes $Y^{(x)}$ have the same semigroup.

Now, we prove that $Y^{(x)}$ is a Feller process on $\mathbb{R}^*$. Let Q_t be the semigroup associated to $Y^{(x)}$. We verify that Q_t is a Feller semigroup, that is,

(i) $Q_tf \in C_0(\mathbb{R}^*)$, for all $f\in C_0(\mathbb{R}^*)$,

(ii) $\lim_{t\downarrow 0} Q_tf(x) = f(x)$, for all $x\in\mathbb{R}^*$.

Let $x\in\mathbb{R}^*$ be fixed. For all $y\in\mathbb{R}^*$ such that $\mathrm{sgn}(y)=\mathrm{sgn}(x)$, by property (1.3), we have

$$Q_tf(y) = \mathbf{E}\left[f\left(Y^{(y)}_t\right)\right] = \mathbf{E}\left[f\left(\frac{y}{x}Y^{(x)}_t\right)\right].$$

The latter expression and the dominated convergence theorem ensure the continuity of $Q_t f$ in x. By (1.3), $(Y_t^{(x)}, t \geq 0) \stackrel{\mathcal{L}}{=} (|x| Y_t^{(\mathrm{sgn}(x))}, t \geq 0)$ for all $x \in \mathbb{R}^*$. Hence,

$$Q_t f(x) = \mathbf{E}\left[f\left(Y_t^{(x)}\right)\right] = \mathbf{E}\left[f\left(|x| Y_t^{(\mathrm{sgn}(x))}\right)\right], \quad x \in \mathbb{R}^*.$$

Using again the dominated convergence theorem, we obtain $\lim_{|x|\to\infty} Q_t f(x) = 0$. For the last part,

$$\mathbf{E}\left[f\left(Y_t^{(x)}\right)\right] = \mathbf{E}\left[f\left(Y_t^{(x)}\right) \Big| T_1^{(x)} > t\right] \mathbf{P}\left(T_1^{(x)} > t\right) + \mathbf{E}\left[f\left(Y_t^{(x)}\right) \Big| T_1^{(x)} \leq t\right] \mathbf{P}\left(T_1^{(x)} \leq t\right).$$

For the first term we have

$$\mathbf{E}\left[f\left(Y_t^{(x)}\right) \Big| T_1^{(x)} > t\right] \mathbf{P}\left(T_1^{(x)} > t\right) = \mathbf{E}\left[f\left(x \exp\{\xi_t^{\mathrm{sgn}(x)}\}\right) \Big| \zeta^{\mathrm{sgn}(x)} > t\right] e^{-q^{\mathrm{sgn}(x)} t}.$$

Letting $t \to 0$, the last expression converges to $f(x)$ by the right continuity of $\xi^{\mathrm{sgn}(x)}$. Thus, it only remains to prove that the second term converges to zero as t tends to zero. Since f is bounded,

$$\left|\mathbf{E}\left[f\left(Y_t^{(x)}\right) \Big| T_1^{(x)} \leq t\right] \mathbf{P}\left(T_1^{(x)} \leq t\right)\right| \leq C\left(1 - e^{-q^{\mathrm{sgn}(x)} t}\right),$$

for some positive constant C. Again, letting $t \to 0$ we obtain the desired result.

The strong Markov property of $Y^{(x)}$ follows from the standard fact that any Feller process is a strong Markov process.

(ii) First note that $\nu^{(x)}(t)$ satisfies

$$\nu^{(x)}(t) = \int_0^t |\mathcal{Y}_s^{(x)}|^\alpha ds, \quad t \geq 0. \tag{1.16}$$

Indeed, if

$$\tau^{(x)}(t) = \int_0^{t|x|^\alpha} |X_s^{(x)}|^{-\alpha} ds,$$

then, since $\tau^{(x)}(\nu^{(x)}(t)|x|^{-\alpha}) = t$, it follows $d\nu^{(x)}(t)/dt = 1/|X_{\nu^{(x)}(t)}^{(x)}|^{-\alpha} = |\mathcal{Y}_t^{(x)}|^\alpha$.

Now, we claim the following: for every $x \in \mathbb{R}^*$ and $n \geq 0$, there exists a Lévy process $\xi^{(x,n)}$ independent of $(X_s^{(x)}, 0 \leq s \leq H_n^{(x)})$ such that,

$$X_{H_n+t}^{(x)} = X_{H_n}^{(x)} \exp\{\xi^{(x,n)}_{\tau^{(x,n)}(t|X_{H_n}^{(x)}|^{-\alpha})}\}, \quad 0 \leq t < H_{n+1}^{(x)} - H_n^{(x)}, \tag{1.17}$$

where

$$\tau^{(x,n)}(t) = \inf\left\{s > 0 : \int_0^s \exp\{\alpha \xi_u^{(x,n)}\} du > t\right\}. \tag{1.18}$$

To verify this, we take $x > 0$ and n even, the other cases can be proved similarly. In this case, $X_{H_n}^{(x)} > 0$. By the strong Markov property, conditionally on $X_{H_n} = y$, we have

$$(X_{H_n+t}, 0 \leq t < H_{n+1} - H_n) \stackrel{\mathcal{L}}{=} \{(X_t, 0 \leq t < H_1), \mathbb{P}_y\}.$$

And since the process on the right hand side of the latter expression is a positive self-similar Markov process, then by Lamperti's representation there exists a Levy process $(\xi^+, \mathbf{P})$ such that

$$\{(X_t, 0 \le t < H_1), \mathbb{P}_y\} \stackrel{\mathcal{L}}{=} \left\{ \left(y \exp\{\xi^+_{\tau^+(ty^{-\alpha})}\}, 0 \le t < A^+(\infty) \right), \mathbf{P} \right\},$$

where

$$A^+(\infty) = \int_0^\infty \exp\{\alpha \xi^+_s\} ds.$$

Furthermore, since $H_1 < \infty$, $\mathbb{P}_y$-a.s., then ξ^+ is a killed Lévy process with lifetime ζ^+, exponentially distributed with parameter $q^+ > 0$ and hence

$$A^+(\infty) = \int_0^{\zeta^+} \exp\{\alpha \xi^+_s\} ds.$$

Note that we chose the superscript + because $\mathrm{sgn}(X_{H_n}) > 0$.

Thus, we have obtained that for all $x > 0$, n even,

$$\{(X_{H_n+t}, 0 \le t < H_{n+1} - H_n), \mathbb{P}_x\} \stackrel{\mathcal{L}}{=} \left\{ \left(X^{(x)}_{H_n} \exp\{\xi^+_{\tau^+(t(X^{(x)}_{H_n})^{-\alpha})}\}, 0 \le t < A^+(\infty) \right), \mathbf{P} \right\}.$$

This shows (1.17). Also, the Lamperti representation ensures that for all $x \in \mathbb{R}^*$, $n \ge 0$,

$$|X^{(x)}_{H_n}|^{-\alpha}(H_{n+1} - H_n) = \int_0^{\zeta^{(x,n)}} \exp\{\alpha \xi^{(x,n)}_u\} du, \tag{1.19}$$

which implies that for all $n \ge 1$

$$H^{(x)}_n = \sum_{k=0}^{n-1} |X^{(x)}_{H_k}|^\alpha \int_0^{\zeta^{(x,k)}} \exp\{\alpha \xi^{(x,k)}_u\} du. \tag{1.20}$$

Now, for $x \in \mathbb{R}^*$ we define the sequence $(U^{(x,n)}, n \ge 0)$ by

$$\exp\{U^{(x,n)}\} = -\frac{X^{(x)}_{H_{n+1}}}{X^{(x)}_{H_{n+1}-}}, \quad n \ge 0.$$

Then, by (1.17) and (1.19) it follows that

$$X^{(x)}_{H_{n+1}-} = X^{(x)}_{H_n} \exp\{\xi^{(x,n)}_\zeta\},$$

and also

$$X^{(x)}_{H_{n+1}} = X^{(x)}_{H_{n+1}-} \frac{X^{(x)}_{H_{n+1}}}{X^{(x)}_{H_{n+1}-}} = -X^{(x)}_{H_n} \exp\{\xi^{(x,n)}_\zeta + U^{(x,n)}\}.$$

Hence, for all $n \ge 0$,

$$X^{(x)}_{H_{n+1}} = x \exp\left\{ \sum_{k=0}^{n} \left(\xi^{(x,k)}_\zeta + U^{(x,k)} \right) + i\pi(n+1) \right\}. \tag{1.21}$$

Note that because of Theorem 1.5, for every $x \in \mathbb{R}^*$, the sequence $(\xi^{(x,n)}, \zeta^{(x,n)}, U^{(x,n)}, n \geq 0)$ satisfies the condition which defines the process $Y^{(x)}$ in (1.11). It only remains to prove that $X^{(x)}$ time changed is of the form (1.11). For that aim, write

$$A^{(x,n)}(t) = \int_0^t \exp\{\alpha \xi_u^{(x,n)}\} du, \quad 0 \leq t \leq \zeta^{(x,n)}.$$

Thanks to (1.17), (1.18) and (1.21), we have

$$\begin{aligned} X^{(x)}_{H_n + |X^{(x)}_{H_n}|^\alpha A^{(x,n)}(t)} &= X^{(x)}_{H_n} \exp\{\xi_t^{(x,n)}\} \\ &= x \exp\{\mathcal{E}^{(x)}_{t+T_n}\}. \end{aligned}$$

On the other hand, by (1.20), for $0 \leq t < \zeta^{(x,n)}$ it follows

$$\begin{aligned} H_n^{(x)} + |X^{(x)}_{H_n}|^\alpha A^{(x,n)}(t) &= \sum_{k=0}^{n-1} |X^{(x)}_{H_k}|^\alpha \int_0^{\zeta^{(x,k)}} \exp\{\alpha \xi_u^{(x,k)}\} du + |X^{(x)}_{H_n}|^\alpha \int_0^t \exp\{\alpha \xi_u^{(x,n)}\} du \\ &= \sum_{k=0}^{n-1} \int_0^{\zeta^{(x,k)}} |x|^\alpha | \exp\{\alpha \mathcal{E}^{(x)}_{u+T_k}\}| du + \int_0^t |x|^\alpha | \exp\{\alpha \mathcal{E}^{(x)}_{u+T_n}|\} du \\ &= \sum_{k=0}^{n-1} \int_{T_k}^{T_{k+1}} |x|^\alpha | \exp\{\alpha \mathcal{E}^{(x)}_{u}\}| du + \int_{T_n}^{t+T_n} |x|^\alpha | \exp\{\alpha \mathcal{E}^{(x)}_{u}\}| du \\ &= \int_0^{t+T_n} |x \exp\{\mathcal{E}^{(x)}_u\}|^\alpha du. \end{aligned}$$

Hence

$$X^{(x)}_{\int_0^{t+T_n} |x \exp\{\mathcal{E}^{(x)}_s\}|^\alpha ds} = x \exp\{\mathcal{E}^{(x)}_{t+T_n}\}, \quad 0 \leq t < \zeta^{(x,n)}.$$

The latter and (1.16) imply that $\mathcal{Y}^{(x)}$ can be decomposed as in (1.11). Furthermore, as a consequence of this decomposition and the converse of the main result in [34], we can conclude that every Lamperti-Kiu process can be constructed as explained in (1.11).

(iii) Let $(\mathcal{G}_t)$ be the natural filtration of $Y^{(x)}$, i.e., $\mathcal{G}_t = \sigma(Y^{(x)}_s, s \leq t)$, $t \geq 0$. Let $\mathcal{F}_t = \mathcal{G}_{\tau(t|x|^{-\alpha})}$, $t \geq 0$. Clearly, $X^{(x)}$ is $(\mathcal{F}_t)$-adapted, and since the strong Markov property is preserved under time changes by additive functionals, $X^{(x)}$ is a strong Markov process. We recall $\mathcal{E}^{(cx)} \stackrel{\mathcal{L}}{=} \mathcal{E}^{(x)}$ for all $c > 0$. Thus, if $c > 0$, then

$$\begin{aligned} \left(cX^{(x)}_{c^{-\alpha}t}, t \geq 0\right) &= \left(cx \exp\{\mathcal{E}^{(x)}_{\tau(t|cx|^{-\alpha})}\}, t \geq 0\right) \\ &\stackrel{\mathcal{L}}{=} \left(cx \exp\{\mathcal{E}^{(cx)}_{\tau(t|cx|^{-\alpha})}\}, t \geq 0\right) \\ &= \left(X^{(cx)}_t, t \geq 0\right). \end{aligned}$$

This proves the self-similar property of $X^{(x)}$. It only remains to prove that all $X^{(x)}$ have the same semigroup. We have

$$X^{(x)}_{t+s} = X^{(x)}_t (Y^{(x)}_{\tau(t|x|^{-\alpha})})^{-1} Y^{(x)}_{\tau((t+s)|x|^{-\alpha})}.$$

On the other hand, for all $s, t \geq 0$,

$$\begin{aligned} \tau((t+s)|x|^{-\alpha}) &= \tau(t|x|^{-\alpha}) + \inf\left\{ r > 0 : \int_0^r |\exp\{\alpha \mathcal{E}^{(x)}_{\tau(t|x|^{-\alpha})+u}\}| du > s|x|^{-\alpha} \right\} \\ &= \tau(t|x|^{-\alpha}) + \inf\left\{ r > 0 : \int_0^r \left| (Y^{(x)}_{\tau(t|x|^{-\alpha})})^{-1} Y^{(x)}_{\tau(t|x|^{-\alpha})+u} \right|^\alpha du > s|X^{(x)}_t|^{-\alpha} \right\}. \end{aligned}$$

Write $\widehat{Y}^{(x)}_s = (Y^{(x)}_{\tau(t|x|^{-\alpha})})^{-1} Y^{(x)}_{\tau(t|x|^{-\alpha})+s}$, $s \geq 0$. Then

$$X^{(x)}_{t+s} = X^{(x)}_t \widehat{Y}^{(x)}_{\widehat{\tau}(s|X^{(x)}_t|^{-\alpha})}.$$

Hence by the strong Markov property of $Y^{(x)}$, Theorem 1.7 *(ii)*, we obtain

$$\begin{aligned} \mathbf{P}(X^{(x)}_{t+s} \in dz \mid \mathcal{F}_t) &= \mathbf{P}(X^{(x)}_t \widehat{Y}^{(x)}_{\widehat{\tau}(s|X^{(x)}_t|^{-\alpha})} \in dz \mid \mathcal{F}_t) \\ &= \mathbf{P}(y \exp\{\mathcal{E}^{(\mathrm{sgn}(y))}_{\tau(s|y|^{-\alpha})}\} \in dz)\big|_{y = X^{(x)}_t} \\ &= \mathbf{P}(X^{(y)}_t \in dz)\big|_{y = X^{(x)}_t}. \end{aligned}$$

This concludes the proof. □

Remark 1.10. Let $A^{(x)} = (A^{(x)}_t, 0 \leq t \leq \infty)$ be the process defined by

$$A^{(x)}_t = \int_0^t |\exp\{\alpha \mathcal{E}^{(x)}_s\}| ds, \quad 0 \leq t \leq \infty.$$

Note that $A^{(x)}$ only depends on x through its sign. From (1.20), (1.21) and Proposition 1.3, under $\mathbb{P}_x$,

$$T = \lim_{n \to \infty} H_n = |x|^\alpha A^{(\mathrm{sgn}(x))}_\infty,$$

i.e., there is a relation between the hitting time of zero for X and the exponential functional of $\mathcal{E}$, similar to the one known for positive self-similar Markov processes. Furthermore, Lamperti's representation can be written as

$$X^{(x)}_t \mathbf{1}_{\{t<T\}} = x \exp\{\mathcal{E}^{(x)}_{\tau^{(x)}(t|x|^{-\alpha})}\} \mathbf{1}_{\{t<|x|^\alpha A^{(\mathrm{sgn}(x))}_\infty\}}, \quad t \geq 0,$$

where $\tau^{(x)}(t) = \inf\{s > 0 : \int_0^s |\exp\{\alpha \mathcal{E}^{(x)}_u\}| du > t\}$, $t < A^{(x)}_\infty$.

Proof of Proposition 1.8. We prove the case $x > 0$, the case $x < 0$ can be proved similarly. Let T_1 and T_2 the first and the second times of sign change for Y, respectively. In the case $x > 0$,

$$T_1 = \inf\{t > 0 : Y_t < 0\}, \quad T_2 = \inf\{t > T_1 : Y_t > 0\}.$$

Since f is bounded, we have

$$\mathbf{E}_x[f(Y_t)] - f(x) = \mathbf{E}_x[f(Y_t)\mathbf{1}_{\{T_1 > t\}} - f(x)] + \mathbf{E}_x[f(Y_t)\mathbf{1}_{\{T_1 \leq t < T_2\}}] + \mathbf{E}_x[f(Y_t)\mathbf{1}_{\{T_2 \leq t\}}],$$

Recall that by construction of Y, (T_1, T_2) are such that under $\mathbf{P}_x$, for $x > 0$, they have the same distribution as $(\zeta^+, \zeta^+ + \zeta^-)$, with ζ^+, ζ^- independent exponential random variables with parameters q^+, q^-, respectively. It is easy to verify that

$$\mathbf{P}_x(T_2 \le t) = \begin{cases} \dfrac{q^-(1-e^{-q^+t}) - q^+(1-e^{-q^-t})}{q^- - q^+}, & q^+ \neq q^-, \\ \\ 1 - e^{-q^+t} - q^+ t e^{-q^+t}, & q^+ = q^-. \end{cases}$$

It follows that $\mathbf{P}_x(T_2 \le t) = o\left(q^+ q^- t^2/2\right)$ as $t \to 0$. Hence, using again that f is bounded, we obtain

$$\frac{1}{t}\mathbf{E}_x[f(Y_t)\mathbf{1}_{\{T_2 \le t\}}] \le \frac{1}{t} C \mathbf{P}_x(T_2 \le t) \to 0, \quad t \to 0.$$

Now we write

$$\frac{1}{t}(\mathbf{E}_x[f(Y_t)\mathbf{1}_{\{T_1 > t\}}] - f(x)) = \frac{1}{t}(\mathbf{E}_x[f(\exp\{\xi_t^+\})] - f(x))e^{-q^+t} + \frac{1}{t} f(x)(e^{-q^+t} - 1),$$

where ξ^+ is a Lévy process such that $\xi_0^+ = \log(x)$, $\mathbf{P}_x$-a.s. The last expression implies

$$\lim_{t\to 0} \frac{1}{t}(\mathbf{E}_x[f(Y_t)\mathbf{1}_{\{T_1 > t\}}] - f(x)) = \mathcal{A}^+(f \circ \exp)(\log(x)) - q^+ f(x).$$

To conclude, observe the identity

$$\mathbf{E}_x[f(Y_t)\mathbf{1}_{\{T_1 \le t < T_2\}}] = \mathbf{E}_x[f(-\exp\{\xi^-_{t-\zeta^+} + \xi^+_{\zeta^+} + U_1^+\}) \mid 0 \le t - \zeta^+ < \zeta^-]\mathbf{P}_x(T_1 \le t < T_2),$$

where ξ^+ is as before and ξ^- is a Lévy process with lifetime ζ^- independent of (ξ^+, ζ^+, U_1^+) and satisfying $\xi_0^- = 0$, $\mathbf{P}_x$-a.s. This together with

$$\lim_{t\to 0} \frac{1}{t}\mathbf{P}_x(T_1 \le t < T_2) = \lim_{t\to 0} \frac{1}{t}\mathbf{P}_x(T_1 \le t) - \lim_{t\to 0} \frac{1}{t}\mathbf{P}_x(T_2 \le t) = q^+,$$

and the convergence

$$\lim_{t\to 0} \frac{1}{t}\mathbf{E}_x[f(-\exp\{\xi^-_{t-\zeta^+} + \xi^+_{\zeta^+} + U_1^+\}) \mid 0 \le t - \zeta^+ < \zeta^-] = \mathbf{E}[f(-x\exp\{U^+\})],$$

which holds by the right continuity of ξ^+ and ξ^-, imply that

$$\lim_{t\to 0} \frac{1}{t}\mathbf{E}_x[f(Y_t)\mathbf{1}_{\{T_1 \le t < T_2\}}] = q^+\mathbf{E}[f(-x\exp\{U^+\})].$$

This ends the proof. □

1.4 Examples

The aim of this section is to characterize the law of $(\xi^\pm, \zeta^\pm, U^\pm)$ which defines the Lamperti-Kiu processes through two examples. The first example is the α-stable process killed at the first hitting time of zero, and the second is the α-stable process conditioned to avoid zero in the case $\alpha \in (1, 2)$.

We start by reviewing some results in the literature about self-similar Markov processes. Through this section X will denote an α-stable process and T its first hitting time of zero ($T = \inf\{t > 0 : X_t = 0\}$, with $\inf\{\emptyset\} = \infty$); and we will denote by X^0 and $X^{\updownarrow}$ the α-stable process killed at T and conditioned to avoid zero, respectively.

In the case $\alpha = 2$, the process X has no jumps and X^0 corresponds to a standard real Brownian motion absorbed at level 0. On the other hand, the Brownian motion conditioned to avoid zero is a three dimensional Bessel process, see e.g. [43]. Thus, depending on the starting point, $X^{\updownarrow}$ is such that $X^{\updownarrow}$ or $-X^{\updownarrow}$ is a Bessel process of dimension 3. Since all Bessel processes are obtained as the images by the Lamperti representation of the exponential of Brownian motion with drift, see e.g. [14] or [51], we obtain the following for $x \in \mathbb{R}^*$,

$$X_t^0 = x \exp\{\xi^0_{\tau(t|x|^{-\alpha})}\}, \quad X_t^{\updownarrow} = x \exp\{\xi^{\updownarrow}_{\tau(t|x|^{-\alpha})}\}, \quad t \geq 0,$$

where ξ^0 and $\xi^{\updownarrow}$ are real Brownian motions with drift, viz., $\xi^0 = (B_t - t/2, t \geq 0)$ and $\xi^{\updownarrow} = (\widetilde{B}_t + t/2, t \geq 0)$, with B, $\widetilde{B}$ real Brownian motions. Therefore, the Lamperti representation is known in the case $\alpha = 2$, so we exclude this case in our examples.

For $0 < \alpha < 2$, let ψ be the characteristic exponent of X: $\mathbb{E}[\exp(i\lambda X_t)] = \exp(t\psi(\lambda))$, $t \geq 0$, $\lambda \in \mathbb{R}$. It is well known that ψ is given by

$$\psi(\lambda) = ia\lambda + \int_{\mathbb{R}} (e^{i\lambda y} - 1 - i\lambda y \mathbf{1}_{\{|y|<1\}})\nu(y)dy, \quad \lambda \in \mathbb{R}, \tag{1.22}$$

where ν is the density of the Lévy measure:

$$\nu(y) = c^+ y^{-\alpha-1}\mathbf{1}_{\{y>0\}} + c^-|y|^{-\alpha-1}\mathbf{1}_{\{y<0\}}, \tag{1.23}$$

with c^+ and c^- being two nonnegative constants such that $c^+ + c^- > 0$. The constant a is $(c^+ - c^-)/(1-\alpha)$ if $\alpha \neq 1$. For the case $\alpha = 1$ we will assume that X is a symmetric Cauchy process, thus $c^+ = c^-$ and $a = 0$.

Another quite well studied positive self-similar Markov process killed at its first hitting time of 0 is the process obtained by killing an α-stable process when it leaves the positive half-line. Formally, if R is the stopping time $R = \inf\{t > 0 : X_t \leq 0\}$, then the process killed at the first time it leaves the positive half-line is $X^{\dagger} = (X_t \mathbf{1}_{\{t<R\}}, t \geq 0)$ where 0 is assumed to be a cemetery state. Caballero and Chaumont in [10] proved that the Lévy process ξ related to X via Lamperti's representation has the characteristic exponent:

$$\Phi(\lambda) = ia\lambda + \int_{\mathbb{R}} [e^{i\lambda y} - 1 - i\lambda(e^y - 1)\mathbf{1}_{\{|e^y-1|<1\}}]\pi(dy) - c^-\alpha^{-1}, \quad \lambda \in \mathbb{R}, \tag{1.24}$$

where the Lévy measure $\pi(dy)$ is

$$\pi(dy) = \left(\frac{c^+ e^y}{(e^y - 1)^{\alpha+1}} \mathbf{1}_{\{y>0\}} + \frac{c^- e^y}{(1 - e^y)^{\alpha+1}} \mathbf{1}_{\{y<0\}} \right) dy. \tag{1.25}$$

Note from (1.24) that the killing rate of the Lévy process ξ is $c^-\alpha^{-1}$.

A further example in the literature appears in [13]. They studied the radial part of the symmetric α-stable process taking values in $\mathbb{R}^d$. In the case $d = 1$, $0 < \alpha < 1$, they proved that the Lévy process in the Lamperti representation for the radial part of the symmetric α-stable process is the sum of two independent Lévy processes ξ_1, ξ_2 with triples $(0, 0, \pi_1)$ and $(0, 0, \pi_2)$ where

$$\pi_1(dy) = \left(\frac{k(\alpha)e^y}{(e^y - 1)^{\alpha+1}} \mathbf{1}_{\{y>0\}} + \frac{k(\alpha)e^y}{(1 - e^y)^{\alpha+1}} \mathbf{1}_{\{y<0\}} \right) dy, \quad \pi_2(dy) = \frac{k(\alpha)e^y}{(e^y + 1)^{\alpha+1}} dy \tag{1.26}$$

and

$$k(\alpha) = \frac{\alpha}{2\Gamma(1 - \alpha) \cos \frac{\pi\alpha}{2}}.$$

In other words, the Lévy process in the Lamperti representation is the sum of a Lévy process with Lévy measure similar to (1.25) and a compound Poisson process. Since the process Y is symmetric in this case, the results in [13] confirm Chybiryakov's results.

The Lévy processes with Lévy measure having the form (1.25) or π_1 in (1.26) are examples of Lamperti-stable processes. For the definition and properties of Lamperti-stable processes, see [12].

1.4.1 The α-stable process killed at zero

The following theorem provides the expression of the infinitesimal generator of the process X^0.

Theorem 1.11. *Let $\alpha \in (0, 2)$ and let $\mathcal{A}$, $\mathcal{A}^0$ the infinitesimal generators of the α-stable process and the α-stable process killed in T, respectively. Then $\mathcal{D}_{\mathcal{A}^0} = \{f \in \mathcal{D}_{\mathcal{A}} : f(0) = 0\}$ and $\mathcal{A}^0 f(x) = \mathcal{A}f(x)$, for $x \in \mathbb{R}^*$. Furthermore, $\mathcal{A}^0 f(x)$ can be written as:*

$$\begin{aligned} \mathcal{A}^0 f(x) &= \frac{1}{|x|^\alpha} \left[sgn(x)axf'(x) + \int_{\mathbb{R}^+} [f(xu) - f(x) - xf'(x)(u-1)\mathbf{1}_{\{|u-1|<1\}}]\nu^{0,sgn(x)}(u)du \right] \\ &\quad + \frac{1}{|x|^\alpha} c^{-sgn(x)} \alpha^{-1} \int_{\mathbb{R}^-} [f(xu) - f(x)]g^0(u)du, \quad x \in \mathbb{R}^*, \end{aligned} \tag{1.27}$$

where

$$\nu^{0,sgn(x)}(u) = \nu(sgn(x)(u-1)), \quad u > 0, \quad g^0(u) = \alpha(1-u)^{-\alpha-1}, \quad u < 0,$$

and ν is given by (1.23).

The proof of the latter theorem will be given at the end of this subsection. The following corollary characterizes the Lamperti-Kiu process associated to the α-stable process killed at its first hitting time of zero and its proof is an immediate consequence of Volkonskii's theorem and the formulas (1.13) and (1.27).

Corollary 1.12. *Let $\xi^{0,\pm}, \zeta^{0,\pm}, U^{0,\pm}$ the random objects in the Lamperti representation of X^0. Then, the characteristic exponent of $\xi^{0,\pm}$ is given by*

$$\psi^{0,\pm}(\lambda) = ia^\pm\lambda + \int_{\mathbb{R}} [e^{i\lambda y} - 1 - i\lambda(e^y - 1)\mathbf{1}_{\{|e^y-1|<1\}}]\pi^{0,\pm}(dy), \quad \lambda \in \mathbb{R},$$

where $a^{\pm} = \pm a$, with a as in (1.22), and $\pi^{0,\pm}(dy) = e^y \nu(\pm(e^y - 1))dy$. The parameters of the exponential random variables $\zeta^{0,\pm}$ are $c^{\mp}\alpha^{-1}$ and the real random variables $U^{0,\pm}$ have density

$$g(u) = \frac{\alpha e^u}{(1+e^u)^{\alpha+1}}, \quad u \in \mathbb{R}.$$

Note that as expected, the Lévy process $\xi^{0,+}$ is the one obtained in [10]. Furthermore, the downwards change of sign rate, which is the death rate in [10], is $c^{-}\alpha^{-1}$. From the triples of $\xi^{0,+}$ and $\xi^{0,-}$ we can observe that both belong to the Lamperti-stable family. In the particular case where X is a symmetric α-stable process with $\alpha \in (0,1)$, the description in Corollary 1.12 coincides with the one in [13], see (1.26). Note that $U^{0,+}$, $U^{0,-}$ are identically distributed and they are such that $U^{0,\pm} \overset{\mathcal{L}}{=} \log V$, where V follows a Pareto distribution with parameter α, viz.,

$$f(x) = \frac{\alpha}{(1+x)^{\alpha+1}}, \quad x > 0.$$

In order to prove the main theorem of this subsection we need the following two lemmas.

Lemma 1.13. *Let X be an α-stable process, $\alpha \in (0,2)$. Then, for any $x \in \mathbb{R}^*$,*

$$\lim_{t\downarrow 0} \frac{1}{t}\mathbb{P}_x(T \le t, X_t \in \mathbb{R}^*) = 0. \tag{1.28}$$

Proof. Since for $\alpha \in (0,1]$ the point zero is polar, then (1.28) is clearly satisfied. Suppose $\alpha \in (1,2)$. For $\delta > 0$, write

$$\mathbb{P}_x(T \le t, X_t \in \mathbb{R}^*) = \mathbb{P}_x(T \le t, |X_t| \in (0,\delta]) + \mathbb{P}_x(T \le t, |X_t| > \delta).$$

First, we verify the following: for $0 < \delta < |x|$ it holds

$$\lim_{t\downarrow 0} \frac{1}{t}\mathbb{P}_x(|X_t| \in (0,\delta]) = \frac{c^{-\mathrm{sgn}(x)}}{\alpha}\mathrm{sgn}(x)(|\delta - x|^{-\alpha} - |\delta + x|^{-\alpha}). \tag{1.29}$$

For this aim, we will use the fact that for every $K > 0$, $(1/t)\mathbb{P}_0(X_t \in dz)$ converges vaguely to $\nu(z)dz$ on $\{z : |z| > K\}$, as $t \downarrow 0$; see e.g. exercise I.1 in [5]. We only show (1.29) in the case $x < 0$, the case $x > 0$ can be proved similarly. For $x < 0$, we have $\delta + x < 0$ and

$$\begin{aligned}
\lim_{t\downarrow 0} \frac{1}{t}\mathbb{P}_x(|X_t| \in (0,\delta]) &= \lim_{t\downarrow 0} \frac{1}{t}\mathbb{P}_0(X_t \in [-\delta - x, \delta - x]) \\
&= \int_{-\delta-x}^{\delta-x} \nu(z)dz \\
&= \frac{c^+}{\alpha}((-\delta - x)^{-\alpha} - (\delta - x)^{-\alpha}),
\end{aligned}$$

which proves the claim. Now, from (1.29) we obtain

$$\limsup_{t\downarrow 0} \frac{1}{t}\mathbb{P}_x(T \le t, |X_t| \in (0,\delta]) \le \frac{c^{-\mathrm{sgn}(x)}}{\alpha}\mathrm{sgn}(x)(|\delta - x|^{-\alpha} - |\delta + x|^{-\alpha}). \tag{1.30}$$

On the other hand, by the strong Markov property

$$\mathbb{P}_x(T \le t, |X_t| > \delta) = \int_0^t \mathbb{P}_0(|X_{t-s}| > \delta)\mathbb{P}_x(T \in ds).$$

Since $(1/t)\mathbb{P}_0(X_t \in dz)$ converges vaguely to $\nu(z)dz$ on $\{z : |z| > K\}$ for every $K > 0$, there exists a constant C such that, for sufficiently small t:

$$\mathbb{P}_0(|X_{t-s}| > \delta) \le \frac{Ct}{\delta^\alpha}, \quad \text{for all } s \in (0,t).$$

Then

$$\mathbb{P}_x(T \le t, |X_t| > \delta) \le \mathbb{P}_x(T \le t)\frac{Ct}{\delta^\alpha}.$$

The latter inequality and (1.30) imply the result. □

Lemma 1.14. *Let $x \in \mathbb{R}^*$, and $\alpha \in (0,2)$. We will denote by $I_1^{(x)}$ and $I_2^{(x)}$ the following integrals*

$$\begin{aligned} I_1^{(x)} &= \int_{\mathbb{R}^+} (u-1)(\mathbf{1}_{\{|u-1|<1\}} - \mathbf{1}_{\{|x(u-1)|<1\}})\nu(sgn(x)(u-1))du, \\ I_2^{(x)} &= \int_{\mathbb{R}^-} (u-1)\mathbf{1}_{\{|x(u-1)|<1\}}\nu(sgn(x)(u-1))du. \end{aligned}$$

The identity

$$I_1^{(x)} - I_2^{(x)} = sgn(x)a(1 - |x|^{\alpha-1}), \quad \textit{holds.}$$

Proof. We will show the case $x < 0$, and $\alpha \neq 1$, the other cases can be proved similarly. First, observe that $|u-1| < 1$ if only if $0 < u < 2$. Thus, if $x = -1$, then $I_1^{(x)} = I_2^{(x)} = 0$ and the lemma is satisfied. Now, suppose that $-1 < x < 0$, then $1 + x^{-1} < 0 < 2 < 1 - x^{-1}$,

$$I_1^{(x)} = -\int_2^{1-x^{-1}} c^-(u-1)^{-\alpha}du = \frac{c^-}{1-\alpha}[1 - (-x)^{\alpha-1}],$$

and

$$I_2^{(x)} = -\int_{1+x^{-1}}^0 c^+(1-u)^{-\alpha}du = \frac{c^+}{1-\alpha}[1 - (-x)^{\alpha-1}].$$

Hence, $I_1^{(x)} - I_2^{(x)} = -a[1 - (-x)^{\alpha-1}]$. Finally, suppose that $x < -1$. In this case, we have $0 < 1 + x^{-1} < 1 < 1 - x^{-1} < 2$, $I_2^{(x)} = 0$ and

$$I_1^{(x)} = -\int_0^{1+x^{-1}} c^+(1-u)^{-\alpha}du + \int_{1-x^{-1}}^2 c^-(u-1)^{-\alpha}du = -a[1 - (-x)^{\alpha-1}].$$

This ends the proof. □

Proof of Theorem 1.11. For any f bounded function such that $f(0) = 0$, we have for $x \in \mathbb{R}^*$

$$\mathbb{E}_x[f(X_t^0) - f(x)] = \mathbb{E}_x[f(X_t) - f(x)] - \mathbb{E}_x[f(X_t)\mathbf{1}_{\{T \le t\}}].$$

On the other hand, by the Lemma 1.13,

$$\lim_{t\to 0}\frac{1}{t}\mathbb{E}_x[f(X_t)\mathbf{1}_{\{T\le t\}}]=0.$$

Then

$$\lim_{t\to 0}\frac{1}{t}\mathbb{E}_x[f(X_t^0)-f(x)]=\lim_{t\to 0}\frac{1}{t}\mathbb{E}_x[f(X_t)-f(x)].$$

Hence, $\mathcal{D}_{\mathcal{A}^0}=\{f\in\mathcal{D}_{\mathcal{A}}:f(0)=0\}$ and $\mathcal{A}^0f(x)=\mathcal{A}f(x)$.

Now we will obtain (1.27). By the first part of the theorem we have that for $x\in\mathbb{R}^*$, $\mathcal{A}^0f(x)$ is given by

$$\mathcal{A}^0f(x)=af'(x)+\int_{\mathbb{R}}[f(x+y)-f(x)-yf'(x)\mathbf{1}_{\{|y|<1\}}]\nu(y)dy. \tag{1.31}$$

Let I be the integral in (1.31). Then, with the change of variables $y=x(u-1)$ we obtain

$$\begin{aligned}
I &= \frac{1}{|x|^\alpha}\int_{\mathbb{R}}[f(xu)-f(x)-xf'(x)(u-1)\mathbf{1}_{\{|x(u-1)|<1\}}]\nu(\mathrm{sgn}(x)(u-1))du\\
&= \frac{1}{|x|^\alpha}\int_{\mathbb{R}^+}[f(xu)-f(x)-xf'(x)(u-1)\mathbf{1}_{\{|u-1|<1\}}]\nu(\mathrm{sgn}(x)(u-1))du\\
&\quad+\frac{1}{|x|^\alpha}\int_{\mathbb{R}^+}[xf'(x)(u-1)(\mathbf{1}_{\{|u-1|<1\}}-\mathbf{1}_{\{|x(u-1)|<1\}})]\nu(\mathrm{sgn}(x)(u-1))du\\
&\quad+\frac{1}{|x|^\alpha}\int_{\mathbb{R}^-}[f(xu)-f(x)-xf'(x)(u-1)\mathbf{1}_{\{|x(u-1)|<1\}}]\nu(\mathrm{sgn}(x)(u-1))du.
\end{aligned}$$

With the help of Lemma 1.14, we can write I as follows

$$\begin{aligned}
I &= \frac{1}{|x|^\alpha}\left[\mathrm{sgn}(x)axf'(x)+\int_{\mathbb{R}^+}[f(xu)-f(x)-xf'(x)(u-1)\mathbf{1}_{\{|u-1|<1\}}]\nu^{0,\mathrm{sgn}(x)}(u)du\right]\\
&\quad+\frac{1}{|x|^\alpha}\int_{\mathbb{R}^-}[f(xu)-f(x)]\nu^{0,\mathrm{sgn}(x)}(u)du-af'(x).
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\mathcal{A}^0f(x) &= \frac{1}{|x|^\alpha}\left[\mathrm{sgn}(x)axf'(x)+\int_{\mathbb{R}^+}[f(xu)-f(x)-xf'(x)(u-1)\mathbf{1}_{\{|u-1|<1\}}]\nu^{0,\mathrm{sgn}(x)}(u)du\right]\\
&\quad+\frac{1}{|x|^\alpha}\left[\int_{\mathbb{R}^-}[f(xu)-f(x)]\nu^{0,\mathrm{sgn}(x)}(u)du\right].
\end{aligned}$$

Finally, note that

$$\frac{\nu^{0,\mathrm{sgn}(x)}(u)}{c^{-\mathrm{sgn}(x)}\alpha^{-1}}=g^0(u),\quad u<0.$$

This ends the proof. □

1.4.2 The α-stable process conditioned to avoid zero

In [48] symmetric Lévy processes conditioned to avoid zero were studied. One of the main results in [48] can be stated as follows. Let X be a Lévy process with characteristic exponent ψ. Consider the following assumptions

H.1 The origin is regular for itself and X is not a compound Poisson process.

H.2 X is symmetric.

Then, under **H.1** and **H.2** the function h, given by

$$h(x) = \frac{1}{\pi}\int_0^\infty \frac{1-\cos\lambda x}{\theta(\lambda)}d\lambda, \quad x \in \mathbb{R},$$

where $\theta(\lambda) = -Re(\psi(\lambda))$, is an invariant function with respect to the semigroup, P_t^0, of the process X killed at T, the first hitting time of 0. Note that if X is an α-stable process with $\alpha \in (0,2)$, **H.1** and **H.2** are satisfied if and only if X is symmetric and $\alpha \in (1,2)$. In this case, the characteristic exponent is given by $\psi(\lambda) = -|\lambda|^\alpha$, h has an explicit form, namely

$$h(x) = C(\alpha)|x|^{\alpha-1}, \quad x \in \mathbb{R},$$

where

$$C(\alpha) = \frac{\Gamma(2-\alpha)}{\pi(\alpha-1)}\sin\frac{\alpha\pi}{2}.$$

In chapter 2 a generalization of the latter fact is considered. There it is proved that for X α-stable process with $1<\alpha<2$, the function h given by

$$h(x) = K(\alpha)(1-\beta\text{sgn}(x))|x|^{\alpha-1}, \quad x \in \mathbb{R}, \tag{1.32}$$

where

$$K(\alpha) = \frac{\Gamma(2-\alpha)\sin(\alpha\pi/2)}{c\pi(\alpha-1)(1+\beta^2\tan^2(\alpha\pi/2))},$$

and

$$c = -\frac{(c^+ + c^-)\Gamma(2-\alpha)}{\alpha(\alpha-1)}\cos(\alpha\pi/2), \quad \beta = \frac{c^+ - c^-}{c^+ + c^-}, \tag{1.33}$$

is an invariant function for the semigroup of X^0. In fact this result is a consequence of a more general result that has been proved in chapter 2 under the sole assumption **H.1**. Since h is invariant for the semigroup P_t^0 and $h(x) \neq 0$, for $x \in \mathbb{R}^*$, then we define the semigroup P_t^h on $\mathbb{R}^*$ by

$$P_t^h(x,dy) := \frac{h(y)}{h(x)}P_t^0(x,dy), \quad x,y \in \mathbb{R}^*, t \geq 0.$$

We denote by $\mathbb{P}_x^h$ the law of the strong Markov process with starting point x and semigroup P_t^h. $\mathbb{P}_\cdot^h$ is Doob's h-transformation of $\mathbb{P}_0$ via the invariant function h as defined in (1.32). Since under the measure $\mathbb{P}_x^h$ it holds $\mathbb{P}_x^h(T=\infty)=1$, then the process X^h can be considered as the process X conditioned to avoid (or never to hit) zero, this has been proved in chapter 2. We use the notation $X^\updownarrow$ instead of X^h to emphasize this fact. Thus, as was mentioned at the beginning of the section, $X^\updownarrow$ is the α-stable process conditioned to avoid zero, when $\alpha \in (1,2)$. In the following lemma we summarize properties of the function h, which follow straightforwardly from its definition and so we omit their proof.

Lemma 1.15. *The function h defined in (1.32) satisfies the following properties*

(i) $h(x) > 0$, *for all* $x \in \mathbb{R}^*$, $h(0) = 0$;

(ii) $h(ux) = |u|^{\alpha-1} h(sgn(u)x)$, *for all* $u \in \mathbb{R}$;

(iii) $(hf)'(x) = h(x)[(\alpha-1)x^{-1}f(x) + f'(x)]$, $f \in C^1$, $x \in \mathbb{R}^*$;

(iv) $h(-x) = h(x) + 2K(\alpha)\beta sgn(x)|x|^{\alpha-1}$, *for all* $x \in \mathbb{R}$.

Using *(ii)* of Lemma 1.15 and (1.1) it is possible to verify that the semigroup of the process $X^{\updownarrow}$ satisfies the self-similarity property. Hence $X^{\updownarrow}$ is real-valued self-similar Markov process. The following theorem provides an expression for the infinitesimal generator of $X^{\updownarrow}$.

Theorem 1.16. *Let* $\mathcal{A}^{\updownarrow}$ *be the infinitesimal generator of* $X^{\updownarrow}$. *For* $x \in \mathbb{R}^*$, $\mathcal{A}^{\updownarrow} f(x)$ *can be written as*

$$\begin{aligned}\mathcal{A}^{\updownarrow} f(x) &= \frac{1}{|x|^{\alpha}}\left[a^{\updownarrow, sgn(x)} x f'(x) + \int_{\mathbb{R}^+} [f(xu) - f(x) - xf'(x)(u-1)\mathbf{1}_{\{|u-1|<1\}}]\nu^{\updownarrow, sgn(x)}(u)du\right] \\ &\quad + \frac{1}{|x|^{\alpha}} c^{sgn(x)}\alpha^{-1}\int_{\mathbb{R}^-}[f(xu) - f(x)]g^{\updownarrow}(u)du, \qquad (1.34)\end{aligned}$$

where

$$a^{\updownarrow, sgn(x)} = sgn(x)a + c^{sgn(x)}\int_0^1 \frac{(1+u)^{\alpha-1}-1}{u^{\alpha}}du - c^{-sgn(x)}\int_0^1 \frac{(1-u)^{\alpha-1}-1}{u^{\alpha}}du \qquad (1.35)$$

and

$$\nu^{\updownarrow, sgn(x)}(u) = u^{\alpha-1}\nu(sgn(x)(u-1)), \quad u > 0; \quad g^{\updownarrow}(u) = \alpha(-u)^{\alpha-1}(1-u)^{-\alpha-1}, \quad u < 0.$$

The following corollary is also a consequence of Volkonskii's theorem and the comparison of (1.13) and (1.34).

Corollary 1.17. *Let* $\xi^{\updownarrow,\pm}, U^{\updownarrow,\pm}, \zeta^{\updownarrow,\pm}$ *the random objects in the Lamperti representation of* $X^{\updownarrow}$. *Then the characteristic exponent of* $\xi^{\pm}$ *is*

$$\psi^{\updownarrow,\pm}(\lambda) = ia^{\updownarrow,\pm}\lambda + \int_{\mathbb{R}}[e^{i\lambda y} - 1 - i\lambda(e^y - 1)\mathbf{1}_{\{|e^y-1|<1\}}]\pi^{\updownarrow,\pm}(dy), \quad \lambda \in \mathbb{R},$$

where $a^{\updownarrow,\pm}$ *is given by (1.35) and* $\pi^{\updownarrow,\pm}(dy) = e^{\alpha y}\nu(\pm(e^y - 1))dy$. *The parameters of the exponential random variables* $\zeta^{\updownarrow,\pm}$ *are* $c^{\pm}\alpha^{-1}$ *and the real random variables* $U^{\updownarrow,\pm}$ *have density*

$$g(u) = \frac{\alpha e^{\alpha u}}{(1+e^u)^{\alpha+1}}, \quad u \in \mathbb{R}.$$

As in the first example, the Lévy processes $\xi^{\updownarrow,+}$, $\xi^{\updownarrow,-}$ belong to the Lamperti-stable family. Furthermore, their Lévy measure, satisfy the relation: $\pi^{\updownarrow,\pm}(dy) = e^{(\alpha-1)y}\pi^{0,\pm}(dy)$. Note that $g(u)$ can be written as

$$g(u) = \frac{\alpha e^{-u}}{(1+e^{-u})^{\alpha+1}}, \quad u \in \mathbb{R}.$$

Hence, $U^{\updownarrow,\pm} \stackrel{\mathcal{L}}{=} -U^{0,\pm} \stackrel{\mathcal{L}}{=} -\log V$, with $U^{0,\pm}$ as in Corollary 1.12 and V is a Pareto random variable.

Proof of Theorem 1.16. Recall that $\mathcal{A}^{\ddagger}f(x) = [h(x)]^{-1}\mathcal{A}^0(hf)(x)$, $x \in \mathbb{R}^*$. Thus, by (1.27) we can write for $x \in \mathbb{R}^*$

$$[h(x)]^{-1}|x|^{\alpha}\mathcal{A}^0(hf)(x) = [h(x)]^{-1}(\mathrm{sgn}(x)ax(hf)'(x) + \mathcal{I}_1^{(x)} + \mathcal{I}_2^{(x)}),$$

where

$$\begin{aligned}
\mathcal{I}_1^{(x)} &= \int_{\mathbb{R}^+}[(hf)(xu) - (hf)(x) - x(hf)'(x)(u-1)\mathbf{1}_{\{|u-1|<1\}}]\nu(\mathrm{sgn}(x)(u-1))du,\\
\mathcal{I}_2^{(x)} &= \int_{\mathbb{R}^-}[(hf)(xu) - (hf)(x)]\nu(\mathrm{sgn}(x)(u-1))du.
\end{aligned}$$

Now, by *(iii)* of Lemma 1.15,

$$[h(x)]^{-1}\mathrm{sgn}(x)ax(hf)'(x) = \mathrm{sgn}(x)axf'(x) + \mathrm{sgn}(x)a(\alpha-1)f(x). \tag{1.36}$$

Also, using *(ii)* and *(iii)* of Lemma 1.15, we have

$$\begin{aligned}
[h(x)]^{-1}\mathcal{I}_1^{(x)} &= \int_{\mathbb{R}^+}[f(xu) - f(x) - xf'(x)(u-1)\mathbf{1}_{\{|u-1|<1\}}]u^{\alpha-1}\nu(\mathrm{sgn}(x)(u-1))du\\
&\quad + \int_{\mathbb{R}^+}(u^{\alpha-1}-1)(u-1)\mathbf{1}_{\{|u-1|<1\}}\nu(\mathrm{sgn}(x)(u-1))du \times xf'(x)\\
&\quad + \int_{\mathbb{R}^+}[u^{\alpha-1} - 1 - (\alpha-1)(u-1)\mathbf{1}_{\{|u-1|<1\}}]\nu(\mathrm{sgn}(x)(u-1))du \times f(x)\\
&= I_1^{(x)} + I_2^{(x)}xf'(x) + I_3^{(x)}f(x)
\end{aligned} \tag{1.37}$$

where

$$\begin{aligned}
I_1^{(x)} &= \int_{\mathbb{R}^+}[f(xu) - f(x) - xf'(x)(u-1)\mathbf{1}_{\{|u-1|<1\}}]u^{\alpha-1}\nu(\mathrm{sgn}(x)(u-1))du,\\
I_2^{(x)} &= \int_{\mathbb{R}^+}(u^{\alpha-1}-1)(u-1)\mathbf{1}_{\{|u-1|<1\}}\nu(\mathrm{sgn}(x)(u-1))du\\
&= c^{\mathrm{sgn}(x)}\int_0^1 \frac{(1+u)^{\alpha-1}-1}{u^{\alpha}}du - c^{-\mathrm{sgn}(x)}\int_0^1 \frac{(1-u)^{\alpha-1}-1}{u^{\alpha}}du,\\
I_3^{(x)} &= \int_{\mathbb{R}^+}[u^{\alpha-1} - 1 - (\alpha-1)(u-1)\mathbf{1}_{\{|u-1|<1\}}]\nu(\mathrm{sgn}(x)(u-1))du.
\end{aligned}$$

And by *(ii)*, *(iv)* of Lemma 1.15 and since $\int_{\mathbb{R}^-}(-u)^{\alpha-1}\nu(\mathrm{sgn}(x)(u-1))du = c^{-\mathrm{sgn}(x)}\alpha^{-1}$, we obtain

$$\begin{aligned}
[h(x)]^{-1}\mathcal{I}_2^{(x)} &= \left(\frac{1+\beta\mathrm{sgn}(x)}{1-\beta\mathrm{sgn}(x)}\right)\int_{\mathbb{R}^-}[f(xu) - f(x)](-u)^{\alpha-1}\nu(\mathrm{sgn}(x)(u-1))du\\
&\quad + \frac{2\beta\mathrm{sgn}(x)}{1-\beta\mathrm{sgn}(x)}c^{-\mathrm{sgn}(x)}\alpha^{-1}f(x).
\end{aligned}$$

Substituting the values of a and β given by (1.22) and (1.33) in the latter equality, it follows

$$[h(x)]^{-1}\mathcal{I}_2^{(x)} = c^{\mathrm{sgn}(x)}\alpha^{-1}I_4^{(x)} - \alpha^{-1}(\alpha-1)\mathrm{sgn}(x)af(x), \tag{1.38}$$

where $I_4^{(x)}$ is the integral

$$\int_{\mathbb{R}^-} [f(xu) - f(x)] g^{\updownarrow}(u) du.$$

Thus, the expressions (1.36), (1.37) and (1.38) imply

$$\begin{aligned} \mathcal{A}^{\updownarrow} f(x) =& |x|^{-\alpha} \left[(\mathrm{sgn}(x) a + I_2^{(x)}) x f'(x) + I_1^{(x)} + c^{\mathrm{sgn}(x)} \alpha^{-1} I_4^{(x)} \right] \\ &+ |x|^{-\alpha} [\alpha^{-1} (\alpha - 1)^2 \mathrm{sgn}(x) a + I_3^{(x)}] f(x). \end{aligned}$$

Finally, since h is an invariant function for the semigroup of X^0, then $f \equiv 1$ belongs to $\mathcal{D}_{\mathcal{A}^{\updownarrow}}$ and it follows that $\alpha^{-1}(\alpha - 1)^2 \mathrm{sgn}(x) a + I_3^{(x)} = 0$. This ends the proof. □

Chapter 2

The Lévy processes conditioned to avoid zero

2.1 Introduction

The main purpose of this work is to construct Lévy processes conditioned to avoid zero. This question is relevant only when 0 is non-polar. Then the event "not hitting zero" has zero probability and hence a standard analytical approach consists on finding an adequate excessive function for the process killed at the first hitting time of zero and then use Doob's h-transformation technique. A good understanding of the associated excessive function allows us to establish analytical and pathwise properties of the constructed process. This is the approach that has been used by Yano [48], under the assumption that the Lévy process is symmetric. So, our results can be seen as a generalization of the results obtained by Yano. A probabilistic approach for constructing Lévy processes conditioned to avoid zero bears on the idea that the construction can be performed by conditioning the process not to hit zero up to an independent exponential time with parameter q, and then make $q \to 0$, so that the conditioning affect the process all over the time interval $[0, \infty)$. This is a generic approach that has been used in several contexts. See for instance Chaumont and Doney [17] and the reference therein, where the case of Lévy processes conditioned to stay positive is investigated. We will prove that in our setting this procedure gives a non-degenerate limit and that both constructions coincide.

2.2 Preliminaries and main results

2.2.1 Notation

Let $\mathcal{D}[0, \infty)$ be the space of càdlàg paths $\omega : [0, \infty) \to \mathbb{R} \cup \{\Delta\}$ with lifetime $\zeta(\omega) = \inf\{s : \omega_s = \Delta\}$, where Δ is a cemetery point. The space $\mathcal{D}[0, \infty)$ is endowed with Skorohod's topology and its Borel σ-field, $\mathcal{F}$. Moreover, let $\mathbb{P}$ be a reference probability measure on $\mathcal{D}[0, \infty)$, under which the coordinate process $X = (X_t, t \geq 0)$ is a Lévy process. We will denote by $(\mathcal{F}_t, t \geq 0)$ the completed, right continuous filtration generated by X. As usual $\mathbb{P}_x$ denotes the law of $X + x$, under $\mathbb{P}$, for $x \in \mathbb{R}$. For notational convenience, we set $\mathbb{P} = \mathbb{P}_0$. We will denote by θ the shift operator and by k the killing operator, i.e., for $\omega \in \mathcal{D}[0,\infty)$, $\theta_t\omega(s) = w(s+t)$, $s \geq 0$, and

$$k_t\omega(s) = \begin{cases} w(s), & s < t, \\ \Delta, & s \geq t. \end{cases}$$

For $t \geq 0$, we use $X \circ \theta_t$, $X \circ k_t$ to denote the functions in $\mathcal{D}[0,\infty)$ given by $\theta_t\omega(\cdot)$ and $k_t\omega(\cdot)$, $\omega \in \mathcal{D}[0,\infty)$, respectively. Throughout the paper $\psi : \mathbb{R} \to \mathbb{C}$ will denote the characteristic exponent of $(X, \mathbb{P})$, which is defined by

$$\psi(\lambda) = -\frac{1}{t}\log(\mathbb{E}[e^{i\lambda X_t}]) = ia\lambda + \frac{\sigma^2}{2}\lambda^2 + \int_{\mathbb{R}} (1 - e^{i\lambda x} + i\lambda x\mathbf{1}_{\{|x|<1\}})\pi(dx), \quad \lambda \in \mathbb{R},$$

where $a \in \mathbb{R}$, $\sigma \geq 0$ and π denotes the Lévy measure, i.e., π is a measure satisfying $\pi(\{0\}) = 0$ and $\int_{\mathbb{R}}(1 \wedge x^2)\pi(dx) < \infty$. We denote by P_t and U_q the transition kernel at time t and the q-resolvent of the process $(X, \mathbb{P})$.

We assume throughout the paper that

H.1 The origin is regular for itself.

H.2 $(X, \mathbb{P})$ is not a compound Poisson process.

We quote the following classical result that provide an equivalent way to verify conditions **H.1** and **H.2** in terms of the characteristic exponent ψ.

Theorem 2.1 (See, e.g., [9] and [32]). *The conditions **H.1** and **H.2** are satisfied if and only if*

$$\int_{\mathbb{R}} Re\left(\frac{1}{q+\psi(\lambda)}\right) d\lambda < \infty, \quad q > 0$$

and

$$\text{either} \quad \sigma > 0 \quad \text{or} \quad \int_{|x|<1} |x|\pi(dx) = \infty.$$

It is known that under these hypotheses, for any $q > 0$, there exists a density of the resolvent kernel that we will denote by $u_q(x, y)$:

$$U_q f(x) = \int_{\mathbb{R}} u_q(x,y)f(y)dy, \quad x \in \mathbb{R},$$

for all bounded Borel functions f. The density $u_q(x, y)$ equals $u_q(y-x)$, where u_q is a continuous function. We refer to chapter II in [5] for a proof of these results. Furthermore, from the resolvent equation

$$U_q - U_r + (q-r)U_qU_r = 0, \quad q, r > 0,$$

it can be deduced that the family of functions $(u_q, q > 0)$ satisfies, for all $q, r > 0$ with $q \neq r$,

$$\int_{\mathbb{R}} u_q(y-x)u_r(z-y)dy = \frac{1}{q-r}[u_r(z-x) - u_q(z-x)], \quad \text{for all } z, x \in \mathbb{R}. \tag{2.1}$$

Let T_0 be the first hitting time of zero for X:

$$T_0 = \inf\{t > 0 : X_t = 0\},$$

with $\inf\{\emptyset\} = \infty$. The process killed at T_0, $X^0 = X \circ k_{T_0}$, is given by

$$X_t^0 = \begin{cases} X_t, & t < T_0, \\ \Delta, & t \geq T_0. \end{cases}$$

For every $x \in \mathbb{R}$, we will denote by $\mathbb{P}_x^0$ the law of the killed process X^0 under $\mathbb{P}_x$. We use the notation P_t^0, U_q^0 for its transition kernel and q-resolvent, respectively. From [5, Corollary 18, p. 64], it is known that,

$$\mathbb{E}_x[e^{-qT_0}] = \frac{u_q(-x)}{u_q(0)}, \quad q > 0, x \in \mathbb{R}. \tag{2.2}$$

Hence, with help of the following well known identity:

$$U_q f(x) = U_q^0 f(x) + \mathbb{E}_x[e^{-qT_0}] U_q f(0),$$

for all bounded Borel functions f and $q > 0$, we obtain the resolvent density for X^0, namely,

$$u_q^0(x, y) = u_q(y - x) - \frac{u_q(-x) u_q(y)}{u_q(0)}, \quad x, y \in \mathbb{R}. \tag{2.3}$$

By $\widehat{\mathbb{P}}_x$ we will denote the law of the dual process $\widehat{X} := -X$ under $\mathbb{P}_{-x}$, $x \in \mathbb{R}$. We will use the notation $\widehat{\ }$ to specify the mathematical quantities related to the dual process $\widehat{X}$. For instance, $(\widehat{P}_t, t \geq 0)$, $(\widehat{U}_q, q > 0)$ are the semigroup and the resolvent of the process $\widehat{X}$, respectively. It is known that the name "dual" comes from the following duality identity. Let f, g be nonnegative and measurable functions. Then, for every $t \geq 0$

$$\int_{\mathbb{R}} P_t f(x) g(x) dx = \int_{\mathbb{R}} f(x) \widehat{P}_t g(x) dx$$

and for every $q > 0$

$$\int_{\mathbb{R}} U_q f(x) g(x) dx = \int_{\mathbb{R}} f(x) \widehat{U}_q g(x) dx.$$

For the semigroup and q-resolvent of the killed process we have as a consequence of *Hunt's switching identity* (see e.g. [5, p. 47, Theorem 5]):

$$\int_{\mathbb{R}} g(x) P_t^0 f(x) dx = \int_{\mathbb{R}} f(x) \widehat{P}_t^0 g(x) dx$$

and for every $q > 0$

$$\int_{\mathbb{R}} g(x) U_q^0 f(x) dx = \int_{\mathbb{R}} f(x) \widehat{U}_q^0 g(x) dx.$$

We observe that $(\widehat{X}, \widehat{\mathbb{P}})$ satisfies also the hypotheses **H.1** and **H.2**. Thus, for any $q > 0$, there exists a continuous density $\widehat{u}_q$ of the resolvent $\widehat{U}_q$. Furthermore, u_q and $\widehat{u}_q$ are related by the equation: $\widehat{u}_q(x) = u_q(-x)$, $x \in \mathbb{R}$. Thereby, for any $q > 0$, $\widehat{\mathbb{E}}_x[e^{-qT_0}]$ and the density of $\widehat{U}_q^0$ can be written in terms of u_q as follows

$$\widehat{\mathbb{E}}_x[e^{-qT_0}] = \frac{u_q(x)}{u_q(0)}, \quad q > 0, x \in \mathbb{R} \tag{2.4}$$

and

$$\widehat{u}_q^0(x,y) = u_q(x-y) - \frac{u_q(x)u_q(-y)}{u_q(0)}, \quad x, y \in \mathbb{R}. \tag{2.5}$$

Since the point zero is regular for itself, there exists a continuous local time at 0 (in fact, at any point $x \in \mathbb{R}$). We denote by $L = (L_t, t \geq 0)$ the local time at zero, which is normalized by $\mathbb{E}(\int_0^\infty e^{-t} dL_t) = 1$, and by n the excursion measure away from zero for X. The measure n has its support on the set of excursions away from zero:

$$\mathcal{D}^0 = \{\epsilon \in \mathcal{D}[0,\infty) : \epsilon(t) \neq 0, 0 < t < \zeta(\epsilon), 0 < \zeta(\epsilon) \leq \infty\}.$$

A nice relation between the excursion measure n and the Laplace transform of the law of T_0 under $\widehat{\mathbb{P}}_x$ can be found in [50, Theorem 3.3] for Lévy processes and in [27, eq. (3.22)], [19, eq. (2.8)] for general Markov processes. This is stated as follows, let f be a nonnegative measurable function, then

$$\int_0^\infty e^{-qt} n(f(X_t), t < \zeta) dt = \int_{\mathbb{R}} f(x) \widehat{\mathbb{E}}_x[e^{-qT_0}] dx. \tag{2.6}$$

In particular, if $f \equiv 1$,

$$\int_0^\infty e^{-qt} n(\zeta > t) dt = \frac{1}{q u_q(0)}, \quad q > 0. \tag{2.7}$$

2.2.2 Main results

Under the assumptions **H.1**, **H.2** and

H.3 $(X, \mathbb{P})$ is symmetric,

Yano [48] showed that the function h defined by

$$h(x) = \lim_{q \to 0} [u_q(0) - u_q(x)], \quad x \in \mathbb{R} \tag{2.8}$$

is a well defined invariant function for the semigroup of the Lévy process killed at its first hitting time of zero. Furthermore, Yano proved that the function h can be expressed in terms of the characteristic exponent of X as

$$h(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{1 - \cos \lambda x}{\theta(\lambda)} d\lambda, \quad x \in \mathbb{R}, \tag{2.9}$$

where $\theta(\lambda) = Re\psi(\lambda)$. Our first main result generalizes (2.8) and (2.9).

Throughout the rest of this paper we assume that H.1 and H.2 are satisfied.

Theorem 2.2. *For $q > 0$, let h_q denote the function defined by*

$$h_q(x) = u_q(0) - u_q(-x), \quad q > 0, \quad x \in \mathbb{R}. \tag{2.10}$$

Then, the function $h : \mathbb{R} \to \mathbb{R}$ defined by

$$h(x) = \lim_{q \to 0} h_q(x), \quad x \in \mathbb{R} \tag{2.11}$$

is such that

(i) for every $x \in \mathbb{R}$, $0 \le h(x) < \infty$. Furthermore, under the condition

$$\int_{\mathbb{R}} \frac{1}{q + Re(\psi(\lambda))} d\lambda < \infty, \quad q > 0, \tag{2.12}$$

it holds

$$h(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} Re\left(\frac{1 - e^{i\lambda x}}{\psi(\lambda)}\right) d\lambda, \quad x \in \mathbb{R}, \tag{2.13}$$

(ii) h is subadditive, continuous function, which vanishes at the point $x = 0$.

(iii) h is invariant with respect to the semigroup of the Lévy process killed at T_0, i.e.,

$$P_t^0 h(x) = h(x), \quad t > 0, \quad x \in \mathbb{R}; \tag{2.14}$$

furthermore

$$n(h(X_t), t < \zeta) = 1, \quad \forall\, t > 0.$$

The proof of (i) and (ii) in Theorem 2.2 will be given in section 2.3.2, as a consequence of analogous results for the sequence of functions $(h_q)_{q>0}$. In order to establish (iii) and other results, and due to technical issues, we will introduce an auxiliary function h^*. The function h^* dominates h and satisfies some integrability conditions. This function, as its name indicates, will help us to prove the main results acting as a dominating function in the dominated convergence theorem. The function h^* is closely related to the local time of the Lévy process $(X, \mathbb{P})$, namely, we have the expression

$$h^*(x) = \mathbb{E}(L_{T_x}) = \lim_{q \to 0} \mathbb{E}\left(\int_0^{T_x} e^{-qt} dL_t\right), \quad x \in \mathbb{R}.$$

The function h^* arises as a particular case of a general function $h(\cdot, \cdot)$ defined by

$$h(x, y) = \mathbb{E}_x(L_{T_y}^x) = \mathbb{E}_0(L_{T_{y-x}}^0) = h(0, y - x) = h^*(y - x),$$

where L_t^x denotes the local time at the point x for the process $(X, \mathbb{P}_x)$. The function $h(\cdot, \cdot)$ is used to establish continuity criteria for local times of Lévy processes, see [2, 3] for this case and [24] for a general Borel right Markov processes.

Besides, in the present context, both Yano's and our results extend the theory of invariant functions for killed Lévy processes that can be found in Section 23 of the treatise by Port and

Stone [42] on the potential theory for Lévy processes in locally compact, non-compact, second countable Abelian groups. The relations with this work will be described in Section 2.3.3 below.

Having constructed the invariant function h, in the following definition, we introduce the associated h-process. We will show that the resulting probability measures are such that the canonical process X never hits the point zero, and thus that we refer to them as the law of the Lévy process conditioned to avoid zero. Theorem 2.5 below summarises these properties. Before to state it, we introduce some notation. Let $\mathcal{H}$ and $\mathcal{H}_0$ be the sets given by positive

$$\mathcal{H} = \{x \in \mathbb{R} : h(x) > 0\}, \quad \mathcal{H}_0 = \mathcal{H} \cup \{0\}.$$

On the set $\mathcal{H}_0$ will be constructed the law of the Lévy process conditioned to avoid zero.

Definition 2.3. *We denote by* $(\mathbb{P}^{\updownarrow}_x, x \in \mathcal{H}_0)$ *the unique family of measures such that for* $x \in \mathcal{H}_0$,

$$\mathbb{P}^{\updownarrow}_x(\Lambda) = \begin{cases} \dfrac{1}{h(x)}\mathbb{E}^0_x(\mathbf{1}_\Lambda h(X_t)), & x \in \mathcal{H}, \\ \\ n(\mathbf{1}_\Lambda h(X_t)\mathbf{1}_{\{t<\zeta\}}), & x = 0, \end{cases}$$

for all $\Lambda \in \mathcal{F}_t$, *for all* $t \geq 0$. *We will refer to it as the law of* X *conditioned to avoid* 0.

Remark 2.4. Note that from this definition, $\mathbb{P}^{\updownarrow}_x(T_0 > t) = 1$, for all $t > 0$, $x \in \mathcal{H}_0$. Hence, $\mathbb{P}^{\updownarrow}_x(T_0 = \infty) = 1$, for all $x \in \mathcal{H}_0$.

Theorem 2.5. *The family of measures* $(\mathbb{P}^{\updownarrow}_x)_{x\in\mathbb{R}}$ *is Markovian and satisfies*

(i) $\mathbb{P}^{\updownarrow}_x(X_0 = x) = 1$, $\forall\, x \in \mathcal{H}_0$.

(ii) $\mathbb{P}^{\updownarrow}_x(T_0 = \infty) = 1$, $\forall\, x \in \mathcal{H}_0$.

The semigroup associated to $(\mathbb{P}^{\updownarrow}_x)_{x\in\mathbb{R}}$ *is given by*

$$P^{\updownarrow}_t(x, dy) := \frac{h(y)}{h(x)}P^0_t(x, dy), \quad x \in \mathcal{H}, \quad t \geq 0.$$

The entrance law under $\mathbb{P}^{\updownarrow}_0$ *is given by*

$$\mathbb{P}^{\updownarrow}_0(X_t \in dy) = n(h(y)\mathbf{1}_{\{X_t\in dy\}}\mathbf{1}_{\{t<\zeta\}}).$$

We propose an alternative construction of the law of the Lévy process conditioned to avoid zero. Our construction is inspired from [4, 16, 17, 18], where Lévy processes conditioned to stay positive are constructed. Lévy processes conditioned to stay positive are constructed in the following way. Let $\underline{L}_t$ be the local time of the process X reflected at its past infimum, that is, $X - \underline{X}$, where $\underline{X}_t := \inf\{X_s : 0 \leq s \leq t\}$. Let $\underline{n}$ be the measure of its excursions away from zero and let $\tau_{(-\infty,0)}$ be the first hitting time of the negative half-line. Denote by $(Q_t(x, dy), t \geq 0, x \geq 0, y \geq 0)$ the semigroup of the process killed at $\tau_{(-\infty,0)}$. In [17, 18] is proven that the function l defined by

$$l(x) := \mathbb{E}\left(\int_{[0,\infty)} \mathbf{1}_{\{\underline{X}_t\geq -x\}} d\underline{L}_t\right), \quad x \geq 0, \tag{2.15}$$

is an excessive or invariant function for the semigroup $(Q_t, t \geq 0)$. The function l is actually an invariant function whenever X does not drift towards infinity. Furthermore, they obtained l as a limit of certain sequence of functions. To be precise, if $\mathbf{e}_q$ is an exponential random time with parameter $q > 0$ and independent of $(X, \mathbb{P})$, then for $x \geq 0$,

$$l(x) = \lim_{q \to 0} \frac{\mathbb{P}_x(\tau_{(-\infty,0)} > \mathbf{e}_q)}{\eta q + \underline{n}(\zeta > \mathbf{e}_q)}, \tag{2.16}$$

where η is such that $\int_0^t \mathbf{1}_{\{X_s = \underline{X}_s\}} ds = \eta \underline{L}_t$ and $\underline{n}(\zeta > \mathbf{e}_q) = \int_0^\infty q e^{-qt} \underline{n}(\zeta > t) dt$. They also showed that the law of Lévy processes conditioned to stay positive can be obtained as a limit, as $q \to 0$, of the law of the process conditioned to stay positive up to an independent exponential time with parameter q (see Proposition 1 in [17]).

The following theorem states that for $x \in \mathcal{H}$, $\mathbb{P}_x^\updownarrow$ is the limit, as $q \to 0$, of the law of the process X under $\mathbb{P}_x$ conditioned to avoid zero, up to an independent exponential time with parameter $q > 0$. Since an exponential random variable with parameter q converges in distribution to infinity as its parameter converges to zero, then this result confirms that, starting at $x \in \mathcal{H}$, we can think of X under $\mathbb{P}_x^\updownarrow$, as the process conditioned to avoid zero on the whole positive real line.

Theorem 2.6. *Let $\mathbf{e}_q$ be an exponential time with parameter $q > 0$, independent of $(X, \mathbb{P})$. Then for any $x \in \mathcal{H}$, and any $(\mathcal{F}_t)_{t \geq 0}$-stopping time T,*

$$\lim_{q \to 0} \mathbb{P}_x(\Lambda, T < \mathbf{e}_q \mid T_0 > \mathbf{e}_q) = \mathbb{P}_x^\updownarrow(\Lambda), \quad \forall \Lambda \in \mathcal{F}_T.$$

In the case $x = 0$, the law $\mathbb{P}_0^\updownarrow$ can also be obtained as a limit involving an independent exponential time. Before stating the result, we point out that for $s > 0$, we will denote by $g_s = \sup\{t \leq s : X_t = 0\}$, the last zero of X before time s.

Proposition 2.7. *Let $\mathbf{e}_q$ be an exponential time with parameter $q > 0$, independent of $(X, \mathbb{P})$. Let $\mathbb{P}^{\mathbf{e}_q}$ be the law of $X \circ k_{\mathbf{e}_q - g_{\mathbf{e}_q}} \circ \theta_{g_{\mathbf{e}_q}}$ under $\mathbb{P}$. Then, for $t > 0$,*

$$\lim_{q \to 0} \mathbb{P}^{\mathbf{e}_q}(\Lambda, t < \zeta) = \mathbb{P}_0^\updownarrow(\Lambda) = n(\mathbf{1}_\Lambda h(X_t) \mathbf{1}_{\{t < \zeta\}}), \quad \forall \Lambda \in \mathcal{F}_t.$$

Another important property of the h-process is its transiency. This is given in the following proposition.

Proposition 2.8 (Transiency property)**.** *The process $(X, \mathbb{P}_x^\updownarrow)_{x \in \mathcal{H}_0}$ is transient.*

In Lemma 2.25 we will prove that for any $x \in \mathcal{H}$, the point x is regular for itself under $\mathbb{P}_x^\updownarrow$. Therefore, there exists a local time at any point $x \in \mathcal{H}$, and we will denote by $n_x^\updownarrow$ the excursion measure away from x for the process $(X, \mathbb{P}_x^\updownarrow)$. In the following proposition we establish a relationship between the excursion measure away from zero for $(X, \mathbb{P})$ and the excursion measure away from x for $(X, \mathbb{P}_x^\updownarrow)$, $x \in \mathcal{H}$.

Proposition 2.9. *For $x \in \mathcal{H}$, let $n_x^\updownarrow$ be the excursion measure out from x for $(X, \mathbb{P}_x^\updownarrow)$ and n the excursion measure out from zero for $(X, \mathbb{P})$. Then, for any measurable and bounded functional $H : \mathcal{D}^0 \to \mathbb{R}$,*

$$n_x^\updownarrow \left(\int_0^\zeta H(\epsilon_u, u < t) q e^{-qt} dt \right) = \frac{1}{h(x)} n \left(\int_0^\zeta H(\epsilon_u + x, u < t) h(X_t + x) \mathbf{1}_{\{T_{\{-x\}} > t\}} q e^{-qt} dt \right).$$

2.3 Proofs

2.3.1 A preliminary result

In order to prove the finiteness of h, we need the following lemma.

Lemma 2.10. *Let $(X,\mathbb{P})$ be a Lévy process with characteristic exponent ψ. Assume that $(X,\mathbb{P})$ satisfies the hypotheses **H.1** and **H.2**, then, $\psi(\lambda)\neq 0$, for all $\lambda\neq 0$ and*

$$\lim_{|\lambda|\to\infty}\psi(\lambda)=\infty.$$

Furthermore,

$$\int_{\mathbb{R}}(1\wedge\lambda^2)Re\left(\frac{1}{\psi(\lambda)}\right)d\lambda<\infty. \tag{2.17}$$

Proof. The first part follows from the fact that $(X,\mathbb{P})$ is not arithmetic (see e.g. [20, Theorem 6.4.7]). Now, since $1/(1+\psi)$ is the Fourier transform of the integrable function u_1, then from the Riemann-Lebesgue theorem it follows $\lim_{|\lambda|\to\infty}\psi(\lambda)=\infty$.

Using that $\lim_{|\lambda|\to\infty}\psi(\lambda)=\infty$, we deduce

$$Re\left(\frac{1}{\psi(\lambda)}\right)\sim Re\left(\frac{1}{1+\psi(\lambda)}\right),\quad |\lambda|\to\infty.$$

The latter and Theorem 2.1 imply that for all $\lambda_0>0$,

$$\int_{|\lambda|>\lambda_0}Re\left(\frac{1}{\psi(\lambda)}\right)d\lambda<\infty. \tag{2.18}$$

On the other hand,

$$\begin{aligned}\left[Re\left(\frac{\lambda^2}{\psi(\lambda)}\right)\right]^{-1} &\geq \frac{Re\psi(\lambda)}{\lambda^2}\\ &\geq \sigma^2+\int_{|y|<1}\frac{(1-\cos\lambda y)}{\lambda^2}\pi(dy)\\ &\longrightarrow \sigma^2+\int_{|y|<1}y^2\pi(dy)>0,\quad \text{as } \lambda\to 0.\end{aligned}$$

The latter limit implies that there exists a λ_0 such that,

$$Re\left(\frac{\lambda^2}{\psi(\lambda)}\right)\leq C,\quad \text{for all } |\lambda|<\lambda_0, \tag{2.19}$$

for some constant positive C. Then, from (2.18) and (2.19), we obtain (2.17). □

2.3.2 Some properties of h_q and h

In order to establish some properties of h, we write h_q in an alternative form, namely in terms of T_0 and the excursion measure n, as follows. Let $\mathbf{e}_q$ be an exponential random variable with parameter $q > 0$ and independent of $(X, \mathbb{P})$. Using (2.2) and (2.7), we can write

$$\begin{aligned} h_q(x) &= u_q(0)(1 - \mathbb{E}_x(e^{-qT_0})) \\ &= \frac{\mathbb{P}_x(T_0 > \mathbf{e}_q)}{n(\zeta > \mathbf{e}_q)}, \end{aligned} \tag{2.20}$$

where

$$n(\zeta > \mathbf{e}_q) = \int_0^\infty q e^{-qt} n(\zeta > t) dt = \frac{1}{u_q(0)}.$$

The expression (2.20) helps us to prove the following lemma, which summarizes some important properties of the sequence $(h_q)_{q>0}$.

Lemma 2.11. *For every $q > 0$, the function h_q is subadditive on $\mathbb{R}$ and it is excessive for the semigroup $(P_t^0, t \geq 0)$.*

Proof. By Proposition 43.4 in [45], we have that for any $q > 0$ and $x, y \in \mathbb{R}$,

$$\mathbb{E}_{x+y}(e^{-qT_0}) \geq \mathbb{E}_x(e^{-qT_0})\mathbb{E}_y(e^{-qT_0}). \tag{2.21}$$

Now, since

$$(1 - \mathbb{E}_x(e^{-qT_0}))(1 - \mathbb{E}_y(e^{-qT_0})) \geq 0,$$

then using (2.21), it follows

$$1 - \mathbb{E}_x(e^{-qT_0}) + 1 - \mathbb{E}_y(e^{-qT_0}) \geq 1 - \mathbb{E}_{x+y}(e^{-qT_0}).$$

Hence, by (2.20)

$$h_q(x + y) \leq h_q(x) + h_q(y), \quad x, y \in \mathbb{R}.$$

This shows that h_q is subadditive on $\mathbb{R}$.

In order to show that h_q is excessive for P_t^0, we claim that

$$\mathbb{P}_x(T_0 > t + \mathbf{e}_q) = \mathbb{E}_x\left(\mathbf{1}_{\{T_0 > t + \mathbf{e}_q\}}\right) = \mathbb{E}_x\left(\mathbb{P}_{X_t}(T_0 > \mathbf{e}_q)\mathbf{1}_{\{t < T_0\}}\right). \tag{2.22}$$

Indeed, we note that for $t > 0$ fixed, $T_0 \circ \theta_t + t = T_0$, on $\{T_0 > t\}$. From this remark and the Markov property, we obtain the following identities

$$\begin{aligned} \mathbb{P}_x(T_0 > t + \mathbf{e}_q) &= \int_0^\infty \mathbb{E}_x(\mathbf{1}_{\{T_0 > t+s\}}) q e^{-qs} ds \\ &= \int_0^\infty \mathbb{E}_x(\mathbf{1}_{\{T_0 > t+s\}} \mathbf{1}_{\{T_0 > t\}}) q e^{-qs} ds \\ &= \int_0^\infty \mathbb{E}_x(\mathbf{1}_{\{T_0 > s\}} \circ \theta_t \mathbf{1}_{\{T_0 > t\}}) q e^{-qs} ds \\ &= \mathbb{E}_x\left(\mathbf{1}_{\{T_0 > \mathbf{e}_q\}} \circ \theta_t \mathbf{1}_{\{T_0 > t\}}\right) \\ &= \mathbb{E}_x\left(\mathbb{P}_{X_t}(T_0 > \mathbf{e}_q)\mathbf{1}_{\{T_0 > t\}}\right). \end{aligned}$$

The identities (2.22) and (2.20) imply

$$\begin{aligned}\mathbb{E}_x(h_q(X_t), t < T_0) &= \mathbb{E}_x\left(\frac{\mathbf{1}_{\{T_0 > t+\mathbf{e}_q\}}}{n(\zeta > \mathbf{e}_q)}\right) \\ &\le \mathbb{E}_x\left(\frac{\mathbf{1}_{\{T_0 > \mathbf{e}_q\}}}{n(\zeta > \mathbf{e}_q)}\right) \\ &= h_q(x).\end{aligned}$$

The above expression also implies that $\lim_{t\to 0} \mathbb{E}_x(h_q(X_t), t < T_0) = h_q(x)$, for $x \in \mathbb{R}$. This shows that h_q is excessive for the semigroup $(P_t^0, t \ge 0)$. □

Before we proceed to the proof of (i) and (ii) in Theorem 2.2 we make a technical remark.

Remark 2.12. Proceeding as in the proof of Theorem 19 p. 65 in [5], it can be shown that

$$u_q(x) = \frac{1}{2\pi}\int_{\mathbb{R}} Re\left(\frac{e^{-i\lambda x}}{q+\psi(\lambda)}\right) d\lambda, \quad x \in \mathbb{R}. \tag{2.23}$$

Then,

$$2u_q(0) - [u_q(x) + u_q(-x)] = \frac{1}{\pi}\int_{\mathbb{R}} (1-\cos\lambda x) Re\left(\frac{1}{q+\psi(\lambda)}\right) d\lambda.$$

On the other hand, making use of the inequality $|1-\cos b| \le 2(1\wedge b^2)$ and (2.17), we obtain

$$\int_{\mathbb{R}} (1-\cos\lambda x) Re\left(\frac{1}{\psi(\lambda)}\right) d\lambda < \infty, \quad x \in \mathbb{R}.$$

Therefore, for all $x \in \mathbb{R}$,

$$\lim_{q\to 0}(2u_q(0) - [u_q(x) + u_q(-x)]) = \frac{1}{\pi}\int_{\mathbb{R}} (1-\cos\lambda x) Re\left(\frac{1}{\psi(\lambda)}\right) d\lambda \tag{2.24}$$

is finite.

Proof of (i) and (ii) in Theorem 2.2. That h is subadditive and excessive follow from Lemma 2.11 (since these properties are preserved under limits of sequences of functions).

To obtain the finiteness of h, we note that for all $q > 0$, $x \in \mathbb{R}$,

$$h_q(x) \le 2u_q(0) - [u_q(x) + u_q(-x)] = \frac{1}{\pi}\int_{-\infty}^{\infty} (1-\cos\lambda x) Re\left(\frac{1}{q+\psi(\lambda)}\right) d\lambda.$$

Then, by (2.24),

$$h(x) \le \frac{1}{\pi}\int_{-\infty}^{\infty} (1-\cos\lambda x) Re\left(\frac{1}{\psi(\lambda)}\right) d\lambda < \infty, \quad \forall x \in \mathbb{R}. \tag{2.25}$$

This proves the finiteness of h.

Now, using (2.23), we obtain

$$h_q(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Re\left(\frac{1-e^{i\lambda x}}{q+\psi(\lambda)}\right) d\lambda$$

and

$$\left| Re\left(\frac{1-e^{i\lambda x}}{q+\psi(\lambda)}\right)\right| \leq \frac{2}{q+Re(\psi(\lambda))}, \quad q>0, \quad \lambda, x \in \mathbb{R}.$$

Then, letting $q \to 0$ and using the dominated convergence theorem, (2.13) is obtained.

Note that

$$(1-\cos\lambda x)Re\left(\frac{1}{\psi(\lambda)}\right) \leq 2(1\wedge\lambda^2)Re\left(\frac{1}{\psi(\lambda)}\right), \quad |x|\leq 1, \quad \lambda\in\mathbb{R}.$$

Then, by (2.17) and dominated convergence theorem, it follows

$$\lim_{x\to 0}\frac{1}{\pi}\int_{-\infty}^{\infty}(1-\cos\lambda x)Re\left(\frac{1}{\psi(\lambda)}\right)d\lambda = 0.$$

Hence, by (2.25), $\lim_{x\to 0} h(x) = 0$. This proves that h is continuous at zero. Furthermore, since h is subadditive on $\mathbb{R}$, the continuity of h at the point zero implies the continuity on the whole real line (see e.g. [29, Theorem 6.8.2]).

□

2.3.3 Another representation for h_q and the behaviour of h at infinity

In this section we make the connection with the results from Section 23 in [42], but before we introduce further notation. For a Borel set B, let T_B be the first hitting time of B, that is, $T_B = \inf\{t>0 : X_t \in B\}$ (with $\inf\{\emptyset\} = \infty$). Let $(P_t^B, t\geq 0)$ be the semigroup of the Lévy process killed at T_B and $U_B(x,A) = \int_0^\infty \mathbb{P}_x(X_t\in A, t<T_B)dt$. For $f:\mathbb{R}\to\mathbb{R}$ an integrable Borel function, we denote

$$J(f) = \int_{\mathbb{R}} f(x)dx, \quad \widehat{f}(\lambda) = \int_{\mathbb{R}} f(x)e^{i\lambda x}dx, \quad \lambda\in\mathbb{R}.$$

Let $\mathfrak{F}^+$ be the class of non-negative, continuous, integrable functions f, whose Fourier transform has compact support and satisfies the following property: there exists a compact set K, a positive and finite constant c, and an open neighbourhood of zero V such that

$$J(f) - Re\widehat{f}(\lambda) \leq c\max_{x\in K}(1-(2\pi)^{-1}\cos\lambda x), \quad \lambda\in V.$$

Let $\mathfrak{F}^*$ be the collection of differences of elements of $\mathfrak{F}^+$. Now, for $q>0$, let A_q and H_q^0 be given by

$$A_q f(x) = c_q J(f) - U_q f(x), \quad H_q^0 f(x) = f(0)\mathbb{E}_x(e^{-qT_0}), \quad x\in\mathbb{R}, \tag{2.26}$$

where c_q is a positive constant. As in [42], the constant c_q is taken to be equal to $U_q g(0)$, with g a symmetric function in $\mathfrak{F}^+$ satisfying $J(g)=1$.

It is said that a function f is essentially invariant if for each $t>0$, $f = P_t^B f$ a.e. Port and Stone proved that the only bounded essentially P_t^B-invariant functions are of the form $C\mathbb{P}_x(T_B=\infty)$ a.e., with C a positive constant (see Theorem 23.1). Furthermore, in the case X

recurrent, if B is such that $U_B(x,A)$ is bounded in x for any compact sets A, then the function $L_B(x)$ given by

$$L_B(x) = \lim_{q\to 0} qc_q \int_0^\infty \mathbb{P}_x(T_B > t)e^{-qt}dt, \quad x \in \mathbb{R}, \tag{2.27}$$

is a P_t^B-invariant function. Port and Stone proved furthermore that the constant c_q above introduced is such that for all $x \in \mathbb{R}$, the limit $\lim_{q\to 0} A_q f(x)$ exists, for $f \in \mathfrak{F}^*$.

The following lemma establishes an identity for h_q in terms of certain classes of functions. This identity is inspired from [42].

Lemma 2.13. *Let $f \in \mathfrak{F}^*$. Then,*

$$A_q f(x) - H_q^0 A_q f(x) = -U_q^0 f(x) + c_q(1 - H_q^0 \mathbf{1}(x))J(f), \tag{2.28}$$

where $\mathbf{1}$ denotes the constant function equal to 1. If $J(f) = 1$, the following holds for $x \in \mathbb{R}$,

$$A_q f(x) - A_q f(0)\mathbb{E}_x(e^{-qT_0}) = -U_q^0 f(x) + \frac{c_q}{u_q(0)} h_q(x). \tag{2.29}$$

Proof. By the strong Markov property, we have

$$\begin{aligned}
\mathbb{E}_x\left(\int_{T_0}^\infty e^{-qt} f(X_t)dt, T_0 < \infty\right) &= \mathbb{E}_x\left(e^{-qT_0}\int_0^\infty e^{-qu} f(X_{u+T_0})du, T_0 < \infty\right) \\
&= \mathbb{E}_x\left(e^{-qT_0}, T_0 < \infty\right)\mathbb{E}\left(\int_0^\infty e^{-qu} f(X_u)du\right) \\
&= H_q^0 \mathbf{1}(x) U_q f(0).
\end{aligned}$$

Thus,

$$\begin{aligned}
U_q f(x) &= \mathbb{E}_x\left(\int_0^{T_0} e^{-qt} f(X_t)dt\right) + \mathbb{E}_x\left(\int_{T_0}^\infty e^{-qt} f(X_t)dt, T_0 < \infty\right) \\
&= U_q^0 f(x) + U_q f(0) H_q^0 \mathbf{1}(x).
\end{aligned} \tag{2.30}$$

On the other hand, since $H_q^0 A_q f(x) = A_q f(0) H_q^0 \mathbf{1}(x)$, we have

$$H_q^0 A_q f(x) + U_q f(0) H_q^0 \mathbf{1}(x) = J(f) c_q H_q^0 \mathbf{1}(x). \tag{2.31}$$

Using (2.30), (2.31) and the definition of $A_q f$ we obtain

$$\begin{aligned}
A_q f(x) - H_q^0 A_q f(x) &= c_q J(f) - U_q^0 f(x) - U_q f(0) H_q^0 \mathbf{1}(x) - H_q^0 A_q f(x) \\
&= -U_q^0 f(x) + c_q(1 - H_q^0 \mathbf{1}(x))J(f),
\end{aligned}$$

which is (2.28).

Now, suppose that $J(f) = 1$. To obtain (2.29), we use the expression (2.26) and (2.28). □

Remarks 2.14. (i) Let $\kappa = \lim_{q\to 0} \frac{1}{u_q(0)}$. From the identity,

$$\mathbb{E}_x[e^{-qT_0}] = 1 - \frac{1}{u_q(0)} h_q(x), \quad q > 0, \quad x \in \mathbb{R},$$

making $q \to 0$, it follows $\mathbb{P}_x(T_0 = \infty) = \kappa h(x)$, $x \in \mathbb{R}$.

(ii) In general by (2.20), we have that $h_q(x)$ can be written as

$$h_q(x) = u_q(0)\mathbb{P}_x(T_0 > \mathbf{e}_q) = u_q(0)\int_0^\infty \mathbb{P}_x(T_0 > t)qe^{-qt}dt, \quad q > 0, \quad x \in \mathbb{R}.$$

Letting $q \to 0$ and taking $B = \{0\}$ in (2.27), we obtain

$$h(x) = kL_{\{0\}}(x), \quad x \in \mathbb{R},$$

where $k = \lim_{q\to 0} \frac{u_q(0)}{c_q}$. Since $0 < h(x_0) < \infty$ and $0 < L_{\{0\}}(x_0) < \infty$, for some $x_0 \in \mathbb{R}$ (Theorem 2.2 and Theorem 18.3 in [42]), it follows that $0 < k < \infty$. Now, taking limit as $q \to 0$ in (2.29), we obtain

$$\frac{1}{k}h(x) = Af(x) - Af(0)\mathbb{P}_x(T_0 < \infty) + U^0 f(x), \quad x \in \mathbb{R}, \quad f \in \mathfrak{F}^*, \tag{2.32}$$

where $Af(x) = \lim_{q\to 0} A_q f(x)$ and $U^0 f(x) = \lim_{q\to 0} U_q^0 f(x)$, $x \in \mathbb{R}$.

To end this section, we establish the behaviour of h at infinity.

Lemma 2.15. *Let* $\kappa := \lim_{q\to 0} \frac{1}{u_q(0)}$. *We have the following*

(i) Suppose that X *is transient. If* $0 < \mu := \mathbb{E}(X_1) \le \infty$, *then*

$$\lim_{x\to\infty} h(x) = \frac{1}{\kappa}, \quad \lim_{x\to-\infty} h(x) = \frac{1}{\kappa} - \frac{1}{\mu};$$

while if $-\infty \le \mu < 0$, *then*

$$\lim_{x\to\infty} h(x) = \frac{1}{\kappa} + \frac{1}{\mu}, \quad \lim_{x\to-\infty} h(x) = \frac{1}{\kappa}.$$

(ii) Suppose that X *is recurrent, then either*

$$\lim_{x\to\infty} h(x) = \frac{1}{k} \quad \text{or} \quad \lim_{x\to-\infty} h(x) = \frac{1}{\kappa}.$$

Remark 2.16. The case where X is transient and $\mathbb{E}(X_1^+) = \mathbb{E}(X_1^-) = \infty$ is not covered in the latter lemma.

Proof of Lemma 2.15. We start by proving (i) in the case $0 < \mu \le \infty$, the other case can be proved similarly. Set $f(x) = u_1(x)$, $x \in \mathbb{R}$. Note that $u_0(x) = \sum_{n=1}^\infty f^{*n}(x)$. Indeed,

$$\begin{aligned}\sum_{n=1}^\infty f^{*n}(x)dx &= \sum_{n=1}^\infty \int_0^\infty \frac{s^{n-1}}{(n-1)!}e^{-s}\mathbb{P}(X_s \in dx)ds \\ &= \int_0^\infty e^{-s}\sum_{n=1}^\infty \frac{s^{n-1}}{(n-1)!}\mathbb{P}(X_s \in dx)ds \\ &= \int_0^\infty \mathbb{P}(X_s \in dx)ds \\ &= u_0(x)dx.\end{aligned}$$

Furthermore, the Fourier transform of f is given by $\widehat{f}(\lambda) = 1/(1 + \psi(\lambda))$, $\lambda \in \mathbb{R}$. Since $h(x) = u_0(0) - u_0(-x)$, it is suffices to compute the limit at infinity of $\sum_{n=1}^\infty f^{*n}(x)$. To that aim, we use the main result in [46], which states that if

(a) $\lim_{|x|\to\infty} f(x) = 0$,

(b) f is in $L_{1+\epsilon}$, for some $\epsilon > 0$,

then

$$\sum_{n=1}^{\infty} f^{*n}(x) \to \frac{1}{\mu}, \text{ as } x \to \infty, \quad \sum_{n=1}^{\infty} f^{*n}(x) \to 0, \text{ as } x \to -\infty.$$

The condition (a) is obtained from the Riemann-Lebesgue theorem. To show that (b) is satisfied we use the Plancherel's theorem (see [44, p. 186], [47, p. 202]). Thus, we will show that $\widehat{f}$ is in L_2. Thereby,

$$\int_{\mathbb{R}} |\widehat{f}(\lambda)|^2 d\lambda = \int_{\mathbb{R}} \frac{1}{|1+\psi(\lambda)|^2} d\lambda \le \int_{\mathbb{R}} \frac{Re(1+\psi(\lambda))}{|1+\psi(\lambda)|^2} d\lambda = \int_{\mathbb{R}} Re\left(\frac{1}{1+\psi(\lambda)}\right) d\lambda < \infty.$$

This concludes the first part of the lemma.

To prove the second part of lemma, we consider the function h^* which is defined in Section 2.3.4. There, it is shown that $h^*(x) = h(x) + h(-x) = \mathbb{E}(L_{T_x})$ (see (2.38) and (2.39) for details). We will prove that $h^*(x)$ tends to infinity as $x \to \infty$ when X is recurrent and thus obtain (ii). The proof is as follows. Let $\mathbf{e}_q$ be an independent exponential time. Observe the elementary inequality

$$h^*(x) = \mathbb{E}(L_{T_x}) \ge \mathbb{E}(L_{T_x} 1_{\{T_x \ge \mathbf{e}_q\}}) \ge \mathbb{E}(L_{\mathbf{e}_q} 1_{\{T_x \ge \mathbf{e}_q\}}),$$

take limit inferior

$$\liminf_{x\to\infty} h^*(x) \ge \mathbb{E}(L_{\mathbf{e}_q} \liminf_{x\to\infty} 1_{\{T_x \ge \mathbf{e}_q\}}).$$

Now, observe that T_x converges towards ∞ in distribution and hence in probability as $x \to \infty$, because by the Riemann-Lebesgue Theorem we have

$$\lim_{x\to\infty} \mathbb{E}(e^{-qT_x}) = \lim_{x\to\infty} \frac{u_q(x)}{u_q(0)} = 0, \quad \forall\, q > 0.$$

We have so proved that

$$\liminf_{x\to\infty} h^*(x) \ge \mathbb{E}(L_{\mathbf{e}_q}) = u_q(0), \quad \forall\, q > 0.$$

We now make q tends to 0 to get

$$\liminf_{x\to\infty} h^*(x) \ge u_0(0).$$

The claim follows because $u_0(0) = \infty$ in the recurrent case. □

2.3.4 An auxiliary function

Let $(h^*_q)_{q>0}$ be the increasing sequence of functions defined by

$$h^*_q(x) = \mathbb{E}\left(\int_0^{T_x} e^{-qt} dL_t\right), \quad q > 0, \quad x \in \mathbb{R},$$

where $T_x = \inf\{t > 0 : X_t = x\}$, the first hitting time of x for X. The sequence $(h^*_q)_{q>0}$ has the properties listed in the following proposition.

Proposition 2.17. *For any $q>0$, the function h_q^* is a symmetric, nonnegative, subadditive continuous function, which can be expressed in terms of the q-resolvent density as*

$$h_q^*(x) = u_q(0) - \frac{u_q(x)u_q(-x)}{u_q(0)}, \quad x \in \mathbb{R}. \tag{2.33}$$

Proof. By definition, h_q^* is a non negative function. The continuity and symmetry of h_q^* is obtained from (2.33). Thus, it only remains to prove (2.33) and that h_q^* is subadditive.

First, we recall an expression that establishes a relation between resolvent densities and local times, (see Lemma 3 and commentary before Proposition 4 in [5, Chapter V]):

$$u_q(-x) = \mathbb{E}_x\left(\int_0^\infty e^{-qt} dL_t\right) = \mathbb{E}\left(\int_0^\infty e^{-qt} dL(x,t)\right), \quad q>0, \quad x \in \mathbb{R}, \tag{2.34}$$

where $(L(x,t), t \geq 0)$ is the local time at point x for $(X, \mathbb{P})$. Thus, using the latter expression, we have

$$u_q(0) = \mathbb{E}\left(\int_0^\infty e^{-qt} dL_t\right) = h_q^*(x) + \mathbb{E}\left(\int_{T_x}^\infty e^{-qt} dL_t, T_x < \infty\right). \tag{2.35}$$

On the other hand, by Markov and additivity properties of local time, it follows

$$\begin{aligned} \mathbb{E}\left(\int_{T_x}^\infty e^{-qt} dL_t, T_x < \infty\right) &= \mathbb{E}\left(e^{-qT_x}\int_0^\infty e^{-qu} dL_{u+T_x}, T_x < \infty\right) \\ &= \mathbb{E}(e^{-qT_x}, T_x < \infty)\mathbb{E}_x\left(\int_0^\infty e^{-qu} dL_u\right) \\ &= \widehat{\mathbb{E}}_x(e^{-qT_0}, T_0 < \infty)\mathbb{E}_x\left(\int_0^\infty e^{-qu} dL_u\right). \end{aligned}$$

Then, using (2.4) and (2.34), the equation (2.35) becomes

$$u_q(0) = h_q^*(x) + \frac{u_q(x)}{u_q(0)} u_q(-x), \quad x \in \mathbb{R}.$$

Hence, (2.33) is obtained.

Now, we prove the subadditivity of h_q^*. The procedure is similar to the one used to prove the subadditivity of h_q in Lemma 2.11. We repeat the arguments for clarity. First, by (2.2) and (2.4) we can write (2.33) as

$$h_q^*(x) = u_q(0)(1 - \mathbb{E}_x(e^{-qT_0})\widehat{\mathbb{E}}_x(e^{-qT_0})). \tag{2.36}$$

Since, for any $x \in \mathbb{R}$, $\mathbb{E}_x(e^{-qT_0}), \widehat{\mathbb{E}}_x(e^{-qT_0}) \leq 1$, it follows

$$(1 - \mathbb{E}_x(e^{-qT_0})\widehat{\mathbb{E}}_x(e^{-qT_0}))(1 - \mathbb{E}_y(e^{-qT_0})\widehat{\mathbb{E}}_y(e^{-qT_0})) \geq 0, \quad x, y \in \mathbb{R}.$$

The latter relation and (2.21) imply

$$\begin{aligned} 1 - \mathbb{E}_x(e^{-qT_0})\widehat{\mathbb{E}}_x(e^{-qT_0}) + 1 - \mathbb{E}_y(e^{-qT_0})\widehat{\mathbb{E}}_y(e^{-qT_0}) &\geq 1 - \mathbb{E}_x(e^{-qT_0})\mathbb{E}_y(e^{-qT_0})\widehat{\mathbb{E}}_x(e^{-qT_0})\widehat{\mathbb{E}}_y(e^{-qT_0}) \\ &\geq 1 - \mathbb{E}_{x+y}(e^{-qT_0})\widehat{\mathbb{E}}_{x+y}(e^{-qT_0}), \end{aligned}$$

for all $x, y \in \mathbb{R}$. Hence, by (2.36)

$$h_q^*(x) + h_q^*(y) \geq h_q^*(x+y), \quad x, y \in \mathbb{R}.$$

This ends the proof. $\square$

Remark 2.18. With help of the expression (2.33), h_q^* can be written in terms of the function h_q as:

$$h_q^*(x) = h_q(x) + h_q(-x) - \frac{1}{u_q(0)} h_q(x) h_q(-x), \quad x \in \mathbb{R}. \tag{2.37}$$

Now, define h^* by

$$h^*(x) = \lim_{q \to 0} h_q^*(x), \quad x \in \mathbb{R}.$$

Since h is finite, then (2.37) implies that $h^*(x)$ is finite for all $x \in \mathbb{R}$. Furthermore, since

$$h_q^*(x) = \mathbb{E}\left(\int_0^{T_x} e^{-qt} dL_t\right) = \mathbb{E}\left(\int_0^{\infty} e^{-qs} \mathbf{1}_{\{X_u \neq -x, 0 \leq u \leq s\}} dL_s\right),$$

then

$$h^*(x) = \mathbb{E}(L_{T_x}) = \mathbb{E}\left(\int_0^{\infty} \mathbf{1}_{\{X_u \neq -x, 0 \leq u \leq s\}} dL_s\right), \quad x \in \mathbb{R}. \tag{2.38}$$

It is known that L_{T_x} is an exponential random variable. Thus, $h^*(x)$ is the expected value of an exponential random variable. We also note that by (2.37), in the recurrent symmetric case, h^* correspond to $2h^Y$, where h^Y is the invariant function given in [48].

Before we give some properties of the function h^*, we have the following technical lemma.

Lemma 2.19. *(i) For any $x \in \mathbb{R}$, $\lim_{q \to 0} q u_q(x) = 0$.*

(ii) For any $q, r > 0$, $x \in \mathbb{R}$,

$$\int_{\mathbb{R}} u_q(y-x) u_r(y) dy = \frac{u_r(x) + u_q(-x)}{r+q}.$$

Proof. Recall the identity

$$\frac{u_q(x)}{u_q(0)} = \widehat{\mathbb{E}}_x(e^{-qT_0}), \quad x \in \mathbb{R}.$$

Hence, $qu_q(x) \sim \widehat{\mathbb{P}}_x(T_0 < \infty) q u_q(0)$ as $q \downarrow 0$. Thus, it is suffices to prove the case $x = 0$. Thanks to (2.23), we have

$$q u_q(0) = \frac{1}{2\pi} \int_{\mathbb{R}} Re\left(\frac{q}{q + \psi(\lambda)}\right) d\lambda.$$

For every $q > 0$, let j_q be the function being integrated in the latter display. Now, we observe the following

$$\begin{aligned} j_q &= Re\left(\frac{q}{q+\psi(\lambda)}\right) \\ &= \frac{q(q + Re(\psi(\lambda)))}{[q + Re(\psi(\lambda))]^2 + [Im(\psi(\lambda))]^2} \\ &= \left[1 + \frac{Re\psi(\lambda)}{q} + \frac{(Im\psi(\lambda))^2}{q(q + Re\psi(\lambda))}\right]^{-1}, \quad q > 0, \quad \lambda \in \mathbb{R}. \end{aligned}$$

Hence, $j_q \downarrow 0$, as $q \downarrow 0$. Thus, $0 \le j_q(\lambda) \le j_1(\lambda)$, $0 < q < 1$, $\lambda \in \mathbb{R}$ and since j_1 is integrable by Theorem 2.1, the dominated convergence theorem implies $\lim_{q\to 0} q u_q(0) = 0$. This shows (i).

Now, let f be a positive, bounded, measurable function. We have

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathbb{R}} u_q(y-x)u_r(y)dyf(x)dx &= \int_{\mathbb{R}}\int_{\mathbb{R}} f(y-z)u_q(z)dzu_r(y)dy \\ &= \frac{1}{rq}\mathbb{E}\left(f(X_{\mathbf{e}_r} - X_{\mathbf{e}_q})\right) \\ &= \frac{1}{rq}\mathbb{E}\left(f(X_{\mathbf{e}_r} - X_{\mathbf{e}_q})\mathbf{1}_{\{\mathbf{e}_r > \mathbf{e}_q\}}\right) \\ &\quad + \frac{1}{rq}\mathbb{E}\left(f(-(X_{\mathbf{e}_q} - X_{\mathbf{e}_r}))\mathbf{1}_{\{\mathbf{e}_q > \mathbf{e}_r\}}\right),\end{aligned}$$

where $\mathbf{e}_q$, $\mathbf{e}_r$ are independent exponential random variables with parameters $q > 0$ and $r > 0$, respectively, which are independent of $(X, \mathbb{P})$. The first term in the latter equation becomes

$$\begin{aligned}\frac{1}{rq}\mathbb{E}\left(f(X_{\mathbf{e}_r} - X_{\mathbf{e}_q})\mathbf{1}_{\{\mathbf{e}_r > \mathbf{e}_q\}}\right) &= \int_0^\infty \int_s^\infty e^{-rt}\mathbb{E}(f(X_t - X_s))dte^{-qs}ds \\ &= \int_0^\infty \int_s^\infty e^{-r(t-s)}\mathbb{E}(f(X_{t-s}))dte^{-(r+q)s}ds \\ &= \int_0^\infty e^{-(r+q)s}U_rf(0)ds \\ &= \frac{1}{r+q}U_rf(0).\end{aligned}$$

In the same way, it can be verified that

$$\frac{1}{rq}\mathbb{E}\left(f(-(X_{\mathbf{e}_q} - X_{\mathbf{e}_r}))\mathbf{1}_{\{\mathbf{e}_q > \mathbf{e}_r\}}\right) = \frac{1}{r+q}\widehat{U}_qf(0).$$

Thus, we have

$$\int_{\mathbb{R}}\int_{\mathbb{R}} u_q(y-x)u_r(y)dyf(x)dx = \int_{\mathbb{R}} \left(\frac{u_r(x) + u_q(-x)}{r+q}\right) f(x)dx,$$

for all positive, bounded, measurable function f. By the continuity of u_r and u_q, we conclude

$$\int_{\mathbb{R}} u_q(y-x)u_r(y)dy = \frac{u_r(x) + u_q(-x)}{r+q},$$

for any $q, r > 0$, $x \in \mathbb{R}$. □

Some properties of the function h^* are summarized in the following lemma.

Lemma 2.20. *The function h^* is a symmetric, nonnegative, subadditive, continuous function which vanishes only at the point $x = 0$ and $\lim_{|x|\to\infty} h^*(x) = \kappa^{-1}$. Furthermore, h^* is integrable with respect to semigroup of the process killed at T_0, i.e., $P_t^0 h^*(x) < \infty$, for all $t > 0$, $x \in \mathbb{R}$.*

Proof. From the definition of h_q^* and (2.33) the non negativity and symmetry of h^* follows. The subadditivity of h^* is obtained from subadditivity of the sequence $(h_q^*)_{q>0}$. We observe that from (2.37), we can write h^* in terms of h as

$$h^*(x) = h(x) + h(-x) - \kappa h(x)h(-x), \tag{2.39}$$

where $\kappa = \lim_{q\to 0} \frac{1}{u_q(0)}$. Hence, h^* is continuous.

Now, we prove that $\lim_{|x|\to\infty} h^*(x) = \kappa^{-1}$. In Lemma 2.15 has been proven that if X is recurrent then $\lim_{x\to\infty} h^*(x) = \infty$. Since h^* is a symmetric function, the same limit is obtained as $x \to -\infty$. Then, $\lim_{|x|\to\infty} h^*(x) = \kappa^{-1}$ when X is recurrent. Suppose that X is transient. In Lemma 2.15 also is obtained that

$$\liminf_{x\to\infty} h^*(x) \geq u_q(0), \quad \forall\, q > 0,$$

without further assumption. Hence, letting q tends to 0, we obtain, $\liminf_{x\to\infty} h^*(x) \geq \kappa^{-1}$. On the other hand, for an exponential independent time with parameter q, $\mathbf{e}_q$, we have

$$\begin{aligned}
\mathbb{E}(L_{T_x}) &= \mathbb{E}(L_{T_x} 1_{\{T_x \geq \mathbf{e}_q\}}) + \mathbb{E}(L_{T_x} 1_{\{T_x < \mathbf{e}_q\}}) \\
&\leq \mathbb{E}\left(L_{T_x}\left(1 - e^{-qT_x}\right)\right) + \mathbb{E}(L_{\mathbf{e}_q}) \\
&= \mathbb{E}\left(L_{T_x}\left(1 - e^{-qT_x}\right)\right) + u_q(0), \quad \forall\, x \in \mathbb{R}.
\end{aligned}$$

Then, since X is transient, $\mathbb{E}(L_{T_x}) < \infty$. Thus, by the dominate convergence theorem

$$h^*(x) = \mathbb{E}(L_{T_x}) \leq \kappa^{-1}.$$

Hence, $\limsup_{x\to\infty} h^*(x) \leq \kappa^{-1}$. Therefore, $\lim_{x\to\infty} h^*(x) = \kappa^{-1}$ if X is transient. Since $h^*(x)$ is a symmetric function the same limit is obtained as $x \to -\infty$. Therefore, $\lim_{|x|\to\infty} h^*(x) = \kappa^{-1}$.

From defintion, $h^*(0) = 0$. To prove that $x = 0$ is the only point where h^* vanishes, we proceed by contradiction. Suppose that $h^*(x_0) = 0$, for some $x_0 \neq 0$. Using the subadditivity of h^* and making induction we get that $h^*(kx_0) = 0$ for all $k \in \mathbb{Z}$. Since $\lim_{|x|\to\infty} h^*(x) = \kappa^{-1} > 0$, the claim $h^*(kx_0) = 0$, for all $k \in \mathbb{Z}$ is a contradiction. Therefore, $h(x) > 0$, for all $x \neq 0$.

Finally, we prove that h^* is P_t^0-integrable. For $x \in \mathbb{R}$, we write $\widehat{h}_q(x) = h_q(-x)$, $q > 0$, and $\widehat{h}(x) = \lim_{q\to 0} \widehat{h}_q(x)$. Let S be the function defined by $S(x) = h(x) + \widehat{h}(x)$. By (2.39), $h^*(x) \leq s(x)$, $x \in \mathbb{R}$. Thus, it is suffices to show that s is P_t^0-integrable.

Now, by (2.1), the following identities hold for $0 < r < q$,

$$\begin{aligned}
U_q h_r(x) &= \int_{\mathbb{R}} u_q(y-x) h_r(y) dy \\
&= \int_{\mathbb{R}} u_q(y-x)\{u_r(0) - u_r(-y)\} dy \\
&= \frac{u_r(0)}{q} - \int_{\mathbb{R}} u_q(y-x) u_r(-y) dy \\
&= \frac{u_r(0)}{q} - \frac{1}{q-r}\{u_r(-x) - u_q(-x)\} \\
&= \frac{h_r(x)}{q} - \frac{r u_r(-x)}{q(q-r)} + \frac{u_q(-x)}{q-r}.
\end{aligned} \tag{2.40}$$

Thanks to Lemma 2.19 (i), $h_r(x) \to h(x)$ as $r \to 0$ and Fatou's lemma, we obtain

$$U_q h(x) \le \frac{h(x) + u_q(-x)}{q}, \quad q > 0, \quad x \in \mathbb{R}. \tag{2.41}$$

On the other hand, by Lemma 2.19 (ii), we have

$$\begin{aligned} U_q \widehat{h}_r(x) &= \int_{\mathbb{R}} u_q(y-x)(u_r(0) - u_r(y))dy \\ &= \frac{u_r(0)}{q} - \frac{u_r(x) + u_q(-x)}{r+q} \\ &= \frac{\widehat{h}_r(x)}{r+q} + \frac{r u_r(0)}{q(r+q)} - \frac{u_q(-x)}{r+q}. \end{aligned} \tag{2.42}$$

Using again Lemma 2.19 (i), $\widehat{h}_r(x) \to \widehat{h}(x)$ as $r \to 0$, and Fatou's lemma, it follows

$$U_q \widehat{h}(x) \le \frac{\widehat{h}(x) - u_q(-x)}{q}, \quad q > 0, \quad x \in \mathbb{R}. \tag{2.43}$$

Adding (2.41) and (2.43), we obtain that for any $q > 0$, $x \in \mathbb{R}$,

$$q U_q S(x) \le S(x). \tag{2.44}$$

Hence, the function s is P_t-integrable and therefore P_t^0-integrable. □

Remarks 2.21. (i) From (2.44) it is deduced that the function S is excessive for the semigroup $(P_t, t \ge 0)$. Since S is a nonnegative function, then S is an excessive function for the semigroup $(P_t^0, t \ge 0)$.

(ii) By (2.20) and (2.36), we have $h_q(x) \le h_q^*(x) \le h^*(x) \le S(x)$, for all $q > 0$, $x \in \mathbb{R}$. On the other hand, the Lemma 2.20 and its proof ensure that h^* satisfies

$$P_t^0 h^*(x) \le S(x), \quad q U_q h^*(x) \le S(x), \quad q > 0, \quad x \in \mathbb{R}. \tag{2.45}$$

These inequalities will be useful in the proofs of Lemma 2.22, assertion (iii) in Theorem 2.2 and Proposition 2.7 .

Lemma 2.22 ensure that the inequality obtained in (2.41) in fact is an equality. This result was established in [48] in the symmetric case.

Lemma 2.22. *For any $q > 0$, $x \in \mathbb{R}$,*

$$U_q h(x) = \frac{h(x) + u_q(-x)}{q}.$$

Proof. Remark 2.21 (ii) states that the function h^* satisfies $h_q(x) \le h^*(x)$, $U_q h^*(x) < \infty$, for all $q > 0$, $x \in \mathbb{R}$. Then, by the dominated convergence theorem and (2.40), it follows

$$U_q h(x) = \lim_{r \to 0} U_q h_r(x) = \frac{h(x) + u_q(-x)}{q}.$$

□

2.4 Proofs of the main results

Proof of (iii) in Theorem 2.2. The first part of the proof is inspired in the proof of Lemma 1 in [17]. Let $\mathbf{e}_q$ be an exponential random variable with parameter $q > 0$ and independent of $(X, \mathbb{P})$. We claim that for $q > 0$, $x \in \mathbb{R}$, it holds,

$$\mathbb{E}_x(\mathbb{P}_{X_t}(T_0 > \mathbf{e}_q)\mathbf{1}_{\{T_0>t\}}) = e^{qt}\left(\mathbb{P}_x(T_0 > \mathbf{e}_q) - \int_0^t \mathbb{P}_x(T_0 > s)qe^{-qs}ds\right). \tag{2.46}$$

Indeed, by (2.22), we have

$$\mathbb{E}_x\left(\mathbb{P}_{X_t}(T_0 > \mathbf{e}_q)\mathbf{1}_{\{t<T_0\}}\right) = \mathbb{P}_x(T_0 > t + \mathbf{e}_q).$$

Now, making the change of variable $u = t + s$, we obtain

$$\begin{aligned}
\mathbb{P}_x(T_0 > t + \mathbf{e}_q) &= \int_0^\infty \mathbb{P}_x(T_0 > t + s)q^{-qs}ds \\
&= e^{qt}\int_t^\infty \mathbb{P}_x(T_0 > u)qe^{-qu}du \\
&= e^{qt}\left(\int_0^\infty \mathbb{P}_x(T_0 > u)qe^{-qu}du - \int_0^t \mathbb{P}_x(T_0 > u)qe^{-qu}du\right) \\
&= e^{qt}\left(\mathbb{P}_x(T_0 > \mathbf{e}_q) - \int_0^t \mathbb{P}_x(T_0 > u)qe^{-qu}du\right).
\end{aligned}$$

Hence, (2.46) follows.

By Remark 2.21 (ii) and Lemma 2.20, we have that the sequence $(h_q)_{q>0}$ is dominated by h^* and h^* is integrable with respect to P_t^0 for any $t > 0$. Then, using dominated convergence theorem, (2.20) and (2.46), it follows

$$\begin{aligned}
\mathbb{E}_x\left(h(X_t), t < T_0\right) &= \mathbb{E}_x\left(\lim_{q\to 0} h_q(X_t), t < T_0\right) \\
&= \lim_{q\to 0}\mathbb{E}_x\left(\frac{\mathbb{P}_{X_t}(T_0 > \mathbf{e}_q)}{n(\zeta > \mathbf{e}_q)}\mathbf{1}_{\{t<T_0\}}\right) \\
&= \lim_{q\to 0} e^{qt}\left(\frac{\mathbb{P}_x(T_0 > \mathbf{e}_q)}{n(\zeta > \mathbf{e}_q)} - \int_0^t \frac{\mathbb{P}_x(T_0 > u)}{n(\zeta > \mathbf{e}_q)}qe^{-qu}du\right) \\
&= h(x) - \frac{1}{n(\zeta)}\int_0^t \mathbb{P}_x(T_0 > u)du,
\end{aligned}$$

where $n(\zeta) = \lim_{q\to 0}\int_0^\infty e^{-qt}n(\zeta > t)dt$. On the other hand, Lemma 2.19 and (2.7) imply $n(\zeta) = \lim_{q\to 0}[qu_q(0)]^{-1} = \infty$. Therefore, we conclude

$$\mathbb{E}_x\left(h(X_t), t < T_0\right) = h(x), \quad t > 0, \quad x \in \mathbb{R}.$$

Now, we prove the second part of (iii) in Theorem 2.2. From (2.6) and Lemma 2.22, we obtain that the Laplace transform of $n(h(X_t), t < \zeta)$ is given by

$$\int_0^\infty e^{-qt}n(h(X_t), t < \zeta)dt = \int_{\mathbb{R}} h(x)\widehat{\mathbb{E}}_x[e^{-qT_0}]dx = \int_{\mathbb{R}} h(x)\frac{u_q(x)}{u_q(0)}dx = \frac{1}{u_q(0)}U_q h(0) = \frac{1}{q}.$$

Hence, the claim follows. □

Proof of Theorem 2.5. The only thing which has to be proved is the fact that $\mathbb{P}_0^{\updownarrow}$ is a Markovian probability measure with the same semigroup as under $\mathbb{P}_x^{\updownarrow}$, $x \in \mathcal{H}$ and that $\mathbb{P}_0^{\updownarrow}(X_0 = 0) = 1$. Since n is a Markovian measure (σ-finite) with semigroup $(P_t^0, t \geq 0)$. Let g be bounded Borel function and $\Lambda \in \mathcal{F}_t$ and $t, s > 0$:

$$\begin{aligned}
\mathbb{E}_0^{\updownarrow}(\mathbf{1}_\Lambda g(X_{t+s})) &= n(\mathbf{1}_\Lambda h(X_{t+s}) g(X_{t+s}) \mathbf{1}_{\{t+s<\zeta\}}) \\
&= n(\mathbf{1}_\Lambda \mathbb{E}_{X_t}^0(h(X_s) g(X_s)) \mathbf{1}_{\{t<\zeta\}}) \\
&= n(\mathbf{1}_\Lambda h(X_t) \mathbb{E}_{X_t}^{\updownarrow}(h(X_s) g(X_s)) \mathbf{1}_{\{t<\zeta\}}) \\
&= \mathbb{E}_0^{\updownarrow}(\mathbf{1}_\Lambda \mathbb{E}_{X_t}^{\updownarrow}(g(X_s))).
\end{aligned}$$

This shows the first part. Now, we prove that $\mathbb{P}_0^{\updownarrow}(X_0 = 0) = 1$. Since X is right continuous at 0, it is suffices to prove that for any $z > 0$,

$$\mathbb{P}_0^{\updownarrow}(|X_\epsilon| < z) \to 1,$$

as $\epsilon \to 0$. The latter is equivalent to prove

$$\lim_{\epsilon \to 0} n(\mathbf{1}_{\{|X_\epsilon| > z\}} h(X_\epsilon) \mathbf{1}_{\{\epsilon < \zeta\}}) = 0.$$

Since $n(h(X_s), s < \zeta) = 1$, $\mathcal{Q}_s(\cdot) := n(\cdot, h(X_s), s < \zeta)$ defines a probability measure. Then, from the Markov property, for all $\epsilon < s$, $\mathbb{P}_0^{\updownarrow}(|X_\epsilon| < z) = \mathcal{Q}_s(\mathbf{1}_{\{|X_\epsilon| < z\}})$. Since the excursions of the Lévy process $(X, \mathbb{P})$ leave 0 continuously, we have $\mathbf{1}_{\{|X_\epsilon| < z\}} \to 1$, $\mathcal{Q}_s$-a.s. as $\epsilon \to 0$. The result follows from the dominated converge theorem. □

Proof of Theorem 2.6. We proceed as in [17]. Let $x \in \mathcal{H}$, T a $(\mathcal{F}_t)_{t \geq 0}$ stopping time and $\Lambda \in \mathcal{F}_T$. With the help of the strong Markov property and since $\mathbf{e}_q$ is independent of $(X, \mathbb{P})$, we can deduce the following

$$\begin{aligned}
\mathbb{E}_x\left(\mathbf{1}_\Lambda \mathbf{1}_{\{T < \mathbf{e}_q\}} \mathbf{1}_{\{T_0 > \mathbf{e}_q\}}\right) &= \int_0^\infty \mathbb{E}_x\left(\mathbf{1}_\Lambda \mathbf{1}_{\{T < T_0\}} \mathbf{1}_{\{T < s\}} \mathbf{1}_{\{T_0 > s\}}\right) q e^{-qs} ds \\
&= \int_0^\infty \mathbb{E}_x\left(\mathbf{1}_\Lambda \mathbf{1}_{\{T < T_0\}} \mathbf{1}_{\{T < s\}} \mathbb{E}_x\left(\mathbf{1}_{\{T_0 > s\}} \circ \theta_T \mid \mathcal{F}_T\right)\right) q e^{-qs} ds \\
&= \int_0^\infty \mathbb{E}_x\left(\mathbf{1}_\Lambda \mathbf{1}_{\{T < T_0 \wedge s\}} \mathbb{P}_{X_T}(T_0 > s)\right) q e^{-qs} ds \\
&= \mathbb{E}_x\left(\mathbf{1}_\Lambda \mathbf{1}_{\{T < T_0 \wedge \mathbf{e}_q\}} \mathbb{P}_{X_T}(T_0 > \mathbf{e}_q)\right) \\
&= n(\zeta > \mathbf{e}_q) \mathbb{E}_x\left(\mathbf{1}_\Lambda \mathbf{1}_{\{T < T_0 \wedge \mathbf{e}_q\}} h_q(X_T)\right) \\
&= \frac{1}{h_q(x)} \mathbb{E}_x\left(\mathbf{1}_\Lambda \mathbf{1}_{\{T < T_0 \wedge \mathbf{e}_q\}} h_q(X_T)\right) \mathbb{P}_x(T_0 > \mathbf{e}_q).
\end{aligned}$$

The latter shows that for $\Lambda \in \mathcal{F}_T$, T stopping time finite a.s.

$$\mathbb{P}_x(\Lambda, T < \mathbf{e}_q \mid T_0 > \mathbf{e}_q) = \frac{1}{h_q(x)} \mathbb{E}_x\left(\mathbf{1}_\Lambda h_q(X_T) \mathbf{1}_{\{T < T_0 \wedge \mathbf{e}_q\}}\right). \tag{2.47}$$

Now, recall that $h_q(x) \leq h_q^*(x) \leq h^*(x)$, $q > 0$, $x \in \mathbb{R}$. Thus,

$$\mathbf{1}_{\{T < T_0 \wedge \mathbf{e}_q\}} h_q(X_T) \leq \mathbf{1}_{\{T < T_0\}} h^*(X_T) \quad \text{a.s.}$$

On the other hand, the first inequality in (2.45) also it is satisfied for stopping times, i.e., $\mathbb{E}_x(h^*(X_T), T < T_0) \leq S(x)$. Then, letting $q \to 0$, with the help of the dominated convergence theorem in (2.47), we obtain the desired result. □

Proof of Proposition 2.7. For every $s > 0$, we consider $d_s = \inf\{u > s : X_u = 0\}$, $g_s = \sup\{u \leq s : X_u = 0\}$ and $G = \{g_u : g_u \neq d_u, u > 0\}$. By definition, for every $q > 0$, $\Lambda \in \mathcal{F}_t$, we have

$$\begin{aligned}
\mathbb{P}^{\mathbf{e}_q}(\Lambda, t < \zeta) &= \mathbb{E}(\mathbf{1}_\Lambda \circ k_{\mathbf{e}_q - g_{\mathbf{e}_q}} \circ \theta_{g_{\mathbf{e}_q}} \mathbf{1}_{\{t < \mathbf{e}_q - g_{\mathbf{e}_q}\}}) \\
&= \mathbb{E}\left(\int_0^\infty \mathbf{1}_\Lambda \circ k_{u-g_u} \circ \theta_{g_u} \mathbf{1}_{\{t<u-g_u\}} q e^{-qu} du\right) \\
&= \mathbb{E}\left(\sum_{s\in G} e^{-qs} \int_s^{d_s} q e^{-q(u-s)} \mathbf{1}_\Lambda \circ k_{u-s} \circ \theta_s \mathbf{1}_{\{t<u-s\}} du\right).
\end{aligned}$$

Now, using the compensation formula in excursion theory (see e.g. [5], [38]) and the strong Markov property of n, we obtain

$$\begin{aligned}
\mathbb{E}\left(\sum_{s\in G} e^{-qs} \int_s^{d_s} q e^{-q(u-s)} \mathbf{1}_\Lambda \circ k_{u-s} \circ \theta_s \mathbf{1}_{\{t<u-s\}} du\right) &= \mathbb{E}\left(\int_0^\infty e^{-qs} dL_s\right) n(\mathbf{1}_\Lambda \mathbf{1}_{\{t<\mathbf{e}_q<\zeta\}}) \\
&= \mathbb{E}\left(\int_0^\infty e^{-qs} dL_s\right) n(\mathbf{1}_\Lambda \mathbb{P}_{X_t}(T_0 > \mathbf{e}_q) \mathbf{1}_{\{t<\zeta\}}).
\end{aligned}$$

Using (2.7) and (2.35) we deduce

$$\mathbb{E}\left(\int_0^\infty e^{-qs} dL_s\right) = u_q(0) = \frac{1}{n(\zeta > \mathbf{e}_q)}.$$

Thus, we see that

$$\mathbb{P}^{\mathbf{e}_q}(\Lambda, t < \zeta) = n(\mathbf{1}_\Lambda h_q(X_t) \mathbf{1}_{\{t<\zeta\}}). \tag{2.48}$$

Now, we prove that $n(S(X_t), t < \zeta) < \infty$, for all $t > 0$, where $S(x) = h(x) + h(-x)$. First, note that since S is excessive for the semigroup $(P_t^0, t \geq 0)$ and n fulfils the Markov property, then $t \mapsto n(S(X_t), t < \zeta)$ is decreasing. This is verified from the following equalities: for $u, t > 0$,

$$\begin{aligned}
n(S(X_{t+u}), t + u < \zeta) &= n((S(X_u)\mathbf{1}_{\{u<\zeta\}}) \circ \theta_t, t < \zeta) \\
&= n(\mathbb{E}_{X_t}(S(X_u), u < \zeta), t < \zeta) \\
&= n(P_u^0 S(X_t), t < \zeta) \\
&\leq n(S(X_t), t < \zeta).
\end{aligned}$$

On the other hand, by (2.6) and remark 2.21 (i), we have

$$\int_0^\infty e^{-t} n(S(X_t), t < \zeta) dt = \int_{\mathbb{R}} S(x) \frac{u_1(x)}{u_1(0)} dx = \frac{1}{u_1(0)} U_1 S(0) \leq \frac{1}{u_1(0)} S(0).$$

Therefore, $n(S(X_t), t < \zeta)$ is finite for every $t > 0$.

Finally, since $\mathbf{1}_\Lambda h_q(X_t) \le S(X_t)$ and $n(S(X_t), t < \zeta) < \infty$, we can apply the dominated convergence theorem in (2.48) to conclude that for $t > 0$ fixed

$$\lim_{q\to 0} \mathbb{P}^{\mathbf{e}_q}(\Lambda, t < \zeta) = n(\mathbf{1}_\Lambda h(X_t)\mathbf{1}_{\{t<\zeta\}}).$$

□

Let $U_q^\updownarrow$ be the q-resolvent for the process $X^\updownarrow = (X, \mathbb{P}_x^\updownarrow)_{x\in\mathcal{H}_0}$, with $U^\updownarrow = U_0^\updownarrow$. To prove that $X^\updownarrow$ is transient, we compute the density of $U^\updownarrow$. For $x, y \in \mathcal{H}$ and $q > 0$, we have

$$u_q^\updownarrow(x,y) = \frac{h(y)}{h(x)} u_q^0(x,y). \tag{2.49}$$

From (2.6) it can be deduced that for $y \in \mathcal{H}$, $q > 0$,

$$\begin{aligned} u_q^\updownarrow(0,y)dy &= \int_0^\infty e^{-qt} n(h(X_t)\mathbf{1}_{\{X_t\in dy\}}, t < \zeta)dt \\ &= h(y)\widehat{\mathbb{E}}_y[e^{-qT_0}]dy \\ &= h(y)\frac{u_q(y)}{u_q(0)}dy. \end{aligned} \tag{2.50}$$

Finally, by Theorem 2.5 (ii), $u_q^\updownarrow(x,0) = 0$, for all $x \in \mathcal{H}_0$. Thus, from the above equations, the density of $U^\updownarrow$ can be obtained . This is stated in the following lemma.

Lemma 2.23. *Let $u_0^\updownarrow(x,y) = \lim_{q\to 0} u_q^\updownarrow(x,y)$, $x, y \in \mathcal{H}_0$. Then $u_0^\updownarrow(x,0) = 0$, for all $x \in \mathcal{H}$,*

$$0 \le u_0^\updownarrow(x,y) = \frac{h(y)}{h(x)}[h(x) + h(-y) - h(x-y) - \kappa h(x)h(-y)], \quad x \in \mathcal{H}, y \in \mathcal{H}, \tag{2.51}$$

and for $y \in \mathcal{H}$,

$$u_0^\updownarrow(0,y) = h(y)(1 - \kappa h(-y)) = h^*(y) - h(-y). \tag{2.52}$$

Proof. An easy computation gives

$$\frac{u_q(-x)u_q(y)}{u_q(0)} = \frac{h_q(x)h_q(-y)}{u_q(0)} - h_q(x) - h_q(-y) + u_q(0), \quad x \in \mathcal{H}, y \in \mathcal{H}.$$

Using this and (2.3) it follows

$$u_q^0(x,y) = h_q(x) + h_q(-y) - h_q(x-y) - \frac{h_q(x)h_q(-y)}{u_q(0)}, \quad x \in \mathcal{H}, y \in \mathcal{H}. \tag{2.53}$$

Letting $q \to 0$ in (2.49) and with help of (2.53), we obtain (2.51). The first equality in (2.52) is obtained from (2.50) recalling that for all y, $\lim_{q\to 0}[u_q(y)/u_q(0)] = \lim_{q\to 0}[1-(u_q(0))^{-1}h_q(-y)] = 1 - \kappa h(-y)$. The second one follows from (2.39). □

Remark 2.24. Note that from (2.37) and (2.53) we have $u_q^\updownarrow(x,x) = u_q^0(x,x) = h_q^*(x)$, $x \in \mathcal{H}$, which implies $u_0^\updownarrow(x,x) = h^*(x)$, $x \in \mathcal{H}$.

Proof of Proposition 2.8. To obtain the transiency property of $X^{\updownarrow}$, we use Theorem 3.7.2 in [21], which states the following. If the conditions:

(i) $U^{\updownarrow}g$ is lower semi-continuous, for any non negative function g with compact support;

(ii) there exists a non negative function f such that $0 < U^{\updownarrow}f < \infty$ on $\mathbb{R}$;

are satisfied, then the process $X^{\updownarrow}$ is transient.

Since h is continuous, from Lemma 2.23 it follows $\lim_{x\to x'} u_0^{\updownarrow}(x,y) = u_0^{\updownarrow}(x',y)$, for all $y \in \mathcal{H}_0$. Let g be a non negative function with compact support K. By Fatou's lemma, we have

$$\liminf_{x\to x'} \int_K g(y)u_0^{\updownarrow}(x,y)dy \geq \int_{\mathbb{R}} g(y)\liminf_{x\to x'}[u_0^{\updownarrow}(x,y)\mathbf{1}_K]dy = \int_K g(y)u_0^{\updownarrow}(x',y)dy.$$

This shows that for any g non negative with compact support, the function

$$x \longmapsto \int_{\mathbb{R}} g(y)u_0^{\updownarrow}(x,y)dy$$

is lower semi-continuous. Thus, condition (i) is satisfied.

Now, we will find a non negative function $f : \mathbb{R} \to \mathbb{R}^+$ such that $0 < U^{\updownarrow}f(x) < \infty$. Let f be given by

$$f(y) = \begin{cases} \dfrac{1}{[h^*(1)]^2}, & |y| \leq 1, \\ \\ \dfrac{1}{y^2[h^*(y)]^2}, & |y| > 1. \end{cases}$$

Since f is continuous and $\lim_{|x|\to\infty} h^*(x) = \kappa^{-1}$, then f, fh^* and $f(h^*)^2$ are integrable with respect to Lebesgue measure. On the other hand, h is dominated by the symmetric function h^*, then the integrability of fh^* and $f(h^*)^2$ imply

$$\int_{\mathbb{R}} f(y)h(y)dy < \infty, \quad \int_{\mathbb{R}} f(y)h(y)h(-y)dy < \infty.$$

Furthermore, since h is subadditive and f is symmetric, it follows,

$$\int_{\mathbb{R}} f(y)h(x-y)dy \leq \int_{\mathbb{R}} f(y)h(x)dy + \int_{\mathbb{R}} f(y)h(y)dy < \infty.$$

Thus, for $x \in \mathcal{H}$,

$$U^{\updownarrow}f(x) = \int_{\mathbb{R}} f(y)u_0^{\updownarrow}(x,y)dy < \infty.$$

Finally,

$$U^{\updownarrow}f(0) = \int_{\mathbb{R}} f(y)u^{\updownarrow}(0,y)dy = \int_{\mathbb{R}} f(y)(h^*(y) - h(y))dy < \infty.$$

This concludes the proof. □

The Lemma 2.25 below states that any $x \in \mathcal{H}$ is regular for itself under $\mathbb{P}_x^{\updownarrow}$. The latter implies the existence of a continuous local time at point $x \in \mathcal{H}$ for the process $(X, \mathbb{P}_x^{\updownarrow})$, see [8, Theorem 3,12, p. 216]. We will denote by $(L^{\updownarrow}(x,t), t \geq 0)$ the local time at point x aforementioned and by $\tau^{\updownarrow}(x,t)$ the right continuous inverse of $L^{\updownarrow}(x,t)$, i.e.,

$$\tau^{\updownarrow}(x,t) = \inf\{s > 0 : L^{\updownarrow}(x,s) > t\}, \quad t \geq 0.$$

It is well known that $(\tau^{\updownarrow}(x,t), t \geq 0)$ is a subordinator killed at an exponential random time independent of $\tau^{\updownarrow}(x,\cdot)$ with Laplace exponent $\Phi^{x,\updownarrow}$ satisfying

$$\mathbb{E}_x(e^{-q\tau^{\updownarrow}(x,t)}) = e^{-t\Phi^{x,\updownarrow}(q)} = e^{-t/u_q^{\updownarrow}(x,x)}, \quad t \geq 0, \tag{2.54}$$

see e.g. [8, Theorem 3.17, p. 218]. Furthermore, using the compensation formula in excursion theory we can be establish that for any $q > 0$,

$$\begin{aligned} \Phi^{x,\updownarrow}(q) = \frac{1}{u_q^{\updownarrow}(x,x)} &= n_x^{\updownarrow}(\zeta > \mathbf{e}_q) + a^x q \\ &= n_x^{\updownarrow}(\zeta = \infty) + a^x q + \int_0^\infty (1 - e^{-qt}) n_x^{\updownarrow}(\zeta \in dt), \end{aligned} \tag{2.55}$$

where a^x satisfies

$$\int_0^t \mathbf{1}_{\{X_s = x\}} ds = a^x L^{\updownarrow}(x,t). \tag{2.56}$$

By Remark 2.24, $\lim_{q\to 0} u_q^{\updownarrow}(x,x) = h^*(x) > 0$, for $x \in \mathcal{H}$, then $(\tau^{\updownarrow}(x,t), t \geq 0)$ is a subordinator killed at an exponential time with parameter $1/h^*(x) > 0$. This also confirms the transiency of $(X, \mathbb{P}_x^{\updownarrow})$, since by (2.55), there exists an excursion of infinite length.

To state the following lemma, we introduce additional notation. For every $x \in \mathbb{R}$, define $d_s^x = \inf\{u > s : X_t = x\}$, $g_s^x = \sup\{u \leq s : X_t = x\}$ and $G^x = \{g_u^x : g_u^x \neq d_u^x, u > 0\}$.

Lemma 2.25. *(i) For $x \in \mathcal{H}$, x is regular for itself for $(X, \mathbb{P}_x^{\updownarrow})$.*

(ii) Let $\mathbf{e}_q$ be an exponential random variable with parameter $q > 0$, independent of $(X, (\mathbb{P}_x^{\updownarrow})_{x\neq 0})$. Then, for every $x \in \mathcal{H}$, the processes $(X_u, u < g_{\mathbf{e}_q}^x)$ and $X \circ k_{\mathbf{e}_q - g_{\mathbf{e}_q}^x} \circ \theta_{g_{\mathbf{e}_q}^x}$ are $\mathbb{P}_x^{\updownarrow}$ independent. Furthermore, their laws are characterized as follows: let F and H be measurable and bounded functionals, then

$$\mathbb{E}_x^{\updownarrow}\left(F(X_u, u < g_{\mathbf{e}_q}^x)\right) = \mathbb{E}_x^{\updownarrow}\left(\int_0^\infty F(X_u, u < s) e^{-qs} dL^{\updownarrow}(x,s)\right)\left[n_x^{\updownarrow}(\zeta > \mathbf{e}_q) + a^x q\right] \tag{2.57}$$

and

$$\mathbb{E}_x^{\updownarrow}\left(H(X \circ k_{\mathbf{e}_q - g_{\mathbf{e}_q}^x} \circ \theta_{g_{\mathbf{e}_q}^x})\right) = u_q^{\updownarrow}(x,x)\left[n_x^{\updownarrow}\left(\int_0^\zeta H(\epsilon_u, u < t) q e^{-qt} dt\right) + a^x q H(\bar{x})\right], \tag{2.58}$$

where a^x is the constant in (2.56).

Proof. Let $x \in \mathcal{H}$. By Fatou's lemma and the definition of $\mathbb{P}_x^\updownarrow$, we have

$$\begin{aligned}
\mathbb{P}_x^\updownarrow(T_x = 0) &= \liminf_{t\to 0} \mathbb{P}_x^\updownarrow(T_x \le t) \\
&\ge \frac{1}{h(x)} \mathbb{E}_x \left(\liminf_{t\to 0} \mathbf{1}_{\{T_x \le t < T_0\}} h(X_t) \right) \\
&= \frac{1}{h(x)} \mathbb{E}_x \left(\mathbf{1}_{\{T_x = 0\}} \mathbf{1}_{\{T_0 > 0\}} h(X_0) \right) \\
&= 1,
\end{aligned}$$

where the latter equality was obtained using the facts that $\{x\}$ is regular for itself under $\mathbb{P}_x$ and $\mathbb{P}_x(T_0 > 0) = 1$. This proves (i).

Before to prove (ii), we recall the following. Since $\tau^\updownarrow(x, \cdot)$ is the inverse of the local time $(L^\updownarrow(x,t), t \ge 0)$ with Laplace exponent given by (2.54), then

$$\mathbb{E}_x^\updownarrow \left(\int_0^\infty e^{-qt} dL^\updownarrow(x,t) \right) = \mathbb{E}_x^\updownarrow \left(\int_0^\infty e^{-q\tau^\updownarrow(x,t)} dt \right) = \int_0^\infty \mathbb{E}_x^\updownarrow(e^{-q\tau^\updownarrow(x,t)}) dt = u_q^\updownarrow(x,x). \tag{2.59}$$

We will denote $\bar{x}$ the path which is identically equal to x and with lifetime zero. Thus, for F and H measurable and bounded functionals, using the compensation formula in excursion theory (see e.g. [5], [38]), it follows

$$\begin{aligned}
&\mathbb{E}_x^\updownarrow \left(F(X_u, u < g_{\mathbf{e}_q}^x) H(X \circ k_{\mathbf{e}_q - g_{\mathbf{e}_q}^x} \circ \theta_{g_{\mathbf{e}_q}^x}) \right) \\
&= \mathbb{E}_x^\updownarrow \left(\sum_{s \in G^x} F(X_u, u < s) e^{-qs} \int_s^{d_s} H(X \circ k_{t-s} \circ \theta_s) q e^{-q(t-s)} dt \right) \\
&\quad + \mathbb{E}_x^\updownarrow \left(\int_0^\infty F(X_u, u < t) H(\bar{x}) q e^{-qt} \mathbf{1}_{\{X_t = x\}} dt \right) \\
&= \mathbb{E}_x^\updownarrow \left(\int_0^\infty F(X_u, u < s) e^{-qs} dL^\updownarrow(x,s) \right) \left[n_x^\updownarrow \left(\int_0^\zeta H(\epsilon_u, u < t) q e^{-qt} dt \right) + a^x q H(\bar{x}) \right],
\end{aligned} \tag{2.60}$$

where a^x is the constant in (2.56). Taking $H \equiv 1$ in (2.60), it follows

$$\mathbb{E}_x^\updownarrow \left(F(X_u, u < g_{\mathbf{e}_q}^x) \right) = \mathbb{E}_x^\updownarrow \left(\int_0^\infty F(X_u, u < s) e^{-qs} dL^\updownarrow(x,s) \right) \left[n_x^\updownarrow(\zeta > \mathbf{e}_q) + a^x q \right].$$

In the same way, if we take $F \equiv 1$ in (2.60) and we use (2.59), we can obtain

$$\mathbb{E}_x^\updownarrow \left(H(X \circ k_{\mathbf{e}_q - g_{\mathbf{e}_q}^x} \circ \theta_{g_{\mathbf{e}_q}^x}) \right) = u_q^\updownarrow(x,x) \left[n_x^\updownarrow \left(\int_0^\zeta H(\epsilon_u, u < t) q e^{-qt} dt \right) + a^x q H(\bar{x}) \right].$$

The latter two displays are (2.57) and (2.58), respectively.

Finally, by (2.55), $u_q^\updownarrow(x,x) = [n_x^\updownarrow(\zeta > \mathbf{e}_q) + a^x q]^{-1}$. Using this fact, (2.57) and (2.58), we conclude

$$\mathbb{E}_x^\updownarrow \left(F(X_u, u < g_{\mathbf{e}_q}^x) H(X \circ k_{\mathbf{e}_q - g_{\mathbf{e}_q}^x} \circ \theta_{g_{\mathbf{e}_q}^x}) \right) = \mathbb{E}_x^\updownarrow \left(F(X_u, u < g_{\mathbf{e}_q}^x) \right) \mathbb{E}_x^\updownarrow \left(H(X \circ k_{\mathbf{e}_q - g_{\mathbf{e}_q}^x} \circ \theta_{g_{\mathbf{e}_q}^x}) \right).$$

This shows the independence property in (ii). □

Now, we will prove that the drift coefficient in (2.55) does not depend on x, and is equal to δ.

Lemma 2.26. *Let δ be the drift coefficient of the inverse local time at the point zero for the Lévy process $(X,\mathbb{P})$. Then for all $x\in\mathcal{H}$, $\mathbb{P}^{\updownarrow}_x$-a.s., $\int_0^t \mathbf{1}_{\{X_s=0\}}ds=\delta L^{\updownarrow}(x,t)$. That is, $a^x=\delta$, for all $x\in\mathcal{H}$.*

Proof. If both δ, a^x are zero, the claim holds. Suppose that $a^x\neq 0$. Using (2.59), the definition of a^x and $\mathbb{P}^{\updownarrow}_x$, we obtain

$$\begin{aligned} a^x u^{\updownarrow}_q(x,x) &= \mathbb{E}^{\updownarrow}_x\left(\int_0^\infty e^{-qt}d[a^xL^{\updownarrow}(x,t)]\right) \\ &= \int_0^\infty \mathbb{E}^{\updownarrow}_x\left(\mathbf{1}_{\{X_t=x\}}\right)e^{-qt}dt \\ &= \int_0^\infty \frac{1}{h(x)}\mathbb{E}_x\left(\mathbf{1}_{\{X_t=x\}}h(X_t)\mathbf{1}_{\{T_0>t\}}\right)e^{-qt}dt \\ &= \mathbb{E}_x\left(\int_0^\infty \mathbf{1}_{\{T_0>t\}}e^{-qt}\mathbf{1}_{\{X_t=x\}}dt\right). \end{aligned} \tag{2.61}$$

Using that $(X,\mathbb{P}_x)$ is equal in distribution to $(X+x,\mathbb{P})$, the definition of δ and the symmetry of $h^*_q(x)$, it follows that the right-hand side in (2.61) is

$$\mathbb{E}\left(\int_0^\infty \mathbf{1}_{\{T_{\{-x\}}>t\}}e^{-qt}\mathbf{1}_{\{X_t=0\}}dt\right)=\delta\mathbb{E}\left(\int_0^\infty \mathbf{1}_{\{T_{\{-x\}}>t\}}e^{-qt}dL_t\right)=\delta h^*_q(x).$$

To conclude the proof recall that $h^*_q(x)=u^{\updownarrow}_q(x,x)$. □

Proof of Proposition 2.9. Let $H:\mathcal{D}^0\to\mathbb{R}$ a bounded and measurable functional. To simplify we write X^q for the path $X\circ k_{\mathbf{e}_q-g^x_{\mathbf{e}_q}}\circ\theta_{g^x_{\mathbf{e}_q}}$. Using the definition of $\mathbb{P}^{\updownarrow}_x$, we obtain

$$h(x)\mathbb{E}^{\updownarrow}_x\left(H(X^q)\right)=\mathbb{E}\left(\int_0^\infty H((X+x)\circ k_{t-g_t}\circ\theta_{g_t})h(X_t+x)\mathbf{1}_{\{T_{\{-x\}}>t\}}qe^{-qt}dt\right). \tag{2.62}$$

We note that $\mathbf{1}_{\{T_{\{-x\}}>t\}}=\mathbf{1}_{\{T_{\{-x\}}\circ\theta_{g_t}>t-g_t\}}\mathbf{1}_{\{T_{\{-x\}}>g_t\}}$ and $h(X_t+x)=h((X_{t-g_t}+x)\circ\theta_{g_t})$. Then, with the help of the compensation formula in excursion theory ([5], [38]), the right-hand side in (2.62) can be written as

$$
\begin{aligned}
&\mathbb{E}\left(\int_0^\infty \mathbf{1}_{\{T_{\{-x\}}>g_t\}} H((X+x)\circ k_{t-g_t}\circ\theta_{g_t}) h((X_{t-g_t}+x)\circ\theta_{g_t}) \mathbf{1}_{\{T_{\{-x\}}\circ\theta_{g_t}>t-g_t\}} q e^{-qt} dt\right) \\
&\quad = \mathbb{E}\left(\sum_{s\in G} \mathbf{1}_{\{T_{\{-x\}}>s\}} \int_s^{d_s} H((X+x)\circ k_{t-s}\circ\theta_s) h((X_{t-s}+x)\circ\theta_s) \mathbf{1}_{\{T_{\{-x\}}\circ\theta_s>t-s\}} q e^{-qt} dt\right) \\
&\qquad + \mathbb{E}\left(\int_0^\infty \mathbf{1}_{\{T_{\{-x\}}>t\}} H(\bar{x}) h(x) q e^{-qt} \mathbf{1}_{\{X_t=0\}} dt\right) \\
&\quad = \mathbb{E}\left(\int_0^\infty \mathbf{1}_{\{T_{\{-x\}}>t\}} e^{-qt} dL_t\right) n\left(\int_0^\zeta H(\epsilon_u + x, u<t) h(X_t+x) \mathbf{1}_{\{T_{\{-x\}}>t\}} q e^{-qt} dt\right) \\
&\qquad + q\delta H(\bar{x}) h(x) \mathbb{E}\left(\int_0^\infty \mathbf{1}_{\{T_{\{-x\}}>t\}} e^{-qt} dL_t\right) \\
&\quad = h_q^*(x)\left[n\left(\int_0^\zeta H(\epsilon_u + x, u<t) h(X_t+x) \mathbf{1}_{\{T_{\{-x\}}>t\}} q e^{-qt} dt\right) + q\delta H(\bar{x}) h(x)\right], \qquad (2.63)
\end{aligned}
$$

where δ is such that $\int_0^t \mathbf{1}_{\{X_s=0\}} = \delta L_t$ under $\mathbb{P}$. Using Lemma 2.26 and $h^*(x) = u_q^{\updownarrow}(x,x)$ in (2.63), we verify

$$
\mathbb{E}_x^{\updownarrow}(H(X^q)) = u_q^{\updownarrow}(x,x)\left[\frac{1}{h(x)} n\left(\int_0^\zeta H(\epsilon_u + x, u<t) h(X_t+x) \mathbf{1}_{\{T_{\{-x\}}>t\}} q e^{-qt} dt\right) + a^x q H(\bar{x})\right]. \qquad (2.64)
$$

Comparing (2.64) with (2.58), the result follows. □

2.5 Two examples

The expression (2.13) in Theorem 2.2 (i) allows us to compute explicitly the function h in the particular case when $(X,\mathbb{P})$ is an α-stable process.

Example 2.27. Suppose that $(X,\mathbb{P})$ is an α-stable process. Then, $(X,\mathbb{P})$ satisfies **H.1** and **H.2** if and only if $\alpha \in (1,2]$. It is well known that the resolvent density of Brownian motion is $u_q(x) = (\sqrt{2q})^{-1} e^{-\sqrt{2q}|x|}$, hence $h(x) = \lim_{q\to 0}[u_q(0) - u_q(-x)] = |x|$. Now, let $\alpha \in (1,2)$. In this case the function h takes the following form

$$
h(x) = K(\alpha)(1 - \beta\,\mathrm{sgn}(x))|x|^{\alpha-1}, \qquad (2.65)
$$

where

$$
K(\alpha) = \frac{\Gamma(2-\alpha)\sin(\alpha\pi/2)}{c\pi(\alpha-1)(1+\beta^2\tan^2(\alpha\pi/2))}
$$

and

$$
c = -\frac{(c^+ + c^-)\Gamma(2-\alpha)}{\alpha(\alpha-1)}\cos(\alpha\pi/2), \quad \beta = \frac{c^+ - c^-}{c^+ + c^-}. \qquad (2.66)
$$

Indeed, recall that the characteristic exponent of $(X,\mathbb{P})$ can be written as

$$
\psi(\lambda) = c|\lambda|^\alpha (1 - i\beta\,\mathrm{sgn}(\lambda)\tan(\alpha\pi/2)),
$$

where c and β are as in (2.66). Now, we have

$$Re\left(\frac{1-e^{i\lambda x}}{\psi(\lambda)}\right)=\frac{1-\cos(\lambda x)+\beta\tan(\alpha\pi/2)\mathrm{sgn}(\lambda)\sin(\lambda x)}{c|\lambda|^{\alpha}(1+\beta^2\tan^2(\alpha\pi/2))}.$$

Since the function $\mathrm{sgn}(\lambda)\sin(\lambda x)$ as function of λ is even, we have

$$h(x)=\frac{1}{c(1+\beta^2\tan^2(\alpha\pi/2))}\left(h^s(x)+\beta\tan(\alpha\pi/2)\frac{1}{\pi}\int_0^\infty\frac{\sin(\lambda x)}{\lambda^\alpha}d\lambda\right),$$

where $h^s(x)$ is the function h obtained in the symmetric case (see example 1.1 in [48]), namely

$$h^s(x)=\frac{\Gamma(2-\alpha)}{\pi(\alpha-1)}\sin(\alpha\pi/2)|x|^{\alpha-1}.$$

On the other hand, with the help of formulae (14.18) of Lemma 14.11 in [45], we obtain

$$\int_0^\infty\frac{\sin(\lambda x)}{\lambda^\alpha}d\lambda=\mathrm{sgn}(x)|x|^{\alpha-1}\int_0^\infty\frac{\sin u}{u^\alpha}du=-\frac{\Gamma(2-\alpha)}{\alpha-1}\cos(\alpha\pi/2)\mathrm{sgn}(x)|x|^{\alpha-1}.$$

The latter two equalities imply the claim.

Recall that L_{T_x} is an exponential random variable with parameter $[h^*(x)]^{-1}$, then by (2.39), in the case when $(X,\mathbb{P})$ is an α-stable process with $\alpha\in(1,2]$, L_{T_x} is an exponential random variable with parameter $1/[2K(\alpha)|x|^{\alpha-1}]$.

When $|\beta|=1$, the function h in (2.65) is equal zero for some values of x. This correspond to a spectrally negative (or positive) alpha-stable process.

The example of a spectrally negative Lévy process will be studied in the following example.

Example 2.28. Suppose that $(X,\mathbb{P})$ is a spectrally negative Lévy process. Let Ψ be the Laplace exponent of the process $(X,\mathbb{P})$, i.e.,

$$\Psi(\lambda)=-\psi(-i\lambda)=-a\lambda+\frac{\sigma^2}{2}\lambda+\int_{(-\infty,0)}(e^{\lambda x}-1-\lambda x\mathbf{1}_{\{|x|<1\}})\pi(dx)$$

It is well known that $\Psi'(0+)=\mathbb{E}(X_1)\in[-\infty,\infty)$ determines the long run behaviour of X. To be precise, if $\Psi'(0+)>0$ then $\lim_{t\to\infty}X_t=\infty$, if $\Psi'(0+)<0$ then $\lim_{t\to\infty}X_t=-\infty$, while $\Psi'(0+)=0$ the process X oscillates.

Now, for $q\geq 0$, let $\Phi(q)$ be the largest root of the equation $\Psi(\lambda)=q$, i.e.,

$$\Phi(q)=\sup\{\lambda\geq 0:\Psi(\lambda)=q\}.$$

For the spectrally negative Lévy processes $(X,\mathbb{P})$ with Laplace exponent Ψ, we consider the q-scale functions $\{W^{(q)},q\geq 0\}$, the family of functions satisfying the following: for every $q\geq 0$, $W^{(q)}=0$, for $x<0$ and $W^{(q)}\geq 0$, for $x\geq 0$. Furthermore, $W^{(q)}$ is determined by its Laplace transform in the following way

$$\int_0^\infty e^{-\theta x}W^{(q)}(x)dx=\frac{1}{\Psi(\theta)-q},\quad \theta>\Phi(q).$$

For notational convenience, we set $W = W^{(0)}$. For a complete account on q-scale functions for spectrally negative Lévy processes see [23].

An important fact on spectrally Lévy processes is that the resolvent density, u_q, can be written in terms of the q-scale function $W^{(q)}$ as follows

$$u_q(x) = \Phi'(q)e^{-\Phi(q)x} - W^{(q)}(-x), \quad q > 0, \quad x \in \mathbb{R}.$$

Furthermore, if $(X, \mathbb{P})$ has unbounded variation, $W^{(q)}(0) = 0$, see Corollary 8.9 and Lemma 8.6 in [35] for details. The latter facts imply that

$$h_q(x) = u_q(0) - u_q(-x) = \Phi'(q)(1 - e^{\Phi(q)x}) + W^{(q)}(x), \quad q > 0, x \in \mathbb{R}.$$

Thus, letting $q \to 0$, we obtain

$$h(x) = \begin{cases} \dfrac{1}{\Psi'(\Phi(0)+)}(1 - e^{\Phi(0)x}) + W(x), & \text{if } \lim\limits_{t\to\infty} X_t = -\infty, \\ \dfrac{-x}{\Psi''(0+)} + W(x), & \text{if } \limsup\limits_{t\to\infty} X_t = -\liminf\limits_{t\to\infty} X_t = \infty, \\ W(x), & \text{if } \lim\limits_{t\to\infty} X_t = \infty. \end{cases}$$

Bibliography

[1] M. T. Barlow. Zero-one laws for the excursions and range of a Lévy process. *Z. Wahrsch. Verw. Gebiete*, 55(2):149–163, 1981.

[2] M. T. Barlow. Continuity of local times for Lévy processes. *Z. Wahrsch. Verw. Gebiete*, 69(1):23–35, 1985.

[3] M. T. Barlow. Necessary and sufficient conditions for the continuity of local time of Lévy processes. *Ann. Probab.*, 16(4):1389–1427, 1988.

[4] J. Bertoin. Splitting at the infimum and excursions in half-lines for random walks and Lévy processes. *Stochastic Process. Appl.*, 47(1):17–35, 1993.

[5] J. Bertoin. *Lévy processes*, volume 121 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1996.

[6] J. Bertoin and M. Yor. Exponential functionals of Lévy processes. *Probab. Surv.*, 2:191–212 (electronic), 2005.

[7] R. M. Blumenthal. *Excursions of Markov processes*. Probability and its Applications. Birkhäuser Boston Inc., Boston, MA, 1992.

[8] R. M. Blumenthal and R. K. Getoor. *Markov processes and potential theory*. Pure and Applied Mathematics, Vol. 29. Academic Press, New York, 1968.

[9] J. Bretagnolle. Résultats de Kesten sur les processus à accroissements indépendants. In *Séminaire de Probabilités, V (Univ. Strasbourg, année universitaire 1969-1970)*, pages 21–36. Lecture Notes in Math., Vol. 191. Springer, Berlin, 1971.

[10] M. E. Caballero and L. Chaumont. Conditioned stable Lévy processes and the Lamperti representation. *J. Appl. Probab.*, 43(4):967–983, 2006.

[11] M. E. Caballero and L. Chaumont. Weak convergence of positive self-similar Markov processes and overshoots of Lévy processes. *Ann. Probab.*, 34(3):1012–1034, 2006.

[12] M. E. Caballero, J. C. Pardo, and J. L. Pérez. On Lamperti stable processes. *Probability and Mathematical Statistics*, 30(1):1–28, 2010.

[13] M. E. Caballero, J. C. Pardo, and J. L. Pérez. Explicit identities for Lévy processes associated to symmetric stable processes. *Bernoulli*, 17(1):34–59, 2011.

[14] P. Carmona, F. Petit, and M. Yor. Exponential functionals of Lévy processes. In *Lévy processes*, pages 41–55. Birkhäuser Boston, Boston, MA, 2001.

[15] L. Chaumont. Sur certains processus de Lévy conditionnés à rester positifs. *Stochastics Stochastics Rep.*, 47(1-2):1–20, 1994.

[16] L. Chaumont. Conditionings and path decompositions for Lévy processes. *Stochastic Process. Appl.*, 64(1):39–54, 1996.

[17] L. Chaumont and R. A. Doney. On Lévy processes conditioned to stay positive. *Electron. J. Probab.*, 10:no. 28, 948–961 (electronic), 2005.

[18] L. Chaumont and R. A. Doney. Corrections to: "On Lévy processes conditioned to stay positive" [Electron J. Probab. **10** (2005), no. 28, 948–961]. *Electron. J. Probab.*, 13:no. 1, 1–4, 2008.

[19] Z.-Q. Chen, M. Fukushima, and J. Ying. Extending Markov processes in weak duality by Poisson point processes of excursions. In *Stochastic analysis and applications*, volume 2 of *Abel Symp.*, pages 153–196. Springer, Berlin, 2007.

[20] K. L. Chung. *A course in probability theory.* Academic Press Inc., San Diego, CA, third edition, 2001.

[21] K. L. Chung and J. B. Walsh. *Markov processes, Brownian motion, and time symmetry*, volume 249 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences].* Springer, New York, second edition, 2005.

[22] O. Chybiryakov. The Lamperti correspondence extended to Lévy processes and semi-stable Markov processes in locally compact groups. *Stochastic Process. Appl.*, 116(5):857–872, 2006.

[23] S. Cohen, A. Kuznetsov, A. E. Kyprianou, and V. Rivero. *Lévy matters II*, volume 2061 of *Lecture Notes in Mathematics.* Springer, Heidelberg, 2012. Recent progress in theory and applications: fractional Lévy fields, and scale functions, With a short biography of Paul Lévy by Jean Jacod, Edited by Ole E. Barndorff-Nielsen, Jean Bertoin, Jacod and Claudia Küppelberg.

[24] N. Eisenbaum and H. Kaspi. On the continuity of local times of Borel right Markov processes. *Ann. Probab.*, 35(3):915–934, 2007.

[25] W. Feller. *An introduction to probability theory and its applications. Vol. I.* Third edition. John Wiley & Sons Inc., New York, 1968.

[26] W. Feller. *An introduction to probability theory and its applications. Vol. II.* Second edition. John Wiley & Sons Inc., New York, 1971.

[27] P. J. Fitzsimmons and R. K. Getoor. Excursion theory revisited. *Illinois J. Math.*, 50(1-4):413–437 (electronic), 2006.

[28] R. K. Getoor. Excursions of a Markov process. *Ann. Probab.*, 7(2):244–266, 1979.

[29] E. Hille and R. S. Phillips. *Functional analysis and semi-groups*. American Mathematical Society Colloquium Publications, vol. 31. American Mathematical Society, Providence, R. I., 1957. rev. ed.

[30] E. Hille and R. S. Phillips. *Functional analysis and semi-groups*. American Mathematical Society, Providence, R. I., 1974. Third printing of the revised edition of 1957, American Mathematical Society Colloquium Publications, Vol. XXXI.

[31] M. Kanda. Two theorems on capacity for Markov processes with stationary independent increments. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 35(2):159–165, 1976.

[32] H. Kesten. *Hitting probabilities of single points for processes with stationary independent increments*. Memoirs of the American Mathematical Society, No. 93. American Mathematical Society, Providence, R.I., 1969.

[33] S. W. Kiu. Two dimensional semi-stable Markov processes. *Ann. Probability*, 3(3):440–448, 1975.

[34] S. W. Kiu. Semistable Markov processes in $\mathbf{R}^n$. *Stochastic Process. Appl.*, 10(2):183–191, 1980.

[35] A. E. Kyprianou. *Introductory lectures on fluctuations of Lévy processes with applications*. Universitext. Springer-Verlag, Berlin, 2006.

[36] J. Lamperti. Semi-stable stochastic processes. *Trans. Amer. Math. Soc.*, 104:62–78, 1962.

[37] J. Lamperti. Semi-stable Markov processes. I. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 22:205–225, 1972.

[38] B. Maisonneuve. Exit systems. *Ann. Probability*, 3(3):399–411, 1975.

[39] P. W. Millar. Germ sigma fields and the natural state space of a Markov process. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 39(2):85–101, 1977.

[40] P. W. Millar. Zero-one laws and the minimum of a Markov process. *Trans. Amer. Math. Soc.*, 226:365–391, 1977.

[41] P. W. Millar. A path decomposition for Markov processes. *Ann. Probability*, 6(2):345–348, 1978.

[42] S. C. Port and C. J. Stone. Infinitely divisible processes and their potential theory. *Ann. Inst. Fourier (Grenoble)*, 21(2):157–275; ibid. 21 (1971), no. 4, 179–265, 1971.

[43] D. Revuz and M. Yor. *Continuous martingales and Brownian motion*, volume 293 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, third edition, 1999.

[44] W. Rudin. *Real and complex analysis*. McGraw-Hill Book Co., New York, third edition, 1987.

[45] K.-i. Sato. *Lévy processes and infinitely divisible distributions*, volume 68 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1999. Translated from the 1990 Japanese original, Revised by the author.

[46] W. L. Smith. Extensions of a renewal theorem. *Mathematical Proceedings of the Cambridge Philosophical Society*, 51:629–638, 9 1955.

[47] D. Widder. *The Laplace transform*. Princeton mathematical series. Princeton university press, 1946.

[48] K. Yano. Excursions away from a regular point for one-dimensional symmetric Lévy processes without Gaussian part. *Potential Anal.*, 32(4):305–341, 2010.

[49] K. Yano. On harmonic function for the killed process upon hitting zero of asymmetric Lévy processes. *J. Math-for-Ind.*, 5A:17–24, 2013.

[50] K. Yano, Y. Yano, and M. Yor. Penalising symmetric stable Lévy paths. *J. Math. Soc. Japan*, 61(3):757–798, 2009.

[51] M. Yor. On some exponential functionals of Brownian motion. *Adv. in Appl. Probab.*, 24(3):509–531, 1992.

[52] V. M. Zolotarev. *One-dimensional stable distributions*, volume 65 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1986. Translated from the Russian by H. H. McFaden, Translation edited by Ben Silver.